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The geometry of Black's single peakedness and related conditions

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Abstract

By using geometry to analyze all three candidate profiles satisfying Black's single peakedness constraint, we characterize all associated election behavior. The same analysis is applied to related profile constraints where some candidate never is top-ranked, or never bottom ranked. © 1999 Elsevier Science S.A. All rights reserved.

Keywords: Black's single peakedness condition; Geometry; Election behavior

1. Introduction

'Black's single peakedness condition' (Black, 1958), a profile restriction widely used to avoid cycles (Thomson, 1993), is where (for three candidates) some candidate never is bottom ranked by the voters. Related restrictions which also prevent cycles are where some candidate never is top, or never middle-ranked (Ward, 1965). (A geometric explanation is in Saari (1995), Section 3.3.) But,

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while avoiding cycles, they allow other troubling, election outcomes. For instance, the 21 voter profile

Number of voters	Preference	Number of voters	Preference
9	$A \succ C \succ B$	8	$B \succ C \succ A$
4	$C \succ B \succ A$		

(1.1)

satisfies both Black's condition (with C) and the 'never middle-ranked' condition (with A) ensuring a transitive pairwise ranking of $C \succ B \succ A$. Nevertheless, A wins with the reversed plurality ranking $A \succ B \succ C$, B wins when 7, 2, and 0 points are assigned, respectively, to a voter's top, second, and bottom ranked candidate, and C wins with the antiplurality method (where a voter votes for two candidates). Even though the profile satisfies two profile restrictions, each candidate can 'win' with an appropriate positional method.

The prominence of Black's condition makes it imperative to understand all associated problems. But the mathematical complexity of the analysis has limited the known results. We remove this barrier by creating an easily used geometric approach which allows us to fairly completely analyze how these restrictions affect procedures with regard to a variety of central issues ranging from the likelihood of outcomes to determining all paradoxes and all strategic behavior.

1.1. Profile sets

By exploiting the limited number of voter types allowed by these restrictions, we find the geometry of the profile spaces. Then, with *elementary algebra*, each space is partitioned into the profile sets causing each possible election outcome. The geometry of the sets determine the election properties; e.g., the weighted volume determine probability information while the shape provides information about strategic action and the effects of groups joining together.

Three candidates define $3! = 6$ strict rankings. Each ranking is denoted by a 'type' number

Type	*Preferences*	*Type*	*Preferences*
1	$A \succ B \succ C$	4	$C \succ B \succ A$
2	$A \succ C \succ B$	5	$B \succ C \succ A$
3	$C \succ A \succ B$	6	$B \succ A \succ C$

(1.2)

determined by the labeling of regions in the *representation triangle* of Fig. 1a. After assigning each candidate a vertex of the equilateral triangle, a point is

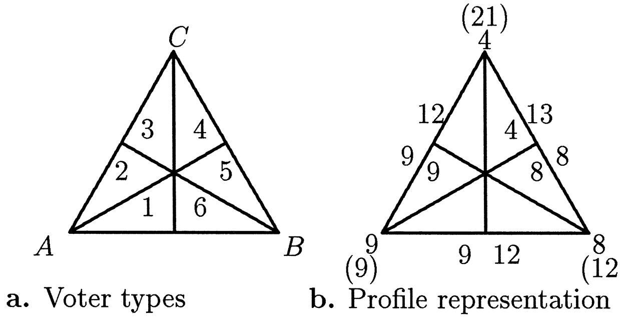


Fig. 1. The representation triangle and ranking regions.

assigned a ranking according to its distance from each vertex where 'closer is better;' e.g., points in region '1' define the $A \succ B \succ C$ ranking of Eq. (1.2). Each point on the vertical line is equal distance from the A and B vertices, so they represent 'indifference' denoted by $A \sim B$.

The representation triangle (introduced in Saari, 1994, 1995) offers a convenient way to represent profiles. As illustrated in Fig. 1b with the Eq. (1.1) profile, list the number of voters with each ranking in the appropriate ranking region. All voters preferring A to B are to the left of the vertical $A \sim B$ line, so A's pairwise tally is the sum of the numbers on the left while B's tally is the sum of those on the right. These tallies are under the A–B edge of the triangle; the other outcomes are next to the appropriate edge. Similarly, A's plurality tally is the number of voters who have her top ranked; it is the sum of entries in the two regions with A as a vertex. Each candidate's plurality Fig. 1b tally is near the assigned vertex. The antiplurality tally for, say B, is the number of voters who have her ranked first or second. Thus the tally is the sum of the entries in the four ranking regions with an edge ending in vertex B; these values are in the parentheses near the vertices. The Borda Count (BC) is the sum of the tallies a candidate receives in her two pairwise elections. So, the BC outcome is $C \succ B \succ A$ with tally $13+12{:}12+8{:}9+9$. (For more details, see Saari, 1995.)

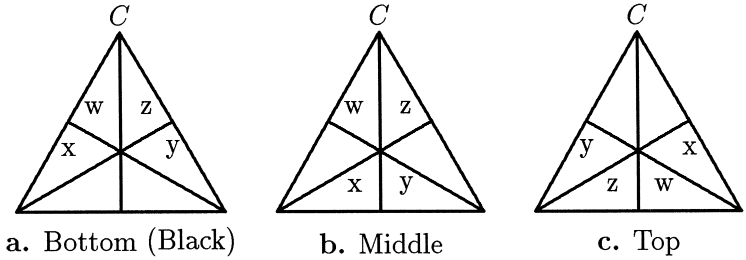


Fig. 2. Profile restrictions.

If n_j is the number of voters with the jth preference, the total number of voters is $n=\sum_{j=1}^{6} n_j$. We use the n_j/n fraction of voters of each type; e.g., for Eq. (1.1) they are $9/21$, $8/21$, and $4/21$.

The profile restrictions are in terms of C; e.g., Black's condition (Fig. 2a) is where C never is bottom ranked, so $x=n_2/n$, $y=n_5/n$, $w=n_3/n$, $z=n_4/n$, Fig. 2b is where C never is middle-ranked, and Fig. 2c is where C never is top ranked. The x, y, z, w definitions, illustrated in the figures, are selected to facilitate comparisons.

2. Pairwise votes with Black's restrictions

The definitions of x, y, z, w from Fig. 2a require all terms to be non-negative and

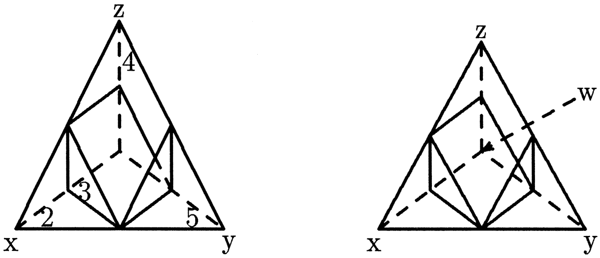
$$x+y+z+w=1. \tag{2.1}$$

By using $w=1-(x+y+z)$, a profile becomes a point in Fig. 3a tetrahedron

$$T_1=\{(x,y,z)\,|\,x,y,z\geq 0,\ x+y+z\leq 1\}. \tag{2.2}$$

Thus, T_1 is the profile space for Black's condition where each (rational) T_1 point represents a unique normalized profile. It is easy to convert a normalized profile into an integer profile; e.g., point $(3/20, 7/20, 4/20)\in T_1$ represents a twenty voter profile where 3, 7, 4, and 6 voters have, respectively, preferences of type 2, 5, 4, and 3. (The missing type-3 fraction is $w=1-(x+y+z)=6/20$).

The almost indistinguishable Fig. 3b represents profiles as (x, y, z, w) in a four dimensional space where each unanimity profile defines a unit vector on a particular R^4 coordinate axis. The vertices are equal distance apart, so Eq. (2.1) forces this space to be the Fig. 3b equilateral tetrahedron denoted by ET_1. In both representations, w increases as profile **p** approaches the rear vertex. T_1 and ET_1 differ in that the T_1 faces are three right and one equilateral triangles while each



a. Profile space T_1 b. Profile space ET_1

Fig. 3. Pairwise comparisons in Profile Space.

ET_1 face is an equilateral triangle. As a linear transformation converts one tetrahedron into the other, statements about the geometric properties and probabilities of events remain the same.

2.1. Pairwise votes

As shown in Fig. 2a, B beats A in the $\{A, B\}$ pairwise vote if and only if $y+z>1/2$. Thus, the $A \sim B$ plane $y+z=1/2$ divides T_1 into two sets determining who beats whom; it is the slanted plane in Fig. 3. Because $y=1$ ensures that $B \succ A$, the side with the y vertex supports $B \succ A$ outcomes. The $A \sim C$ and $B \sim C$ divisions correspond, respectively, to $x=1/2$ and $y=1/2$ planes—the vertical planes in Fig. 3. As it is easy to determine which rankings are on each side of an indifference plane (by computing what happens at a vertex), we have identified all profiles which define each possible pairwise outcome. The four strict outcomes are denoted by Eq. (1.2) numbers.

Let σ interchange A and B in a ranking and let $\sigma(\mathbf{p})$ be the profile created from $\mathbf{p}$ by interchanging A and B for each voter. If $f(\mathbf{p})$ is the election ranking, then *neutrality* requires

$$f(\sigma(\mathbf{p})) = \sigma(f(\mathbf{p})). \tag{2.3}$$

The importance of Eq. (2.3) for our purposes is the following.

Proposition 1. *If* $\mathbf{p}$ *satisfies Black's condition, then so does* $\sigma(\mathbf{p})$.

