Geometry of Chaotic and Stable Discussions

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1. INTRODUCTION. It always seems to be the case. No matter how hard you might work on a proposal, no matter how polished and complete the final product may be, when it is presented to a group for approval, there always seems to be a majority who wants to "improve it." Is this just an annoyance or is there a reason? The mathematical modeling provides an immediate explanation in terms of some interesting and unexpected mathematics. Even more: the mathematics describing this behavior underscores the reality that it can be surprisingly easy even for a group sincerely striving for excellence to make inferior decisions. Indeed, these difficulties are so pervasive and can arise in such unexpected ways that it is realistic to worry whether groups you belong to have been inadvertently victimized by these mathematical subtleties based on the orbits of symmetry groups. These problems can occur even if all decisions are reached by consensus, such as a committee discussing the selection of a new calculus book.

This paper addresses deliberations by discussing a branch of voting theory where Euclidean geometry models an "issue space." When describing how it is possible to unintentionally make inferior choices, we will encounter mathematical behaviors remarkably similar to "attractors" and "chaotic dynamics" from dynamical systems. Since the coexistence of chaotic and stable behavior is common in the Newtonian N-body problem and dynamical systems, it is interesting to find that this combination also coexists in the "dynamics of discussions." Another connection arises when configurations central to the N-body problem play a suggestive role in the analysis; at another step we use singularity theory. What adds to the delight of this topic is that, while the mathematics can be intricate, the issues can be described at a classroom level where some even lead to student-level research projects.

2. SYMMETRY AND SOME OF ITS CONSEQUENCES. A convenient way to introduce the mathematical structures that cause problems is with an example (see Saari [**14**], [**15**]) explicitly designed to underscore the reality that an election outcome need not reflect the views of the voters. To emphasize that outcomes can drastically change with the choice of an election procedure, I often joke that

> For a price, I will come to your organization before your next election. You tell me who you want to win. After talking with members of your group, I will design a voting procedure that involves all candidates in which your designated choice will be the sincere winner.

To illustrate, suppose that in a department of thirty, the voter preferences for a slate of candidates $\{A, B, C, D, E, F\}$ (where "$A \succ B$" means that A is strictly preferred to B) are as in Table 1.

Table 1.

Number	Ranking
10	$A \succ B \succ C \succ D \succ E \succ F$
10	$B \succ C \succ D \succ E \succ F \succ A$
10	$C \succ D \succ E \succ F \succ A \succ B$

(1)

It is trivial to find a way to elect C. The real challenge is to elect F: this is because *everyone* prefers C, D, and E to F, so this group clearly views F as the inferior choice. But notice what would happen should candidates be compared pairwise, with the loser eliminated and the winner advanced to be compared with the next candidate. If we start by comparing D and E, D wins unanimously; comparing D with C, C wins unanimously; comparing C with B, B wins with a two-thirds vote; comparing B with A, A wins with a two-thirds vote; and in the final comparison of A with F, F is the final winner with a two-thirds vote. Nobody likes F, but each comparison is decided with a vote of landslide proportions, so who would dare argue with the final outcome?

Such behavior can, and probably often does occur in discussions where the goal is to reach a consensus. Imagine trying to select a calculus book. Let's see: book D costs less than E; C does a better job of describing limits than D; B has a better selection of problems than C; ... The rest of the story is apparent; after selecting book F *everyone* leaves the meeting disappointed not only with the selection but with the tastes and standards of his or her colleagues.

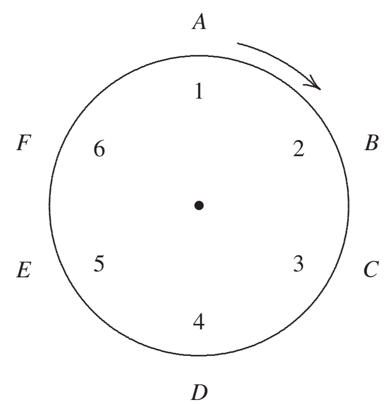


Figure 1. Ranking wheel and Z_6 orbits.

To understand this phenomenon, notice that Table 1 consists of three of the six terms from the Z_6-orbit of the ranking $A \succ B \succ C \succ D \succ E \succ F$. A convenient way to introduce this structure to nonmathematicians and students is with what I call a *ranking wheel*. Equally spaced, place the numbers 1, 2, ... up to the number of candidates (six in our case) along the edge of a freely rotating wheel attached to a wall. Then, as illustrated in Figure 1, write on the wall the names of the candidates as given in an initial ranking. Next, rotate the wheel to move the "1" under the next candidate, in this instance B, and read off the second ranking: $B \succ C \succ D \succ E \succ F \succ A$. Continue creating rankings in this manner until each candidate is in first place precisely once. While the construction does not favor any candidate, any three or more rankings from this orbit create a pairwise voting cycle. With a little experimentation, it becomes obvious how to compare candidates to make any designated choice the "final winner."

This structure does not merely provide *a* way to generate cyclic outcomes, it is the *only* way. It was recently proved (see Saari [**13**]) that all possible pairwise comparison anomalies with n alternatives, whether used to describe surprising outcomes from tournaments, agendas, pairwise cycles, comparison of pairwise outcomes with other procedures, and so forth, arise because the data includes components of Z_n-orbits of the alternatives.[1] An extension of these comments explains all those mysteries that occur with other aggregation methods such as the well known difficulties of pairwise comparisons in probability and statistics. More generally, we now know that all of those mysterious voting paradoxes, which have been described in many delightful papers but which can cause serious problems with actual elections, occur because embedded within the data (voter preferences) are components of orbits of a wide variety of symmetry groups (see (Saari [**13**], [**14**]). Moreover, by imposing a coordinate system on the space of data where some coordinate directions correspond to these symmetry configurations, it follows that almost all data sets must be tainted by these symmetry structures.

If so many unexpected difficulties are caused by the symmetries defined by a finite number of objects, imagine what might occur with a continuum of alternatives constrained only by residing in Euclidean n-space R^n. An aspect of this issue is explored for the remainder of this paper.

3. ISSUE SPACE. What complicates a selection process are the competing issues. Even when selecting a calculus book we worry about the book size, the cost, the graphics, the exposition, the exercises, and so forth. In national legislation, the issues might involve balancing the amount of money dedicated toward foreign aid and domestic issues including NSF-sponsored research. A department's graduate committee may worry about the level of a TA's stipend combined with the expected number of hours of work.

