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Economic Theory

Negative externalities and Sen's liberalism theorem*

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Summary. Sen's seminal, negative theorem about minimal liberalism has had a profound effect on economics, philosophy, and the social sciences. To address concerns raised by his result, we show how Sen's assumptions *must* be modified to obtain positive conclusions; e.g., one resolution allows an agent to be decisive only if his choice does not impose "strong negative externalities" on others. We also uncover a significantly different interpretation of Sen's societal cycles: rather than describing the rights of individuals to choose, the cycles identify when these choices impose difficulties on others. Other ways to address Sen's difficulties come from game theory.

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1 Introduction

In 1859, John Stuart Mill succinctly articulated the basic issue with his declaration, "There is a circle around every human being which no government, ... ought to be permitted to overstep... [T]hat there is, or ought to be, some space in human existence thus entrenched around... no one who professes the smallest regard to human freedom or dignity will call in question." His comment underscores a basic tenet of liberalism, which asserts that certain issues and choices naturally belong

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within an individual's sphere of influence. Restated in terms of the societal structure, rights of individuals to decide constitutes a natural "decentralization" of a portion of societal processes. Because the admirable intent of this timeless objective is generally assumed to be true, it is understandable why Sen's "Minimal Liberalism Theorem" (Sen 1970a, b), which seemingly asserts that no such decentralization exists, has generated considerable concern. More precisely, Sen proved that no decision rule allows individuals to take these actions while protecting society against the discord caused by decision cycles. (To see some of the links between economics and philosophy that are created by Sen's result, see Sen, 1987; Broome, 1991; Hausman and McPherson, 1996; Kolm, 1996.)

Yet, seemingly in direct conflict with Sen's result, individuals do make personal decisions on a daily basis, and these decisions need not cause societal problems. This reality creates a puzzle; it suggests that something deeper must occur. What can it be? Standard approaches from social choice theory do not seem to shed light on this mystery. For instance, a traditional approach of checking whether an impossibility assertion persists after tinkering with the assumptions may be more of an analysis in logic than an attempt to reconcile differences between theory and observation.

To search for a more meaningful understanding of the problem, the approach taken here is to identify not what *might* be done, but what *must* be done to circumvent the perplexing difficulties Sen raised. To support our conclusions, we appeal to the theory of economic decentralization and mechanism design introduced by Hurwicz (1960): this theory characterizes those organizational ways that can achieve a specified societal goal. With this theory, we indicate why our way to address Sen's theorem is almost mandated. Among our conclusions is a new interpretation of Sen's result and a recognition (which addresses the above puzzle) that our ways to sidestep Sen's problem have parallels in daily practice. We then create arguments to support Sen's (1970a) suggestion that "the ultimate guarantee for individual liberty may rest not on rules for social choice but on developing individual values that respect each other's personal choice."

What significantly aids our program is that the source of Sen's negative conclusion now is understood (Saari, 1997, 1998, 2001). The surprisingly simple explanation shows that when a decision rule satisfying Sen's condition of minimal liberalism (ML) is used with a sufficiently heterogeneous society, the merger effectively eliminates the crucial assumption that the individuals have transitive preferences.[1] Obviously, if the assumption of individual rationality is ignored, then Sen's conclusion must be anticipated.

What makes this comment unexpected is that Sen's theorem explicitly requires the individuals to have transitive preferences. This conflict means that ML forces a gap between the actual preferences of individuals and how decision rules use and interpret this information. As described (Sect. 3), ML inadvertently requires the decision rules to treat certain transitive profiles as though the data comes from individuals with cyclic preferences. But should a rule mistakenly try to service

[1] A similar argument (Saari, 2001) explains Arrow's Impossibility Theorem (Arrow, 1951) (also see Saari, 2000).

cyclic, rather than the actual transitive individuals, it is readily understandable why cyclic societal outcomes occur. As an aside, this explanation of Sen's result nicely demonstrates how natural assumptions can, unintentionally but effectively, jettison other crucial assumptions that we mistakenly believe are being used.

It follows from this description that to replace Sen's result with positive conclusions, we must discover how to reclaim the intended transitivity information for the decision process. There are many ways to do this: one offered here (Sect. 4) extracts an informational aspect about transitive preferences that, when used to modify ML, converts Sen's negative conclusion into a positive assertion. This condition relies on a new interpretation for Sen's societal cycles: an interpretation that significantly shifts the emphasis about who is being wronged. Rather than describing a person's right to select, we show that Sen's cycles can be interpreted as reflecting societal conflicts where the actions of decisive agents strongly impose upon the interests and wishes of others.[2] This observation has interesting consequences. For instance, rather than contradicting Mill's statement, which opened this paper, we will explain why it may be more accurate to view Sen's result as modeling and capturing dysfunctional societal settings where the actions of agents hurt the interests of all others.

Finding a positive replacement for Sen's result, then, requires finding ways to avoid this infringement on the interests, and maybe rights, of others. To do so, the informational facet we identify and then use – a "strong negative externality constraint" – limits when an agent can be decisive. Interestingly, as the spirit of our condition mimics what actually happens in society, our result provides theoretical insights into centuries old, pragmatic societal practices that have evolved to avoid these concerns. Then we indicate why these types of constraints are almost demanded by decentralization theory.