The Fig. 3 symmetry reflects Proposition 1; e.g., if $\mathbf{p}$ elects A, $\sigma(\mathbf{p})$ elects B; if $\mathbf{p}_1$ has C winning with A in second plate, $\sigma(\mathbf{p}_1)$ has C winning with B in second place. To demonstrate how conclusions follow from Fig. 3, we find the likelihood of each pairwise election outcome. If $F(x, y, z, w)$ is the probability distribution, then $P(j)$, the probability of outcome j, $j=2, 3, 4, 5$, is

$$P(j) = \int\int\int\int_{R_j} F(x, y, z, w)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}w\,\mathrm{d}z \tag{2.4}$$

where R_j is the Fig. 3 region defining a type j outcome. So, if each Fig. 3 profile is equally likely (F is a constant), elementary geometry (used to compute the ratio of volumes of the different regions and of T_1) shows that $P(2)=P(5)=1/8$ while $P(3)=P(4)=3/8$. More generally, for any *centrally distributed probability distribution*, where the distribution uses the number of voters of each type but not the type names, the σ symmetry of the figure (about $(1/4, 1/4, 1/4)$) ensures that $P(2)=P(5)$, $P(3)=P(4)$. If the probability emphasizes the central point $(1/4, 1/4, 1/4)$, such as the central limit theorem, the $P(3)=P(4)$ likelihood increases at the expense of $P(2)=P(5)$. These new, useful results follow immediately from the T_1 geometry.

Theorem 1. *With Black single peakedness condition where each T_1 profile is equally likely,*

$$P(2) = P(5) = \frac{1}{8}, P(3) = P(4) = \frac{3}{8}. \tag{2.5}$$

With n voters, as $n \to \infty$ the Eq. (2.5) *values are limits approached at least with order n^{-1}. This limit also holds for the setting where the number of voters is bounded by n.*

The probability of a type 2 or 5 outcome for precisely n odd or even voters is, respectively,

$$\frac{1}{8}\left[1 + \frac{3}{n+2}\right] \to \frac{1}{8} \text{ and } \frac{1}{8}\left[1 - \frac{3}{n(n+4)}\right] \to \frac{1}{8}. \tag{2.6}$$

If the number of voters is less than or equal to n, the two respective probabilities are

$$\frac{1}{8}\left[\frac{(n^2+7n+16)(n+7)}{(n^2+5n+10)(n+5)}\right] \to \frac{1}{8} \text{ and } \frac{1}{8}\left[\frac{(n+4)(n+1)}{n^2+5n+10}\right] \to \frac{1}{8} \tag{2.7}$$

The probability of a type 3 or 4 outcome for precisely n odd or even voters is, respectively,

$$\frac{3}{8}\left[1 - \frac{1}{n+2}\right] \to \frac{3}{8} \text{ and } \frac{3}{8}\left[1 - \frac{4n+3}{(n+1)(n+3)}\right] \to \frac{3}{8}. \tag{2.8}$$

The respective values if the number of voters is less than or equal to n are

$$\frac{3}{8}\left[\frac{n^3 + (26/3)n^2 + 25n + (88/3)}{(n^2+5n+10)(n+5)}\right] \to \frac{3}{8}$$

$$\text{and } \frac{3}{8}\left[\frac{(n^2 + (11/3)n + (4/3))(n+1)}{(n^2+5n+10)(n+5)}\right] \to \frac{3}{8}.$$

So, it is more likely to have a strict outcome with an odd than with an even number of voters. The following illustrates this oddity for a type 3 outcome by computing the $3/8$ multiples of Eq. (2.8). The oscillations, showing the rapid and slower approaches to unity for odd and even n values, capture the impossibility of a tie with an odd number of voters; odd values of n have no boundary points.

n	$Multiple$	n	$Multiple$	n	$Multiple$	n	$Multiple$
1	$\frac{2}{3}$	2	$\frac{4}{15}$	3	$\frac{4}{5}$	4	$\frac{16}{35}$
5	$\frac{6}{7}$	6	$\frac{4}{7}$	7	$\frac{8}{9}$	8	$\frac{64}{99}$

The probabilities described later in this paper are computed as in the following, so only the limiting values are given hereafter. This proof proves that these limits are approached at least with order n^{-1} as $n \to \infty$; it manifests the distribution of rational points in T_1 and ET_1.

Proof. The limiting values follow from the geometry. The probabilities for n use the equalities

$$\sum_{j=1}^{k} j = \binom{k+1}{2} = \frac{k(k+1)}{2}, \quad \sum_{j=1}^{k} j^2 = \frac{1}{6}\left[k(k+1)(2k+1)\right],$$
$$\sum_{j=1}^{k} j^3 = \binom{k+1}{2}^2 \tag{2.9}$$

To compute the number of T_1 points with n voters, if $w = \alpha/n$ (α of the n voters have type 3 preferences), then $x = (n-\alpha)/n$ requires $y = z = 0$—there is one point. Similarly, the number of points with $x = (n-\alpha-j)/n$ is the number of ways a y numerator can be selected with denominator n and $y + z = 1 - x - w = j/n$; this is $j+1$. Continue until $x = 0 = [n - \alpha - (n-\alpha)]/n$ which is supported by $(n-\alpha)+1$ points. Thus the total number of points is

$$\sum_{\alpha=0}^{n} \sum_{j=1}^{n-\alpha+1} j = \sum_{\alpha=0}^{n} \frac{(n-\alpha+1)(n-\alpha+2)}{2} = \frac{(n+1)(n+2)(n+3)}{6}. \tag{2.10}$$

The number of T_1 points with common denominator less than or equal to n is

$$\frac{1}{6}\sum_{k=1}^{n}(k+1)(k+2)(k+3) = \frac{1}{24}\left[(n^2+5n+10)(n+5)n\right]. \tag{2.11}$$

When n is odd, the number of points satisfying $w = \alpha/n$ and $x = (n-\alpha-j)/n$ again is $j+1$. The difference in the computation from Eq. (2.10) is that condition $x > 1/2$ restricts the computations to $x = (n+1)/2n$, or until $j = [(n-1)/2] - \alpha$. Thus the summation is

$$\sum_{\alpha=0}^{(n-1)/2} \sum_{j=1}^{(n+1)/2-\alpha} j = \frac{1}{48}(n+1)(n+3)(n+5).$$

Eq. (2.6) follows by dividing this value by that of Eq. (2.10). To find the probability value for all fractions satisfying $x > 1/2$ with common denominator $k \le n$, we compute

$$\frac{1}{48}\sum_{k=1}^{n}(k+1)(k+3)(k+5) = \frac{1}{8(24)}\left[(n^2+7n+16)(n+7)n\right],$$

which leads to the first of the Eq. (2.7) expressions.

For a the type 3 outcome, a fixed x restricts the admissible y, z values to a right triangle; when $y=[(n-1)/2n]-(j/n)$, there are $j+1$ admissible z numerators. So, when n is odd, each x value is associated with $\sum_{j=1}^{(n+1)/2} j=[(n+1)(n+3)]/8$ points. As x ranges from $x=0/n$ to $x=(n-1)/2n$, there are $(n+1)/2$ x values leading to $[(n+1)^2(n+3)]/16$ points. The result follows. The remaining expressions are similarly obtained. □

2.2. Other pairwise properties

To see how geometry dictates properties of procedures, observe that each set is convex. (For any two points in a region, the connecting line also is in the region.) To see the significance, suppose two subcommittees with six and 24 voters have respective preferences $\mathbf{p}_1=(x, y, z)=(3/6, 2/6, 0)$ (so $w=1/6$) and $\mathbf{p}_2=(12/24, 0, 8/24)$ (so $w=4/24$). The normalized profile defined by combining the committees is $\mathbf{p}_3=(15/30, 2/30, 8/30)=1/5\ \mathbf{p}_1+(1-(1/5))\mathbf{p}_2$; $w=5/30$. More generally, if m_j is the number of voters in profile $\mathbf{p}_j$, $j=1, 2$, the normalized form the combined profile is

$$\mathbf{p}_3=\mu\mathbf{p}_1+(1-\mu)\mathbf{p}_2;\ \mu=\frac{m_1}{m_1+m_2}. \tag{2.12}$$

According to Eq. (2.12) *the normalized combined profile is on the line connecting the two original profiles.* So, if $\mathbf{p}_1$ and $\mathbf{p}_2$ define the same outcome, this common outcome holds for the combined profile $\mathbf{p}_3$. Conversely, if a profile region is not convex, examples exist where $\mathbf{p}_1$ and $\mathbf{p}_2$ define a common outcome different from $\mathbf{p}_3$. Implications arise by postulating conditions where the same outcome should persist when profiles are combined; this includes monotonicity and the following.

Definition 1. An agenda $\langle A, B, C\rangle$ is where the winner of $\{A, B\}$ majority election is advanced to be compared with C. A procedure is weakly consistent (Saari, 1995) if when $\mathbf{p}_1$ and $\mathbf{p}_2$ have the same outcome, that is the $\mathbf{p}_3$ outcome. Positive involvement (Saari, 1995) is where if c_j wins with $\mathbf{p}_1$, and if $\mathbf{p}_2$ is a profile of the same type of new voters where c_j is top-ranked, then c_j still wins with $\mathbf{p}_3$.

An agenda does not satisfy weak consistency or positive involvement, so it suffers the 'no-show' paradox from Moulin, 1988 where by *not voting*, a voter forces a better outcome. (Also see Fishburn and Brams, 1983.) The convexity ensured by Black's condition, however, permits both conditions to hold.

