Sen's Theorem: Geometric Proof and New Interpretations

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Abstract

Sen's classic social choice result supposedly demonstrates a conflict between standard welfare concepts and even minimal forms of liberalism. By providing what appears to be the first direct mathematical proof of this seminal result, we reinforce a significantly different interpretation: rather than a conflict among rights, Sen's result occurs because subtle features of certain assumptions totally negate the effects of another one. Thus, what is stated need not be what we get. This explanation enriches interpretations of Sen's conclusion by allowing new and radically different kinds of societal conflicts. We also describe the problems Sen identified in terms of the likelihood that they can occur. By clarifying what causes Sen's result, our proof suggests new and practical ways to sidestep these difficulties. [1] [2]

1 Introduction

Problems central to decision and social choice theory were aptly characterized with Amartya Sen's comment (in his 1998 Nobel Prize lecture, also see Sen 1999) that

> a camel is a horse designed by a committee [because] a committee that tries to reflect the diverse wishes of its different members in designing a horse could very easily end up with something far less congruous, half a horse and half something else—a mercurial creation combining savagery with confusion.

Expanding on his quote, a basic objective in choice theory should be to discover societal decision rules that avoid creating camels when horses are intended. Is this possible? Sen, a leader in identifying subtle but important barriers that prevent achieving this objective, discovered a fundamental difficulty in his 1970 "Impossibility of a Paretian Liberal" (Sen, 1970a, b, 1976c). By demonstrating it is impossible to satisfy even a surprisingly minimal aspect of liberalism when combined with the Pareto condition, Sen's Theorem appears to describe a fundamental conflict between standard welfare concepts and liberalism. While his surprising result spawned an extensive literature about individual rights, his theorem is, in fact, wider reaching by capturing central concerns across disciplines. To indicate this universality, we illustrate Sen's Theorem with an example as common as an academic department hiring a new faculty member.

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A traditional way to analyze an impossibility result from choice theory, such as Sen's Theorem, is to modify the basic assumptions until a positive conclusion emerges. But to find positive conclusions and to explore the basis for the ongoing discussion of Sen's result, we needed to go beyond knowing *that* Sen's assumptions create a conflict to determine *why* they conflict. To do so, we derived what appears to be the first general, direct proof of Sen's assertion.[3]

Interesting corollaries follow from our approach. For instance, by knowing why Sen's theorem occurs, we can enrich the discussion of his seminal conclusion by identifying new and very different interpretations. Also, while any rule satisfying Sen's assumptions can experience societal cycles, we tend to forget that this behavior need not always occur. Do these cycles dominate the societal outcomes, or do they constitute a minority? As our proof catalogues *all possible profiles* that support a particular Sen outcome, it was easy to answer this question by deriving what appears to be the first "probabilistic" description of Sen's conclusion. We find that while the problems Sen identified are definitely in a minority, they occur often enough to be troubling. Finally, to advance our goal of finding positive assertions, we identify the features that must be part of any successful approach to sidestep the fundamental difficulties.

1.1 Sen's Theorem

To indicate why the problems Sen identified enjoy a universality extending beyond discussions about individual rights, suppose a two-member search committee for an economics department is charged with hiring one of the final candidates Amy, Bill, and Cindy. As part of the evaluation, each candidate's citation index and quality of published papers are examined; assume this leads to the following listing of relevant information:

Candidate	**IQ**	**Macro citations**	**Micro citations**	**Years from PhD**
Amy	120	60	10	4
Bill	110	55	80	3
Cindy	100	65	70	5

(1)

The rules for assembling the committee ranking are natural:

- **Unrestricted Domain.** Each committee member can rank the candidates in any desired manner as long as the ranking is complete and transitive.

- **Pareto.** If everyone ranks a pair in the same manner, this common ranking will be the committee's ranking.

- **Minimal Liberalism (ML)—or Division of Expertise.** Committee members were selected because of their expertise. As Garrett's expertise is macroeconomics—an area in which both Amy and Bill claim ability—it is natural to defer to Garrett's knowledge by asserting that how he ranks Amy and Bill will be the committee ranking. Similarly, Sandy is an expert

[3] All proofs of Sen's result that we have seen are based on creating examples; our proof appears to be the first direct verification of this result.

in microeconomics where both Bill and Cindy claim abilities: Sandy's ranking of Bill and Cindy determines their committee ranking. (ML is defined in Thm. 1.)

- **No Cycles.** In order to make a decision, the committee ranking must be cycle free.

By identifying this example with Sen's Theorem, it follows that with three or more candidates and two or more committee members, *no ranking rule will always satisfy these four conditions.*

Theorem 1 *(Sen, 1970), Assume that each voter has a complete, binary, transitive preference ranking with no restrictions on the preferences. There does not exist a decision rule that ranks the alternatives and always satisfies the following conditions:*

1. *(Minimal Liberalism) There are at least two agents each of whom is decisive over at least one assigned pair of alternatives. Their ranking of the assigned pairs of alternatives determine the societal ranking of the pairs.*

2. *(Weak Pareto) If for any pair of alternatives, all voters rank the pair in the same manner, then this unanimous ranking is the societal ranking of the pair.*

3. *The outcome does not have any cycles.*

Without the transitivity assumption, Sen's result loses all surprise: the conclusion becomes obvious and immediate. After all, if the voters can have cyclic preferences, then we must expect cyclic societal outcomes. In other words, the transitivity of individual preferences goes beyond being a tradition assumption to become a crucial one.