A different perspective is based on Fine's (1975) identification of Sen's (1970a) "Prude and Lascivious" example with the Prisoner's Dilemma game. This fine observation introduced a new research direction: instead of a direct mechanism approach where individuals report their sincere preferences, Sen's structures are described in terms of games with an emphasis on strategies and finding a choice; e.g., see the articles in Arrow et al. (1996) such as Hammond (1996) and Pattanaik (1996).

It is easy to show (as we do) that associated with any example illustrating Sen's result is a game, and, conversely, any game satisfying certain simple conditions generates an example illustrating Sen's theorem. This connection suggests examining whether game theory can provide ways to address the problems of Sen's theorem. The connection we stress is how, with appropriate assumptions (e.g., an appropriate discount rate), the 'Tit-for-Tat" strategy for infinitely repeated games leads to cooperative solutions for the Prisoner's Dilemma. As we show, this strategy, which clearly has parallels with commonly used societal practices, always is applicable within Sen's structure.

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[2] All of the papers about Sen's theorem that we have seen emphasize an individual's sphere of influence and the rights of the decisive agent. Thus this new explanation is unexpected.

2 Sen's theorem

Sen's theorem requires individual preferences to be complete, transitive and unrestricted over the alternatives. The only condition imposed on the societal outcome is that it does not admit cycles. For the decision rule, Sen assumes the (weak) Pareto condition: if the individuals are unanimous in their ranking of any pair, that is the pair's societal ranking. The remaining condition is a minimal version of the rights of individuals as associated with liberalism.

Definition 1. Minimal Liberalism *(ML) is where at least each of two individuals is assigned at least one pair of alternatives. These individuals are "decisive" over the assigned pair in that the way they rank the pair is the pair's societal ranking.*

Although these conditions seem to be reasonable and innocuous, Sen proves the surprising conclusion that with three or more alternatives, no decision rule can always satisfy them. As described in Sen's writings, this outcome suggests a fundamental divide between welfarism and liberalism. In an interesting paper Gaertner et al. (1991) expand on this point by commenting that "this problem persists under virtually every plausible concept of individual rights that [they] can think of." We add support to their comment because our explanation for Sen's theorem holds for all choices of individual rights that have certain natural properties.

In Sen's formulation, each individual ranks even those pairs of alternatives that, by being assigned to decisive individuals, presumably belong to someone else's personal sphere of influence: these rankings create externalities. Not only do these externalities play a key role in our explanation, but they also add interest to Sen's result. This is because externalities are an economic and political reality, so Sen's theorem provides a natural setting to examine their consequences; e.g., as we will see, externalities play a central role when describing Sen's theorem from the perspective of decentralization.

3 The source of Sen's result

It is customary to prove Sen's theorem, while demonstrating its relevancy, by creating examples similar to what occurs in daily life. The personal decisions the individuals make in these examples force cyclic societal rankings. Subsequent to Sen's proof, others have found particularly clever examples that generate not one, but several societal cycles, e.g., Brunel and Salles (1998) and Salles (1997) did so by nicely extending Sen's "Prude and Lascivious" story.

A recently developed approach (Saari, 2001) converts the construction of examples from an art form that requires deep insights about human interactions into a simple constructive process. Indeed, this approach makes it easy to generate any number of examples that illustrate Sen's theorem with any number of societal cycles that are intertwined in any desired manner, as well as creating examples that do not use the Pareto condition. Arguments that are central to our discussion are outlined next.

Start with any desired societal outcome, where AB means that alternative A is preferred to alternative B. Assign the societal cycle as the (temporary) preferences for each individual: initially, then, individuals have cyclic preferences. To illustrate by creating examples where the societal outcome has the two cycles AB, BC, CD, DA and BC, CE, EA, AB, this assignment defines the following "informational table." With the unanimity of individuals, the Pareto condition mandates the specified societal outcome.

individual	$\{A, B\}$	$\{B, C\}$	$\{C, D\}$	$\{A, D\}$	$\{C, E\}$	$\{A, E\}$
Anne	AB	BC	CD	DA	CE	EA
Barb	AB	BC	CD	DA	CE	EA
Connie	AB	BC	CD	DA	CE	EA
Outcome	AB	BC	CD	DA	CE	EA

(3.3.1)

Next, assign pairs of alternatives to decisive individuals. The only condition imposed on this assignment rule is that

> *for each individual and each societal cycle, there is at least one pair where* another *individual is decisive.*

This condition holds for *each individual* whether or not the individual is decisive over any pair. In our example, BC occurs in both cycles, so an easy way to satisfy this condition for both cycles and for both Barb and Connie is to let Anne be decisive over $\{B, C\}$. It remains to satisfy this condition for Anne; i.e., pairs from the first and second societal cycles, which differ from $\{B, C\}$, must be assigned to other decisive agents. Because AB is in both cycles, we could let Barb be decisive over $\{A, B\}$. But to illustrate the flexibility of the approach, let Barb be decisive over $\{A, D\}$ and Connie over $\{C, E\}$. These assignments are reflected in the next information table where the dashes indicate that a person's ranking is irrelevant because the outcome is determined by a decisive agent. The individual preferences are not changed, so the societal outcome must also remain unchanged. The only difference is that the BC, DA, EA societal outcomes are now justified by minimal liberalism rather than the Pareto condition.