Following the lead of Hotelling [**4**], as extended by others including Enelow and Hinich [**2**], Kramer [**5**], McKelvey [**7**], and Plott [**10**], the obvious way to model n issues is to assign each issue to an axis of R^n. Designate a voter's level of support for the various issues by a point in R^n called the voter's "ideal point." In the graduate student TA example, an ideal point in R^2 represents a voter's desired level of (stipend, hours of work). Similarly, since a proposal, or an item of legislation, describes a particular combination of the issues, it also is represented as a point of R^n. As for voter preferences, it is reasonable to assume that the closer a proposed alternative is to a voter's ideal point, the more the voter likes it. The first goal is to determine which alternatives will be adopted by a majority vote for a specified set of voters' ideal points.

To illustrate with a simple example, consider three voters and two issues such that the ideal points define the vertices of a triangle in R^2. Which alternative should these voters adopt? The baricenter? How about favoring a particular voter by selecting her ideal point? The surprising fact is that, whatever point is selected, a majority of the voters can successfully offer a competing counterproposal! Rephrasing this assertion in terms of frustration that many of us have experienced, no matter how refined and complete a proposal may be, during a meeting some majority can propose an "improving amendment" that will pass. For instance, at almost any MAA business meeting when bylaw changes and other legislation are introduced, one can expect amendments.

The geometry explaining this situation is demonstrated in Figure 2, where the ideal points for the three voters indicated by bullets define the vertices of a triangle and the proposal is depicted by the diamond in the interior of the triangle (it is hidden behind the intersection of the three circles). Each circle has its center at a particular voter's ideal point and passes through the diamond proposal. As all points inside a particular circle are closer to the associated voter's ideal point, this voter prefers any of these points to the diamond proposal.

[1] By using this fact, a wide selection of interesting problems can be designed for students. By experimenting with terms from Z_n-orbits they can create several examples using pairwise comparisons with counterintuitive outcomes.

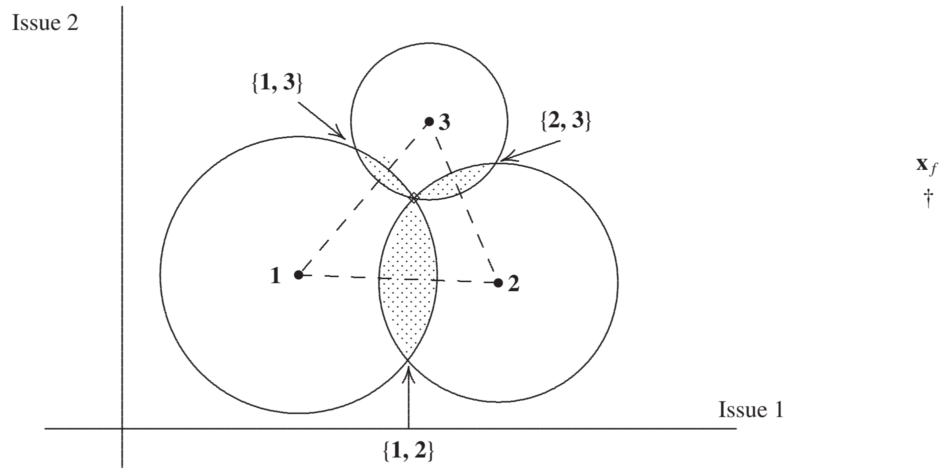


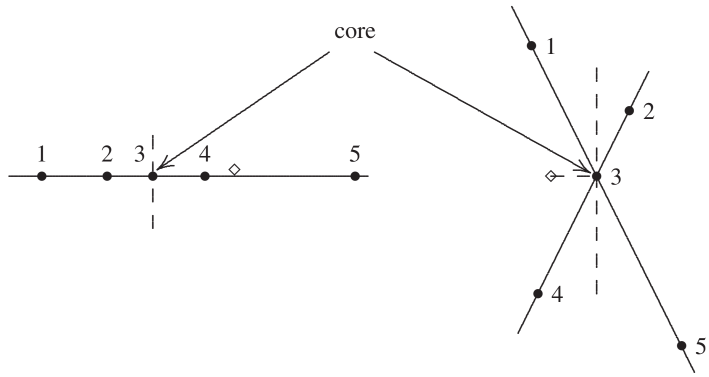
Figure 2. Forming coalitions.

The intersections of the circles in Figure 2 define the trefoil shaded regions. Each shaded leaf identifies points that a majority of the voters prefers to the diamond proposal. The largest region, for instance, consists of alternatives that coalition $\{1, 2\}$ strictly prefers to the proposed diamond, while the upper-right leaf identifies all options strictly preferred by coalition $\{2, 3\}$. In other words, with a surprisingly wide variety of possibilities, any majority can force an "improvement" over the original diamond proposal.

The discouraging observation is that, unless the ideal points lie along a line (where an appropriate combination of the issues defines a single issue), circles constructed as suggested intersect for *any proposal*. Consequently, whatever the new proposal, there is always a majority coalition that can offer an "improved alternative." The dynamic continues; another majority can be found to propose an even better "improvement" to the just approved "improvement," and for this proposal ....

This dynamic forces us to wonder whether, similar to the selection of F in the earlier Z_6 example, a group might adopt an outcome that everyone dislikes more than the original proposal. This is the case. Notice, for instance, the dagger hiding in the far right of Figure 2. As described later in this paper, there is a sequence of "improvements," each approved by a majority vote, starting at the diamond and ending at the dagger. As mathematicians, it is reasonable to explore this setting in order to understand how bad the situation can be and whether other voting procedures offer help. But before addressing these concerns, let me identify what it takes for a proposal to be "durable" in the sense that there are no successful counterproposals.

Core and attractors. Should the ideal points lie on a line, as in Figure 3a, certain proposals could never be undone by any majority. As simple experimentation using the "circle geometry" proves, the only durable proposal in the five voter setting of Figure 3a is voter three's ideal point—the views of the *median voter*. For instance, select any proposal to the right of this ideal point, say the diamond. While voters four and five support the new proposal, all voters with ideal points on or to the left of the dashed vertical line, the majority coalition of $\{1, 2, 3\}$, prefer voter three's ideal point. More generally, the definition of "median" ensures that any point on one side or the other of the median voter's ideal point must be supported by less than a majority of the voters; i.e., it cannot replace the median voter's ideal point. This geometry makes



a. Single issue setting **b.** Special two issue setting

Figure 3. Finding the core.

it easy to understand why this "median voter" phenomenon is often used to explain the similarity of the political platforms for the major political parties.

In game theory, these durable configurations are called "core points." This term is used in what follows.

