A Weak Formulation for Solving Elliptic Interface Problems Without Body Fitted Grid

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Abstract

A typical elliptic interface problem is casted as piecewise defined elliptic partial differential equations (PDE) in different regions which are coupled together with interface conditions, such as jumps in solution and flux across the interface. In many situations, such as the interface is moving, the challenge is how to solve such a problem accurately, robustly and efficiently without generating a body fitted mesh. The key issue is how to capture complex geometry of the interface and jump conditions across the interface effectively on a fixed mesh while the interface is not aligned with the mesh and the PDE is not valid across the interface. In this work we present a systematic formulation and further study of a second order accurate numerical method proposed in [16] for elliptic interface problem. The key idea is to decompose the solution into two parts, a singular part and a regular part. The singular part captures the interface conditions while the regular part belongs to an appropriate space in the whole domain, which can be solved by a standard finite element formulation. In a general setup the two parts are coupled together. We give an explicit study of the construction of the singular part and the discretized system for the regular part. One key advantage of using weak formulation is that one can avoid assuming/using more regularity than necessary of the solution and the interface. We present the numerical algorithm and numerical tests in 3D to demonstrate the accuracy and other properties of our method.

Keywords: elliptic equations, interface, non-body fitted mesh, finite element method, jump condition.

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1 Introduction

Interface problems occur in many multi-physics and multi-phase applications in science and engineering. In this work we study a typical elliptic interface problem which is casted as piecewisely defined elliptic partial differential equations (PDE) in different regions which are coupled together with interface conditions, such as jumps in solution and flux across the interface. A natural numerical approach is to generate a mesh that fit the interface, i.e., a body fitted mesh that does not allow the interface to cut across a cell. Then each piecewise elliptic PDE and the jump conditions across the interface can be naturally put into a standard finite element formulation, e.g., [5]. However, in many situations, such as when the interface is moving, generating a body fitted mesh following a highly dynamically moving interface may be both computational expensive and challenging, especially in 3D. A more desirable approach is to solve the interface problem effectively on a fixed mesh which does not fit the interface in general. The key challenge is how to capture complex geometry of the interface and jump conditions across the interface to achieve both accuracy and robustness on a non body fitted mesh. Quite a few numerical methods in this category has been proposed and studied. For examples, immersed boundary method, immersed interface method, ghost fluid method, extended finite element method.

In [31, 32], in order to simulate the flow patten of blood in the heart, Peskin proposed the "immersed boundary" method, which used numerical approximation of δ-function for singular sources on the interface. In [33], in order to compute two-phase flow, a level-set method was combined with the "immersed boundary" method. These methods are simple to use and but difficult to achieve high-order accuracy.

In [25, 26], the solution is extended to a rectangular region by using Fredholm integral equations. The proposed method can deal with interface conditions $[u] \neq 0$ and $[u_n] = 0$ and when Greens function is available. The discrete Laplacian was evaluated using these jump conditions and a fast Poisson solver can be used to compute the extended solution. It can achieve second or higher-order accuracy.

A large class of finite difference methods have been proposed. The main idea is to use difference scheme and stencils carefully near the interface to incorporate jump conditions and achieve high order local truncation error using Taylor expansion. Using finite difference scheme typically requires taking high order derivatives of jump conditions and interface in Taylor expansion. Also property of the discretized linear system is hard to analyze for interface problem with general jump condition. The "immersed interface" method was proposed in [18]. This method incorporates the interface conditions into the finite difference scheme near the interface to achieve second-order accuracy based on a Taylor expansion in a local coordinate system. Second order differentiation of the interface is needed.

Various applications and extensions of the "immersed interface" method are provided in [21]. In [3], instead of using Taylor expansions, a variational method is used to define numerical stencils near the interface and a Lagrange multiplier approach is used to enforce jump conditions. A boundary condition capturing method based on dimension splitting, the ghost fluid method, was proposed in [22, 9]. The method extends the solution from one side across the interface using the jump conditions. In [34], the boundary condition capturing method is improved with a multi-grid method. A weak formulation was used in [23] to prove the convergence. However, the method in [22] can only get first-order accuracy due to simple dimension by dimension extrapolation. It is in recent work [27] that for smooth interfaces the result was improved to second-order accuracy. Other dimension splitting type of methods based on finite difference include [41, 6, 35].

The existing finite element schemes on nonfitted meshes are usually designed by modifying the finite element basis near the interface. Examples are immersed finite element method [20, 10], adaptive immersed interface method [4], extended finite element method [28, 36, 7]. The penalty finite element method [1, 13] modifies the bilinear form near the interface by penalizing the jump of the solution value (with no general flux jump) across the interface. Recently a few unfitted mesh methods were developed based on the discontinuous Galerkin method using well known interior penalty technique to deal with jump and flux conditions for elliptic interface problem [24, 2, 12, 37].

The Matched Interface and Boundary (MIB) method was developed in [41] and improved to handle sharp-edged interfaces in two dimension in [39] and in three dimension in [40]. Also, there has been a large body of work from the finite volume perspective for developing high order methods for elliptic equations in complex domains, such as [8, 29] for two-dimensional problems and [30] for three-dimensional problems. Another recent work in this area is a class of kernel-free boundary integral (KFBI) methods for solving elliptic BVPs, presented in [38].

In this work we present a new formulation and further study of the second order accurate numerical method proposed in [16] for elliptic interface problem (1) with general matrix coefficient. The method can be viewed as a type of unfitted finite element method. The key idea in our formulation is to decompose the solution into two parts, a singular part and a regular part. The singular part is constructed explicitly in terms of the interface conditions and/or the regular part only locally near the interface. While the regular part lives in an appropriate space through the whole domain, which can be solved by a standard finite element formulation. In general, the coupling of singular and regular part leads to a non-symmetric discretized system. We give a explicit study of the construction of the singular part and the discretized system for the regular part in different setups. One key advantage of using weak formulation is that one can avoid assuming/using more regularity than necessary of the solution and the interface. Our method is also quite simple and easy to implement. The starting point is similar to that of the extended finite element method except that the interface conditions are enforced strongly for the local equations. Moreover, only standard finite element basis are introduced in our formulation. The method was also developed in [17] for sharp-edged interfaces, in [14] for elasticity interface problems and in [15] for multi-domain interface problems.

2 The equation and weak formulation

Consider an open bounded domain $\Omega \subset R^d$. Let Γ be an interface of co-dimension one, which divides Ω into disjoint open subdomains, Ω^- and Ω^+, hence $\Omega = \Omega^- \bigcup \Omega^+ \bigcup \Gamma$. Assume that the boundary $\partial\Omega$ and the boundary of each subdomain $\partial\Omega^\pm$ are Lipschitz continuous. Since $\partial\Omega^\pm$ are Lipschitz continuous, so is Γ. A unit normal vector of Γ can be defined a.e. on Γ, see Section 1.5 in [11].

We seek the solution to the following elliptic equation with piecewise smooth variable coefficient

$$-\bigtriangledown \cdot(\beta(x) \bigtriangledown u(x)) = f(x), \quad x \in \Omega \setminus \Gamma \tag{1}$$

in which $x = (x_1, ..., x_d)$ denotes the spatial variables and $\bigtriangledown$ is the gradient operator. The coefficient $\beta(x)$ is assumed to be a $d \times d$ matrix that is uniformly elliptic and its components are continuously differentiable on each disjoint sub-domain, Ω^- and Ω^+, but they may be discontinuous across the interface Γ. The right-hand side $f(x)$ is assumed to be in $L^2(\Omega)$.