Person	$\{A, B\}$	$\{B, C\}$	$\{C, D\}$	$\{A, D\}$	$\{C, E\}$	$\{A, E\}$
Anne	AB	BC	CD	$--$	CE	$--$
Barb	AB	$--$	CD	DA	CE	$--$
Connie	AB	$--$	CD	$-$	CE	EA
Outcome	AB	BC	CD	DA	CE	EA

(3.3.2)

The important point is that the Eq. 3.3.2 information table also arises if, instead of the cyclic preferences, Anne, Barb, and Connie have, respectively, the transitive preferences

$$ABCDE, \quad CDEAB, \quad CDEAB. \tag{3.3.3}$$

The third step, then, is to construct transitive rankings that generate the same information table: as explained later, this always can be done. When presenting the end-product as a "two societal cycles" example that illustrates Sen's theorem, ignore the initial cyclic preference and use only the constructed Eq. 3.3.3 transitive rankings.

To further illustrate, instead of the above assignment process, adopt the original option where Barb is decisive over $\{A, B\}$. The information table now becomes

Person	$\{A,B\}$	$\{B,C\}$	$\{C,D\}$	$\{A,D\}$	$\{C,E\}$	$\{A,E\}$
Anne	$--$	BC	CD	DA	CE	EA
Barb	AB	$--$	CD	DA	CE	EA
Connie	$--$	$--$	CD	DA	CE	EA
Outcome	AB	BC	CD	DA	CE	EA

(3.3.4)

where associated choices of transitive preferences for Anne, Barb, and Connie could be

$$BCDEA, \quad CDEAB, \quad CDEAB. \tag{3.3.5}$$

Again, by using the derived transitive preferences rather than the initial cyclic ones, we have a different "two societal cycles" example that illustrates Sen's result.

The reason this simple approach works, and why transitive preferences always can be created at the end of this process, is that the way decisive agents are assigned to pairs makes it irrelevant how each individual ranks at least one pair from each societal cycle. But reversing the ranking of just one pair in a cycle (from the initial cyclic preferences) creates a transitive ranking. The insight, in other words, is that ML makes it impossible to distinguish cyclic from transitive preferences (for more details, see Saari, 2001).

This approach offers a simple and easy way to construct a wide variety of examples illustrating Sen's theorem. Even stronger, the construction makes it arguable that, at least for those examples that can be created in this manner, the cyclic societal outcomes reflect an intent of the decision rule to satisfy the needs of the original cyclic preferences rather than the later constructed transitive preferences. After all, the societal cycle for the initial setting reflects the original unanimity among the group: the construction makes it clear that the initial cyclic setting cannot be distinguished from the transitive preferences selected at the end of the process.

These last comments lead to one of our central points. As asserted next, *all possible examples illustrating Sen's theorem can be constructed in this manner.* This includes Gibbard's (1974) troubling extension, and any other choice of rights that we have examined (also see the "colored shirt" example in Gaertner et al., 1991). This next assertion makes it arguable that, for *all possible examples* illustrating Sen's theorem, the decision rule attempts to respect the wishes of individuals with associated cyclic preferences rather than the actual intended individuals with transitive preferences.

Theorem 1 (Saari, 2001). *Any example illustrating Sen's theorem, which is based on the rankings of decisive agents and unanimity among individuals, can be constructed in the above manner by first assigning the societal cyclic rankings to each person. In this way, the Pareto condition requires the societal outcome to be as specified.*

The proof (Saari, 2001) is straightforward. To see the idea, notice that any example illustrating Sen's theorem has societal cycles. For each pair of alternatives, either the voters are in complete agreement, or the outcome is determined by a decisive agent. In the latter case, change each agent's ranking to agree with that of the decisive agent: the rule ignores this information, so these changes have no effect on the information used to determine the outcome. By doing so, the newly imposed preferences for all of the agents are cyclic and agree with the societal outcome.

The important message of Theorem 1 is that Sen's conclusion occurs because minimal liberalism emasculates the assumption that individuals have rational preferences. Consequently, to find resolutions of Sen's problem, we must find ways to allow decision rules to use the explicitly required information that the preferences are transitive. This search starts in the next section.

4 Regaining transitivity through negative externalities

To sidestep the problems identified by Sen's theorem, we must find ways to allow a decision rule to differentiate transitive from cyclic rankings. One approach uses the notion of a "strong preference" given in Saari (1995, 2001) and used in Brunel (1998). ("Strong" is in the sense of *ordinal rankings*.) Others (e.g., Luce and Raiffa, 1957) have used versions of this natural notion, but our usage differs significantly because we use it to distinguish unrelated binary rankings from those with a transitive structure.

The idea is simple: when listing a binary ranking coming from a transitive ranking, also specify how many other alternatives separate the two choices. Thus the $ABCDE$ ranking has a $[BC, 0]$ binary ranking because no alternative separates B and C, but a stronger $[AE, 3]$ ranking because three alternatives separate A and E. When binary rankings are not intended to be related, pairs are not separated. For instance, if $AE, AC, CE, AD, DE, CB, ...$ are binary rankings with no intended relationship among them (even though $\{A, C, E\}$ accidentally satisfy transitivity), we have $[AE, 0]$.