Definition 1. For a specified decision rule that compares two points, if **p** is such that no other point is preferred by the decision rule to **p**, then **p** is called a *core point.* The *core* is the set of all core points. If a voter's ideal point is a core point, then it is called a *bliss point.*

In Figure 3a, voter three's ideal point is the bliss point. Figure 3b depicts a two-dimensional setting that also has a core point: this symmetric five voter arrangement pairs ideal points on opposite sides of straight lines that pass through the third voter's ideal point. To construct a "Plott configuration" (see Plott [**10**]), start with an odd number of points on the line: the core is the median voter's ideal point. Next, partition the remaining points on the line into pairs where one is to the left, and the other to the right, of the median voter's ideal point. Rotate this pair about the median voter's ideal point.

The Plott configuration always admits a core. As true with Figure 3a, voter three's ideal point is the Figure 3b bliss point. To see this, suppose the diamond in Figure 3b is a counterproposal. Let the line from the diamond to voter three's ideal point (the horizontal dashed line in the figure) be the normal vector for the vertical dashed line passing through this point. All voters with ideal points on and to the right of this vertical dashed line, a majority coalition of {2, 3, 5}, prefer three's ideal point to the diamond.

As Plott's construction can be used in any dimension, for any number of issues there always exists a core point whenever there are an odd number of voters. The same statement holds for an even number of voters. To see this, add a fictitious "median voter" to create a Plott construction. While this point is not a true ideal point, it is a core point for the even number of voters. (In Figure 3b, for instance, a core point for voters 1, 2, 4, and 5 is where voter three—who is now fictitious—had an ideal point.)

The core—in Figures 3a and 3b this is a bliss point given by voter three's ideal point—enjoys properties reminiscent of an attractor from dynamical systems. To explain, notice that the diamond proposal in Figure 3a can be beaten by any alternative closer to voter three's ideal point. In turn, the new proposal can be beaten by any proposal even closer to the median voter's ideal point. Similarly, proposals in Figure 3b that are closer to voter three's ideal point on the horizontal dashed line beat the diamond proposal. It is this succession of successful proposals and counterproposals converging to the core that resembles an attractor in dynamics.

A natural generalization of the majority rule requires a specified super-majority, such as a two-thirds or four-fifths vote, for victory. This is called a "q-rule."

Definition 2. For n voters, a *q-rule* is one for which an alternative wins if and only if it receives at least q-votes.

Let $\lceil x \rceil$ be the function that rounds a real number up to the nearest integer. Majority rule is where $q = \lceil (n+1)/2 \rceil$. The decision procedure currently used to select a pope for the Catholic Church requires over a two-thirds vote of the cardinals: this is the q rule for $\lceil (2n+1)/3 \rceil$. Unanimity is where $q = n$. An obvious relationship follows.

Proposition 1. *If* $\mathbf{p}$ *is a core point for a* q_1*-rule, then it is a core point for any* q*-rule where* $q > q_1$.

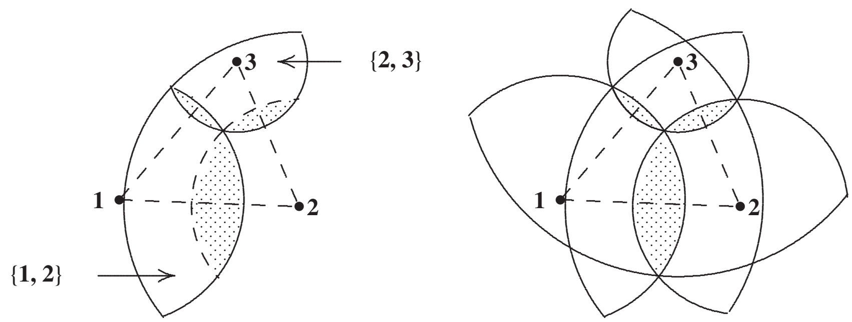
Implications of an empty core. If a core ensures stability and even serves as an attractor for the dynamic of proposals and counterproposals, what happens when the core is empty, as with the ideal points in Figure 2? Problems must arise because, by definition, an empty core means that any proposal can be beaten by some other proposal. But even without the stability ensured by a core, it is reasonable to expect these proposals to satisfy some nice property such as remaining within a reasonable distance of the ideal points. This is not the case: the democratic process permitting counterproposals allows discussions to resemble "chaotic dynamics." While counterintuitive, this assertion may have been anticipated by veterans of departmental politics.

To explain, McKelvey [**7**] used arguments from differential topology to prove the remarkable fact that, if the core is empty, it is possible to go, via majority votes, from *any proposal* to *any other proposal*. There are no restrictions on the beginning and final choices.

Theorem 1 (McKelvey). *Suppose the ideal points of the voters have an empty core for the majority vote. For any two proposals* $\mathbf{p}_b$ *and* $\mathbf{p}_f$ *there exists a sequence of counterproposals* $\{\mathbf{p}_j\}_{j=1}^N$ *that starts at the beginning proposal* $\mathbf{p}_b = \mathbf{p}_1$ *and progresses to the final proposal* $\mathbf{p}_f = \mathbf{p}_N$ *by majority votes; i.e., for each* $j = 1, \dots, N-1$, *proposal* $\mathbf{p}_{j+1}$ *beats* $\mathbf{p}_j$ *by a majority vote.*

An immediate consequence of this so-called chaos theorem is that there exist proposals beginning at $\mathbf{p}_b$ that pass through any number of specified proposals in any specified order, only to return to $\mathbf{p}_b$. To illustrate with the ideal points in Figure 2, a sequence of proposals can be found starting at the diamond, moving high into the second quadrant (assuming that negative values for coordinates have interpretations), jumping next over to the dagger, then progressing all the way to a point in the extreme right side in the fourth quadrant before ending at the original diamond proposal. No wonder some departmental meetings seem interminable and cyclic in substance! (Richards [**11**] relates the "voting chaos theorem" with "dynamical chaos" by mimicking mathematical approaches used to demonstrate dynamical chaos to recover the voting result.)

A way to develop insight into the mathematical structure at play here is to compute some of these agendas. To provide intuition, the shaded region in Figure 4a contains all points that can beat the original proposal from Figure 2. To find the possible second



a. First iterate by {1, 3} **b.** All second iterates

Figure 4. Second iterate.

counterproposals, start with the points in the upper left shaded leaf. The points that can be reached in two steps passing through this leaf are inside the solid curved arcs of Figure 4a. This region resembles a pinched pea pod and includes the shaded region as well as voter three's ideal point. Notice the dynamic: the first step, a successful amendment made by coalition {1, 3}, defines a point in this leaf; the second step requires either voter one or voter three to create a new coalition with voter two, the possible proposals of which are determined by a circle using voter two's ideal point as the center that passes through the choice determined at the first step. Figure 4a describes the extreme situation by using the far tip of the upper left leaf.