Given functions $v(x)$ and $w(x)$ along the interface Γ, we prescribe the jump conditions in the solution and flux across the interface Γ:

$$\begin{cases} [u]_\Gamma(x) \equiv u^+(x) - u^-(x) = v(x) \\ \left[(\beta \bigtriangledown u) \cdot n\right]_\Gamma(x) \equiv n \cdot (\beta^+(x) \bigtriangledown u^+(x) - \beta^-(x) \bigtriangledown u^-(x)) = w(x) \end{cases} \tag{2}$$

where " $\pm$ " superscripts refer to limits taken from the subdomains $\Omega^\pm$. For specificity, we prescribe Dirichlet boundary condition at the domain boundary

$$u(x) = g(x), \quad x \in \partial\Omega \tag{3}$$

for a given function g on the boundary $\partial\Omega$. The setup of the problem is illustrated in Figure 1.

The key idea in our formulation is to decompose the solution $u(x)$ into two parts: $u(x) = u_r(x) + u_s(x)$, where $u_r(x)$ is the regular part that has no jumps in its value and first derivatives across the interface, and $u_s(x)$ is the singular part that captures those jump conditions of $u(x)$ across the interface. We define $u_s(x)$ to satisfy

$$u_s(x) = 0, \quad x \in \partial\Omega, \tag{4}$$

$$[u_s](x) = v(x), \quad x \in \Gamma, \tag{5}$$

$$[\bigtriangledown u_s \cdot \beta \cdot n](x) = w(x) - [\bigtriangledown u_r \cdot \beta \cdot n], \quad x \in \Gamma. \tag{6}$$

An important remark here is that our decomposition is not the same as requiring $u_r(x)$ to satisfy homogeneous jump conditions in general (except for the case where $\beta(x)$ is continuous across the interface). Instead, we require that $[u_r] = 0, [\bigtriangledown u_r \cdot n] = 0$ and equation (6) couples u_s and u_r together. Using the above equations, if one can construct $u_s(x) \in H^1(\Omega^+) \cup H^1(\Omega^-)$ that is linearly

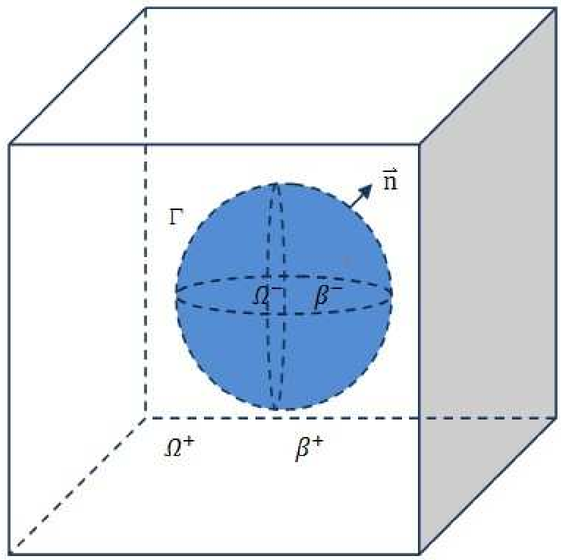

Figure 1: Setup of the problem

dependent on $u_r(x)$: $u_s = \mathcal{L}(u_r)$, the original elliptic problem can be casted into the following weak formulation: find $u_r(x) \in H^1(\Omega)$ such that

$$\int_\Omega \bigtriangledown u_r \cdot \beta \cdot \bigtriangledown \phi dx + \int_{\Omega^+ \cup \Omega^-} \bigtriangledown \mathcal{L}(u_r) \cdot \beta \cdot \bigtriangledown \phi dx = \int_\Omega f \phi dx - \int_\Gamma w \phi ds \quad (7)$$

$$u_r(x) = g(x), \quad x \in \partial\Omega, \quad (8)$$

for all $\phi(x) \in H_0^1(\Omega)$. Then the solution to the original elliptic problem is $u = u_r + u_s$. If $\beta(x)$ is continuous across the interface, then $[\bigtriangledown u_r \cdot \beta \cdot n]_\Gamma = 0$, $u_s(x)$ can be constructed independent of u_r and the above weak formulation is reduced to a simpler form: find $u_r(x) \in H^1(\Omega)$ such that

$$\int_\Omega \bigtriangledown u_r \cdot \beta \cdot \bigtriangledown \phi dx = -\int_{\Omega^+ \cup \Omega^-} \bigtriangledown u_s \cdot \beta \cdot \bigtriangledown \phi dx + \int_\Omega f \phi dx - \int_\Gamma w \phi ds \quad (9)$$

$$u_r(x) = g(x), \quad x \in \partial\Omega, \quad (10)$$

for all $\phi(x) \in H_0^1(\Omega)$.

The key issue is how to construct $u_s(x)$ and we give an explicit construction formula and some analysis in next Section.

3 Construction of the jump function

3.1 The two dimensional case

We use the same condition as proposed in [16] to construct the singular part u_s. Suppose $\mathcal{T} = \{T_i\}_{i \in I}$ is a fixed mesh for Ω that is not body fitted to the interface

Γ. There are two kinds of triangles depending on whether they intersect the interface or not. Let $\{T_j\}_{j\in J}$ be the triangles intersecting with Γ, which is called interface triangle, and $\{T_k\}_{k\in K}$, $K = I\backslash J$. The regular part $u_r(x) \in H^1(\Omega)$ will be approximated using standard piecewise linear finite element basis on $\mathcal{T}$, denoted by $u_r^h(x) = \sum_i u_i\phi_i$, where ϕ_i is the nodal basis on $\mathcal{T}$. We construct the singular part $u_s(x)$, denoted by $u_s^h(x)$, with support only on interface triangles $\{T_j\}_{j\in J}$ i.e.,

$$u_s^h(x) = 0, \quad \text{if } x \in T_k,\ k \in K, \tag{11}$$

Thus (4) is satisfied.

Here is our construction of $u_s^h(x)$ at the interface triangles $\{T_j\}_{j\in J}$. Let $T_j = \Delta ABC$, where A, B and C are the vertices of the triangle. Suppose the intersection points of the interface and ΔABC are D and E, on the edges $\overline{AB}$ and $\overline{AC}$ respectively (see Figure 2). We use the line $\overline{DE}$ to approximate the interface in the triangle. Let M be the middle point of $\overline{DE}$, and n is the unit normal to $\overline{DE}$.

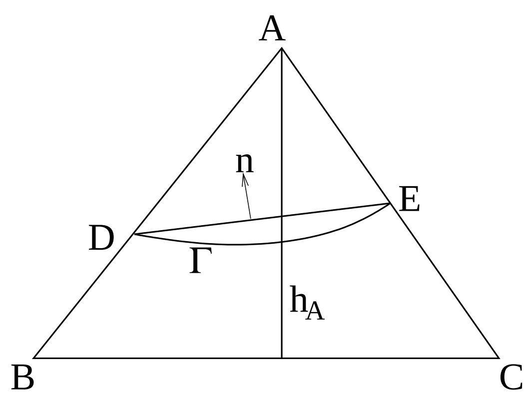

Figure 2: An interface triangle.