Sen's assertion mandates that situations exist where the search committee cannot satisfy the specified requirements. Indeed, according to Table 1, Garrett prefers Amy over Bill because of their relative performances in macroeconomics. His one disappointment is that Cindy, who has the best performance, no longer is interested in this area: Garrett's ranking is $C \succ A \succ B$. Sandy, on the other hand, is impressed with Bill's citation record based on his theory papers, so she ranks Bill above Cindy. But with her negative opinion of Amy, Sandy's ranking is $B \succ C \succ A$.

Using a dash to represent where information from a person is irrelevant (because the decision is determined by another member), the information used to assemble the committee ranking follows:

Member	**Ranking**	$\{A, B\}$	$\{B, C\}$	$\{A, C\}$
Garrett	$C \succ A \succ B$	$A \succ B$	–	$C \succ A$
Sandy	$B \succ C \succ A$	–	$B \succ C$	$C \succ A$
	Committee Ranking	$A \succ B$	$B \succ C$	$C \succ A$

(2)

Sen's assertion is demonstrated by the cyclic outcome, and, in practice, by the need to hold more committee meetings. Obvious modifications can be made; e.g., each committee member could be replaced with several members where decisions are made by majority vote, the committee could use a wider assortment of information including letters of recommendation, etc.

1.2 Outline

Sen (1970a, b) asserts that his result demonstrates the inconsistency of ML with the Pareto condition. But Sen's result traditionally is established via examples, rather than a formal proof, where it then is *assumed* that the cyclic societal rankings are caused by a conflict between the ML and Pareto conditions. Is this assumption correct? What if these cyclic societal rankings are *not* due to these conditions? The consequences are significant; if *all* cyclic societal outcomes are caused by a different subtle feature, then we must re-examine the large literature generated by Sen's theorem.

In fact, Saari (1998, 2001) claims that the true cause for Sen's seminal result is not the nature of ML and Pareto, but rather because *ML and Pareto require societal rankings to be made over pairs.* By specifying what happens with each pair, connecting information among pairs, including transitivity of individual preferences, is dismissed. But if the rule cannot use the transitivity of individual preferences, the de facto setting reverts to that described after the statement of Sen's Theorem where cyclic outcomes are anticipated. In other words, rather than a conflict between societal needs and individuals rights, Sen's result reflects how conditions that concentrate on pairs can ignore the individual rationality assumption. Indeed, the proof of the above comments (Sects. 2, 5) make it clear that ML and Pareto cannot be the basic problem because they can be replaced with many other conditions *that force attention on individual pairs,* and the same conflict arises.

What makes this "cyclic voters" comment surprising is that Sen explicitly requires the agents to have *transitive preferences;* he must because, as mentioned, without transitive preferences or the ability of a decision rule to use this transitivity information Sen's Theorem loses substance. But if this transitivity assumption is ignored, then we must wonder whether, rather than reflecting a conflict among rights, Sen's cycles occur because the decision rule treats the *actual transitive preferences as being cyclic.* As we show, this is precisely what happens. To demonstrate this phenomenon with the search committee example, notice that all of the information needed by the Sen decision rule to assemble the committee ranking is

Evaluator	$\{A, B\}$	$\{B, C\}$	$\{A, C\}$
Garrett	$A \succ B$	–	$C \succ A$
Sandy	–	$B \succ C$	$C \succ A$
Committee Ranking	$A \succ B$	$B \succ C$	$C \succ A$

(3)

It is obvious from this table that each committee member's *full* ranking of the candidates is irrelevant. We do not know, and it does not matter, whether Garrett's ranking of $\{B, C\}$ is $C \succ B$, which would make his ranking transitive, or $B \succ C$, which would make his preferences cyclic. This suggests that the real conflict generated by Minimal Liberalism is not one of rights, but rather that ML emphasizes specific pairs, with the consequence of ignoring any connecting information. This ML feature is what vitiates the crucial assumption of transitive preferences.

To use an analogy to explain how this affects the explicitly made transitivity assumption, notice that the explicitly stated information in the "Years-from-PhD" and "IQ" columns of Table 1 played no role in determining the committee outcome. Consequently this irrelevant information can be safely removed without affecting the analysis. Applying the same standards, once we prove that transitivity of preferences plays no role in Sen's result, we can safely remove the assumption of "transitive preferences" from Sen's Theorem. But by doing so, it becomes questionable whether Sen's result captures a conflict between societal and individual rights. As we show in Sect. 3.3, it can: we describe how to re-establish this "rights" connection but with a very different interpretation.

As it is crucial for our analysis to determine whether "transitivity" is, or is not, ignored, we must consider all possible profiles whether transitive or not. This formulation underscores that the intent of the individual rationality assumption is to restrict the decision rule to transitive profiles. We must determine whether Sen's assumptions force the decision rule to respect, or to ignore, this restriction: we prove that the rules ignore it.