Theorem 2. *In general, an agenda does not satisfy positive involvement and weak consistency. With Black's single peakedness condition, however, an agenda satisfies both conditions.*

Proof. A wins with $\langle A, B, C \rangle$ iff $\mathbf{p}$ is in region 2; B wins iff $\mathbf{p}$ is in region 5; C wins iff $\mathbf{p}$ is in the 3, 4 region. As all sets are convex, the conclusion follows Saari (1995). □

2.3. Strategic behavior

As the Gibbard, 1973, Satterthwaite, 1975 Theorem requires all three candidates (non-dictatorial) procedures to admit situations where some voter can successfully manipulate the system, an agenda can be manipulated. But does the Gibbard–Satterthwaite Theorem hold for agendas when Black's condition is satisfied? To be a meaningful, both the sincere and strategic profiles must satisfy Black's condition; i.e., strategic voters choose strategies which satisfy Black's condition with respect to C.

Theorem 3. *With Black's single peakedness, there never is a situation where a voter, by choosing a strategy consistent with Black's restraint, can successfully manipulate the outcome of an agenda.*

Proof. Let $\mathbf{p}_1$ and $\mathbf{p}_2$ be, respectively, the sincere and the strategic profile; to change the outcome, $\mathbf{p}_2$ must be in a different profile set. This requires (Chap. 5, Saari, 1995) the profile change $\mathbf{v} = \mathbf{p}_2 - \mathbf{p}_1$ to cross a Fig. 3 indifference boundary. To explain $\mathbf{v}$, if a type 2 voter votes like type 5, x decreases and y increases by $1/n$ to define $\mathbf{v} = \mathbf{d}_{2\to 5} = (1/n)(-1, 1, 0)$ in the direction from the x to the y vertex. Similarly, because $w = 1-(x+y+z)$, a change from type 3 to type 4 defines $\mathbf{d}_{3\to 4} = (1/n)(0, 0, 1)$ in the direction from the w to the z vertex. Since each voter type has three options, Black's condition define 12 strategies—the 12 directions from one vertex to another. Six follow; the others use the relationship $\mathbf{d}_{j\to k} = -\mathbf{d}_{k\to j}$; e.g., a type 5 voter voting as though type 2 defines $\mathbf{d}_{5\to 2} = -\mathbf{d}_{2\to 5} = (1/n)(1, -1, 0)$.

From	*To*	$\mathbf{v}$	*From*	*To*	$\mathbf{v}$
2	5	$\mathbf{d}_{2\to5} = \frac{1}{n}(-1,1,0)$	2	4	$\mathbf{d}_{2\to4} = \frac{1}{n}(-1,0,1)$
2	3	$\mathbf{d}_{2\to3} = \frac{1}{n}(-1,0,0)$	3	5	$\mathbf{d}_{3\to5} = \frac{1}{n}(0,1,0)$
3	4	$\mathbf{d}_{3\to4} = \frac{1}{n}(0,0,1)$	4	5	$\mathbf{d}_{4\to5} = \frac{1}{n}(0,1,-1)$

(2.13)

For a manipulation to be successful, the new outcome must be preferred by the strategic voter. With Black's condition, only the $x = 1/2$, $y = 1/2$ boundaries separate agenda winners. With Fig. 3, we can determine which $\mathbf{d}_{j\to k}$ vectors cross the boundaries and where this improves the voter's outcome. (So the geometry identifies *who* can manipulate and *when*.) For instance, a vector crosses the $x = 1/2$ boundary iff the x component of $\mathbf{d}_{j \to k}$ is nonzero where the sign determines the direction of change. For instance, if a type 2 voter votes as type 5, the $n\mathbf{d}_{2 \to 5} = (-1, 1, 0)$ change can cross the $x = 1/2$ boundary. But, the sincere $\mathbf{p}_1$ must be on the $x > 1/2$ side; e.g., $\mathbf{p}_1$ elects A. As the new C outcome changes the type 2 voter's top-ranked alternative to his second one, this move is counterproductive. A similar analysis using all $\mathbf{d}_{j \to k}$ with the boundaries proves the theorem. □

For any procedure depending on pairwise voting, issues including strategic action, monotonicity, etc., can similarly be analyzed with Fig. 3. The resolution of these issues, however, now is possible and easy because of the geometry (see Saari, 1995).

3. Positional voting with Black's condition

Positional voting assigns w_j points to a voter's jth ranked candidate, $w_1 \geq w_2 \geq w_3 = 0$, and ranks the candidates according to the sum of assigned points. A normalization assigns one point to a voter's first choice to define the *voting vector* $\mathbf{w}_\lambda = (1, \lambda, 0)$, $0 \leq \lambda \leq 1$; e.g., the normalized version of the system where 7, 2, and 0 points are assigned to candidates is $\mathbf{w}_{2/7} = (7/7, 2/7, 0)$. Important systems are the plurality $\mathbf{w}_0 = (1, 0, 0)$, BC $\mathbf{w}_{1/2} = (1, 1/2, 0)$, and antiplurality $\mathbf{w}_1 = (1, 1, 0)$ votes.

To find A's $\mathbf{w}_\lambda$ tally from Fig. 1a, she receives one point from voters who have her top ranked (the number of voters in regions 1 and 2) and λ points from voters who have her second ranked (the number of the voters in regions 3 and 6). For Black's Fig. 2a condition, A's tally is $x + w\lambda$. Each candidate's tally (after the $w = 1 - x - y - z$ substitution) is given next.

$Candidate$	$\mathbf{w}_\lambda$ $Tally$
A	$x + \lambda(1 - x - y - z)$
B	$y + \lambda z$
C	$1 - (x + y) + \lambda(x + y)$

(3.1)

3.1. Geometry of $\mathbf{w}_\lambda$ outcomes

To analyze $\mathbf{w}_\lambda$, mimic the approach for pairwise elections. Namely, to find the $\mathbf{w}_\lambda$ relative ranking of the pair $\{A, B\}$, set their tallies equal to find that the $A \sim B$ plane is $(x - y) + \lambda(\mathbf{1} - x - y - 2z) = 0$; this plane divides T_1 into profile sets with relative $A \succ B$ and $B \succ A$ $\mathbf{w}_\lambda$ rankings. For each λ, the equation is satisfied by the line $(t,\ t,\ 1/2 - t)$, $-\infty < t < \infty$, so the $A \sim B$ $\mathbf{w}_\lambda$ planes rotate about this line—*the rotation axis*—with λ changes. A plane is determined by a line and another point, so each $\mathbf{w}_\lambda$ plane is determined by the rotation axis and where it intersects the y-axis. The information for all pairs follows.

Pair	*Equation*	*Rotation Axis*	*y Point*
$A \sim B$	$x - y + \lambda(1 - x - y - 2z) = 0$	$(t, t, \frac{1}{2} - t)$	$\frac{\lambda}{1+\lambda}$
$A \sim C$	$1 - 2x - y + \lambda(2x + 2y + z - 1) = 0$	$(\frac{1-t}{2}, t, -t)$	$\frac{1-\lambda}{1-2\lambda}$
$B \sim C$	$1 - x - 2y + \lambda(x + y - z) = 0$	$(1 - 2t, t, 1 - t)$	$\frac{1}{2-\lambda}$

(3.2)

To appreciate how the rotating planes affect election outcomes, notice when a plane passes over a profile, the election ranking changes. *All conflict is caused by this rotation effect*. As each indifference plane has a different rotation axis, radical changes in election rankings must occur with changes in λ; i.e., a wide selection of profiles change election outcomes with different $\mathbf{w}_\lambda$ as apparent from Fig. 4 which depicts the plurality $(\lambda = 0)$, BC $(\lambda = 1/2)$, and antiplurality $(\lambda = 1)$ behavior.

The $\lambda = 0$ equations for the $A \sim B$, $A \sim C$, $B \sim C$ planes (respectively, $y = x$, $y = 1 - 2x$, $2y = 1 - x$), depend only on x and y values, so the plurality figure depicts where they meet the $x - y$ coordinate plane; the actual planes extend vertically in the z direction. The respective equations of these planes for $\lambda = 1$ are $y + z = 1/2$, $y + z = 0, y + z = 1$, so only the antiplurality $A \sim B$ plane passes through the T_1 interior. (The $A \sim C$ plane is the x-axis, the $B \sim C$ plane is the slanted T_1 boundary line in the $x = 0$ plane.) By being the 'middle' transition stage, the Borda planes, $x - 3y - 2z = -1$, $z = 2x - 1$, $x + 3y + z = 2$, involve all three variables.

The dotted Fig. 3 lines represent ranking regions of pairwise outcomes. Thus, $\mathbf{p}$ on one side of the pairwise plane for a particular pair and on the other for a λ

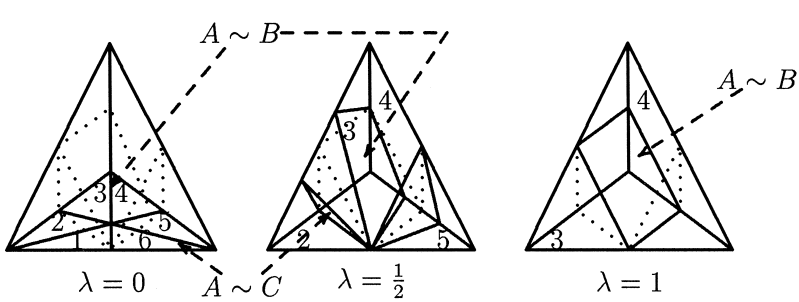


Fig. 4. Positional comparisons.

plane generates conflicting outcomes. The significant difference in these planes ensures wide spread differences in outcomes. The BC planes, however, more closely approximate the dotted pairwise planes; this supports the fact (e.g., Saari, 1997a,b) that BC outcomes most closely reflect the associated pairwise outcomes.