We use these $[XY, \alpha]$ terms, where α measures the *intensity of the binary XY ranking*, to distinguish transitive rankings from general binary ones. This distinction occurs because a transitive ranking always has some positive intensity values, while α always equals zero for unrelated binary rankings. To illustrate, while each binary ranking for each cyclic preference in the Eq. 3.3.1 information table has an $\alpha = 0$ intensity level, a difference emerges when the Eq. 3.3.2 table reflects the Eq. 3.3.3 transitive rankings: because Anne is decisive, she determines the societal $\{B, C\}$ outcome with her weak $[BC, 0]$ preference even though Barb and Connie disagree as manifested by their strongly opposing $[CB, 3]$ rankings. Similarly, Barb determines the $\{A, D\}$ outcome with her $[DA, 1]$ preference, but

Anne disagrees with her stronger[3] $[AD, 2]$ views. Finally, Connie's decisive agent's choice of EA from her $[EA, 0]$ preferences affects Anne's opposite and stronger $[AE, 3]$ views. A similar pattern emerges with the Eq. 3.3.4 information table and the Eq. 3.3.5 transitive preferences: Barb's decisive $[AB, 0]$ selection is opposed by Anne's strongly opposing $[BA, 3]$ views, and Anne's decisive $[BC, 0]$ choice counters Barb's strongly opposing $[CB, 3]$ views.

4.1 Dysfunctional societies

This transitivity information, then, suggests that a decisive agent's choice can impose a strong, negative externality for someone else. The observation that a decisive agent's choice can be met with disagreement is not new; it has been recognized by many including Fine (1975), Campbell and Kelly 1997, Saari and Brunel as reported in (Brunel, 1998), and Hillinger and Lapham (1998). What is new (as shown below) is that this disagreement must occur in *all possible examples* that illustrate Sen's theorem; that beyond someone disagreeing with the outcome, in *each cycle someone strongly* disagrees with the decisive agent's choice; that this strong disagreement indicates that ML is obscuring whether the individual preferences are, or are not, transitive; and that it provides a means to understand how to resolve these difficulties. For purposes of this paper, this central term is defined as follows.

Definition 2. *For any pair of alternatives $\{X, Y\}$, a decisive agent's choice of X creates a strong, negative externality if another agent's sincere ranking is $[YX, \alpha]$ with a positive α intensity.*

A way to allow the societal outcome to reflect the rationality of the individuals is to change ML so that a decisive agent can make a decision only if the choice does not impose a strong negative externality on others. This change in ML, which now allows the rule to use information about the transitivity of individual preferences, converts Sen's negative conclusion into a positive one. Moreover, this conclusion is loosely connected with what we experience in daily life; e.g., consider those common noise abatement laws that limit how loudly music can be played in public. Even should music be played loudly, these laws are enforced only should someone complain. In other words, enforcement requires a complaint, so the personal cost of reporting an infringement tacitly determines whether the negative externality is strong. As another example, the color of a shirt a person wears should be a personal decision even if others disagree. A possible exception is if this choice causes strong negative externalities, as in some large cities where certain colors indicate support for a rival gang: here a reaction manifesting a strong negative disagreement can be lethal.

Theorem 2. *Suppose a decisive agent can determine the societal outcome of an assigned pair only when the choice does not create a strong negative externality for some other agent. The pairwise outcomes determined by the decisive agents and the Pareto condition do not generate cycles.*

[3] Remember, these intensity comparisons are being made with ordinal, not cardinal, rankings.

This theorem, which is proved in Section 7, is not targeted toward assisting only one or two individuals: surprisingly, the consequences affect *everyone.* This is because for any societal cycle in any example illustrating Sen's theorem, *each person* suffers at least one strong negative externality. Sen's societal cycles, then, require *everyone to be strongly and adversely affected by the actions of others.*

Theorem 3. *In each societal cycle caused by Pareto and the choices of decisive agents, the choice from at least two pairs of alternatives create a strong negative externality for someone. Indeed, for each cycle,* each individual *suffers at least one strong negative externality.*

These two theorems, both proved in Section 7, expose an alternative explanation for Sen's theorem: the societal cycles capture dysfunctional settings where *everyone* is negatively and strongly affected by what someone else does. While discussions of Sen's theorem have traditionally focussed on the rights of individuals to decide within their personal spheres of influence, Theorems 2 and 3 promote a radically different message. They suggest that these personal spheres are not as "personal" as normally assumed; after all, rather than strictly personal, actions taken are strongly (in the above intensity sense) affecting others in a negative manner. It is our reading of Mill's statement, for instance, that the circle about each human excludes actions that create victims by strongly and negatively affecting others: this interpretation of Mills and of Sen suggests the two are addressing very different concerns. Thus rather than concentrating on the "rights" of decisive individuals, maybe an appropriate way to analyze Sen's result is to emphasize the rights of the victims. This is the spirit of Theorem 2 and those described in Section 5.