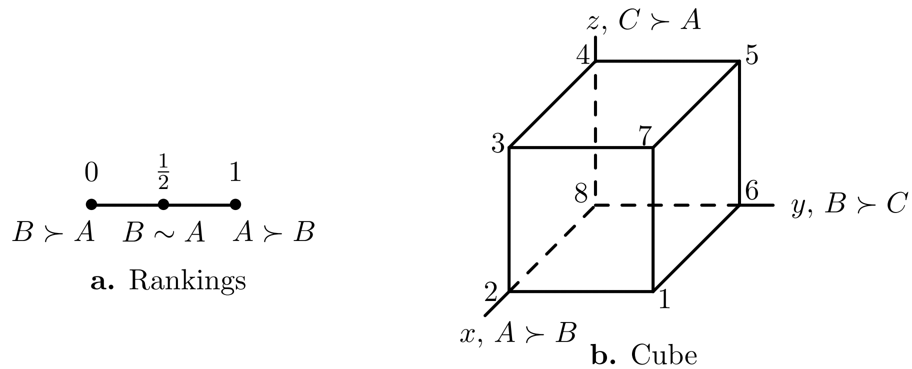


Fig. 1. Cube and rankings

2 Notions of the Geometric Proof

While the proof[4] of Sen's Theorem is in Sect. 5, we describe a special case with three alternatives $\{A, B, C\}$ to develop intuition about the structure of Sen's result and to identify other corollaries and consequences. For a specified pair, say $\{A, B\}$ in Fig. 1a, let the points $\{0, \frac{1}{2}, 1\}$ on the line interval $[0, 1]$ represent, respectively, $B \succ A$ (B is preferred to A), $A \sim B$ (indifference between A and B), and $A \succ B$. To represent all three pairs, use the unit cube (Saari 1995) where the x axis represents the $\{A, B\}$ rankings as described, the y direction represents $\{B, C\}$ rankings where $y = 1$ corresponds to $B \succ C$, and the z direction represents the $\{A, C\}$ rankings where $z = 1$ represents $C \succ A$. This is displayed in Fig. 1b where, for convenience, the vertices are labeled 1 to 8. The rankings associated with the eight vertices are

Vertex	**Ver. No.**	**Ranking**	**Vertex**	**Ver. No.**	**Ranking**
$(1, 1, 0)$	1	$A \succ B \succ C$	$(0, 0, 1)$	4	$C \succ B \succ A$
$(1, 0, 0)$	2	$A \succ C \succ B$	$(0, 0, 1)$	5	$B \succ C \succ A$
$(1, 0, 1)$	3	$C \succ A \succ B$	$(0, 1, 0)$	6	$B \succ A \succ C$
$(1, 1, 1)$	7	$B \succ C, C \succ A, A \succ B$	$(0, 0, 0)$	8	$B \succ A, C \succ B, A \succ C$

(4)

Vertices 7 and 8 correspond to cyclic rankings while the other six represent transitive rankings. Two vertices on a cube edge differ only by the ranking of a single pair: e.g., the rankings assigned

[4] A sketch for this proof first appeared as Li's answer for an exam problem while she was a student in a course taught by her coauthor. Li's outline was motivated by Saari's proof (2001) of Arrow's Theorem.

to vertices 1 ($A \succ B \succ C$) and 2 ($A \succ C \succ B$) differ only in the $\{B, C\}$ ranking; this choice reflects that the connecting edge is parallel to the y axis (the $\{B, C\}$ ranking edge).

Important for our proof are the geometric connections among transitive and cyclic rankings. Each cyclic vertex, for instance, is adjacent to three transitive rankings. This geometry indicates that a change in one particular pair from any of these transitive rankings creates the cyclic one; e.g., as vertices 5 and 7 are on an edge in the x-direction, reversing the $B \succ A$ ranking of vertex 5 ($B \succ C \succ A$) leads to the cyclic ranking of vertex 7. Also, each transitive ranking is "adjacent" to a cyclic one; e.g., each vertex with an "odd" name is adjacent (connected by an edge) to vertex 7, each vertex with an "even" name is adjacent to vertex 8. Notice an interesting feature: if an edge connects two transitive "vertices," then the binary ranking change involves two adjacently ranked alternatives; e.g., vertices 1 and 2 differ in $A \succ (B \succ C)$ and $A \succ (C \succ B)$. But an edge connecting a transitive and cyclic ranking involves changing a binary ranking where the alternatives are *not* adjacent in the transitive ranking—they are separated by another alternative; e.g., vertices 3 and 7 differ in the $\{B, C\}$ ranking where 3 has $C \succ A \succ B$—with A separating C and B—while 7 has $B \succ C$. This "adjacency" geometry, which extends to any number of alternatives, plays a major role in our analysis.

2.1 Creating an example

To illustrate how to use this geometry to prove Sen's Theorem, start with the decisive choices in the search committee example and assume only that each agent ranks each pair; i.e., do not assume that an agent's ranking is transitive. Garrett is decisive over $\{A, B\}$, so his choice forces the outcome to be on either the front ($x = 1$) face of the cube, with vertices $\{1, 2, 3, 7\}$ for the $A \succ B$ ranking, or on the back ($x = 0$) face with vertices $\{4, 5, 6, 8\}$ for the $B \succ A$ ranking. Similarly, Sandy's choice of $\{B, C\}$ allows her to select either the side face of $y = 1$ representing $B \succ C$, or the $y = 0$ side face representing $C \succ B$. Garrett's and Sandy's choices determine the societal outcome for these two pairs, so the societal outcome must be on both selected faces; i.e., it is a vertex on the connecting edge for the two faces. For each of the four possible cases, the two selected faces are connected by a vertical edge that can be identified by its vertices.