Let $\phi_A(x)$, $\phi_B(x)$ and $\phi_C(x)$ be the three linear base functions defined on this triangle:

$$\phi_A(A) = 1,\ \phi_A(B) = 0,\ \phi_A(C) = 0. \tag{12}$$

The same way to define $\phi_B(x)$ and $\phi_C(x)$. We construct $u_s^h(x)$ as a linear function on the two parts ΔADE and $BCED$. Hence $u_s^h(x)$ is the linear combination of ϕ_A, ϕ_B and ϕ_C with different coefficients on the two parts. Another condition for $u_s^h(x)$ is: $u_s^h(x) = 0$ on A, B and C, thus $u_s^h(x)$ should be in the form:

$$u_s^h(x) = \begin{cases} c_2\phi_B(x) + c_3\phi_C(x) & x \in \Delta AED \\ c_1\phi_A(x) & x \in BCED \end{cases} \tag{13}$$

We enforce the jump in u at the two end points D and E, and flux jump at the middle point M, which leads to the following equations for c_1, c_2 and c_3:

$$\begin{aligned} c_2\phi_B(D) + c_3\phi_C(D) - c_1\phi_A(D) &= v(D), &(14)\\ c_2\phi_B(E) + c_3\phi_C(E) - c_1\phi_A(E) &= v(E), &(15)\\ n\cdot\left(c_2 \bigtriangledown \phi_B \cdot \beta^+ + c_3 \bigtriangledown \phi_C \cdot \beta^+ - c_1 \bigtriangledown \phi_A \cdot \beta^-\right) &= \tilde{w}(M), &(16)\end{aligned}$$

where

$$\tilde{w}(x) = w(x) - [\bigtriangledown u_r \cdot \beta \cdot n]\,. \tag{17}$$

The relations of ϕ_A, ϕ_B and ϕ_C are

$$\begin{aligned} &\phi_C(D) = \phi_B(E) = 0, &(18)\\ &\phi_B(D) + \phi_A(D) = \phi_C(E) + \phi_A(E) = 1, &(19)\\ &\phi_A(x) + \phi_B(x) + \phi_C(x) = 1. &(20)\end{aligned}$$

The linear system of c_1, c_2 and c_3 is

$$\begin{bmatrix} -\phi_A(D) & 1-\phi_A(D) & 0 \\ -\phi_A(E) & 0 & 1-\phi_A(E) \\ -n\cdot\beta^-\cdot\bigtriangledown\phi_A & n\cdot\beta^+\cdot\bigtriangledown\phi_B & n\cdot\beta^+\cdot\bigtriangledown\phi_C \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} v(D) \\ v(E) \\ \tilde{w}(M) \end{bmatrix} \tag{21}$$

If this linear system can be solved, u_s^h can be constructed linearly dependent on u_r^h which then results in a linear system for u_r^h using (7). Denote the coefficient matrix in (21) to be P. We have the following theorem.

Theorem 3.1. If (1) β is a symmetric positive definite matrix and has no jump across the interface, or (2) $\beta > 0$ is a scalar and the triangle is non-obtuse, the coefficient matrix P is non-singular.

Proof.

Let $\lambda_1 = \frac{|AD|}{|AB|}$, $\lambda_2 = \frac{|AE|}{|AC|}$, and z be the unit vector perpendicular to the 2-D plane pointing out of the paper, then

$$\begin{aligned} \bigtriangledown\phi_B &= \frac{\overrightarrow{AC}\times z}{2S}, &(22)\\ \bigtriangledown\phi_C &= \frac{\overrightarrow{BA}\times z}{2S}, &(23)\end{aligned}$$

and

$$\begin{aligned} \det(P) &= \lambda_1 n\cdot\beta^+\cdot\bigtriangledown\phi_C + \lambda_2 n\cdot\beta^+\cdot\bigtriangledown\phi_B - \lambda_1\lambda_2 n\cdot\left(\beta^- - \beta^+\right)\cdot\bigtriangledown\phi_A \\ &= n\cdot\beta^+\cdot\frac{(\lambda_1\overrightarrow{BA} + \lambda_2\overrightarrow{AC})\times z}{2S} - \lambda_1\lambda_2 n\cdot\left(\beta^- - \beta^+\right)\cdot\bigtriangledown\phi_A \\ &= -\frac{|DE|}{2S} n\cdot\beta^+\cdot n - \lambda_1\lambda_2 n\cdot\left(\beta^- - \beta^+\right)\cdot\bigtriangledown\phi_A. &(24)\end{aligned}$$

Using h_A and $\hat{h}_A$ to denote the distance from A to BC and DE respectively, we can get

$$\begin{aligned} h_A &= \frac{2S}{|BC|}, &(25)\\ \hat{h}_A &= \lambda_1\lambda_2\frac{2S}{|DE|}, &(26)\end{aligned}$$

and

$$\bigtriangledown\phi_A = \frac{1}{h_A} n_A, \tag{27}$$

where S is the area of ΔABC and n_A is the unit normal direction of the line BC. The determinant of P can be simplified as

$$\det(P) = -\lambda_1\lambda_2 \left(\frac{1}{\hat{h}_A} n \cdot \beta^+ \cdot n + \frac{1}{h_A} n \cdot \left(\beta^- - \beta^+\right) \cdot n_A \right) \tag{28}$$

If β is a matrix and has no jump, i.e., $\beta^+ = \beta^- = \beta$. The determinant of P is

$$\det(P) = -\frac{\lambda_1\lambda_2}{\hat{h}_A} n \cdot \beta \cdot n, \tag{29}$$

While β is positive definite, we have

$$\det(P) < 0. \tag{30}$$

If $\beta > 0$ is a scalar with jump cross the interface, the determinant of P is

$$\det(P) = -\lambda_1\lambda_2 \left(\frac{\beta^+}{\hat{h}_A} - \left(\beta^+ - \beta^-\right) \frac{n \cdot n_A}{h_A} \right) \tag{31}$$

Since ΔABC is non-obtuse, we can find a point F on BC such that AF is perpendicular to BC, thus $|AF| = h_A$. There must exist an intersection point G between AF and DE, see Figure 2. The distance from A to DE should be smaller than or equals to $|AG|$, and $|AG| \leq h_A$, so

$$\hat{h}_A \leq h_A. \tag{32}$$

The angle between n and n_A equals to the angle between DE and BC, which is an acute angle or right angle in ΔABC. Thus $0 \leq n \cdot n_A \leq 1$. By $\beta^+ > (\beta^+ - \beta^-)$, we have $\det(P) < 0$.

By this theorem, c_1, c_2 and c_3 can be solved uniquely and hence $u_s^h(x)$ can be constructed on the interface triangle ΔABC. Another important issue is whether this construction of jump function is stable with respect to the shape of the triangle and interface cut. For the two cases stated in Theorem 3.1, we have the formulae for the determinant (29) and (31). We can see the factor $\frac{\lambda_1\lambda_2}{\hat{h}_A}$ is introduced by the interface cut which will be discussed in detail below. The remaining term in (29) for case 1 has no dependence on the shape of the interface triangle. The remaining term in (31) is

$$\beta^+ - (\beta^+ - \beta^-) \frac{n \cdot n_A \hat{h}_A}{h_A} \geq \beta^- \tag{33}$$

if the triangle is not obtuse. If we consider another possibility of β^+ and β^- arrangement with respect the cut, we have

$$|\det(P)| \geq \frac{\lambda_1\lambda_2}{\hat{h}_A} \min\{\beta^+, \beta^-\} \tag{34}$$

for case 2.

Now let us look at how the local construction is affected by the cut. Actually we can explicitly solve the linear system for c_1, c_2, c_3. For simplicity of discussion, we assume homogeneous jump conditions. Inhomogeneous jump conditions will only add bounded terms in the numerator.

$$c_1 = \frac{\nabla u_r \cdot (\beta^+ - \beta^-) \cdot n}{\frac{n \cdot \beta^+ \cdot n}{\hat{h}_A} - \frac{n \cdot (\beta^+ - \beta^-) \cdot n_A}{h_A}}, \qquad c_2 = \frac{1 - \lambda_1}{\lambda_1} c_1, \qquad c_3 = \frac{1 - \lambda_2}{\lambda_2} c_1,$$

When the cut tends to degeneracy, although $\det(P) \to 0$, c_1, c_2, c_3 are all bounded uniformly.

1. $\lambda_1 \to 0$ and $\lambda_2 \to 0$,

$$c_1 \to 0, \qquad c_2 \to \frac{\nabla u_r \cdot (\beta^+ - \beta^-) \cdot n}{n \cdot \beta^+ \cdot n} h_A^1, \qquad c_3 \to \frac{\nabla u_r \cdot (\beta^+ - \beta^-) \cdot n}{n \cdot \beta^+ \cdot n} h_A^2,$$

   where $h_A^1 = \frac{\hat{h}_A}{\lambda_1} = \frac{\hat{h}_A}{|AD|} |AB| \sim O(h), h_A^2 = \frac{\hat{h}_A}{\lambda_2} = \frac{\hat{h}_A}{|AE|} |AC| \sim O(h)$.