There are many ways to refine Theorem 2. According to Theorem 3, for example, at least two pairs in each cycle cause strong negative externalities, but only one ranking needs to be reversed. So different criteria – maybe a condition comparing how many individuals are negatively affected in each pair – can be used to make the selection. Also, cardinal rankings could be used. But rather than exploring these refinements, our main interest in Theorem 2 is to demonstrate how the structural source of Sen's theorem helps to identify natural solutions for these problems. Indeed, as described next, the externalities required by Sen's formulation come close to mandating that any resolution reflects the spirit of Theorem 2.

4.2 Decentralization

To explain the last comment, building on the theory of decentralization as started by Hurwicz (1960), arguments motivated by results in Hurwicz, Reiter, and Saari (1978) were developed in Saari (1984) to characterize all organizational ways to achieve a specified societal outcome (for certain classes of problems). As we should expect, the admissible forms of "decentralization" are governed by what is being modeled. Central to our purposes is that an agent can have unfettered rights to make decisions only if the consequences of this agent's actions are separated from that of all others. While the admissible kinds of "separation" can be surprisingly subtle, natural examples have the $h(x)g(y_1, y_2)$ or $h(x)+g(y_1, y_2)$ form (where the choice of x has consequence $h(x)$, that of y_1, and of y_2 is $g(y_1, y_2)$, etc.). Notice, "separability" captures the fact that the $h(x)$ consequences are not constrained, in any mannner, by the acts of others. But as discussed next, Sen's "no societal cycles" condition prohibits this separability. Consequently, as with Theorem 2, actions taken by "decisive agents" must be constrained.

To avoid introducing complicated technicalities, we illustrate the basic idea with a simple example where two agents select, respectively, x and y values with the respective consequences $u_1(x) = x^2$, $u_2(y) = y^3$. So far, the agents' actions are unrestricted. But if a constraint must be satisfied, such as $u_1(x) + u_2(y) \leq 1$, then the act of one person obviously limits what the other can do.

A strikingly similar situation arises in Sen's framework. When considering only a particular pair, or when no constraints are imposed on the societal ranking, the actions of the decisive agents are unfettered. But just as the model problem changes by imposing the $u_1(x)+u_2(y) \leq 1$ constraint, the choice theory setting is changed by imposing Sen's "no societal cycle" condition. This is because, according to Theorems 1, 3, the rankings for at least two pairs in each societal cycle are determined by different decisive agents. To avoid or break the cycle, at least one of these rankings must be reversed. Thus, to achieve a non-cyclic outcome, the actions taken by these decisive agents must be coordinated in some manner. [Notice: replacing the "no cycle" condition with any other constraint on societal outcomes that requires coordination of pairwise rankings creates another "impossibility theorem;" e.g., this kind of argument also explains why IIA causes Arrow's conclusion. What adds considerable support to the earlier Gaertner et al. (1991) comment about individual rights is that if a definition of "individual rights" involves, in any manner or structure, an unconstrained action, then conflict will arise.] As true with Theorem 2, this decentralization argument shows that a coordination among the decisive agents' actions is needed to resolve the difficulties raised by Sen's theorem.

Theorem 2 demonstrates one way to reintroduce information about transitivity of preferences to achieve this coordination: other approaches have been proposed. While developed independent of our structural argument, they clearly reflect our argument as they require coordination. As examples we point to the interesting contractual solutions explored by Hardin (1998), Sugden (1978, 1985) and others. The resulting mechanisms can experience other difficulties, such as with incentives [e.g., see Sen's (1986) comments about the contractual approach], but the first objective must be to understand what needs to be done to get around Sen's perplexing problems before addressing other consequences. This exploration is continued with game theory.

5 Seeking help from games

One might be left with the sense that the "no strong externality" condition of Theorem 2 must be enforced through laws. While reasonable, we now explore whether self-enforcing methods exist that capture the message of Theorem 2 and the spirit of the earlier quoted Sen's comment about individual values. Here we turn to elementary aspects of game theory.

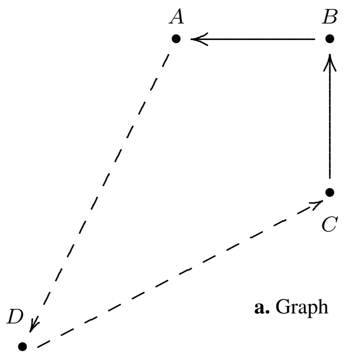
After Fine (1975) established a connection between the Prisoner's Dilemma (PD in what follows) and Sen's "Lascivious and Prude" example, game theory became a valued venue for studying individual rights (e.g., see Hammond, 1996; Pattanaik, 1996; and their references). These papers adopt a game theoretic approach: an overly simplified description is that in Sen's original version, the agents report their sincere preferences, while the generalizations permit other strategic behavior. Our goal, however, is to understand how to resolve the problems (the strong, negative externalities everyone suffers) from Sen's original formulation, so our use of game theory appears to differ.