To find all points that can be reached in two steps, compute similar regions for the other two shaded leafs of the original trefoil; these three superimposed pinched pea pods are in Figure 4b. As an illustration of what we learn from this geometry, these ideal points make it possible to go from the diamond proposal to any specified voter's ideal point with just two "amendments." Notice the opportunities provided to a sufficiently crafty individual who could channel the discussion and change coalitions in a manner to ensure that the final outcome is precisely what he wants: his ideal point.

To compute what can happen in three steps, use the same approach. Notice from Figure 4b, where the shaded region contains the proposals that can win with one step and the superimposed peapods identify the alternatives that can be adopted in two steps, how the regions of successful counterproposals expand quite rapidly. For three steps, the regions expand even more.

This chaos theorem supports my earlier "for a price ..." claim. Namely, with an empty core and any initial proposal, there are always ways to make counterproposals, each accepted by a majority vote, so as to reach eventually any specified point. A particularly worrisome interpretation is that, even with the best intensions, voters could keep "improving" a proposal, each by a majority vote, only to reach a final version to which *everyone* strongly prefers the original.

Rate of convergence. But what is the minimum value of N? For instance, although some meetings may seem incessant, they never allow $N = 10^{100}$ proposals, if only because this would require more time than allowed by the age of the universe. Consequently, should the minimal value of N required to go from $\mathbf{p}_b$ to $\mathbf{p}_f$ be sufficiently large, the chaos theorem would lose practical significance.

One of my former Ph.D. students, Maria Tataru, investigated this growth rate question in her dissertation. But first she extended the McKevley "chaos theorem" from the majority vote to any q-rule. (See Tataru [**20**], [**21**].)

Theorem 2 (Tataru). *Suppose that the ideal points in R^n for a finite number of voters fail to admit a core for a q-rule. For any two proposals $\mathbf{p}_b$ and $\mathbf{p}_f$ there exist N*

proposals $\{\mathbf{p}_j\}_{j=1}^N$ *such that* $\mathbf{p}_b = \mathbf{p}_1$, $\mathbf{p}_f = \mathbf{p}_N$, *and* $\mathbf{p}_{j+1}$ *beats* $\mathbf{p}_j$ *with the* q*-rule for* $j = 1, \dots, N-1$.

Tataru's proof differs from McKelvey's in that, rather than using differential topology, she emphasizes the set-orbit structure introduced by the symmetry of the circles: the symmetries form a group, and she determined the associated orbits. Her orbit structure approach had the advantage of giving her a handle on the number of iterations N needed to go from one point to another. In this manner she showed that upper and lower bounds on the minimum value of N depend linearly on the distance between $\mathbf{p}_b$ and $\mathbf{p}_f$. As suggested by Figure 4, the locations of the ideal points determine the size of regions of counterproposals; e.g., size matters, larger regions allow smaller values of N. In turn, a smaller bound on the minimum value of N means that the setting is more chaotic [**20**],[**21**].

Theorem 3 (Tataru). *With a finite number of voters whose ideal points fail to admit a core for a* q*-rule, upper and lower bounds for the value of* N *can be found that are linear in the (Euclidean) distance* $\|\mathbf{p}_b - \mathbf{p}_f\|_2$*. These bounds are determined by the locations of the ideal points.*

In her thesis [**20**], Tataru examined how the configuration defined by the ideal points determines the coefficients of the linear expression. To describe her result in terms of three voters, in Figure 5 let $h = \min(h_1, h_2, h_3)$ be the minimum of the three altitudes defined by the triangle. While she has even sharper estimates, Tataru proved that

$$C + \frac{\|\mathbf{p}_b - \mathbf{p}_f\|_2}{h} \leq \text{Minimum } N \leq C + 3\frac{\|\mathbf{p}_b - \mathbf{p}_f\|_2}{h}, \tag{2}$$

in which C is a constant needed to handle situations where $\mathbf{p}_b$ and $\mathbf{p}_f$ are close to each other. For example, if in Figure 2 $\mathbf{p}_b$ is the diamond proposal and $\mathbf{p}_f$ is a point close to $\mathbf{p}_b$ but outside the trefoil, it takes $N = 2$ steps, rather than one iterate, to achieve the goal. (Tataru's result is a limit theorem, so it is more applicable for large values of $\|\mathbf{p}_b - \mathbf{p}_f\|_2$.)

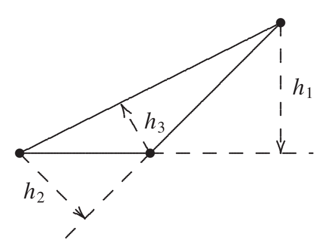


Figure 5. The minimum altitude and growth rates.

According to inequality (2), the smallest bounds on N occur when the ideal points define an equilateral triangle; this setting permits the maximum chaotic behavior. At the other extreme, $h \to 0$ requires that the location of the ideal points approximate a straight line setting in which a core exists (here $N \to \infty$). Consequently, the closer the ideal points approximate settings where a core exists, the more subdued the successful proposals and counterproposals. While it remains theoretically possible to reach any desired point, it requires an unrealistic number of steps. Tataru's growth estimates establish a nice link between admissible chaotic behavior and the stability of a core point. Incidentally, while it is easy to form an agenda moving from any beginning $\mathbf{p}_b$ to a final $\mathbf{p}_f$, I do not believe that an algorithm specifying the "optimal" path (i.e., with a minimum N) has been found or even investigated. But by following the lead of Figure 4, finding such an algorithm seems to be a doable and reasonable project.

4. GENERIC PROPERTIES OF THE CORE. The core plays a central role in understanding voting behavior for q-rules. As asserted earlier, the combination of Plott's construction (see Figure 3) and Proposition 1 ensures that a core exists for any q-rule with any number of issues and voters. But what about structural stability? Will the core persist with slight changes to the ideal points? Namely, can a trivial shift in just one individual's preferences push a group's discussion from the stability setting of a core to McKelvey's chaotic framework?

To demonstrate that a core need not persist, I use the easily proved fact that with preferences defined by the Euclidean distance the points preferred by q voters are the points in the convex hull defined by their ideal points. Thus, for $n = 5$ and $q = 3$, the core is the intersection of the convex hulls defined by the $\binom{5}{3} = 10$ triplets. We use this fact to show that the configuration in Figure 6, where voter two's ideal point is only slightly changed from that in Figure 3b, has an empty core.

By moving voter two's ideal point, the convex hull defined by voters $\{1, 2, 4\}$ in Figure 6 (denoted by the dashed lines) meets the convex hull defined by $\{3, 4, 5\}$ only in voter four's ideal point. Similarly, this hull meets the hull defined by $\{2, 3, 5\}$ only in voter two's ideal point. Thus, the common intersection of all hulls—the core—is empty. To see, for instance, that voter three's ideal point is no longer a core point, notice that the majority coalition of $\{1, 2, 4\}$ prefers any point on the line segment starting from and perpendicular to the line connecting the ideal points of voters two and four and ending in voter three's ideal point.