The rotation of planes from the $\lambda=0$ position to the $\lambda=1$ final location identifies new results; e.g., we determine how likely it is for **p** to retain the same ranking with the pairwise and all $\mathbf{w}_\lambda$ methods. (From the figures, this compatibility occurs only with type 3 and 4 outcomes.) Also, we determine the likelihood these procedures have the same candidate top-ranked. (The answer involves the volume of the region with a type 3 *or* 4 outcome for all methods.)

Theorem 4. *If all profiles in T_1 are equally likely, the probability that a profile has the same ranking with the pairwise and all $\mathbf{w}_\lambda$ is 1/9. Thus, 8/9 of the profiles change rankings with procedure. The likelihood that the same candidate is top-ranked with any of these procedures is 11/18 ≈ 0.611.*

Proof. The region where all procedures have a type-3 ranking is bounded above by the $y=z$ plane (the profiles giving a relative $A \sim B$ antiplurality ranking) and the region in the x–y plane defined by the type-3 plurality outcome. This volume is

$$\int_0^{1/3}\int_y^{(1-y)/2} y\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{108}.$$

By σ symmetry, the region for a type-4 outcome has the same volume. Therefore, the probability is determined by doubling this common value and dividing by $1/6$ (the volume of T_1).

Similarly, the antiplurality outcomes ensure that if the same candidate is elected, she is C. To find the likelihood, double the volume of the region which has both a type-3 plurality and pairwise outcome and divide by $1/6$. The volume is

$$\int_0^{1/3}\int_y^{(1-y)/2} [1-x-y]\,\mathrm{d}x\,\mathrm{d}y = \frac{11}{216}.$$

□

As most (about 88.9%) profiles change rankings, it may appear that Black's condition does not provide much assistance. Black's condition, however, ensures the same 'winner' for over 60% of the profiles. When compared with Merlin et al., 1997 where they find that with *no* restrictions, this 'common winner' phenomenon occurs with slightly over half of the profiles, it follows that Black's condition does provide some relief.

The advantages offered by Black's condition come from the A–B symmetry of Fig. 4 (used to prove Theorem 4). This σ symmetry reflects Proposition 1 and the neutrality of positional methods. Namely, if $f(\mathbf{p}, \mathbf{w}_\lambda)$ is the $\mathbf{w}_\lambda$ ranking for profile $\mathbf{p}$, then

$$f(\sigma(\mathbf{p}), \mathbf{w}_\lambda) = \sigma(f(\mathbf{p}, \mathbf{w}_\lambda)) \tag{3.3}$$

That is, for each $\mathbf{p}$, $\sigma(\mathbf{p})$ satisfies Black's condition and defines the $\mathbf{w}_\lambda$ outcome which reverses A and B. So, the likelihood a procedure has a type-3 outcome equals the likelihood it has a type-**4** outcome. This is true for any probability distribution which respects the σ symmetry.

3.2. Plurality and pairwise conflicts

The $\lambda = \mathbf{0}$ portion of Fig. 4 displays considerable conflict between the pairwise and plurality outcomes. For instance, the profile line defining an $A \sim B \sim C$ plurality outcome (extending vertically from where the three solid lines cross in the $\lambda = 0$ figure) is in the interior of the sets of pairwise outcomes with C top-ranked. But profiles supporting any desired plurality election abut this line, so *any of the 13 plurality rankings can accompany a pairwise ranking where C is top ranked*. In the two regions where the pairwise outcomes have C middle-ranked, the plurality ranking either agrees with the pairwise outcomes, or it relegates C to last place. The geometry also proves that all discrepancies are robust.

Particularly troubling conflicts are the reversed plurality and pairwise rankings; e.g., the Eq. (1.1) profile. But as this occurs in the smallest Fig. 4 regions (e.g., a type 3 pairwise $(C \succ A \succ B)$ and the reversed type 6 plurality rankings), this perverse event is the least likely to occur. Indeed, the volumes of the T_1 region with a type 3 pairwise outcome is $1/16$ and that of the 'type 3 pairwise outcome and type 6 plurality' region is $1/(54 \times 16)$. Consequently, *the likelihood the plurality ranking reverses a given pairwise ranking where C is the Condorcet winner is* $1/54 \approx 0.0185$. Assuming all T_1 profiles are equally likely, Black's condition allows only about 2% of the elections to experience this perverse behavior. It also follows from the geometry that this behavior requires at least 12 voters.

Theorem 5. *With Black's condition, assume that all T_1 profiles are equally likely. If C is the Condorcet winner, the likelihood a plurality ranking completely reverses the pairwise ranking is $1/54 \approx 0.0185$. Any such profile involves at least twelve voters; one example has five voters with preferences $A \succ C \succ B$, four with $B \preceq C \succ A$, and three with $C \succ B \succ A$.*

If C is the Condorcet winner, the likelihood C is plurality bottom, middle, and top ranked are, respectively, 0.0648, 0.1204, 0.8148.

Proof. By integration, the volume of the {type 3 pairwise} $\cap$ {type 6 plurality} region is

$$\int_{1/3}^{1/2} \int_{(1-y)/2}^{y} \left(\frac{1}{2} - y \right) \mathrm{d}x\, \mathrm{d}y = \frac{1}{54 \times 16}. \tag{3.4}$$

The conditional probability assertion follows from the $1/16$ volume of a profile set with type 3 pairwise ranking. As a similar statement holds for the {type 4 pairwise} ∩ {type 1 plurality} region, the assertion follows by combining probabilities.

A **p** with the minimum number of voters illustrating a specified behavior is a point in the T_1 region with the smallest common denominator. To find $\mathbf{p} \in$ {type 4 pairwise} ∩ {type 1 plurality}, the pairwise outcomes require $y+z>1/2$, $y<1/2$ while the plurality requires (Eq. (3.2)) $x>y$ (so $A \succ B$) and $1<x+2y$ (so $B \succ C$). Expressing fractions as $x=X/d$, $y=Y/d$, $z=Z/d$, with integers, we have

$$2(Y+Z)>d, \quad 2Y<d, \quad X>Y, \quad d<X+2Y \text{ and } X+Y+Z \le d.$$

The $X>Y$ (or $X-1 \ge Y$) and $d<X+2Y$ conditions require $d<3X-2$, or $(d+2)/3<X$. So, for $d=10$, 11, 12, it follows that $X \ge 5$. The other equations prove that no example exists for $d \le 11$; a $d=12$ example is the indicated profile. □

While Theorem 5 shows that consistency between the pairwise and plurality winners occurs about 81% of the time, it carries the troubling message that even Black's condition permits conflict in about one out of five plurality elections. But the conflict between the pairwise and plurality outcomes is, in general, more severe (Merlin et al., 1997; Saari and Tataru, 1999), so Black's condition reduces the likelihood of its occurrence. Also, compare the simplicity of our approach with that used by these papers and, say, Gehrlein, 1981 and Lepelley, 1993.

3.3. Different λ's

Fig. 4 shows that Black's condition restricts the antiplurality $(\lambda=1)$ strict outcomes to $C \succ B \succ A$ or $C \succ A \succ B$ where each has the same limiting $1/2$ probability. The only conflict with pairwise rankings, which interchange the antiplurality's two top ranked candidates, occurs with probability $1/4$. All other $\mathbf{w}_\lambda$ outcomes depend on how the indifference planes change; e.g., the $A \sim B$ λ-plane rotates about the line connecting $(0, 0, 1/2)$ with $(1/2, 1/2, 0)$—the line connecting the midpoint on the z-axis with the midpoint of the line in the x–y plane. The monotonic progress of this plane is captured by the movement of its y-axis intercept which starts at the origin when $\lambda=0$ and moves monotonically to $y=1/2$ when $\lambda=1$. So (Fig. 4), the $A \sim B$ plane tilts from its initial vertical position until it reaches its final $\lambda=1$ orientation parallel to the x-axis. The $B \sim C$ plane rotates about the line connecting $(1, 0, 1)$ and $(0, 1/2, 1/2)$; the y-axis point monotonically moves from $y=1/2$ $(\lambda=0)$ to $y=1$ when $\lambda=1$ where (Fig. 4) it is parallel to the x-axis. The more dramatic $A \sim C$ plane motion rotates about the axis defined by $(1/2, 0, 0)$ and $(0, 1, -1)$, but its y-axis point moves rapidly. At $\lambda=0$, it is at $y=1$, but as $\lambda \to 1/2$, it moves to infinity (at the BC the plane is parallel to the y-axis), to re-emerge from negative infinity when $\lambda > 1/2$ to approach zero as $\lambda \to 1$.

Theorem 6. *With Black's single peakedness, if C is the Condorcet winner and each profile in T_1 is equally likely, then the likelihood C is $\mathbf{w}_\lambda$ bottom ranked monotonically decreases from 0.0648 for the plurality vote to zero as $\lambda \to 1/2$. For $\lambda \geq 1/2$, C never is bottom ranked. The likelihood C is $\mathbf{w}_\lambda$ second ranked monotonically decreases from 0.1204 for the plurality vote to zero as $\lambda \to 1$. The likelihood that C is $\mathbf{w}_\lambda$ top ranked monotonically increases from 0.8148 for the plurality vote to unity as $\lambda \to 1$. Only the antiplurality vote always has C top ranked with Black's conditions.*

3.4. Runoff elections

Other geometric consequences of Fig. 4 are illustrated with $\mathbf{w}_\lambda$ runoff elections where the two $\mathbf{w}_\lambda$ top-ranked candidates advance to a runoff to determine the 'winner.' For a candidate to win, the profile must be in a T_1 region where she is at least second ranked and wins the associated two candidate election. The geometry of this profile set provides new results.