2. $\lambda_1 \to 0$,

$$c_1 \to 0, \qquad c_2 \to \frac{\nabla u_r \cdot (\beta^+ - \beta^-) \cdot n}{n \cdot \beta^+ \cdot n} h_A^1, \qquad c_3 \to 0,$$

3. $\lambda_2 \to 0$,

$$c_1 \to 0, \qquad c_2 \to 0, \qquad c_3 \to \frac{\nabla u_r \cdot (\beta^+ - \beta^-) \cdot n}{n \cdot \beta^+ \cdot n} h_A^2.$$

In the degenerate cases, (1) $\lambda_1 = 0, \lambda_2 \neq 0$, c_1 is decoupled from c_2, c_3; (2) $\lambda_1 \neq 0, \lambda_2 = 0$, c_2 is decoupled from c_1, c_3; (3) $\lambda_1 = \lambda_2 = 0$, e.g., vertex A just touches the interface, then the numerical treatment becomes simply enforcing the jump condition in u at A.

Remark 1. If one uses standard triangulation based on rectangular mesh, which is the main point for this method, the non-obtuse triangle condition in Theorem 3.1 is automatically satisfied. On a general triangular mesh, one can solve the local linear system in the least squares sense which will not be discussed here.

Now we look at the approximation error of our constructed piecewise linear function for piecewise smooth functions in a triangle. Let ψ be a piecewise smooth function on a non-obtuse and shape regular triangle ΔABC. ψ is only discontinuous across the smooth interface Γ, see Figure 2. Denote ψ^+ and ψ^- to be the two pieces respectively. Denote $v(D) = [\psi](D), v(E) = [\psi](E), w(M) = [\beta \nabla \psi \cdot n](\tilde{M})$, where β is given , $[\cdot]$ denote jumps across the interface. Here we use $\tilde{M}$ to denote some point on Γ. Denote u_r to be the linear function that interpolates $\psi(A), \psi(B), \psi(C)$. Denote u_s to be the piecewise linear function constructed by (13)-(16). Then $u_r + u_s$ is a piecewise linear function which can be extended to two linear functions on ΔABC denoted as u^+ and u^- respectively.

Theorem 3.2. The constructed piecewise linear function $u_r + u_s$ with proper extension (defined above) approximates the piecewise smooth function ψ to the second order, i.e., $\forall x \in \Delta ABC$, $|\psi^{\pm}(x) - u^{\pm}(x)| = O(h^2)$, where h denotes the size of the non-obtuse and shape regular triangle ΔABC.

Proof. Without loss of generality we assume $\psi(A) = \psi(B) = \psi(C) = 0$, i.e., $u_r = 0$. Let ψ_h^+ be some linear approximation of ψ^+ in ΔADE and is extended to ΔABC, such as a linear interpolation through ADE, and let ψ_h^- be some linear approximation of ψ^+ in $BCED$ and is extended to ΔABC, such as a linear interpolation through BCE or BCD or a least square fitting of $BCED$, such that $\psi_h^+(A) = \psi_h^-(B) = \psi_h^-(C) = 0$, $|\psi^{\pm}(x) - \psi_h^{\pm}(x)| = O(h^2)$ and $|\nabla\psi^{\pm}(x) - \nabla\psi_h^{\pm}(x)| = O(h)$ $\forall x \in \Delta ABC$.

It is obvious that both of the two piecewise linear approximations, $u^{\pm}$ and $\psi_h^{\pm}$ satisfy the linear system (14)-(16) with $v(D), v(E)$ differed by $O(h^2)$ and $w(M)$ differed by $O(h)$. Solving the linear system (14)-(16), we have

$$c_1 = \frac{-\hat{h}_A w(M) + \hat{h}_A \beta^+ \left[v(D) n \cdot \frac{\overrightarrow{AE} \times z}{2\lambda_1\lambda_2 S} + v(E) n \cdot \frac{\overrightarrow{DA} \times z}{2\lambda_1\lambda_2 S} \right]}{\beta^+ - \frac{(\beta^+ - \beta^-) n \cdot n_A \hat{h}_A}{h_A}} \tag{35}$$

Moreover, we have

$$\hat{h}_A \left[n \cdot \frac{\overrightarrow{AE} \times z}{2\lambda_1\lambda_2 S} + n \cdot \frac{\overrightarrow{DA} \times z}{2\lambda_1\lambda_2 S} \right] = (\overrightarrow{AE} + \overrightarrow{DA}) \cdot \frac{\overrightarrow{ED}}{|\overrightarrow{ED}|^2} = -1, \tag{36}$$

and hence $\frac{|\overrightarrow{AE}|}{|\overrightarrow{ED}|}$ and $\frac{|\overrightarrow{DA}|}{|\overrightarrow{ED}|}$ are bounded since ΔABC is shape regular. Let c_1^u and c_1^{ψ} be the coefficients for the two piecewise linear approximations respectively, we have $|c_1^u - c_1^{\psi}| = O(h^2)$. Since $u^-(x) = c_1^u \phi_A(x)$ and $\psi_h^-(x) = c_1^{\psi}\phi_A(x)$, we have $|u^-(x) - \psi_h^-(x)| = O(h^2)$ and $|\nabla u^-(x) - \nabla\psi_h^-(x)| = O(h)$, $\forall x \in \Delta ABC$.

We also have

$$c_2 = \frac{1-\lambda_1}{\lambda_1} c_1 + \frac{v(D)}{\lambda_1}, \quad c_3 = \frac{1-\lambda_2}{\lambda_2} c_1 + \frac{v(E)}{\lambda_2} \tag{37}$$

which implies

$$c_2\phi_B(D) = (1-\lambda_1)c_1 + v(D), \quad c_3\phi_C(E) = (1-\lambda_2)c_1 + v(E). \tag{38}$$

With $u^+(D) = c_2^u \phi_B(D), \psi_h^+(D) = c_2^{\psi}\phi_B(D)$, $u^+(E) = c_3^u\phi_C(E), \psi_h^+(E) = c_3^{\psi}\phi_C(E)$, and $u^+(A) = \psi_h^+(A) = 0$, we have $|u^+(x) - \psi_h^+(x)| = O(h^2)$ and $|\nabla u^+(x) - \nabla\psi_h^+(x)| = O(h)$, $\forall x \in \Delta ABC$. So we prove the theorem.

Similar results is true for 3D. The above theorem shows that the construction of piecewise linear approximation near the interface can approximate a piecewise smooth function to second order. Together with the use of linear elements in our variational formulation, the numerical solution can potentially approximate a piecewise smooth solution of the elliptic interface problem to the second order pointwise in the whole domain. As a simple corollary, if the regular part of our numerical solution, u_r^h, approximates the true solution at grid points to second order, then the full solution $u_r^h + u_s^h$ approximates the true solution to second order at the interface cut too. Extensive numerical tests in 2D and 3D demonstrate that our method does achieve second order accuracy.

Remark 2. Note that our local construction of the piecewise linear approximation at interface triangles does not guarantee a match at the interface cut point on a common edge of two adjacent interface triangles on the same side of the interface, which is the case for the piecewise smooth solution to the PDE. Numerical results show that our method is second order accurate in L^∞ norm at grid points which implies that our numerical solution is also second order accurate at interface cuts from Theorem 3.2 which also implies that the mismatch for our numerical solution at the interface cut point on a common edge of two adjacent interface triangles on the same side of the interface is no more than $O(h^2)$. The result is similar in three dimension.