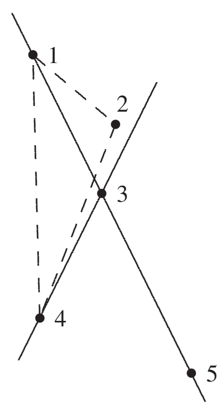
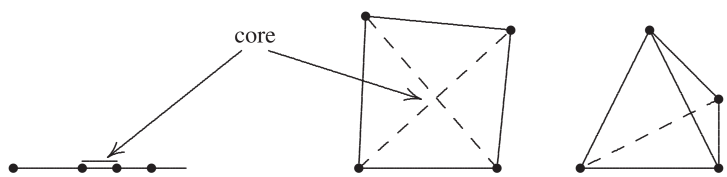


Figure 6. Dropping the core.

Geometry and the existence of the core. The geometry of convex hulls helps us appreciate when a core can, or cannot, persist under small changes in voter preferences. Intuitively, the more hulls there are, the more difficult it is for them to have a nonempty intersection. But since a q-rule with $q \geq \lceil (n+1)/2 \rceil$ defines $\binom{n}{q}$ possible coalitions and convex hulls and since $\binom{n}{q}$ increases as q decreases to $n/2$ (the majority rule), we must anticipate that the closer q is to majority rule, the more difficult it is for a core to exist or persist.

The dimensionality k of issue space also plays a crucial role. With $k = 1$ (a single issue), the core is nonempty because any convex hull involving $q \geq \lceil (n+1)/2 \rceil$ points must include the median voter's ideal point. This need not be true when $k \geq 2$. Indeed, as the majority rule for $n = 3$ requires $q = 2$, the convex hulls are the edges of the triangle defined by the ideal points. Because these edges never have a common intersection, a core never exists.

To trace what happens to the core with changes in dimensions, consider the special case of $n = 4$ and $q = 3$. When $k = 1$, the convex hull of any three points (Figure 7a) must include the two interior ideal points, so the core is the closed interval defined by these two points. This core clearly persists with changes in these points. If $k = 2$ (Figure 7b), the convex hulls are now triangles consisting of any two edges of the quadrilateral sharing a common vertex and a diagonal. These hulls meet at the intersection of the diagonals to create a core in a two-dimensional issue space; again, changes in the ideal points only move the diagonals, so the core persists. (Similarly, an ideal point in the interior of the triangle defined by the other three ideal points must be a bliss point.) But for $k = 3$, the four ideal points define a tetrahedron whose faces are the hulls in question. As the faces never have a common intersection, a core does not exist. So, with changes in k, the core for this example progresses from an interval, to a point, to the empty set.



a. $k = 1$; core = interval **b.** $k = 2$; core = point **c.** $k = 3$; core = $\emptyset$

Figure 7. Changes in the core with dimensions.

In general, larger q values define fewer hulls. Since these hulls require more points, their larger size and dimension make it is easier to have a stable core. Indeed, as the hulls for a q_1-rule are subsets of hulls for a q-rule with $q > q_1$, results about the geometric structure of the core, and others such as Proposition 1, follow immediately. Incidentally, these notions probably can be combined to determine which issue space dimensions allow a structurally stable core, but I doubt whether anyone has tried this approach.

Stability of a core. The natural problem is to understand when a core will persist with small changes in the ideal points. Restating the dimensionality comments in common terms, more issues provide more reasons for voters to disagree, so it is more likely for chaos to ensue. Using these terms, this concern was brought to my attention by R. Kieckhefer, a professor of the history of religions at Northwestern University. He pointed out that several times when a simple majority ($q = \lceil (n+1)/2 \rceil$) was used to elect a pope for the Catholic Church, the precarious instability of the voting system, generated by raising new issues (so k increases) to induce voters to change opinion, caused the church to erupt into dissension and conflict, with a pope and an antipope vying for power. To achieve stability, in 1179 the Third Lateran Council adopted the current $q = \lceil (2n+1)/3 \rceil$ rule; stability was achieved—for a while. This history suggests that the persistence of the core involves finding an upper bound on k, the number of issues, in terms of the values of q and n. As shown next, this is the case.

The importance of this problem caused it to attract considerable research attention. Schofield [**18**] and then McKelvey and Schofield [**8**], [**9**] obtained some bounds on k values. While their conclusion described only a subset of the relevant k values, it was correctly greeted as a major advance. Banks [**1**] found an error in these papers that, unfortunately, invalidated the conclusions and reopened the problem. The problem was finally completed in Saari [**12**]; the result is described next.

But first, why should a voter's preferences be defined in terms of Euclidean distances? After all, a voter placing more importance on one issue than another might measure "closeness" with ellipses. More generally, rather than the Euclidean distance where a person with ideal point $\mathbf{q}$ strictly prefers $\mathbf{x}$ to $\mathbf{y}$ if and only if $\|\mathbf{x}-\mathbf{q}\|_2 < \|\mathbf{y}-\mathbf{q}\|_2$, that is, $-\|\mathbf{x}-\mathbf{q}\|_2 > -\|\mathbf{y}-\mathbf{q}\|_2$, we determine what happens if the jth person's preferences are defined in terms of a *utility function* $U_j : R^k \to R$, where $\mathbf{x}$ is strictly preferred to $\mathbf{y}$ precisely when $U_j(\mathbf{x}) > U_j(\mathbf{y})$, and $\mathbf{x}$ is *indifferent* to $\mathbf{y}$ if and only if $U_j(\mathbf{x}) = U_j(\mathbf{y})$. Assume that these utility functions are smooth and strictly convex. (Just the assumption of smoothness is sufficient for much of what follows.) The Euclidean preferences become the special case $U_j(\mathbf{x}) = -\|\mathbf{x}-\mathbf{q}\|_2$. Other choices might define ellipsoids for level sets to capture individual scaling effects for certain issues. But notice: by generalizing to utility functions, the geometry defining the core may change.