Theorem 7. *With Black's condition, a $\mathbf{w}_\lambda$ runoff satisfies positive involvement and weak consistency with respect to A and B. Weak consistency holds for C if and only if $\lambda \geq 1/2$.*

For a $\mathbf{w}_\lambda$ runoff election where $\lambda < 1/2$, there exist situations where each of two committees elect C, but when they combine as a full group, the sincere outcome is A.

Proof. A wins only if $\mathbf{p} \in \{$type 2 pairwise$\} \cap \{$type 1, 2, or 3 $\mathbf{w}_\lambda\}$ region. In Fig. 4, $\mathbf{p}$ is in the portion of the small tetrahedron with vertex $x = 1$ defined by the dotted lines (type 2 pairwise outcome) with a type 1, 2, or 3 $\mathbf{w}_\lambda$ outcome. It is clear from Fig. 4 (and the interpolation for other λ) that this region is convex for any λ. A similar assertion holds for B. However C wins if $\mathbf{p} \in \{$type 3 or 4 pairwise$\}$ and $\{\mathbf{w}_\lambda$ has anything *other* than a type 1 or 6 outcome$\}$. For $\lambda < 1/2$, this exclusion of $\mathbf{w}_\lambda$ outcomes creates a dent in the profile region that denies convexity and forbids 'weak consistency.' The more forgiving 'positive involvement' (Saari, 1995) restricts $\mathbf{p}_2$ to either the w (all voters are type 3) or z (all voters are type 4) vertices. It is obvious from the geometry that if $\mathbf{p}_1$ elects C with this runoff, the line connecting $\mathbf{p}_1$ with either $\mathbf{p}_2$ choice remains in the region.

Examples exploit the lack of convexity of the region where C is elected; i.e., $\mathbf{p}_1$ is in a type 2 $\mathbf{w}_\lambda$ region near the $B \sim C$ $\mathbf{w}_\lambda$ plane where C is the Condorcet winner. To illustrate with $\lambda = 0$, the profile (x, y, z) must satisfy $x < 1/2$, $x + 2y < 1$, $2x + y > 1$; e.g., $(x, y, z) = (9/20, 5/20, 0)$. Select $\mathbf{p}_2$ in the region electing C with a type 5 $\mathbf{w}_\lambda$ outcome near the $A \sim C$ boundary; e.g., (5/20, 9/20, 0). Use Eq. (2.12) to select the committee sizes so that $\mathbf{p}_3$, the full committee, elects whomever we choose. For $\lambda = 0$, a choice to elect A has $3/5$ of all voters in the $\mathbf{p}_1$ subcommittee. Thus $\mathbf{p}_1$ has 27 of type 2, 15 of type 5, and 18 of type 3 while $\mathbf{p}_2$ has 10 of type 5, 18 of type 5, and 12 of type 3. □

3.5. Strategic behavior

The manipulative behavior of $\mathbf{w}_\lambda$ elections and runoffs is determined by the geometry of boundaries. To use the scalar product approach of Saari (1995), the gradients of the $A \sim B$, $C \sim A$, $C \sim B$ planes for $\mathbf{w}_\lambda$ are, respectively,

$$(1-\lambda, -1-\lambda, -2\lambda), (-2+2\lambda, -1+2\lambda, \lambda),$$
$$\text{and } (-1+\lambda, -2+\lambda, -\lambda).$$

Each gradient points in the profile direction helping the first listed candidate. (The first vector supports A over B; the other two directions support C over the 'other' candidate.) To determine the effects of a strategic action, compute the scalar product of each gradient with each $\mathbf{d}_{j\to k}$; positive or negative values mean, respectively, that the change helps or hurts the indicated candidate (either A or C); a zero value means the change has no impact. The values follow.

$\mathbf{d}_{j\to k}$	$A \sim B$	$C \sim A$	$C \sim B$	$\mathbf{d}_{j\to k}$	$A \sim B$	$C \sim A$	$C \sim B$
$n\mathbf{d}_{2\to 5}$	-2	1	-1	$n\mathbf{d}_{2\to 4}$	$-\lambda-1$	$2-\lambda$	$-1-2\lambda$
$n\mathbf{d}_{2\to 3}$	$\lambda-1$	$2-2\lambda$	$1-\lambda$	$n\mathbf{d}_{3\to 5}$	$-1-\lambda$	$-1+2\lambda$	$-2+\lambda$
$n\mathbf{d}_{3\to 4}$	-2λ	λ	$-\lambda$	$n\mathbf{d}_{4\to 5}$	$-1+\lambda$	$-1+\lambda$	$-2+2\lambda$

(3.5)

So, if a type 2 voter uses $\mathbf{d}_{2\to 5}$ or $\mathbf{d}_{2\to 4}$, the outcome is personally worse because the change counters the voter's true preferences. For instance, the positive value for $\mathbf{d}_{2\to 5}$ helps C over the voter's top choice of A; the two negative values indicate strategies that help his lower ranked candidate at the expense of a higher ranked one. The only strategic action is if $\lambda < 1$ where the voter uses $\mathbf{d}_{2\to 3}$ to assist C over B; the sincere $\mathbf{p}_1$ must be near the $B \sim C$ plane with a $\mathbf{w}_\lambda$ type 1 or 5 outcome. According to Fig. 4, a type **1** outcome requires $\lambda < 1/2$. A similar analysis determines when each type of voter can successfully manipulate the outcome.

Theorem 8. *With Black's single peakedness condition, a type 2 agent can successfully vote strategically only by voting like a type 3 voter with* $\mathbf{w}_\lambda$, *for* $\lambda < 1$, *when the sincere* $\mathbf{w}_\lambda$ *outcome has B narrowly ahead of C. Similarly, the* *only time a type 5 agent can successfully vote strategically by voting like a type 4 agent is when the sincere* $\mathbf{w}_\lambda$ *outcome,* $\lambda < 1$*, has C narrowly ahead of B.*

A type 3 voter can vote as though a type 2 voter for a better $\mathbf{w}_\lambda$ *outcome,* $\lambda < 1$*, when the sincere outcome has B narrowly ahead of A. This voter can vote either as though type 4 (for* $0 < \lambda < 1$*) or as though type 5 (for* $1/2 < \lambda$*) when the sincere* $\mathbf{w}_\lambda$ *outcome has A narrowly ahead of C.*

A type 4 voter voting as a type 5 voter can have a better $\mathbf{w}_\lambda$ *outcome,* $\lambda < 1$*, when the sincere* $\mathbf{w}_\lambda$ *outcome has A narrowly ahead of B. This voter can vote either as though type 3 (for* $0 < \lambda < 1$*) or as though type 2 (for* $1/2 < \lambda$*) when the sincere* $\mathbf{w}_\lambda$ *outcome has B narrowly ahead of C.*

Proof. A $\mathbf{d}_{j \to k}$ profile change crosses a particular boundary when the sign of the appropriate Eq. (3.5) entry is non-zero. Values of '$-1 + 2\lambda$' change direction depending upon whether λ is larger or smaller than $1/2$. (As the change occurs with the BC, the associated zero value means that the BC has no strategic advantage; this is compatible with the general result (Saari, 1995) that the BC minimizes the likelihood of a successful manipulation by a small number of voters.) The sign of a Eq. (3.5) entry gives the directional change, so, to alter the election outcome, $\mathbf{p}_1$ must be on the appropriate side of the boundary. The last step involves determining whether a change in a pairwise ranking is compatible, or contrary, to the strategic voter's sincere preferences. □

The more complicated strategic analysis of $\mathbf{w}_\lambda$ runoffs requires determining which two candidates are $\mathbf{w}_\lambda$ top ranked as they are advanced to the runoff and then the runoff outcomes.

Theorem 9. *Even with profiles satisfying Black's single peakedness condition where a strategic voter must vote consistent with this constraint, there exist situations where a voter can obtain a personally better outcome by voting strategically in a* $\mathbf{w}_\lambda$ *runoff election when* $\lambda < 1/2$*. But, Black's condition makes it impossible for a voter to successfully manipulate the* $\mathbf{w}_\lambda$ *runoff when* $\lambda \geq 1/2$*.*

Proof. From Eq. (3.5), when a type 2 voter votes as type 3, the outcome crosses the $B \sim C$ $\mathbf{w}_\lambda$ surface if $\lambda < 1$ and if $\mathbf{p}_1$ is near the $B \sim C$ plane with a type 5 or 1 $\mathbf{w}_\lambda$ outcome. The first case only interchanges whether B or C is top ranked, so it does not change the candidates in a runoff and has no effect on the outcome. According to Fig. 4 the second setting (sincere type 1 outcome) prohibits B from advancing to the runoff only if $\lambda < 1/2$. As this strategic action advances A and C, rather than A and B to the runoff, the final outcome depends upon the pairwise elections. If $\mathbf{p}_1$ has a type 2 pairwise outcome, the strategic action has no effect—A wins with or without the strategic action. If $\mathbf{p}_1$ is in the region with type 3 pairwise outcomes, C wins by beating A. But the sincere vote has A beating B, so the strategic vote is counterproductive for the type 2 voter. The remaining possibility is if $\mathbf{p}_1$ is in a region of type 4 pairwise outcomes; here C beats A whereas the sincere vote has B beating A. Thus, in a narrowly defined T_1 region which exists for $\lambda < 1/2$, a type 2 voter can be strategically successful. (With a slight miscalculation of the pairwise outcomes, the strategy can be counterproductive.) All remaining strategic actions follow similarly from Eq. (3.5) and Fig. 4.