As a player's moves in a game constitute the actions of a decisive agent, it is easy to establish connections between multiplayer games and Sen's theorem. To indicate how to do so with examples, consider only the part of the Eq. 3.3.4 information table that involves the AB, BC, CD, DA cycle and the two decisive agents Anne and Barb: this creates the abridged information table

Person	$\{A, B\}$	$\{B, C\}$	$\{C, D\}$	$\{A, D\}$
Anne	–	BC	CD	DA
Barb	AB	–	CD	DA
Outcome	AB	BC	CD	DA

(5.5.1)

Letting solid vertical and horizontal arrows indicate, respectively, the preferred alternative in each pair where Anne and Barb are decisive, and letting dashed arrows indicate a Pareto improved choice, the information table of Eq. 5.5.1 is captured by Figure 1a.



a. Graph

Barb

Anne $\begin{pmatrix} - & A & B \\ - & - & C \\ D & - & - \end{pmatrix}$

b. Associated game

Figure 1a,b. From a Sen example to a game

The Figure 1b game matrix mimics the direction of these arrows: the decisive moves are either horizontal or vertical reflecting which player can make which moves. So, substitute appropriate values for A, B, C, D in the Figure 1 matrix that, in the appropriate coordinate, satisfy the designated ranking inequality: the dashes indicate where the entry is immaterial. The particular choice made here,

$$\mathcal{G}_1 = \begin{pmatrix} - & 1,1 & 4,0 \\ - & - & 3,3 \\ 2,2 & - & - \end{pmatrix},$$

is not of the PD form. Using the Eq. 3.3.2 information table, or restoring Connie to this example, leads to a three player game

To go from a game to a Sen example, first select matrix entries that create a cycle, and then draw solid and dashed lines in the above manner. Two illustrations follow where $\mathcal{G}_2$ is the PD and $\mathcal{G}_3$ is a game with a mixed strategy solution. Anne and Barb are, respectively, the row and column players, and the generic representation for the matrix is $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

$$\mathcal{G}_2 = \begin{pmatrix} 5,5 & -1,6 \\ 6,-1 & 0,0 \end{pmatrix}, \quad \mathcal{G}_3 = \begin{pmatrix} 3,0 & 0,2 \\ 1,3 & 2,1 \end{pmatrix} \qquad (5.5.2)$$

Figure 2 shows each game's arrow structure that is used to construct the associated information tables.

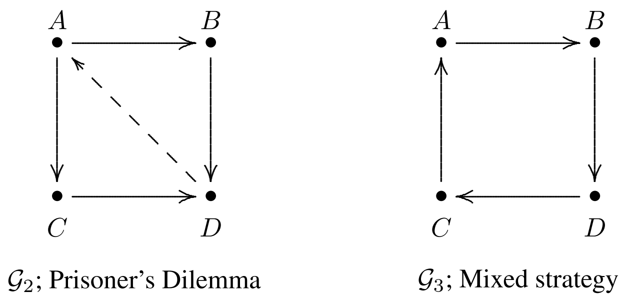


Figure 2. From games to Sen examples

For the PD game $\mathcal{G}_2$, Anne and Barb have, respectively, preferences $CADB$ and $BADC$. The information table for Sen's framework is

	$\{A,B\}$	$\{B,D\}$	$\{A,D\}$	$\{A,C\}$	$\{C,D\}$
Anne	$--$	DB	AD	CA	$--$
Barb	BA	$--$	AD	$--$	DC
Outcome	BA	DB	AD	CA	DC

(5.5.3)

For the mixed strategy $\mathcal{G}_3$ game, Anne and Barb have, respectively, preferences $ADCB$ and $CBDA$; the associated Sen societal cycle in Eq. 5.5.4 does not use the Pareto condition.

	$\{A,B\}$	$\{B,D\}$	$\{C,D\}$	$\{A,C\}$
Anne	$--$	DB	$--$	AC
Barb	BA	$--$	CD	$--$
Outcome	BA	DB	CD	AC

(5.5.4)

While there are connections between games and Sen's theorem, the significant differences complicate a transfer of insights from game theory to choice theory. Game theory, for example, seeks a specific solution while Sen's result seeks a societal ranking. Moreover, the choice of a strategy in game theory requires other options to disappear; e.g., if Anne selects the middle row in the Figure 1 game, Barb faces a triplet of options, but only these options. In Sen's framework, agents make decisions for each pair separately.

Yet, certain solution concepts, such as a mixed strategy, do transfer for special settings of Sen's theorem. To identify which solution concepts may be useful, recall from Theorem 2 that the goal is to incorporate information about individual rationality, as manifested by the strongly negative externality condition, into the analysis. This suggests considering the "Tit-for-Tat" strategy from an infinitely repeated PD. What makes "Tit-for-Tat" applicable for the PD is that while a player can exploit his opponent, and by doing so creates a strong negative externality, the repeated structure of the game allows his opponent to retaliate – this ability to retaliate encourages cooperation. This strategy may seem not to be applicable to Sen's setting because, as illustrated, most examples (this can be made precise) illustrating Sen's result do not have the PD game structure. But Theorem 3 establishes the needed connection by asserting that, for each societal cycle, any decisive agent who imposes a strong externality on someone else *is vulnerable* to the punishment of having a strong negative externality imposed on him. Consequently, the structure needed to impose a "Tit-for-Tat" strategy *always exists.* (With several decisive agents, implementing this strategy may involve coordinated action among several agents.)