Rather than changing ideal points, we must now determine when a core exists for an open set of utility functions. The topology imposed on the utility functions is the Whitney topology. (There are several references for this topology and the singularity theory used next; for example, see Golubitsky and Guillemin [**3**] or Saari and Simon [**17**].) In the following theorem, "generic" means that an assertion holds for a residual set of utility functions in this topology; that is, it holds for a countable intersection of open, dense sets. As one can show (see Saari and Simon [**17**]), if the proposals are restricted to a compact subset of R^k, the residual sets can be replaced with open dense sets. To interpret these comments in simpler terms, if k satisfies the specified bounds, then, in general, a core persists even with small changes in preferences. On the other hand, if k does not satisfy the bounds, then slight changes in preferences cause the core to disappear. To state the theorem, recall that a "bliss point" is a core point that also is a voter's ideal point. After the formal statement (see [**12**]), an easily used, intuitive description is given.

Theorem 4 (Saari).

a. *For a q-rule, bliss-core points exist generically if and only if*

$$k \leq 2q - n. \tag{3}$$

b. *Nonbliss core points exist, generically, for $k \leq 2$ when $q = 3$ and $n = 4$. If $n \geq 5$ and $4q < 3n + 1$, then nonbliss core points exist generically if and only if*

$$k \leq 2q - n - 1. \tag{4}$$

For super-majorities in which $4q \geq 3n + 1$, let α be the largest odd integer such that $q/n > \alpha/(\alpha + 1)$. Nonbliss core points exist generically if and only if

$$k \leq 2q - n - 1 + \frac{\alpha - 1}{2}. \tag{5}$$

c. *For any k and n there exists a q-rule where core points exist generically. In particular, the unanimity rule $q = n$ exists in all dimensions.*

Unless $4q \geq 3n + 1$, which is a super-majority where a successful vote requires more than three-fourths support, expect stable cores of some sort to exist whenever $k \leq 2q - n$. To interpret this inequality, notice that the maximal value of $k = 2[q - (n/2)]$ corresponds to the number of voters who would have to change their minds to reverse the outcome. In other words, the bound on k identifies the number of issues, one per voter, that need to be raised to change the election outcome. To illustrate with the two-thirds vote adopted by the Catholic Church, the current values are $n = 135$ cardinals and $q = 91$. For the $n - q = 44$ voters on the losing side to reverse the outcome, they would have to convince $91 - 44 = 47$ voters to change their minds. This number of required defections agrees with $k = 2q - n = 2(91) - 135 = 47$.

To illustrate further we consider the majority vote and an odd number of voters. As it takes only one voter to change the conclusion, a stable core exists only with a single issue. (Here $q = (n + 1)/2$, so $k = 2[(n + 1)/2] - n = 1$.) The situation improves only slightly with an even number of voters. It takes two voters to change an outcome, so the core persists for up to two issues. (Here $q = (n/2) + 1$ so $k \leq 2[(n/2) + 1] - n = 2$ issues.) As it is difficult to imagine elections with only two issues, these assertions underscore the precarious nature of this standard voting method.

It is interesting to wonder whether this instability is observed in actual elections. For instance, if the voters' general perception of a candidate places her at a core point, she will win. To defeat her, her opposition must destroy the core: they must change the voters' perception of the winning candidate. Theorem 4 describes how to do this: introduce new issues in a way that will perturb voter preferences. This flurry of new issues is common in the closing days of any closely contested election. In effect, this activity—that may, or may not, be manifested by negative campaigning—increases the dimension k.

Yet, in other ways, the conclusion of Theorem 4 seems to be in conflict with reality: we can enjoy stability in two-candidate elections for, say, mayor. To explain, notice that Theorem 4 describes what happens if there is a freedom to advance different proposals. In contrast, a mayoral or gubernatorial election involves specific candidates, so it imposes stability. Thus, instability requires permitting any proposal, or candidate, to join. An illustration is the 2003 California recall election for governor, where 135 competing proposals (candidates) including a self-described "porn-queen" were thrown into the mix. Unintentionally reflecting the mathematics, the press commonly described the situation as "chaotic"!

It remains to discuss the "super-majorities" described by inequality (5). This inequality shows that a super-majority election provides a slight stability bonus by adding extra dimensions to the "number of voters who need to reverse opinions" computation. With a three-fourths rule, the bonus allows the number of issues permitting stability for a bliss and nonbliss core point to agree. With a five-sixths rule, the nonbliss core points exist for a dimension $k = 2q - n - 1 + (5 - 1)/2 = 2q - n + 1$.

The number of extra dimensions, however, is "slight" when using reasonable super-majorities. After all, the extreme 90% rule, where $q/n > 9/10$, adds only four extra dimensions to permit the core to persist with $k = 2q - n - 1 + 4 = 2q - n + 3$ independent issues. The mathematical significance of these bonus dimensions is that as the q-rule approaches unanimity, the issue space dimension k grows without bound as it must. But for the more widely used super-majorities and for Euclidean preferences, the appropriate choice is the $k = 2q - n$ value corresponding to the "number of voters who need to reverse opinions" computation. (For Euclidean preferences, an exception occurs for $q = n - 1$ to reflect partially the transition to $k = \infty$ for $q = n$.) Notice a peculiarity: these bonus dimensions never involve bliss points. The reason is given in the next section.

5. OUTLINE OF THE PROOF. Although the proof of Theorem 4 is technical and long (it requires about thirty published pages), the basic ideas are natural: this is where singularity theory and configurations from the Newtonian N-body problem play a role. To start, with n voters and a specified proposal $\mathbf{x}$, alternatives that are preferred by the jth voter are determined by the gradient $\nabla U_j(\mathbf{x})$. Actually, it is not the gradient that matters, but its direction $\nabla U_j(\mathbf{x})/\|\nabla U_j(\mathbf{x})\|$. (The strict convexity assumption forces the preferred choices to be strictly in the half-plane that includes the normal vector $\nabla U_j(\mathbf{x})$.) If at $\mathbf{x}$ the gradient directions of a majority of the voters point in the same general direction, then this majority can successfully propose an alternative. Therefore proposal $\mathbf{x}$ is a core point only when a limited number of these gradient directions point in the same general direction. Only the gradient directions are needed, so the gradient conditions for a core can be described as an arrangement of points on the sphere S^{k-1}.

To achieve the objective, which is to determine when the core condition is or is not robust, we use singularity theory. To motivate the approach, consider the well-known fact that, generically, the critical points of a smooth mapping $F : R^2 \to R^1$ are isolated. In words, if (x, y) is a critical point for F, then in any sufficiently small neighborhood of (x, y), there are no other criticial points of F. As with the core this assertion about critical points being isolated combines domain points and the gradient. So, to outline a proof that motivates the approach used to study the core, we use the five dimensional jet space $J^1 = (x, y; z; A, B)$ where x, y, z, A, and B belong to R^1 and J^1 is endowed with the appropriate topology. This space J^1 is intended to capture the domain, range, and first derivative terms, so

$$j^1(F)(x, y) = \big(x, y; F(x, y); \nabla F(x, y)\big) \tag{6}$$

is a mapping from R^2 to the five-dimensional J^1.

