HOMEWORK 3: ITERATIVE METHODS

DUE DATE: NOV 13

(1) The product of two symmetric matrices is not necessarily symmetric but will be in a different inner product.
   (a) Suppose A and B are SPD, prove that BA is SPD with respect to the inner product $(\cdot\,,\cdot)_A$ or $(\cdot\,,\cdot)_{B^{-1}}$.
   (b) Prove that
$$I - \bar{B}A = (I - BA)^*(I - BA).$$
(2) Prove that the convergence rate of Richardson, weighted Jacobi method, and Gauss-Seidal method for the 5-point stencil finite difference method of the Poisson equation on a uniform mesh with size h, is like
$$\rho \le 1 - Ch^2.$$
Thus when $h \to 0$, we will observe slow convergence of those classical iterative methods.
(3) Consider k-steps of Richardson methods with different parameters $\omega_1, \ldots, \omega_k$. Then the error equation is
$$e^k = (I - \omega_k A) \cdots (I - \omega_1 A)e^0.$$
Consider the optimization problem of choosing k-parameters:
$$\min_{\omega_i \in \mathbb{R}, i=1,\ldots,k} \left\{ \max_{\lambda \in [\lambda_{\min}(A), \lambda_{\max}(A)]} |(I - \omega_k \lambda) \cdots (I - \omega_1 \lambda)| \right\}. \tag{1}$$
Find the solution of (1) and derive the rate. This trick is known Chebyshev acceleration.
(4) Let us consider the matrix equation
$$Au = f,$$
where A is an $N \times N$ SPD matrix. Let us take the trivial decomposition of $\mathbb{R}^N = \sum_{i=1}^N \operatorname{span}\{e_i\}$, where $\{e_i, i = 1, \ldots, N\}$ is the canonical basis of $\mathbb{R}^N$. Prove that
   - for $R_i = \omega I$, PSC is Richardson method;
   - for $R_i = A_i^{-1}$, PSC is Jacobi method;
   - for $R_i = A_i^{-1}$, SSC is the Gauss-Seidal method.

(5) Prove that PSC using local solvers $R_i = A_i^{-1}$ is equivalent to the Jacobi method for solving the large system $\tilde{A}\tilde{u} = \tilde{f}$.