Garrett	**Sandy**	**Edge Vertices**	**Garrett**	**Sandy**	**Edge Vertices**	
$x = 0$	$y = 0$	$4, 8$	$x = 1$	$y = 1$	$1, 7$	(5)
$x = 0$	$y = 1$	$5, 6$	$x = 1$	$y = 0$	$2, 3$	

According to Table 5, two of these choices (the bottom row) must have a transitive outcome (represented by either of the two specified vertices) independent of whether the agents have transitive or cyclic preferences and independent of the societal $\{A, C\}$ ranking. The choices from the top row, however, allow a cyclic outcome as defined by either vertex 7 or 8. To ensure the vertex 7 cyclic societal outcome, accompany the $x = 1, y = 1$ choices with *any condition* (including any replacement of Pareto that selects a ranking for the pair)[5] that places the societal outcome on the

[5]For example, the Pareto condition could be replaced with the majority vote over a pair or by a host of other

$z = 1$ surface. Sen's Theorem specifies two choices: either assign the $\{A, C\}$ pair to a decisive agent, or use the Pareto condition. In the former case, one agent must be decisive over $\{A, C\}$. So to have a $z = 1$ societal outcome, this agent must have the $z = 1$ ($C \succ A$) ranking. Alternatively, to ensure a $z = 1$ societal ranking (with the vertex 7 cyclic societal outcome) with the Pareto condition, both Sandy and Garrett must have a $z = 1$ ($C \succ A$) choice. Similarly, to ensure the vertex 8 societal conclusion, supplement the $x = 0, y = 0$ choices from decisive agents with any condition (Pareto or an agent is decisive over two pairs) that forces a $z = 0$ outcome.

It remains to find transitive preferences that yield either the vertex 7 or 8 cyclic outcome. (As the construction never uses the transitivity of individual preferences, we must show that at least one supporting profile involves transitive preferences.) The analysis is immediate because selecting a ranking for a pair and a cube face are equivalent — the societal outcome is sole vertex on all three selected faces. An agent's preferences are also identified with those specified cube faces that the agent helped to select. But with two or more decisive agents, each agent's possible preferences are on the connecting edge of at most *two* faces. With Garrett and the vertex 7 outcome, for instance, the choices for his profiles are vertices on the connecting edge of the $x = 1, z = 1$ faces: vertices 3 and 7. The following table lists all supporting profiles when the z outcome is determined by the Pareto condition.

Vertex outcome	Garrett	Garrett's edge	Sandy	Sandy's edge
7	$x = 1,\ z = 1$	$3, 7$	$y = 1,\ z = 1$	$5, 7$
8	$x = 0,\ z = 0$	$6, 8$	$y = 0,\ z = 0$	$2, 8$

(6)

According to the top row of Table 6, the cyclic societal outcome of vertex 7 is supported by four possible profiles where in each pair Garrett's preference is listed first: $\{(3, 5),\ (3, 7), (7, 5), (7, 7)\}$. Only the $(3, 5)$ choice endows each agent with a transitive preference to create an example. Similarly, all possible profiles with a vertex 8 cyclic societal outcome come from the bottom row; they are $\{(6, 2), (6, 8), (8, 2), (8, 8)\}$ where $(6, 2)$ represents the only transitive preferences.

2.2 Observations

The complete proof of Sen's result for three alternatives just modifies the above to handle all possible choices for which agents are decisive, over what pairs, and whether the Pareto condition is, or is not, needed. The general proof (Sect. 5) involving any number of agents and alternatives is similar where the geometry of the cube is replaced with the geometry of hypercubes. Transitive preferences come into the analysis only to verify that among the several supporting profiles, a transitive one can be found. In other words, as claimed, *the individual rationality assumption plays no role in Sen's theorem.*

possibilities. The same analysis, showing that the Pareto condition is not important to obtain a Sen-type conclusion, holds for any number of alternatives and agents. *Any rule,* which ranks a pair independent of the choices made by the decisive agents and preferences over other pairs, suffices. ML can be similarly replaced.

Certain conclusions follow immediately from our geometric construction. For instance, the ranking of a pair, as determined either by a decisive agent or by all agents through the Pareto condition, defines a face of the hypercube. The societal outcome is determined by the intersection of all selected faces. But with at least two decisive agents, each agent's preferences must be on *some*, but not all, of these faces. This observation leads to a geometric proof (Sect. 5) of the following assertion made by Saari.

Theorem 2 *(Saari, 2001) Allow voter preferences to include all ways to rank pairs, rather than just the transitive rankings. Any example of cyclic societal rankings created by choices of decisive agents and the Pareto condition includes the unanimity profile where each voter's preference agrees with the cyclic societal ranking over these pairs.*

Theorem 2 provides a particularly simple way to construct *all possible examples* illustrating Sen's Theorem (Saari 2001), including those in (Salles 1997). Even more: as the theorem holds for all possible examples illustrating Sen's result, it supports the comment that Sen's conditions force a decision rule to ignore the individual rationality assumption. Instead, the rule tries to service the wishes of non-existent voters with cyclic preferences. (See (Saari 2001) and (Saari and Petron 2004).)