To see why a strategic voter cannot successfully manipulate a $\mathbf{w}_\lambda$ runoff method for $\lambda \geq 1/2$, notice that the limited number of $\mathbf{w}_\lambda$ outcomes allow a change between a type 3 and 4 outcome to alter which candidates are in the runoff. However, these locations for the sincere profile also are where C wins all pairwise elections. Thus, the strategic action does not alter the outcome. □

3.6. Line of indifference

A slight change in an $A \sim B \sim C$ outcome creates any other ranking, so subtle $\mathbf{w}_\lambda$ behavior is detected by determining how the 'completely tied' $\mathbf{w}_\lambda$ outcomes change with λ. To find this $\mathbf{w}_\lambda$ point, solve the $A \sim B \sim C$ equations to obtain

$$(x, y, z) = \left(\frac{1+\lambda(1-3w)}{3}, \frac{1+(\lambda^2-\lambda)(1-3w)}{3(1-\lambda)}, \frac{(1-3w)+\lambda(3w-2)}{3(1-\lambda)} \right). \tag{3.6}$$

An equivalent expression using $x = [1+\lambda(1-3w)]/3$ is

$$(x, y, z) = \left(x, \frac{1-2w+3(w-1)x+3x^2}{2-3(x+w)}, \frac{1-2x^2+3w(x-1+w)}{2-3(x+w)} \right) \tag{3.7}$$

where $x \in [1/3, (2-3w)/3)$. The solutions decrease from a T_1 line segment $(1/3, 1/3, 1/3-w)$; $0 \leq w \leq 1/3$, for the plurality vote to the T_1 point $(1/2, 1/2, 0)$ for the BC. It *leaves* T_1 for $\lambda > 1/2$ to approach infinity (Eq. (3.6)) as $\lambda \to 1$; this causes the Fig. 4 antiplurality properties. This geometry discloses a surprisingly wide variety of election conflicts allowed by Black's condition.

Theorem 10. *With Black's condition, for any integer k between one and seven, there exist* $\mathbf{p}$ *with precisely k* $\mathbf{w}_\lambda$ *election rankings as* λ *varies. Indeed, for some* $\mathbf{p}$*, each candidate wins with an appropriate* $\mathbf{w}_\lambda$*, while other* $\mathbf{p}$ *require each candidate to be bottom ranked with an appropriate* $\mathbf{w}_\lambda$*.*

If all profiles in T_1 *are equally likely and C is the Condorcet winner, the probability* $\mathbf{p}$ *has seven different* $\mathbf{w}_\lambda$ *outcomes where each candidate wins with some* $\mathbf{w}_\lambda$ *is no more than 0.0062. Any example requires at least 19 voters; one such example is where 8 have* $A \succ C \succ B$*, 7 have* $B \succ C \succ A$*, and 4 have* $C \succ B \succ A$*.*

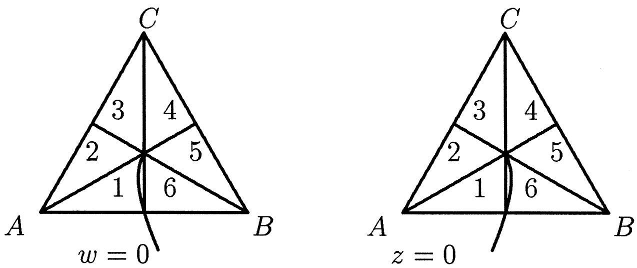


Fig. 5. ET_1 curves of completely tied outcomes.

The probability $\mathbf{p}$ *has seven different* $\mathbf{w}_\lambda$ *outcomes where each candidate loses with some* $\mathbf{w}_\lambda$ *is at least 0.0586. An example requires at least seven voters; one such example is where 4 have* $A \succ C \succ B$, *3 have* $B \succ C \succ A$, *and 3 have* $C \succ B \succ A$.

Proof. Profiles supporting a $\mathbf{w}_\lambda$ complete tie (Eqs. (3.6) and (3.7)) define a line, so it suffices to locate the end points. Fig. 5 displays the curve of endpoints on the $w = 0$ and $z = 0$ ET_1 faces.Each λ defines a point on each curve equal distance from the $\lambda = 0$ point. Each curve leaves its face at the BC point. For each λ, the profiles causing a $\mathbf{w}_\lambda$ complete tie is the line connecting the indicated λ points on the two faces. The Fig. 5 faces are subdivided into plurality ranking regions; i.e., the line of $A \sim B \sim C$ plurality outcomes connects the two center points and the BC point is where the curves leave the edge at $(1/2, 1/2, 0)$. The faces are similar (after reversing directions) because Black's condition is invariant with σ (Proposition 1, Eq. (3.3)). Indeed, the σ symmetry requires the $\mathbf{w}_\lambda$ $A \sim B \sim C$ line to pass through the $A \sim B$ plurality plane when $z = w$.

To explain the narrow region between the curve and $A \sim B$ plurality line, Fig. 6 (an enlarged $w = 0$ face) has dashed lines where $\mathbf{w}_{1/4}$ has tied pairs; the lines

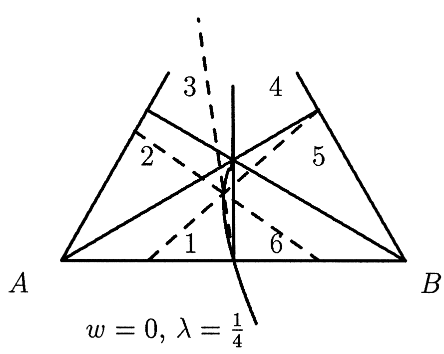


Fig. 6. ET_1 curve of completely tied outcomes.

intersect on the curve where $\mathbf{w}_{1/4}$ has the $A \sim B \sim C$ outcome. The Fig. 6 numbers identify which profiles define the six $\mathbf{w}_{1/4}$ strict election rankings. (A symmetric picture holds for the $z=0$ face.) By differing from plurality lines, the dashed lines define profiles with different plurality and $\mathbf{w}_{1/4}$ outcomes. As a $\mathbf{p}$ with a $\mathbf{w}_{1/4}$ $A \sim B \sim C$ outcome is in the interior of the type 1 plurality set, slight changes in $\mathbf{p}$ change the $\mathbf{w}_\lambda$ outcome to any of 13 three-candidate rankings while the $A \succ B \succ C$ plurality outcome remains. This conflict allowed by Black's condition is more serious than suggested by the introductory example.

Now consider a $\mathbf{p} \in \{\text{type } \mathbf{1} \text{ plurality}\} \cap \{\text{type } \mathbf{4}\ \mathbf{w}_{1/4}\}$ region as λ varies from 0 to $1/4$. Each λ defines dashed lines similar to Fig. 6, so as λ varies $\mathbf{p}$'s strict election rankings vary from type 1 to 6 to 5 to 4 allowing each candidate to 'win' with an appropriate $\mathbf{w}_\lambda$; with ties, $\mathbf{p}$ admits *seven* different election rankings. Moreover, these properties hold for all $\mathbf{p}$ between the curve and the $A \sim B$ plurality line. In the $z=0$ face, the plurality ranking is type 6 and the $\mathbf{w}_\lambda$ outcomes range from type 6 to 1 to 2 to 3, so the same properties hold with changed type numbers (required by σ). More generally, the set of $\mathbf{p}$ with these properties is defined by the plane of $A \sim B$ plurality outcomes and the curved surface obtained by connecting λ points on the curves in each face.

A $\mathbf{p}$ in the type 1 plurality region of the $w=0$ face *outside* the curve has seven $\mathbf{w}_\lambda$ election outcomes as λ varies, but each candidate *loses* with some $\mathbf{w}_\lambda$ rather than wins. Similarly, $\mathbf{p}$ in the plurality type 2 region has five $\mathbf{w}_\lambda$ outcomes, $\mathbf{p}$ in the plurality $A \succ B \succ C$ region has six rankings, etc. In the $w=0$ face, only $\mathbf{p}$ in the plurality type 4 region have a fixed election outcome for all λ; in the $z=0$ face, all $\mathbf{p}$ with a fixed outcome are in the type 3 plurality region.

Finding a $\mathbf{p}$ with the minimum number of voters requires finding a point with the smallest common denominator in the indicated region. The cone shape geometry (e.g., Fig. 5) of these regions requires $\mathbf{p}$ to be where z or w equals zero; let $w=0$. If each candidate can win, the $\mathbf{w}_\lambda$ rankings move from type 1 to 6 (see Fig. 6); e.g., if $\mathbf{p}$ has an $A \sim B$ $\mathbf{w}_\lambda$ ranking, the ranking is $A \sim B \succ C$. With integer profiles, the $\mathbf{w}_\lambda$ tally is $(X, Y+\lambda Z, Z+\lambda(X+Y))$, so an $A \sim B$ relative ranking occurs when

$$\lambda^* = \frac{X-Y}{Z}. \tag{3.8}$$

Because the $\mathbf{w}^*_\lambda$ ranking is $A \sim B \succ C$, we have that

$$X > Z + \lambda^*(X+Y), \tag{3.9}$$

or, by substituting the value from Eq. (3.8),

$$Z(X-Z) > (X-Y)(X+Y) \tag{3.10}$$

Eq. (3.10) requires examples with small numbers of voters to have $Z \approx X/2$ and $X - Y = 1$. So, start with small X and Y values, and determine whether $Z \approx X/2$ satisfies Eq. (3.10). If $X=7$, $Y=6$, and $Z=3$ or 4, the left-hand side of

Eq. (3.10) is 12 while the right is 13, so an example with $X=7$ (and smaller X values) does not exist. The next choice of $X=8$, $Y=7$, $Z=4$ works; this is the example stated in the theorem. Thus, the smallest number of voters is $X+Y+Z=19$.