Theorem 4. *If a decision problem with decisive agents is infinitely repeated, a "Tit-for-Tat" strategy, where action is taken to impose a strong negative externality on an agent who imposes a strong negative externality on someone else, always is applicable.*

Whether the "Tit-for-Tat" strategy enforces cooperation depends on how individuals discount the future and the values they assign to their choice and negative externalities: the analysis and proof of the theorem becomes a standard exercise. Our main point is that this strategy always exists, so it identifies an alternative way to achieve the needed coordination among agents described earlier, and a way to address the strong negative externalities (by retaliating on another issue). Clearly, the retaliatory behavior has parallels in practice.

To further explore the connection between Sen's result and daily life, consider those situations where actions of individuals impose upon others, but the decisive agents are immune from retaliation. An example from the 1980s is the choice to smoke in a restaurant. In terms of the structure of Sen's theorem, smokers were immune from sanctions because the non-smokers did not have sufficient power or rights. But once the non-smokers gained power, a cyclic societal outcome arose as manifested by claims and counter claims and even legislative debates. In other words, the societal cycles from Sen's theorem may manifest a transitory state during the emergence of retaliatory power (Theorem 4) for agents previously suffering from the acts of unrestrained decisive agents. Brunel (1998) explores related issues.

A weakness with "Tit-for-Tat" is that it assumes a continual (infinite) interaction among players. Thus, as our final comment about the "dysfunctional society" interpretation of Sen's theorem, we discuss another natural way to counter the strong negative externalities that cause Sen's result. Rather than using a "Tit-for-Tat" strategy, to avoid the PD consequences, we follow the spirit of mechanism design to allow the players adopt actions that change the game. Namely, a way to implement some version of Theorem 2 is to create the appropriate societal structures.

Rather than using changes determined by formal society, it is informative to consider settings where societal changes evolve out of a need to avoid PD type consequences. Our motivation comes from Sieberg's (2001) use of game theory to explain criminal activities, where she shows that many settings, prostitutes without a pimp, dealers without a gang, are captured by the PD. Using her prostitution example, if a customer pays in advance, the prostitute may not provide the expected service; if he is to pay after, he may leave without paying. Both are involved in illegal activity, so neither agent can appeal to authorities. To avoid these problems, a surrogate government – an enforcement policy policed by a pimp to ensure an atmosphere where customers have faith they will not be bilked (so they will return) and prostitutes are ensured payment – is created. To avoid the strong negative externalities, then, the players, as indicated in Eq. 5.5.5 (Sieberg, 2001, p. 66), take actions to convert the PD into a game where the Nash and Pareto optimal point agree.

$$\begin{pmatrix} (1,1) & (-\beta,\alpha) \\ (\alpha,-\beta) & (0,0) \end{pmatrix} \to \begin{pmatrix} (1-C,1-C) & (-\beta+P,\alpha-J) \\ (\alpha-J,-\beta+P) & (0,0) \end{pmatrix} \tag{5.5.5}$$

In Eq. 5.5.5, $\alpha > 1, \beta > 0$, which creates a PD. The changed environment is modeled by $0 < C < 1$, which can be viewed as an enforcement charge levied on each party, $J > \alpha$, which is the penalty imposed on the cheating party, and $P > \beta$, which is a payment to the injured party. The first matrix leads to a Sen cycle given by the Eq. 5.5.3 information table, while (with the realistic assumption that $-\beta + P < 1 - C$) the second matrix defines the Anne and Barb preferences as, respectively, $ABDC$ and $ACDB$. The associated information table is

	$\{A,B\}$	$\{B,D\}$	$\{A,D\}$	$\{A,C\}$	$\{C,D\}$
Anne	–	BD	AD	AC	–
Barb	AB	–	AD	–	CD
Outcome	AB	BD	AD	AC	CD

(5.5.6)

with the associated transitive societal ranking of $ABCD$. While this institutional approach does not completely fulfill Sen's sense that "individual values [will emerge] that respect each other's personal choice," it is comes close in the sense that enforcement approaches, either surrogate or legally established, guarantee it. Actions are taken by the players to avoid the strong negative externalities caused by others.

6 Conclusion

Although well examined for over three decades, Sen's theorem rightfully remains a source of considerable discussion and insight. As demonstrated here, the structure that explains the source of Sen's theorem is surprisingly rich; e.g., it demonstrates that rather than describing whether an individual has a right to make a personal decision within his "sphere of influence," the societal cycles reflect a "conflict among rights" causing a dysfunctional society. This dysfunctional society arises because the actions of the decisive agents create strong negative externalities for the others, and nobody is immune from these difficulties. Once this notion is identified, ways to address these conflicts are forthcoming.

7 Proofs

Proof (Theorem 2). Assume that the conclusion is false. This means that an example illustrating Sen's example can be created with a societal cycle, but where no individual strongly disagrees with any decisive agent's choice. According to Theorem 1, this example can be created by using the approach described in Section 3; this is where each individual starts with cyclic preferences and where, if necessary, the individual's ranking of a pair assigned to a different decisive individual is reversed to convert the ranking from the initial cyclic one to a transitive one. But if two alternatives are adjacent in a transitive ranking, reversing them keeps the ranking transitive. As at least the ranking of one pair from each cycle for each individual needs to be interchanged to generate a transitive ranking, and as this pair has to be one assigned to a decisive agent, for each individual and each cycle the transitive ranking has to have at least one strong disagreement. This contradiction proves the theorem.