3.2 The three dimensional case

Now we extend the above construction of $u_s^h(x)$ to 3D. In 3D, the interface may cut through a tetrahedron in two ways, see Fig. 3. For case 1, the intersection of the interface with the tetrahedron T can be approximated by a triangle. Let A_i be the point i, G be the barycenter of the interface triangle $A_5A_6A_7$, and ϕ_i, $i = 1, 2, 3, 4$, be the linear finite element basis on the tetrahedron,

$$\phi_i(A_j) = \begin{cases} 1, & j = i \\ 0, & j \neq i \end{cases} \qquad i, j = 1, 2, 3, 4 \tag{39}$$

Similar to 2D case, u_s^h satisfies the following jump conditions:

$$\begin{aligned} \left[u_s^h\right](A_i) &= v(A_i),\ i = 5, 6, 7, &(40)\\ \left[n \cdot \beta \cdot \bigtriangledown u_s^h\right](G) &= \tilde{w}(G), &(41) \end{aligned}$$

where $\tilde{w}(x) = w(x) - \left[n \cdot \beta \cdot \bigtriangledown u_r^h\right](x)$. Assume that point A_1 belongs to Ω^+ and the other three vertices A_i $(i = 2, 3, 4)$ belong to Ω^-. We have

$$u_r^h = \sum_{i=1}^{4} u^i \phi_i, \tag{42}$$

and

$$u_s^h = \begin{cases} c_2\phi_2 + c_3\phi_3 + c_4\phi_4, & \text{in } T \cap \Omega^+, \\ c_1\phi_1, & \text{in } T \cap \Omega^-. \end{cases} \tag{43}$$

Substitute (43) into (40) and (41), we can obtain the linear system for the coefficients d_i $(i = 1, 2, 3, 4)$

$$\begin{pmatrix} -\phi_1(A_5) & \phi_2(A_5) & \phi_3(A_5) & \phi_4(A_5) \\ -\phi_1(A_6) & \phi_2(A_6) & \phi_3(A_6) & \phi_4(A_6) \\ -\phi_1(A_7) & \phi_2(A_7) & \phi_3(A_7) & \phi_4(A_7) \\ -n \cdot \beta^- \cdot \bigtriangledown\phi_1 & n \cdot \beta^+ \cdot \bigtriangledown\phi_2 & n \cdot \beta^+ \cdot \bigtriangledown\phi_3 & n \cdot \beta^+ \cdot \bigtriangledown\phi_4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} v(A_5) \\ v(A_6) \\ v(A_7) \\ \tilde{w}(G) \end{pmatrix} \tag{44}$$

Again, using P to denote the coefficient matrix of the linear system (44), we have the following theorem:

Theorem 3.3. For case 1, if (1) the coefficient β is a symmetric positive definite matrix and has no jump across the interface, or (2) $\beta > 0$ is a scalar and the angles between any two faces of the tetrahedron are non-obtuse, the matrix P is invertible.

Proof. By the definition of ϕ_i ($i = 1, 2, 3, 4$), they have the following relations

$$\begin{aligned}
&\phi_2(A_6) = \phi_2(A_7) = 0, \\
&\phi_3(A_5) = \phi_3(A_7) = 0, \\
&\phi_4(A_5) = \phi_4(A_6) = 0, \\
&\phi_2(A_5) = 1 - \phi_1(A_5), \\
&\phi_3(A_6) = 1 - \phi_1(A_6), \\
&\phi_4(A_7) = 1 - \phi_1(A_7), \\
&\sum_{i=1}^{4} \phi_i \equiv 1.
\end{aligned}$$

The matrix P can be simplified as

$$P = \begin{pmatrix} -\phi_1(A_5) & 1-\phi_1(A_5) & 0 & 0 \\ -\phi_1(A_6) & 0 & 1-\phi_1(A_6) & 0 \\ -\phi_1(A_7) & 0 & 0 & 1-\phi_1(A_7) \\ -n\cdot\beta^-\cdot\triangledown\phi_1 & n\cdot\beta^+\cdot\triangledown\phi_2 & n\cdot\beta^+\cdot\triangledown\phi_3 & n\cdot\beta^+\cdot\triangledown\phi_4 \end{pmatrix} \tag{45}$$

Let $\lambda_5 = \frac{|A_1A_5|}{|A_1A_2|}$, $\lambda_6 = \frac{|A_1A_6|}{|A_1A_3|}$, $\lambda_7 = \frac{|A_1A_7|}{|A_1A_4|}$, then the determinant of P is

$$\begin{aligned}
\det(P) &= [1-\phi_1(A_5)]\,[1-\phi_1(A_6)]\,[1-\phi_1(A_7)]\, n\cdot\beta^-\cdot\triangledown\phi_1 \\
&\quad -\phi_1(A_5)\,[1-\phi_1(A_6)]\,[1-\phi_1(A_7)]\, n\cdot\beta^+\cdot\triangledown\phi_2 \\
&\quad -\phi_1(A_6)\,[1-\phi_1(A_5)]\,[1-\phi_1(A_7)]\, n\cdot\beta^+\cdot\triangledown\phi_3 \\
&\quad -\phi_1(A_7)\,[1-\phi_1(A_5)]\,[1-\phi_1(A_6)]\, n\cdot\beta^+\cdot\triangledown\phi_4 \\
&= \lambda_5\lambda_6\lambda_7 \left[n\cdot\left(\beta^- - \beta^+\right)\triangledown\phi_1 - n\cdot\beta^+\left(\frac{\triangledown\phi_2}{\lambda_5} + \frac{\triangledown\phi_3}{\lambda_6} + \frac{\triangledown\phi_4}{\lambda_7}\right)\right]
\end{aligned}$$

Let V be the volume of the tetrahedron. We can calculate the gradient of the basis function as

$$\begin{aligned}
\nabla\phi_2 &= \frac{\overrightarrow{A_1A_4}\times\overrightarrow{A_1A_3}}{6V}, \\
\nabla\phi_3 &= \frac{\overrightarrow{A_1A_2}\times\overrightarrow{A_1A_4}}{6V}, \\
\nabla\phi_4 &= \frac{\overrightarrow{A_1A_3}\times\overrightarrow{A_1A_2}}{6V}.
\end{aligned}$$

Using these relations, we can get

$$\begin{aligned}\det(P) &= \lambda_5\lambda_6\lambda_7 n\cdot\left(\beta^- - \beta^+\right)\bigtriangledown\phi_1 - \frac{1}{6V}n\cdot\beta^+\cdot(\overrightarrow{A_7A_6}\times\overrightarrow{A_7A_5}) \\ &= \frac{\hat{S}}{3V}n\cdot\beta^+\cdot n - \lambda_5\lambda_6\lambda_7 n\cdot\left(\beta^+ - \beta^-\right)\cdot\bigtriangledown\phi_1\end{aligned}$$

where $\hat{S}$ is the area of the triangle $A_5A_6A_7$. This formula is similar to Eq. (24). Let h_1 and $\hat{h}_1$ be the distance from A_1 to $A_2A_3A_4$ and $A_5A_6A_7$ respectively, and n_1 be the unit normal direction of the plane $A_2A_3A_4$. We also have the relations

$$\bigtriangledown\phi_1 = \frac{1}{h_1}n_1, \tag{46}$$

$$\hat{h}_1 = \lambda_5\lambda_6\lambda_7\frac{3V}{\hat{S}}. \tag{47}$$

The determinant of P can be simplified as

$$\det(P) = \lambda_5\lambda_6\lambda_7\left[\frac{1}{\hat{h}_1}n\cdot\beta^+\cdot n - \frac{1}{h_1}n\cdot\left(\beta^+ - \beta^-\right)\cdot n_1\right]. \tag{48}$$

If β is positive definite and has no jump, the determinant of P is

$$\det(P) = \frac{\lambda_5\lambda_6\lambda_7}{\hat{h}_1}n\cdot\beta^+\cdot n, \tag{49}$$

which is positive. Therefore P is invertible.

If $\beta^+ = \beta^- = \beta > 0$ is a scalar with jump cross the interface ,the determinant of P is

$$\det(P) = \lambda_5\lambda_6\lambda_7\left[\frac{\beta^+}{\hat{h}_1} - \frac{(\beta^+ - \beta^-)}{h_1}n\cdot n_1\right]. \tag{50}$$

Similar to the 2D case, since the angles between any two faces of the tetrahedron are non-obtuse, there must be $\hat{h}_1 < h_1$ and $0 \leq n\cdot n_1 \leq 1$. By $\beta^+ > (\beta^+ - \beta^-)$, we have $\det(P) > 0$.