The difference in constructing $\mathbf{p}$ $(w=0)$ where each candidate loses with some $\mathbf{w}_\lambda$ is that $A \sim B$ mandates a $C \succ A \sim B$ ranking. (Fig. 6 requires the rankings to go from types 1 to 2 to 3 to 4.) So, Eq. (3.8) remains in effect, but Eq. (3.9) is reversed to $X < Z + \lambda^*(X+Y)$ to define

$$Z(X-Z) < (X-Y)(X+Y). \tag{3.11}$$

Here, the inequalities are satisfied with small Z values. As the plurality and antiplurality outcomes are, respectively, $A \succ B \succ C$ and $C \succ B \succ A$, we have $X > Y > Z$, $X+Y+Z > Y+Z > X$, or

$$Z+Y > X > Y > Z. \tag{3.12}$$

Thus the smallest Z value is $Z=2$ when $X-Y=1$, so the smallest example satisfying Eqs. (3.12) and (3.11) is $X=4$, $Y=3$, $Z=2$. A direct computation shows that C is the Condorcet winner.

To prove the probability statements compute the volume of the region between the curved surface and the $A \sim B$ plurality plane. To bound this value, notice that if the curve for the $z=0$ portion of Fig. 5 is a direct replica of the $w=0$, rather than a reversal, the ratio of volumes of the region and the tetrahedron is less than the ratio of areas of that bounded by the curve and the $w=0$ surface. (This region occurs near the A–B edge of the tetrahedron where the volume is 'pinched.') The reversal of these curves creates a cone where the actual relative volume (probability) is less than $1/4$ of the relative area of the areas. This area is given by

$$\int_{1/3}^{1/2}\left[2x-\frac{1}{3}+\frac{1}{3(3x-2)}\right]\mathrm{d}x=\frac{1}{12}-\frac{1}{9}\ln(2)$$

(Saari and Valognes, 1998). By considering only profiles in the square defined by the Condorcet condition (with area $1/4$), the limiting probability is four times this value, or $1/3 - 4/9\ \ln(2) \approx 0.0253$. Thus, the probability for our problem is bounded by $0.0253/4 \approx 0.0062$. The regions where C is the Condorcet winner and the plurality outcome is of type 1 or 6 (where C is bottom ranked) are divided into two regions by the surface of completely tied outcomes. According to Theorem 5, the likelihood of this occurring with Black's condition is 0.0648. Therefore, the likelihood $\mathbf{p}$ has seven outcomes where each candidate loses with some $\mathbf{w}_\lambda$ is no less than 0.0648–0.0062 ≈ 0.0586. □

These probability estimates confirm the graphics which suggest it is unlikely for Black's condition to allow each candidate to be a 'winner;' it occurs in less than 1% of the profiles. On the other hand, the polarizing flavor of the 19 voter and all other examples, where one candidate (A) is strongly favored by a segment of the population but disapproved of by the rest, is a kind observed in actual elections; e.g., see examples cited in Saari (1995; Saari and Tataru (1999). So, probability estimates can be misleading; by identifying the actual profiles, however, we can determine whether this behavior is more likely due to, say, the game theoretic maneuvering of the candidates.

4. Top-ranked restrictions

Surprisingly, all conclusions with the 'never top-ranked' constraint (Fig. 2c) follow directly from the analysis of Black's condition. This translation is a direct consequence of Eq. (4.1) where ρ represents the reversal of a ranking; e.g., $\rho(A \succ B \succ C) = C \succ B \succ A$.Let $\rho(\mathbf{p})$ be the profile constructed by reversing each voter's ranking in $\mathbf{p}$.

Proposition 2. (Saari, 1994, 1995) *For all profiles* $\mathbf{p}$ *and all* λ, $0 \leq \lambda \leq 1$, *we have that*

$$f(\mathbf{p},\mathbf{w}_\lambda) = \rho\left[f(\rho(\mathbf{p}),\mathbf{w}_{1-\lambda})\right]. \tag{4.1}$$

To illustrate, the reversal of the Eq. (1.1) profile has 9, 8, 4 voters with the respective rankings of $B \succ C \succ A$, $A \succ C \succ B$, $A \succ B \succ C$. According to Eq. (4.1), the election rankings of this profile with $\mathbf{w}_1$, $\mathbf{w}_{1-2/7}$, $\mathbf{w}_0$ reverse, respectively, the Eq. (1.1) profile outcomes with $\mathbf{w}_0$, $\mathbf{w}_{2/7}$, $\mathbf{w}_1$.

To convert statements about Black's condition to the 'top-ranked' restriction, the (x, y, z, w) representation in Fig. 2c is the reversal of profile (x, y, z, w) in Fig. 2a. This is one of the three Eq. (4.1) steps, so to translate results from Black's condition to conclusions about the top-ranked condition, carry out the two steps:

- Election rankings are reversed. By listing outcomes in terms of 'type', this involves adding or subtracting 3 from each type number for results using Black's condition.
- Results concerning $\mathbf{w}_\lambda$ in Black's condition now hold for $\mathbf{w}_{1-\lambda}$ for the top-ranked restriction.

These conditions are illustrated with Fig. 7, a converted form of Fig. 6, where the solid lines divide the $w = 0$ face into *antiplurality* (rather than plurality) ranking regions. In Fig. 6, the dashed lines define $\mathbf{w}_{1/4}$ ranking regions; Fig. 7 is divided into $\mathbf{w}_{3/4}$ ranking regions. Figs. 3–6 can be similarly transferred to identify new properties of the top-ranked profile restriction.

The geometry remains the same after re-identifying the regions, so each assertion about Black's condition has a reversed conclusion for the top-ranked condition. For instance, the figure corresponding to Fig. 3 shows that the 'never top ranked constraint' allows only A and B to win with an agenda. The geometry of agenda profile sets differ but remain convex so an agenda still satisfies positive

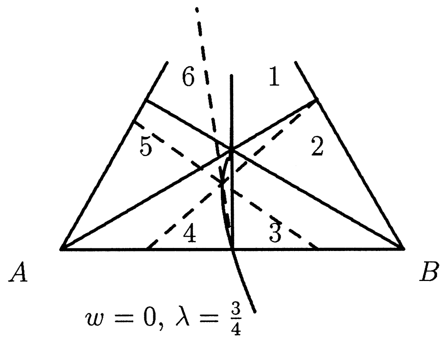


Fig. 7. ET_2 curve of completely tied outcomes.

involvement and weak consistency. With positional rankings, the antiplurality (not the plurality) method has all 13 rankings with a particular pairwise ranking. Instead of problems occurring for $\lambda < 1/2$, they occur for $\lambda > 1/2$; only the BC ($\lambda = 1/2$) has reasonable properties in both settings. Reversing Theorem 10, it is more likely for each candidates to win, rather than lose, with some $\mathbf{w}_\lambda$.

5. Never middle-ranked restriction

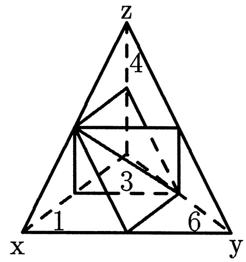
Black's condition reflects a natural situation where voters never have a popular candidate bottom ranked, while the 'never top-ranked' condition corresponds to settings involving an unpopular candidate. The last 'C never middle-ranked' condition captures a polarized setting where the voters either 'love or hate C.' This constraint represented by Fig. 2b, has added theoretical interest because if $\mathbf{p}$ satisfies the middle-ranked conditions, then so do profiles $\sigma(\mathbf{p})$, $\rho(\mathbf{p})$, $\sigma(\rho(\mathbf{p}))$, and $\rho(\sigma(\mathbf{p}))$. These symmetry conditions determine new election properties.

5.1. Pairwise vote

All results use the same approach, so only new kinds of assertions are described. From Fig. 2b, the planes of pairwise votes for $A \sim B$, $A \sim C$, $B \sim C$, which are respectively,

$$y + z = \frac{1}{2}, \quad x + y = \frac{1}{2}, \quad x + y = \frac{1}{2},$$

partition profile space T_3 as in Fig. 8. The two planes divide T_3 into regions with strict pairwise outcomes of types 1, 3, 4, 6. The vertical plane, $x + y = 1/2$, serves the $\{A, C\}$ and $\{B, C\}$ rankings; the slanted plane divides the $\{A, B\}$ rankings. So, C is the Condorcet winner for all profiles on the side of the vertical



a. Profile space T_3

Fig. 8. Pairwise comparisons.

plane containing the vertical axis. The profiles sets where A and B are Condorcet winners are on the other side of this vertical plane; the two regions are determined by the slanted plane. All regions where a candidate is the Condorcet winner are convex.

Theorem 11. *For the middle-ranked profile restriction, each of the four strict outcomes are equally likely with any probability distribution which depends upon the number of candidates of each type rather than the names of the types. An agenda satisfies both positive involvement and weak consistency. A voter cannot successfully manipulate an agenda.*

Proof. The first assertion follows from symmetry; e.g., if $\mathbf{p}$ causes a type 1 outcome, then $\rho(\mathbf{p})$, $\sigma(\mathbf{p})$, and $\rho(\sigma(\mathbf{p}))$ cause, respectively, types 4, 6, and 3 outcome. Thus, each $\mathbf{p}$ from a profile set uniquely defines profiles with the same number of voters for each of the three other profile sets. By assumption, each is equally likely. The conclusion now follows from summation.