As our geometric proof identifies *all possible profiles* supporting a Sen example, we can significantly expand on earlier observations by providing a new statistical interpretation for Sen's Theorem; one that supports this "non-existent cyclic preferences" comment. We start by determining how often a Sen-type phenomenon can be expected to occur.

3 How often and why do societal cycles occur?

To appreciate why Sen's assumptions force cyclic societal outcomes, we adopt the natural approach of examining the data—voter preferences. In this way we can determine when, why, and how often societal cycles occur.

3.1 Likelihoods

With the emphasis on the Sen cycles, we tend to forget that these cycles not not always occur. Are the cycles the dominant effect, or are they in a minority. We answer this question here.

Table 5 identifies all supporting profiles for all possible societal conclusions, so it can be used to construct examples supporting any of the six transitive and two cyclic rankings as societal outcomes. Even more: with assumptions about the likelihood of each binary choice, we can determine the likelihood of each societal outcome. In doing so we discover that societal cyclic outcomes are distinctly less likely than transitive ones—even when voters have cyclic preferences.

Theorem 3 *With three alternatives and n-agents ($n \geq 2$), suppose there are two decisive agents where each is decisive over precisely one of two designated pairs. Assume that the voters indepen-*

dently rank each pair, but not necessarily in a transitive manner. Assume that, for each agent, each of the two rankings for a pair is equally likely. Assuming that the Pareto condition is satisfied for the one pair not assigned to decisive agents, the likelihood of a transitive ranking is $\frac{3}{4}$*, and the likelihood of a cyclic outcome is* $\frac{1}{4}$*. But if each voter's preferences are given by strict transitive rankings, where each is selected independently and with equal likelihood, then, subject to the condition that the Pareto condition is satisfied for the one pair, the likelihood of a transitive societal outcome is* $\frac{8}{9}$ *while the likelihood of a cyclic outcome is* $\frac{1}{9}$*.*

With three decisive agents, each decisive over a different pair, the likelihood of a cyclic outcome is $\frac{1}{4}$*; this is true whether the voters can only rank pairs, or have transitive rankings.*

Although cyclic behavior is not prominent (Thm. 3), it occurs often enough to be bothersome. A similar statement holds for any number of alternatives and decisive voters. With more decisive (rational) voters and alternatives, however, expect the likelihood of a Sen cycle to increase. The proof of Thm. 3 is in Sect. 5, but a sense of these assertions follows from the Table 5 list of profiles. The actual proof reduces to a counting argument.

The next natural question is to characterize those settings where a transitive, or a cyclic, outcome occurs. By emphasizing the data, the structure of a level set (i.e., all profiles supporting a specified societal outcome) becomes a valuable tool. For instance, if cyclic preferences are not in the level set of a societal outcome, then cyclic preferences play no role in determining that outcome. Conversely, if cyclic preferences dominate the level set, they play a major role. Thus to understand whether and how ML undercuts the assumption of transitive preferences we need to analyze when, and how often, cyclic preferences are in the level sets of outcomes.

Proposition 1 *Assume there are three alternatives and two agents, where each is decisive over a different pair and they agree on the ranking of the remaining pair. If the societal outcome is transitive, then either all choices of preferences for the two agents are transitive, or half of the profiles require a transitive preference for one agent. It is impossible for a transitive societal outcome to be supported by a profile consisting strictly of cyclic preferences. On the other hand, should the outcome be cyclic, then three-fourths of the possible supporting profiles include a voter with cyclic preferences, and both rankings are cyclic for one choice.*

The proof is in Sect. 5, but a sense for this result comes by examining all profiles for the search committee example that lead to different kinds of outcomes. While a similar result holds in general, the approach developed next provides an easier way to capture this fact.

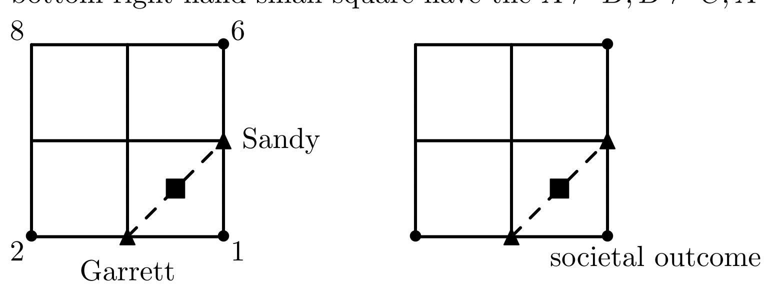
3.2 Interpretation

As Prop. 1 shows, it is reasonable to associate a transitive societal outcome with transitive profiles. After all, all voters have transitive profiles for most supporting profiles. Only one-third of the profiles allow a voter with cyclic preferences, and even here the other person must have transitive preferences.[6] So in an undefined sense (maybe just accidentally), the individual rationality of voters affects transitive societal outcomes.