To describe the generic properties of critical points, we need to find a subspace of J^1 that characterizes these points. This is easy: critical points are determined by the three-dimensional subspace $\Sigma = (x, y; z; 0, 0)$ where the gradient is zero. Thus, all critical points of F are given by $[j^1(F)]^{-1}(\Sigma)$. Since Σ has dimension three, or codimension two, the inverse function theorem tells us that, provided $j^1(F)$ satisfies the appropriate determinant conditions, $[j^1(F)]^{-1}(\Sigma)$ is a codimension two (zero-dimensional) set in R^2; that is, the critical points of F are isolated.

To describe the relevant determinant conditions, first consider a smooth mapping $G : R^k \to R^m$, and let Σ be a smooth s-dimensional submanifold of R^m. The goal is to find appropriate conditions on G so that the inverse function theorem can be used to describe $G^{-1}(\Sigma)$. To do so, the problem of finding $G^{-1}(\Sigma)$ is converted into a problem where the answer is well known: the inverse image of a point. The first step is to notice that Σ is given locally by $g^{-1}(\mathbf{0})$ where g is an appropriately chosen smooth mapping $g : R^m \to R^{m-s}$. By composition of maps, it follows that when $G(\mathbf{x})$ is in Σ, we have that $g(G(\mathbf{x})) = \mathbf{0}$ in R^{m-s}. Consequently, the inverse image $G^{-1}(\Sigma)$ of the manifold Σ can be described locally as the inverse image of a point $[g \circ G]^{-1}(\mathbf{0})$. In order to use the inverse function theorem, $Dg(G)$ evaluated at $\mathbf{x}$ must have full rank $m - s$. By use of the chain rule, this full rank condition translates into the condition that the product of the matrices, $D_y g D_\mathbf{x} G$, where $y = G(\mathbf{x})$, must have full rank $m - s$. In this expression, the Jacobean Dg is evaluated at $y = G(\mathbf{x})$ and DG at $\mathbf{x}$.

The next step is to determine conditions G must satisfy to ensure that the matrix product $D_y g D_\mathbf{x} G$ has maximal rank. Stated in another manner, we need to determine how $D_y g D_\mathbf{x} G$ acts on a vector $\mathbf{v}$ in R^k so that it is, or is not, in the kernel. The first leg of $\mathbf{v}$'s journey is determined by $D_\mathbf{x} G(\mathbf{v})$: the matrix $D_\mathbf{x} G$ maps $\mathbf{v}$ from R^k to R^m. The next task is to determine those locations of $D_{\mathbf{x}}G(\mathbf{v})$ such that, in the second leg of $\mathbf{v}$'s journey, the matrix $D_y g$ will, or will not, map $D_{\mathbf{x}}G(\mathbf{v})$ to zero.

The answer for this question comes from the definition of $g^{-1}(\mathbf{0})$. Because $g^{-1}(\mathbf{0})$ defines a local portion of Σ, matrix Dg maps the tangent vectors of Σ to the point $\mathbf{0}$. In other words, the kernel of $D_y g$ is the s-dimensional tangent space of Σ at $y = G(\mathbf{x})$. Denote this space by $T_{G(\mathbf{x})}\Sigma$.

Combining these arguments, we see that if $D_{\mathbf{x}}G(\mathbf{v})$ is in the tangent space $T_y\Sigma$, for $y = G(\mathbf{x})$, then $\mathbf{v}$ is in the kernel of $D_y g D_{\mathbf{x}} G$. Thus, to satisfy the rank condition, the image of $D_{\mathbf{x}}G$ must include a $(m - s)$-dimensional linear subspace that is *not* in the tangent space $T_y\Sigma$. An argument putting these statements together now shows that for $D_y g D_{\mathbf{x}} G$ to have maximal rank, it must be that

$$\mathrm{Span}\big[D_{\mathbf{x}}G(R^n) \cup T_y\Sigma\big] = R^m \tag{7}$$

for $y = G(\mathbf{x})$. When condition (7) is satisfied, we say that G has a *transverse intersection with* Σ at $G(\mathbf{x})$.

Let me illustrate this condition with our motivating example about the critical points of F. Here the jet mapping $j^1(F)$ takes the role of G, so we need to determine whether $j^1(F)$ has a transverse intersection with $\Sigma = (x, y; z; 0, 0)$. The first step, which is to find $Dj^1(F)$, requires differentiating the three terms in (6) to obtain

$$Dj^1(F) = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \nabla F; D^2 F\right).$$

According to this expression, if $\mathbf{v}$ is in R^2, then

$$Dj^1(F)(\mathbf{v}) = \big(\mathbf{v}; \nabla F \cdot \mathbf{v}; D^2 F(\mathbf{v})\big).$$

The other term is the tangent space $T\Sigma = (x, y; z; 0, 0)$ (for real x, y, and z) that can be identified with Σ. Therefore, the transverse intersection condition (7) is satisfied if and only if D^2F maps R^2 to R^2 in a manner that will compensate for the zeros in the component directions of $T\Sigma$ that represent the gradients. In other words, this condition is satisfied if and only if D^2F has rank two.

Although the analysis for the motivating problem is not difficult and reduces to a familiar condition, in general the computation to verify (7) can be messy and complicated. This reality underscores the importance of the following result by René Thom [**22**] (also see [**3**]). To afford a compact statement of it, the notion of a transverse intersection must be slightly modified.

Definition 3. A smooth mapping $G : R^k \to R^m$ has a *transverse intersection* with a submanifold Σ of R^m if either (1) $\mathrm{Image}(G) \cap \Sigma = \emptyset$ or (2) condition (7) is satisfied for each $\mathbf{x}$ in $G^{-1}(\Sigma)$.

We now can state Thom's theorem.

Theorem 5 (Thom). *Let Σ be a smooth submanifold of R^m. Generically, a mapping $F : R^k \to R^m$ has a transverse intersection with Σ (i.e., this is true for a countable intersection of open dense sets in the space of such mappings F).*

The "empty set" condition means that, generically, the mappings miss Σ. To illustrate this condition, suppose that we wish to find the generic property of functions $F : R^2 \to R^1$ whose critical points occur along the line $x = 1$. The appropriate J^1 submanifold is $\Sigma_1 = \{1, y; z; 0, 0\}$, which has codimension three. According to Thom's theorem, generically, mappings have a transverse intersection with Σ_1. If condition (7) were satisfied, then the set $(j^1(F))^{-1}(\Sigma_1)$ of critical points would have codimension three in R^2; i.e., it would be a set of dimension $2 - 3 = -1$, which cannot exist. Consequently, generically, a smooth function $F : R^2 \to R^1$ does not have a critical point along the line $x = 1$. The point is that while examples with this property are easy to create, slightly perturbing them destroys the property. As shown next, Plott's plots share this mathematical structure.