Note that the condition 'the angles between any two faces of the tetrahedron are acute angles' is a sufficient condition for case 1, not necessary. It can be weakened to 'the distance from the vertex A_1 to the interface $A_5A_6A_7$ is smaller than to the opposite face $A_2A_3A_4$. Cartesian grid based tetrahedrons satisfying this angle condition are used in our numerical tests.

For simplicity of discussion, we assume homogeneous jump conditions. Inhomogeneous jump conditions will only change the numerator and will not introduce extra singularities. Using the same computation as in 2D, one can find explicit expression for c_1, c_2, c_3, c_4

$$c_1 = \frac{\nabla u_r\cdot(\beta^+ - \beta^-)\cdot n}{\frac{n\cdot\beta^+\cdot n}{\hat{h}_1} - \frac{n\cdot(\beta^+ - \beta^-)\cdot n_1}{h_1}}, \quad c_2 = \frac{1-\lambda_5}{\lambda_5}c_1, \quad c_3 = \frac{1-\lambda_6}{\lambda_6}c_1, \quad c_4 = \frac{1-\lambda_7}{\lambda_7}c_1,$$

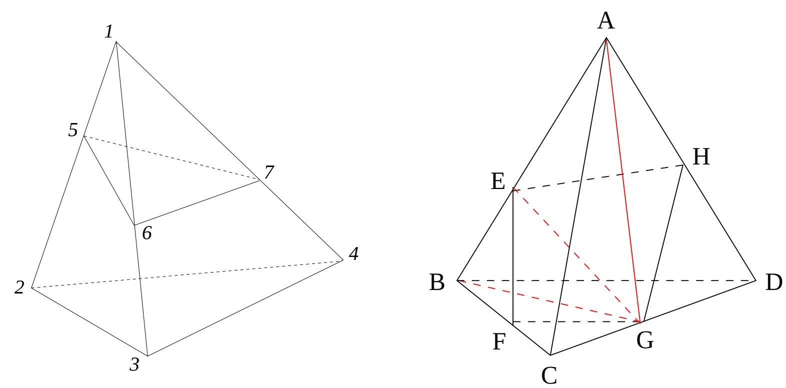

Figure 3: Interface tetrahedron, left: case 1, right: case 2

and show that they are bounded even when the interface cut tends to degenerate. For a true degenerate case, the simple treatment as in 2D can be used.

For the case 2 (Figure 3) the difference is that the intersection of the interface with the tetrahedron is approximated by a quadrilateral instead of a triangle. We still enforce $[u]$ at the four vertices labeled as 5, 6, 7, 8 and $[\nabla u \cdot \beta \cdot n]$ at the centroid location 9,10 of two triangles. There are more equations than unknowns. We use the least squares method to solve the local system. In this case it is non-conforming. Extensive numerical examples show our construction work nicely in both cases. As in 2D, if one uses regular tetrahedralization based on cubic mesh, like in our implementation, the angle condition in Theorem 3.3 is automatically satisfied. On a general tetrahedron mesh, one can solve the local linear system in the least squares sense for both case 1 and case 2.

3.3 Further discussion

In this section we discuss a few issues about the numerical solution u_r^h and the corresponding fully discretized linear system from the weak formulation (7). Since we are using piecewise linear finite element space, our construction of the singular part u_s^h and regular part u_r^h is equivalent to (1) piecewise linear approximation of the interface, (2) piecewise linear approximation of the jump condition in u, (3) piecewise constant approximation of the jump condition in flux, and (4) piecewise linear finite element approximation of the regular part. Assume enough regularity of the interface Γ, the jump conditions $v(x), w(x)$ along Γ, and $\beta^{\pm}(x)$ in $\Omega^{\pm}$, our numerical algorithm can achieve second order accuracy in L^{∞} which is verified by our numerical tests.

Another important issue is the behavior of the full linear system for u_s^h resulting from the weak formulation (7). We already showed that the local linear system for the construction of the singular par u_s^h is stable with respect to the interface cut. Further, it is desirable to have the full linear system behaves similarly to that of a standard elliptic problem with no interface , i.e., the linear system is positive definite and the condition number of the the full linear system only depends on $\beta^{\pm}(x)$ and the underlying mesh size h but not on how the interface cut through the mesh. If β is continuous, it is obvious that the discretized linear system for $u_r^h(x)$ from the weak formulation (9) is obviously symmetric positive definite and behaves the same way as elliptic problem with no interface since the singular part $u_s^h(x)$ can be constructed independent of $u_r^h(x)$. However, if β is discontinuous across the interface Γ, the construction of $u_s^h(x)$ depends on $u_r^h(x)$ linearly. Define the bilinear form

$$B\left[u,v\right] = \int_{\Omega} \nabla u \cdot \beta \cdot \nabla v dx + \int_{\Omega^+ \cup \Omega^-} \nabla \mathcal{L}^0(u) \cdot \beta \cdot \nabla v dx, \tag{51}$$

where $\mathcal{L}^0(u) = u_s^0(x)$ is a jump function satisfying homogeneous jump condition, i.e.,

$$\begin{aligned} \left[u_s^0\right](x) &= 0, \quad x \in \Gamma, & (52)\\ \left[\bigtriangledown u_s^0 \cdot \beta \cdot n\right](x) &= -\left[\bigtriangledown u \cdot \beta \cdot n\right], \quad x \in \Gamma. & (53) \end{aligned}$$

The matrix A for the full linear system comes from the discretization of this bilinear form. For any vector $c \in R^n$, define $u_r^h = \sum_{i=1}^n c_i \phi_i$ and $\phi = \sum_{i=1}^n c_i \phi_i$, where $\phi_i, i = 1, \ldots, n$ are the piecewise linear nodal basis corresponding to a triangulation of the domain, then

$$c^T A c = B[u_r^h, \phi] = \int_{\Omega} \nabla u_r^h \cdot \beta \cdot \nabla \phi dx + \int_{\Omega^+ \cup \Omega^-} \nabla \mathcal{L}^0(u_r^h) \cdot \beta \cdot \nabla \phi dx, \tag{54}$$

where $\mathcal{L}^0(u_r^h)$ comes from the local reconstruction of the singular part on interface triangles/tetrahedrons. Hence the linear system for $u_r^h(x)$ using the weak formulation is not symmetric in general. In 1D one can prove the following result.

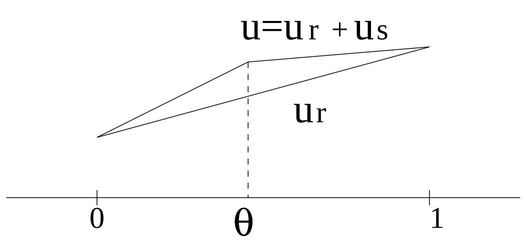

Figure 4: 1D case.

Theorem 3.4. If $\beta^{\pm}$ are positive, then the discretized linear system for the above bilinear form in one dimension is positive definite. Further, the condition number is of order $O(h^{-2})$.

Proof. One only has to look at the weak formulation in a cell cut by an interface. Suppose the interface cuts through the cell $I = [0, 1]$ at $, \theta, 0 < \theta < 1$, see Figure 4. Let $u(x) = u_r(x) + u_s(x)$ be the piecewise linear function in I and $u_{r,x} = \frac{du_r}{dx} = \gamma$ in I.