Proof (Theorem 3). The above proof also proves Theorem 3. This is because to go from the original cyclic preferences to transitive ones, for each cycle each individual has to reverse the binary ranking for a pair that is assigned to a decisive individual. But, reversing this pair makes it a strong negative externality. Hence, for each cycle, each individual suffers at least one positive externality. Also, for each cycle, the construction requires at least two people to be decisive.

Proof (Theorem 4). This follows from the above discussion.

References

1. Arrow, K.: Individual values and social choice, 2nd edn. New York: Wiley 1951/1963
2. Arrow, K., Sen, A., Suzumura, K.: Social choice re-examined, vol. 2. New York: St. Martin's Press 1996
3. Broome, J.: Weighting goods. Oxford: Basil Blackwell 1991
4. Brunel, A.: Contribution à lánalyse des droits en théorie du choix social. Ph.D. Thesis, Université de Caen (1998)
5. Brunel, A., Salles, M.: Interpretative, semantic and formal difficulties of the social choice rule approach to rights. In: Laslier, J.-F., Fleurbaey, M., Laslier, N., Trannoy, A. (eds.) Freedom in economics, new perspectives in normative analysis, pp. 101–111. New York: Routledge 1998
6. Campbell, D., Kelly, J.S.: Sen's theorem and externalities. Economica **64**, 375–386 (1997)
7. Fine, B.: Individual liberalism in a paretian society. Journal of Political Economy **83**, 1277–1281 (1975)
8. Gaertner, W., Pattanaik, P., Suzumura, K.: Individual rights revisited. Economica **59**, 161–177 (1991)
9. Gibbard, A.: A pareto-consistent libertarian claim. Journal of Economic Theory **7**, 388–410 (1974)
10. Hammond, P.: Game forms versus social choice rules as models of rights. In: Arrow, K., Sen, A., Suzumura, K. (eds.) Social choiced re-examined, pp. 82–95. New York: St. Martin's Press 1996
11. Hardin, R.: Morality within the limits of reason. Chicago: University of Chicago Press 1998
12. Hausman, D.M., McPherson, M.S.: Economic analysis and moral philosophy. Cambridge: Cambridge University Press 1996
13. Hillinger, C., Lapham, V.: Freedom in economics: new perspectives in normative analysis, pp. 139–157. London and New York: Routledge 1998
14. Hurwicz, L.: Optimality and informational efficiency in resource allocation processes. in: Arrow, K.J., Karlin, S., Suppes, P. (eds.) Mathematical methods in the social sciences, pp. 27–46. Stanford, CA, Stanford University Press 1960
15. Hurwitz L., Reiter, S., Saari, D.: On constructing mechanisms with message spaces of minimal dimension for smooth performance functions. Mimeo, Northwestern University, University of Minnesota (1978)
16. Kolm, S.C.: Modern theories of justice. Cambridge, MA: MIT Press 1996
17. Luce, R.D., Raiffa, H.: Games and decisions. New York: Wiley 1957
18. Pattanaik, P.: On modelling individual rights: some conceptual issues. In: Arrow, K., Sen, A., Suzumura, K. (eds.) Social choiced re-examined, pp. 100–128. New York: St. Martin's Press 1996
19. Saari, D.G.: A method for constructing message systems for smooth performance functions. Journal of Economic Theory **33**, 249–274 (1984)
20. Saari, D.G.: Basic geometry of voting. Berlin Heidelberg New York: Springer 1995
21. Saari, D.G.: Are individual rights possible? Mathematics Magazine **70**, 83–92 (1997)
22. Saari, D.G.: Connecting and resolving Sen's and Arrow's theorems. Social Choice & Welfare **15**, 239–261 (1998)
23. Saari, D.G.: Mathematical structure of voting paradoxes 1; pairwise vote. Economic Theory **15**, 1–53 (2000)
24. Saari, D.G.: Mathematical structure of voting paradoxes 2: positional voting. Economic Theory **15**, 55–101 (2000)
25. Saari, D.G.: Elections and decisions; explaining the unexpected. Cambridge, UK: Cambridge University Press 2001
26. Salles, M.: On modelling individual rights: some conceptual issues: discussion. In: Arrow, K.J., Sen, A.K., Suzumura, K. (eds.) Social choice re-examined, vol. 2, pp. 129–133. New York: St Martin's Press 1997
27. Sen, A.: The impossibility of a Paratian liberal. Journal of Political Economy **78**, 152–157 (1970a)
28. Sen, A.: Collective choice and social welfare. San Francisco, CA: Holden-Day 1970b
29. Sen, A.: Liberty and social choice. The Journal of Philosophy **80**, 5–28 (1983)
30. Sen, A.: Foundations of the social choice theory: an epilogue. In: Elster, J., Hylland, A. (eds.) Foundations of social choice theory. Cambridge: Cambridge University Press 1986
31. Sen, A.: On ethics and economics. Oxford: Basil Blackwell 1987

32. Sieberg, K.: Criminal dilemmas. Berlin Heidelberg New York: Springer 2001
33. Sugden, R.: Social choice and individual liberty. In: Martis, M., Norbay, A. (eds.) Contemporary economic analysis. London: Croom Helm 1978
34. Sugden, R.: Liberty, preference and choice. Economics and Philosophy **1**, 213–219 (1985)