The situation drastically changes with a cyclic societal outcome: only one-fourth of the possible supporting scenarios endow all agents with transitive preferences. In contrast, half of the profiles have at least one cyclic voter and one-fourth require both voters to have cyclic preferences. These comments suggest that cyclic outcomes for a Sen decision rule occur because, by ignoring the individual rationality assumption, the rule is capturing the views of non-existing cyclic voters rather than the actual transitive ones. As this interpretation differs significantly from the "rights" discussion in the literature, it must be made precise, and we do so next. More precisely, we prove that all Sen rule outcomes, whether transitive or cyclic, admit a common interpretation.

To accomplish this goal, we modify notions that Saari and Sieberg (2001) developed to explain pairwise voting paradoxes. By examining all possible profiles supporting any specified set of pairwise outcomes, they proved that a societal outcome always and accurately reflects the *average* of all supporting profiles—but not necessarily a specific one. "Paradoxes," then, are merely situations where a specified profile deviates from the average of *all* supporting profiles, so the rule ignores it. As an illustration, the "paradox of voting" given by profile $A \succ B \succ C$, $B \succ C \succ A$, $C \succ A \succ B$ is where, by 2:1 votes, the pairwise cyclic outcome is $A \succ B$, $B \succ C$, $C \succ A$. Of the five profiles that support this outcome, the "average" is a cyclic profile that agrees with the cyclic outcome.

In a similar way, we show that all Sen rule outcomes are "averages over all possible supporting profiles." To motivate our approach, where we must modify the Saari and Seiberg argument to capture Sen's assumptions, suppose the societal outcome is $A \succ B \succ C$ "vertex 1," because of Garrett's decisive $A \succ B$ choice and Sandy's $B \succ C$: both have the $A \succ C$ ranking. Garrett's preference ranking is given either by vertex 1 or 2, and Sandy's by either vertex 1 or 6 — they are the bullets in Fig. 2a, which is the bottom face of the Fig. 1b cube. The square is subdivided into four smaller squares where the ranking associated with a point is determined by the closest vertex; i.e., all points in the bottom right-hand small square have the $A \succ B, B \succ C, A \succ C$ ranking.



a. A specific example **b.** In general

Fig. 2. Explaining Sen's outcomes

Continuing our theme of analyzing the data, notice that of the four possible profiles, $(1, 1)$, $(1, 6)$, $(2, 1)$, $(2, 6)$, the societal outcome is conclusively supported by $(1, 1)$ with its unanimous support of the vertex 1 ($A \succ B \succ C$) outcome. Only one voter in next two profiles prefers

[6]Four profiles are associated with each pair of edges. With three pairs of edges, there are twelve possible profiles. Eight choices are transitive; four choices have one cyclic preference.

$A \succ B \succ C$, so they do not provide as strong support as profile $(1, 1)$, but the outcome remains reasonable. Only the outlier $(2, 6)$ profile, where Garrett prefers $A \succ C \succ B$ while Sandy prefers $B \succ A \succ C$, might raise doubts about the $A \succ B \succ C$ conclusion.

To analyze all examples, we devise an argument that involves all supporting profiles for any specified Sen outcome. To do so, notice that the flexibility in Garrett's and Sandy's choices for a pair occur when the other agent is decisive over that pair. A natural way to handle the unknown preferences of a "non-decisive" agent is to replace them with the average of their possible choices. In Fig. 2a, then, average Garrett's choices of $B \succ C$ and $C \succ B$ to obtain the $B \sim C$ ranking as indicated by the triangle on the bottom edge of Fig. 2a. Similarly, averaging Sandy's two choices for the $\{A, B\}$ ranking leads to a $A \sim B$ ranking indicated by the triangle on the right edge of Fig. 2a. The "average of the averaged rankings" is represented by the dark square; it averages the "averaged points" (the triangles) of both agents. To describe this "averaging of averaged rankings" approach with coordinates, the Fig. 1b coordinates for Garrett are $(1, 0, 0)$ and $(1, 1, 0)$, so the averaged value is $\frac{1}{2}(1, 0, 0) + \frac{1}{2}(1, 1, 0) = (1, \frac{1}{2}, 0)$. Similarly, the averaged coordinates for Sandy are $(\frac{1}{2}, 1, 0)$. In turn, the averaged outcome of the two averaged rankings is $\frac{1}{2}(1, \frac{1}{2}, 0) + \frac{1}{2}(\frac{1}{2}, 1, 0) = (\frac{3}{4}, \frac{3}{4}, 0)$, which is the $A \succ B$, $B \succ C$, $A \succ C$ rankings of vertex 1. The next definition makes this precise for any number of agents, pairs, any assignment of decisive agents to pairs, etc.

Definition 1 *All rankings for n specified pairs can be represented in an "orthogonal cube" in R^n; it is the cube defined by the 2^n vertices with entries consisting of zeros and ones. Each coordinate direction designates the rankings for a particular pair where 0 and 1 represent the pair's two strict rankings. For a point in the orthogonal cube, the ranking for each pair is determined by the point's proximity to each vertex where "closer is better."*

Let $\mathcal{K}_j$ be a set of k n-vectors where each coordinate entry of a vector consists of zeros and ones; $\mathcal{K}_j$ represents k possible strict rankings, over all n pairs, that the jth agent can choose. The agent's "averaged preference ranking" is the average of the k-vectors.