From $[u(\theta)] = 0, [\beta u_x(\theta)] = 0$, where $\beta^{\pm} > 0$, we have $\beta^+ u_x^+ = \beta^- u_x^-$. Therefore $u_x^+ = \frac{\beta^-}{\beta^+} u_x^-$. Let $u_x^- = \alpha$, then $\alpha\theta + \frac{\beta^-}{\beta^+}\alpha(1-\theta) = \gamma$, $\alpha = \frac{\beta^+}{\beta^+\theta + \beta^-(1-\theta)}\gamma$. We have

$$B[u_r, u_r] = \int_0^\theta \beta^- u_x^- u_{r,x} + \int_\theta^1 \beta^+ u_x^+ u_{r,x} = \frac{\beta^- \beta^+}{\beta^+\theta + \beta^-(1-\theta)}\gamma^2$$

In another word, β can be regarded as a harmonic average of β^- and β^+ in the cell I.

In higher dimensions, we can not prove a similar result although numerical tests strongly suggest so. We will present numerical examples to demonstrate both order of convergence and condition number of the matrix resulted from the weak formulation in next section.

4 Numerical Experiments

Since extensive 2D numerical results have been provided in previous work [16, 17] for this method, we present numerical results in 3D in this Section. All the errors are measured in the L^∞ norm. We consider the following standard elliptic interface problem in 3D

$$\begin{aligned} -\nabla\cdot(\beta\nabla u) &= f, \ \texttt{in}\ \Omega^{\pm}, \\ [u] &= v, \ \texttt{on}\ \Gamma, \\ [(\beta\nabla u)\cdot n] &= w, \ \texttt{on}\ \Gamma, \\ u &= g, \ \texttt{on}\ \partial\Omega, \end{aligned}$$

on the rectangular domain $\Omega = (x_{min}, x_{max}) \times (y_{min}, y_{max}) \times (z_{min}, z_{max})$.

We assume that β and f are smooth on Ω^+ and Ω^-, but they may be discontinuous across the interface Γ, which is assumed to be Lipschitz continuous. We a level-set function ϕ on Ω to represent Γ, i.e., $\Gamma = \{\phi = 0\}$, $\Omega^- = \{\phi < 0\}$ and $\Omega^+ = \{\phi > 0\}$. $n = \frac{\nabla\phi}{|\nabla\phi|}$ is a unit normal vector of Γ pointing from Ω^- to Ω^+. β is a 3×3 matrix that is uniformly elliptic in each subdomain. In our numerical tests, the mesh are all based on Cartesian grid. We cut every cubic cell $[x_i, x_{i+1}] \times [y_j, y_{j+1}] \times [z_k, z_{k+1}]$ into six tetrahedrons, see Fig.5. These tetrahedrons satisfy the angle condition in Theorem 3.3. The interface cut is approximated using linear interpolation of the level set function.

In all numerical experiments below, the level-set function $\phi(x, y, z)$, the coefficient matrix $\beta^{\pm}(x, y, z)$ and the solutions

$$\begin{aligned} u &= u^+(x, y, z), \ \texttt{in}\ \Omega^+, \\ u &= u^-(x, y, z), \ \texttt{in}\ \Omega^- \end{aligned}$$

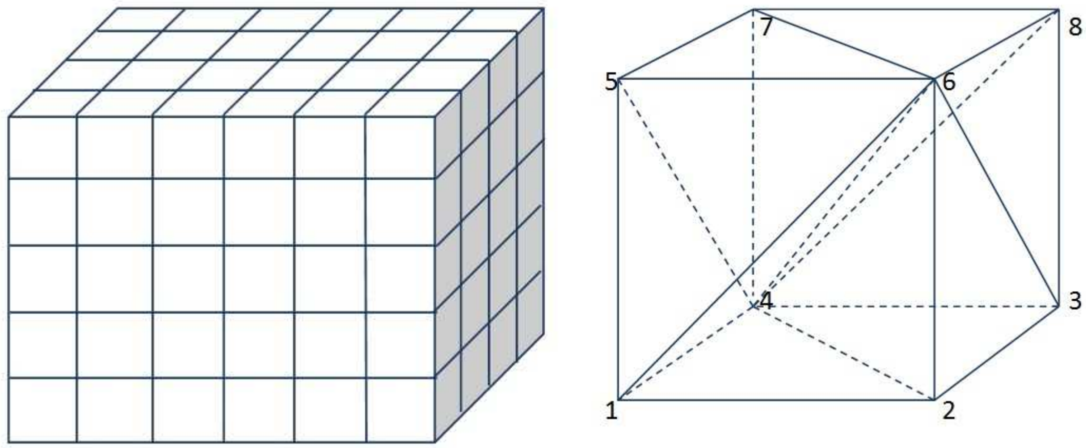

Figure 5: Cubic mesh and the tetrahedralization

are given. Hence

$$\begin{aligned} f &= -\nabla \cdot (\beta \nabla u), \\ v &= u^+ - u^-, \\ w &= (\beta^+ \nabla u^+) \cdot n - (\beta^- \nabla u^-) \cdot n \end{aligned}$$

on the whole domain Ω. g is obtained as a proper Dirichlet boundary condition, since the solutions are given. All errors are measured in L^∞ norm in the whole domain Ω.

Example 1. We use this simple example, where β is piecewise constant, to demonstrate that the condition number for the fully discretized system only depends on $\beta^\pm$ but on on the interface. The level-set function ϕ, the coefficients $\beta^\pm$ and the solution $u^\pm$ are given as follows:

$$\phi(x, y, z) = 0.25 - x^2 - y^2 - z^2,$$

$$\begin{aligned} \beta^+(x, y, z) &= 1, \\ \beta^-(x, y, z) &= 1, 50, 100, 150, 200, 250, 300, \\ u^+(x, y, z) &= 5 - \sin(x^3) + 3y^2 + z, \\ u^-(x, y, z) &= -\cos(x) + 3y + z^3 \end{aligned}$$

We plot the condition number VS the ratio between β^- and β^+ for 48-by-48-by-48 grid in Figure 6. The correlation coefficient for 12-by-12-by-12, 24-by-24-by-24, 48-by-48-by-48 grids are 0.99997, 0.99998 and 0.99995, which are clearly linear relations.

Example 2. We test both order of convergence and the condition number for the discretized linear system for a piecewise smooth scalar β. The level-set

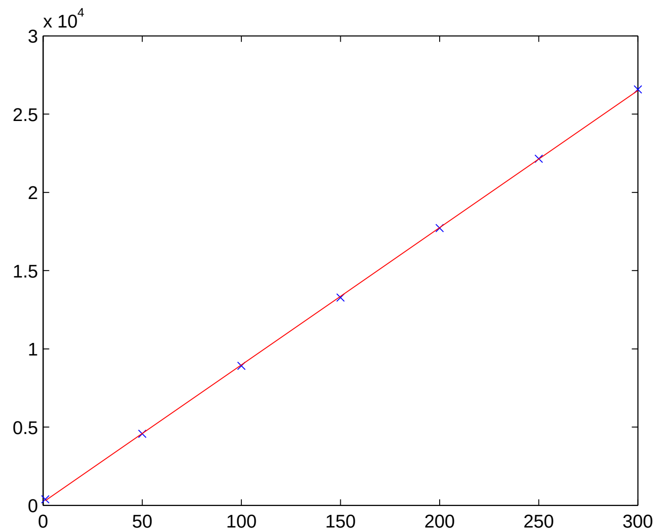

Figure 6: Example 1: condition number VS the ratio between β^- and β^+

function ϕ, the coefficients $\beta^{\pm}$ and the solution $u^{\pm}$ are given as follows:

$$\phi(x, y, z) = 0.25 - x^2 - y^2 - z^2,$$

$$\begin{aligned}
\beta^+(x, y, z) &= x^2 + 2y^2 + 1, \\
\beta^-(x, y, z) &= x^2 + 3y^2 + 1, \\
u^+(x, y, z) &= 5 - \sin(x^3) + 3y^2 + z, \\
u^-(x, y, z) &= -\cos(x) + 3y + z^3
\end{aligned}$$

Table 1 shows the error and condition numbers on different grids. Second order accuracy is achieved. The condition number grows approximately with the rate of $O(h^{-2})$, just like the case for elliptic problem without interface.