Core and the *N*-body problem. To describe the core conditions, we use the mapping $\mathcal{U} : R^k \to R^m$ whose components are the utility functions for all n agents:

$$\mathcal{U}(\mathbf{x}) = \big(U_1(\mathbf{x}), \dots, U_n(\mathbf{x})\big), \tag{8}$$

where the appropriate jet space is

$$J^1 = (\mathbf{x}; \mathbf{y}; \mathbf{A}_1, \dots, \mathbf{A}_n) \quad (\mathbf{x} \in R^k; \mathbf{y} \in R^n,\ \mathbf{A}_j \in R^k,\ j = 1, \dots, n)$$

and the jet map is

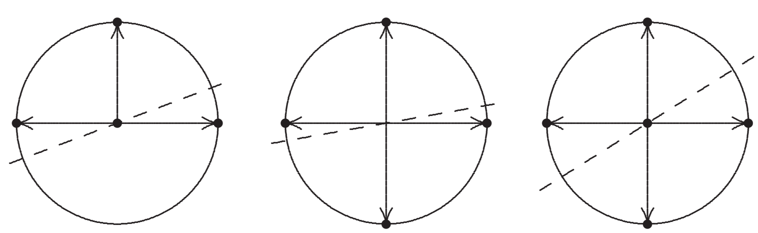
$$j^1(\mathcal{U})(\mathbf{x}) = \big(\mathbf{x};\ \mathcal{U}(\mathbf{x});\ \nabla U_1(\mathbf{x}), \dots, \nabla U_n(\mathbf{x})\big).$$

The approach is to define first the subspaces Σ that characterize the core conditions and then determine the codimension of these Σ choices.

The construction is based on the positioning of the gradients, which are treated as points on S^{k-1}. To see what is involved, suppose that $n = 4$, $q = 3$ (the majority rule), and $k = 2$. For a point to be a core point, any line passing through it cannot have more than $q - 1 = 2$ points on one side; if q or more points were on one side, they would define a winning coalition that has a commonly preferred alternative somewhere on that side of the line. This argument dictates that the points must be positioned on the circle so that for any line passing through the origin (the dashed line in Figure 8) at most two points are on either side. With a bliss point, as depicted in Figure 8a, this condition is satisfied with the open condition that any two of the remaining three vectors are separated by more than $\pi/2$. This leads to the definition

$$\Sigma_2 = \big\{(\mathbf{x}; \mathbf{y}; \mathbf{A}_1, \dots, \mathbf{A}_n) \mid \mathbf{A}_1 = \mathbf{0}, \mathbf{A}_j \cdot \mathbf{A}_k < 0 \text{ for } j, k = 2, 3, 4\big\}.$$

Since the restrictions on the vectors $\mathbf{A}_j$ for $j = 2, 3, 4$ represent an open condition, Σ_2 has codimension two (determined by $\mathbf{A}_1 = \mathbf{0}$). Thus, generically, the core consists of isolated points. This means, for instance, that generically the location of bliss points cannot define a curve.



a. $n = 4$, bliss point **b.** $n = 4$, symmetry **c.** $n = 5$, Plott

Figure 8. Finding Σ.

I leave it to the reader to show that for the nonbliss point setting of Figure 8b this condition, in which no three points are on the same side of any dashed line, forces pairs of the points to be positioned precisely opposite one another. As this imposes conditions such as $\mathbf{A}_1 = -\mathbf{A}_3$ and $\mathbf{A}_2 = -\mathbf{A}_4$, the corresponding Σ has codimension two. In this case the core consists of isolated points and persists with small changes in preferences.

The Plott configuration for $n = 5$ and $q = 3$ depicted in Figure 8c leads to a different conclusion. Since no more than $q - 1 = 2$ points can be on the same side of any line drawn through the center, even placing points equally spaced $2\pi/5$ radians apart (a codimension four construction) fails to meet the requirement: the condition allowing a core can be satisfied only with a bliss point. Namely, the corresponding Σ must have $\mathbf{A}_1 = \mathbf{0}$, a codimension two condition, with a symmetry condition imposed on the remaining points $\mathbf{A}_2 = -\mathbf{A}_4$ and $\mathbf{A}_3 = -\mathbf{A}_5$. Notice how these conditions define a codimension four setting. Generically, then, the core is empty. This determines a situation similar to requiring a critical point at $x = 1$, which can happen but will not persist.

This construction answers a question about the $\alpha/2$ bonus dimensions of Theorem 4: can they occur with bliss points? Yes, just place a voter's ideal point at the core point. But, while such core points exist, they do not exist generically. To explain, because a bliss point contributes codimension k to the Σ structure, all constraints on the remaining vectors must define open conditions. The constraints defining the "bonus" dimensions, however, are lower dimensional.

Part of the challenge in proving the theorem, then, is to determine whether n points can be positioned on S^{k-1} so that no more than $q - 1$ of them are on the same side of any hyperplane passing through the origin *and* the codimension of the associated Σ is no more than k. For $k = 2$, the construction uses a circle, so the analysis is easy. For $k = 3$ the analysis is slightly harder. The condition for $n = 4$ and $q = 3$ is easy because the symmetric positioning of four points on S^2 gives rise to a regular tetrahedron. But how should five, or six, points be positioned and can this be done using "open" conditions?

For $k > 3$, the challenge is more interesting. This is where insights from my research in the Newtonian N-body problem has helped. To suggest the connection, key to the N-body problem are configurations known as "central configurations:" this is where the position vector for each particle is the same scalar multiple of the particle's acceleration vector. An interesting fact is that there exist solutions that maintain these equilibrium configurations for all time. Now think of N equal masses placed on S^{k-1} though whose center a plane is passed. Should there be many more points on one side of the plane, then the configuration most surely would not be in "equilibrium." The particles would move to form a more balanced configuration. Considerations of this type motivated the final constructions needed to prove the theorem.

6. CONCLUSION. Voting is something we all do often, yet, as shown here and in Saari [**14**], the process is fraught with dangers. But voting is only one of many mathematical concerns that surface in the social and behavioral sciences. While the consequences and modeling of these difficulties belong to the social sciences, the mathematics underlying many of these issues can be quite sophisticated. Indeed, I expect that the only way many of these crucial problems will ever be resolved is through the muscle power of mathematics. In other words, more mathematicians need to get involved. There are delightful rewards: the mathematics can be fascinating, and the results often prove to be of interest and importance to a wide audience.

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