If a vector representing a ranking, or an averaged ranking, is assigned to each of a agents, then the "averaged ranking over the agents" is the average of the a vectors.

As indicated in Fig. 2b, the Fig. 2a geometric argument explains any societal outcome—even a cyclic one. Rather than a ML and Pareto conflict, a Sen outcome reflects an average over the voters' choices. As asserted next, this is always true (proof in Sect. 5); the averaged ranking of the averaged choices for the voters always agrees with the Sen outcome. Consequently a paradox, such as when a Sen cyclic ranking occurs with transitive profiles, means that the rule is essentially ignoring the specified profile because it differs in nature from the averages of all supporting profiles.

Theorem 4 *For any number of alternatives, for any specified number of decisive agents where each is assigned a specified number of (distinct) pairs, let $\mathcal{S}$ represent the portion of a decision rule's societal outcome that is determined by the choices made by the decisive agents and by the Pareto condition. For any societal outcome, one of the supporting profiles is the unanimity profile* *where each voter's preference (for the $\mathcal{S}$ portion of the rankings) agrees with the societal outcome. Moreover, this $\mathcal{S}$ portion of the societal outcome always coincides with the ranking determined by the average of the averaged profile (with no assumption of transitivity) for each agent.*

While is it too much to expect a decision rule to accurately reflects the intentions of each and every profile, the $\mathcal{S}$ part of the societal outcome accurately reflects either a particular unanimity profile, or the average of all possible complete binary rankings (transitivity need not hold). In particular, Thm. 4 means that with a cyclic societal Sen outcome, the decision rule does *not* involve, in any manner, the crucial assumption of individual rationality. Instead, the actual transitive profile is being statistically dismissed as an unlikely outlier where the outcome is determined by the cyclic preferences of non-existent (by assumption) voters.

3.3 Individual liberties?

As the above arguments make clear, the cyclic outcomes are not caused by those properties of ML and Pareto that concern rights, but rather because the outcomes for pairs are determined independent of outcomes for other pairs. It is this feature that causes the rule to ignore the explicit individual rationality assumption. It now is easy to replace ML and Pareto with any number of other conditions—they need not involve "rights"—and the same conclusion occurs.

But as indicated in the introduction, Sen's conditions capture a selection of actual behavior: people do make individual decisions and there is a tendency to respect the Pareto condition. So, in light of the above analysis, how should ML and Pareto be interpreted? In addressing this question, Saari and Petron (2004) found surprisingly different interpretations of Sen's result: rather than addressing individual liberties, it is arguable that there are situations where Sen's cycles more accurately model a contentious, or maybe even a dysfunctional, society where *everyone* is in strong conflict with someone else. Theorem 4 and the geometry of the cube helps us identify when it is arguable that a Sen outcome captures individual liberties and when it identifies a contentious society. Our argument uses the following definition.

Definition 2 *(Saari 2001, Saari and Petron 2006) In a transitive ranking where alternative X is ranked above Y, let $[X \succ Y, \alpha]$ represent the relative ranking where α designates the number of alternatives in the transitive ranking that separate X and Y.*

A strict ranking for a pair of alternatives, (X, Y), that is determined by a decisive agent imposes a strong negative externality *on another agent if that other agent's transitive preference ranks the pair in the opposite manner, say $[Y \succ X, \alpha]$, and the alternatives are separated by at least one other alternative, that is $\alpha > 0$.*

To illustrate the definition, if Sandy is decisive over $\{B, C\}$ and selects $B \succ C$, this does not create a strong negative externality for Ann's $C \succ B \succ D \succ A$ preferences because her $[C \succ B, 0]$ ranking shows that C and B are not separated by another alternative. Jane, with a

$C \succ A \succ B \succ D$ ranking, or $[C \succ B, 1]$, does experience a *strong negative externality* by Sandy's choice as her opposing $C \succ B$ ranking is separated by alternative A. In the search committee example, Garrett's $[C \succ B, 1]$ ranking displays his strong disagreement with Sandy's $B \succ C$ choice, and Sandy's $[B \succ A, 1]$ ranking demonstrates the strong negative externality caused by Garrett's $A \succ B$ choice. As shown next, with a Sen cycle, *everyone* strongly disagrees with someone else. The second part of the following theorem is in (Saari 2001) and (Saari, Petron 2004) but a new geometric proof is given. The first part, used to interpret transitive societal outcomes, is new.

Theorem 5 *For any number of agents, where each has transitive preferences, assume that a specified number (two or more) of them are decisive where each is assigned at least one (distinct) pair of alternatives. If the $\mathcal{S}$ part of a societal ranking is transitive, then there are choices of transitive preferences for each agent that support the outcome and ensure that the agent does not suffer a strong negative externality due to choices made by the decisive agents. If, however, this $\mathcal{S}$ part of the societal outcome is cyclic, then, for each cycle,* each *agent suffers a strong negative externality for a choice made by some decisive agent.*

With a transitive societal outcome, then, there is no apparent reason to question whether minimal liberalism models individual liberties. Other agents could, of course (Thm. 5), disagree with the choices made by the decisive agents, but the disagreement need not be severe. Instead, the average of the averages of their possible choices coincides with the transitive outcome (Thm. 4). The situation differs significantly with a cyclic societal outcome: as *everyone* suffers a strong negative externality imposed by someone else, we might wonder whether there are times where this contentious society is caused by some decisive agents abusing their absolute freedom of choice.