An agenda satisfies positive involvement and weak consistency due to the convexity of the profiles sets for each winner. The strategic behavior assertion uses the scalar product method and Eq. (5.1). The gradients are (1, 1, 0), which helps A over C and B over C, and (0, 1, 1), which helps B over A.

$n\mathbf{d}_{j\to k}$	$(1,1,0)$	$(0,1,1)$	$n\mathbf{d}_{j\to k}$	$(1,1,0)$	$(0,1,1)$
$n\mathbf{d}_{1\to 6}=(-1,1,0)$	0	1	$n\mathbf{d}_{1\to 4}=(-1,0,1)$	-1	1
$n\mathbf{d}_{1\to 3}=(-1,0,0)$	-1	0	$n\mathbf{d}_{4\to 3}=(0,0,-1)$	0	-1
$n\mathbf{d}_{4\to 6}=(0,1,-1)$	1	0	$n\mathbf{d}_{6\to 3}=(0,-1,0)$	-1	-1

(5.1)

To illustrate, a type 1 voter has strategies to cross both planes. The only positive value for $\mathbf{d}_{1\to 6}$, shows that such a change helps the type 1 voter's second ranked candidate, B by hurting his top ranked A; this is counterproductive. A similar analysis, comparing how the change in the pairwise rankings contrasts with the voter's sincere preferences, holds for all other situations. □

5.2. Positional voting

It follows from Fig. 2b that the A, B, C $\mathbf{w}_\lambda$ tallies are, respectively,

$$x+\lambda(1-x-z), \quad y+\lambda(x+z), \quad 1-(x+y).$$

Therefore, the equations dividing profiles into regions for the relative $\mathbf{w}_\lambda$ rankings are as follows.

$Pair$	$Equation$	$Axis$	$x\ pt$
$A\sim B$	$x-y+\lambda(1-2(x+z))=0$	$(t,t,\frac{1}{2}-t)$	$\frac{\lambda}{2\lambda-1}$
$C\sim A$	$1-2x-y+\lambda(x+z-1)=0$	$(1-\frac{t}{2},t-1,\frac{t}{2})$	$\frac{\lambda-1}{\lambda-2}$
$C\sim B$	$1-x-2y-\lambda(x+z)=0$	$(t,\frac{1-t}{2},-t)$	$\frac{1}{1+\lambda}$

(5.2)

Fig. 9 displays the three planes of $\mathbf{w}_\lambda$ tie votes for $\lambda=0,\ 1/2$. While the representation for the plurality vote agrees with that from Black's condition (Fig. 4a), the dotted lines representing the pairwise votes differ. Thus, the middle-ranked condition admits new plurality and pairwise ranking relationships. (Results about plurality runoffs also differ.) To simplify the $\lambda=1/2$ figure, the dotted pairwise outcomes lines are suppressed; and the $A\sim B$, $A\sim C$, $B\sim C$ BC planes are denoted, respectively, by the solid, dashed, and dotted lines. As the profile line of completely tied BC outcomes connects point $(1/2,\ 0,\ 1/2)$ (the midpoint of the left downward slanting edge) to $(0,\ 1/2,\ 0)$ (the midpoint of the y-axis), the three BC planes connect on this line and form a pinwheel. For Theorem 12, notice that this $A\sim B\sim C$ line is in the plane of $A\sim C$, $B\sim C$ pairwise votes. Also, the plane of profiles defining an $A\sim B$ BC outcome agrees with the plane of $A\sim B$

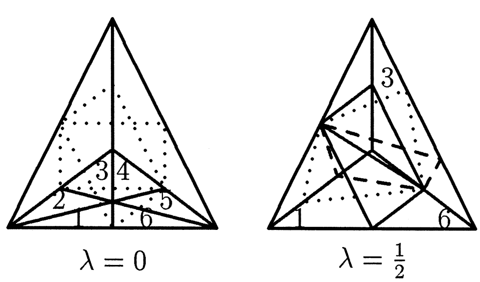


Fig. 9. ET_3 positional comparisons.

pairwise outcomes. This compatibility creates BC election relationships. Again, the BC profile division more closely approximates the pairwise outcomes than that of any other $\mathbf{w}_\lambda$.

With minor changes, the representation for the antiplurality vote is essentially that of Fig. 9a. This is a consequence of Eq. (4.1) and the invariance of the middle-ranked restriction with respect to ρ. Thus, instead of the substitution $w = 1 - (x + y + z)$, use the 'reversed' (relative to Fig. 2b) substitution $y = 1 - (x + w + z)$. Relabel the x-, y-, z-axes of Fig. 9a, respectively, as the z-, w-, x-axes and either add or subtract 3 from the type numbers for all outcomes. The resulting figure (including the dotted lines for the pairwise rankings) of Fig. 9a holds for the antiplurality procedure. A sample of the kinds of outcomes available from this geometry follows.

Theorem 12. *With the middle-ranked restriction, a pairwise outcome defining a type 1 or 6 outcome can be accompanied by any of the 13 plurality rankings. Similarly, a pairwise ranking of type 3 or 4 can be accompanied by any of the 13 antiplurality rankings.*

For all $\mathbf{p}$, *the pairwise and relative {A, B} BC ranking always agree. Each strict pairwise outcome can be accompanied by three BC outcomes where one agrees with the pairwise rankings, one involves a tie with one of the candidates and C, and the last reverses this pairwise ranking with C.*

Proof. The assertions about the plurality and antiplurality rankings follow directly from Fig. 9a. Elementary computations determine the limiting probabilities.

The assertion about the BC uses the fact that the line of completely tied BC election outcomes is in the plane of $A \sim C$, $B \sim C$ pairwise outcomes. At any profile in the interior of this line, draw a plane $\mathscr{P}$ passing through and orthogonal to this line. In $\mathscr{P}$, the axis is a point $\mathbf{n}$. The intersection of each plane of pairwise outcomes with $\mathscr{P}$ is a line passing through $\mathbf{n}$; this forms a figure similar to a coordinate axis. Each BC plane also intersects $\mathscr{P}$ defining a line, but the BC $A \sim B$ line is on one of the coordinate axis; the other two are not. Thus each of the four open regions (corresponding to strict pairwise rankings) is bisected by one of these lines (see Fig. 10). The conclusion now follows. □

For other properties, notice from Fig. 9a that the union of type {1, 2, 3, 6} plurality regions is non-convex; this is where A is one of the two top-ranked candidates. The portions with an $A \succ B \succ C$ pairwise outcome is where A wins a plurality runoff, so the non-convexity means a plurality runoff is not weakly consistent. A similar picture holds for C winning the runoff. On the other hand, connecting any point in this region with the $x = 1$ vertex (the only vertex where A

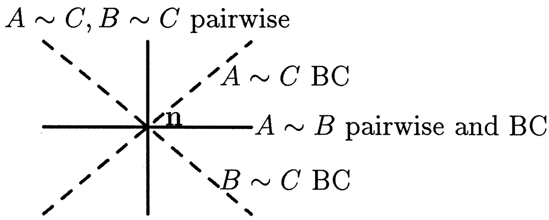


Fig. 10. The division of plane $\mathscr{P}$.

is top-ranked) defines a line in this region. Consequently, the plurality runoff is positively involved.

Corollary 1. *With the middle-ranked profile restriction, the plurality runoff is positively involved but not weakly consistent.*

To find the graph (Fig. 11) for $w=0$ of $\mathbf{w}_\lambda$ $A \sim B \sim C$ outcomes, solve the $A \sim C$ and $B \sim C$ equations to obtain

$$(x,y,z) = \left(\frac{1-\lambda+\lambda^2}{3(1-\lambda)}, \frac{1-2\lambda}{3(1-\lambda)}, \frac{1+\lambda}{3} \right),\ 0 \le \lambda \le \frac{1}{2} \tag{5.3}$$

Symmetry with σ and ρ requires a similar figure to occur when the x and y and the w and z roles are interchanged. Thus the other curve of $\mathbf{w}_\lambda$ $A \sim B \sim C$ outcomes is in the $z=0$ face and continues to the midpoint of the $y-w$ edge when $\lambda = 1/2$. In Fig. 11, the curve passes through the type 3 plurality region, so in the $z=0$ face, the σ symmetry requires the accompanying curve to pass through the type 4 plurality region. The continuation of each curve for $1/2 \le \lambda \le 1$ involves the fundamental Eq. (4.1) and ρ. Namely, since ρ makes the changes $x \to z$, $z \to x$, $y \to w$, $w \to y$, the curve in Fig. 11 continues from the $w=0$ face into the $y=0$ face of ET_3.

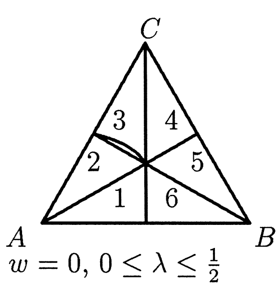


Fig. 11. ET_3 curve of completely tied outcomes.

Theorem 13. *With the middle-rank restriction, for integer k, $1 \le k \le 7$, there exist* $\mathbf{p}$ *with precisely k* $\mathbf{w}_\lambda$ *election outcomes as* λ *varies. There exist* $\mathbf{p}$ *where each candidate wins with an appropriate* $\mathbf{w}_\lambda$, *and other* $\mathbf{p}$ *where each candidate is bottom ranked with an appropriate* $\mathbf{w}_\lambda$. *With the exception of the BC, all 13* $\mathbf{w}_\lambda$ *election outcomes can accompany the same pairwise outcome.*

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