$n_x \times n_y \times n_z$	Err in U	Order	Condition Numbers
$6 \times 6 \times 6$	0.02384		31.55259
$12 \times 12 \times 12$	0.00759	1.6506	1.81615e+002
$24 \times 24 \times 24$	0.00202	1.9118	8.76619e+002
$48 \times 48 \times 48$	0.00051	1.9716	3.98244e+003

Table 1: Example 2: Sphere interface, scalar β

Example 3. In this example, β is a piecewise smooth tensor. The level-set function ϕ, the coefficients $\beta^{\pm}$ and the solution $u^{\pm}$ are given as follows:

$$\phi(x, y, z) = 0.25 - x^2 - y^2 - z^2,$$

$$\beta^+(x,y,z) = \begin{pmatrix} 4x^2+6 & \sin(y+x) & yx \\ \sin(y+x) & 2z^2+3 & 0.5\sin(x) \\ yx & 0.5\sin(x) & \cos(xy+z)^2+5 \end{pmatrix},$$
$$\beta^-(x,y,z) = \begin{pmatrix} \cos(x+y)^2+3 & z & 0.2\sin(z-x) \\ z & z^2+5 & y \\ 0.2\sin(z-x) & y & \sin(z)^2+2 \end{pmatrix},$$
$$u^+(x,y,z) = 5-\sin(x^3)+3y^2+z,$$
$$u^-(x,y,z) = -\cos(x)+3y+z^3$$

Table 2 shows the error and condition number on different grids. Again second order accuracy is achieved and the condition number grows approximately with the rate of $O(h^{-2})$.

$n_x \times n_y \times n_z$	Err in U	Order	Condition Numbers
$6 \times 6 \times 6$	0.01579		31.02191
$12 \times 12 \times 12$	0.00512	1.6243	1.34866e+002
$24 \times 24 \times 24$	0.00140	1.8742	5.55680e+002
$48 \times 48 \times 48$	0.00035	1.9884	2.25701e+003

Table 2: Example 3: Sphere interface, matrix β

Example 4. In this example β is a piecewise smooth tensor while the interface is only Lipschitz. The coefficients $\beta^\pm$ and the solution $u^\pm$ are given as follows:

$$\beta^+(x,y,z) = \begin{pmatrix} 4\sin(x)^2+6 & \sin(y+x)z & yx \\ \sin(y+x)z & 2z^2+\cos(x^2)^2+3 & 0.5\sin(xy) \\ yx & 0.5\sin(xy) & \cos(xy+z)^2+5 \end{pmatrix},$$
$$\beta^-(x,y,z) = \begin{pmatrix} xz+\cos(x+y)+3 & x & 0.2\sin(y-x) \\ x & z^2+5 & yz \\ 0.2\sin(y-x) & yz & \sin(z)^2+2 \end{pmatrix},$$
$$u^+(x,y,z) = 10-x^3+2y^2-2z+\sin(x+y+z)+\sin(x)+z,$$
$$u^-(x,y,z) = z^3+y^2-2x$$

The level-set function ϕ is given as:

$$\phi(x,y,z) = \min(x^2+y^2+(z+0.5)^2-0.25, \min(\min((x-0.4)^2+y^2+z^2-0.25, (x+0.3)^2+y^2+z^2-0.25), x^2+(y+0.5)^2+z^2-0.25)),$$

which is a the boundary of four intersected balls.

Figure 7 shows the numerical solution (the value from outside at the interface) with our method using 24 grid points in both x, y and z directions. Table 3 shows the error on different grids.

Example 5. In our last example, the solution is chosen to be singular, i.e., the second derivative blows up. The level-set function ϕ, the coefficients $\beta^\pm$ and

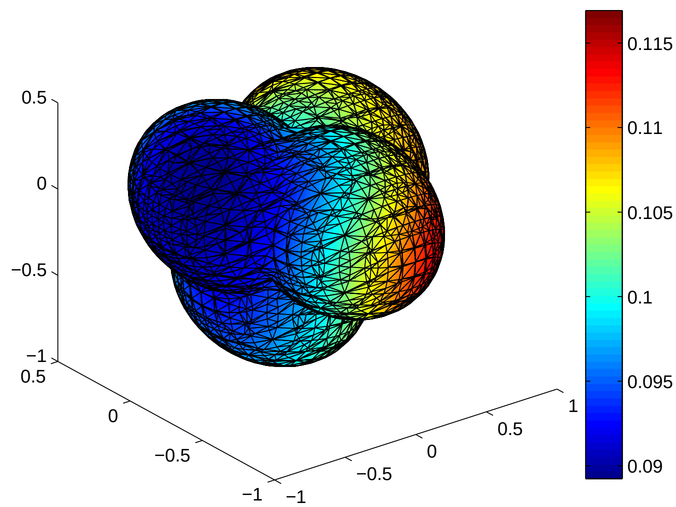

Figure 7: Example 4: Four Balls

$n_x \times n_y \times n_z$	Err in U	Order
$6 \times 6 \times 6$	0.04193	
$12 \times 12 \times 12$	0.01426	1.5556
$24 \times 24 \times 24$	0.00370	1.9467
$48 \times 48 \times 48$	0.00100	1.8939
$96 \times 96 \times 96$	0.00025	2.0183

Table 3: Example 4: Four Balls

the solution $u^{\pm}$ are given as follows:

$$\phi(x,y,z) = (x-0.4)^2 + y^2 + z^2 - 0.16,$$

$$\begin{aligned}
\beta^+(x,y,z) &= \begin{pmatrix} 4x^2+6 & \sin(y+x) & yx \\ \sin(y+x) & 2z^2+3 & 0.5\sin(x) \\ yx & 0.5\sin(x) & \cos(xy+z)^2+5 \end{pmatrix}, \\
\beta^-(x,y,z) &= \begin{pmatrix} \cos(x+y)^2+3 & z & 0.2\sin(z-x) \\ z & z^2+5 & y \\ 0.2\sin(z-x) & y & \sin(z)^2+2 \end{pmatrix}, \\
u^+(x,y,z) &= (x^2+y^2+z^2)^{5/6}, \\
u^-(x,y,z) &= \sin(x+y)
\end{aligned}$$

Figure 8 shows the numerical solution (the value from outside at the interface) with our method using 24 grid points in both x, y and z directions. Table

4 shows the error on different grids.

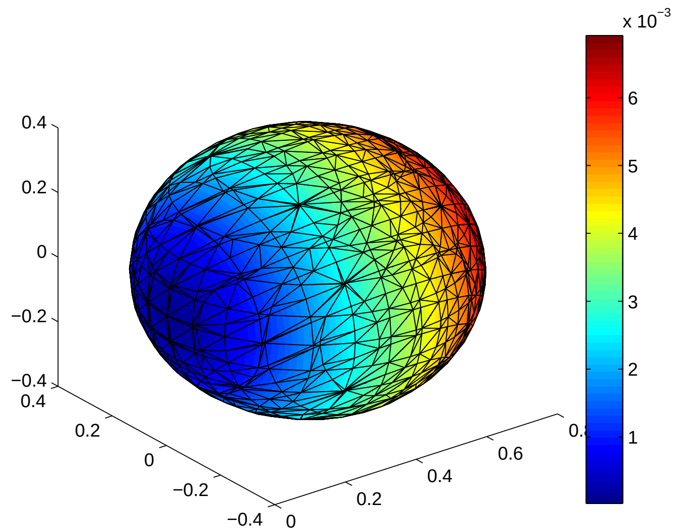

Figure 8: Example 5: singularity

$n_x \times n_y \times n_z$	Err in U	Order
$6 \times 6 \times 6$	0.02227	
$12 \times 12 \times 12$	0.00722	1.6262
$24 \times 24 \times 24$	0.00225	1.6816
$48 \times 48 \times 48$	0.00069	1.6951
$96 \times 96 \times 96$	0.00019	1.8700

Table 4: Example 5: singularity

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