4 Sidestepping Sen-type consequences

The ubiquity of Sen's assumptions suggests that ways to sidestep his negative conclusions must be situation-specific. If, for instance, Sen's cycles occur because some decisive agents abuse their power by inflicting strong negative externalities on others, then an obvious solution is to abridge the rights of the abusive decisive agents. Saari and Petron (2004) develop this approach by allowing a decisive agent to be decisive only if the decision does not impose a strong negative externality on someone else. But they did not identify what should be the outcome for this pair because a resolution requires information beyond whether a strong negative externality exists. Before explaining the nature of this added information, we give a partial answer.

Theorem 6 *Limit a decisive voter's rights in the following manner: if a decisive voter's ranking of a pair imposes a strong negative externality on another agent, then replace the decisive voter's choice with a tie ranking for the alternatives. The portion of the societal outcome made by this modified form of decisive voters and the Pareto condition does not create cycles.*

While this modification of minimal liberalism avoids cyclic societal outcomes, the outcome need not be transitive. To see this with the search committee example, because Garrett's $A \succ B$ choice creates a strong negative externality for Sandy with her $B \succ C \succ A$ preferences, Garrett's choice is replaced with the tie $A \sim B$. For similar reasons, Sandy's $B \succ C$ choice is replaced by $B \sim C$. As such, the societal ranking is the $C \succ A$, $A \sim B$, $B \sim C$ quasi-transitive outcome leading to the hiring of Cindy.

While Thm. 6 provides one approach, we need to understand how, in general, to sidestep Sen's problems. Any approach must recognize that Sen's problems occur because *over the societal rankings determined by decisive agents and the Pareto condition, the decision rule never uses information about the transitivity of individual preferences. Thus transitive outcomes are accidents, or, as true with "single-peakedness," reflect the structure of restricted choices of preferences.*

It follows from these comments that determining the strict ranking of a pair by blindly using the strong negative externality condition, or any condition that is independent of other societal events, need not resolve the difficulty. Even worse, a transitive societal ranking could even be converted into a cyclic one! For an example, let Ann, Barb, and Carl be decisive over, respectively, $\{A, B\}$, $\{B, C\}$, and $\{A, C\}$ where everyone except David, has the $A \succ B \succ C$ preference ranking; David prefers $C \succ B \succ A$. The transitive societal outcome of $A \succ B \succ C$ imposes negative externalities of $[C \succ B, 0], [B \succ A, 0]$ and $[C \succ A, 1]$ for David. If a rule reversed an outcome should someone suffer a strong negative externality, then disgruntled David's choices would convert the transitive $A \succ B \succ C$ into the cyclic $A \succ B, B \succ C, C \succ A$. To avoid cycles, a pair's societal ranking must be coordinated with what happens with other pairs.

A way to correct a cycle is to identify and reverse the ranking of a particular pair; actually, this is about the only approach. While only one ranking needs to be reversed, criteria need to be specified to determine which one. To illustrate with the special setting where decisive agents abuse their rights, we could select the decisive agent whose action creates the most egregious violation of the rights of others. A natural approach is for each agent suffering a negative externality $[X \succ Y, \alpha]$ to cast a ballot listing the α value. Arguably, the decisive agent with the largest tally causes the strongest problem, so this agent's rights are restricted.

Theorem 7 *If a decision rule satisfying Sen's conditions creates a societal cycle with the actions of decisive agents and the Pareto condition, then the actions of at least two decisive agents are involved. For the choice made by each decisive agent in the cycle, compute the following tally: sum the alpha values for all voters with a negative externality $[X \succ Y, \alpha]$. Reverse the ranking of the agent with the largest tally by replacing the agent's choice with indifference. (Break ties in some manner.) The societal outcome will not have a cycle.*

While the Thm. 7 procedure would never be used as specified, it captures the spirit of what happens in practice. Examples include laws such as sound abatement. To be enforced, someone must complain. Not only must the sound create a negative externality for the complainant, but, because making a complaint involves a cost, it must be a strong negative externality. Now suppose complaints have been lodged against several individuals. If not all problems can be handled, then we need to establish enforcement priorities. Theorem 7 provides a way to decide: act against the agent who received the most and loudest complaints.

To illustrate with an example, suppose the information is

Person	**Preferences**	$\{A,B\}$	$\{B,C\}$	$\{C,D\}$	$\{A,D\}$
Anne	$ABCD$	AB	BC	CD	–
Barb	$CDAB$	AB	–	CD	DA
Connie	$CDAB$	AB	–	CD	–
Outcome		AB	BC	CD	DA

(7)

The $\{B,C\}$ choices have both Barb and Connie with $[C \succ B, 2]$, so the level of discontentment with Anne is 4. For the $\{A,D\}$ choices, Anne has $[A \succ D, 2]$, but Connie agrees with the choice. Anne had the highest level of complaints, so her ranking is reversed to obtain $C \succ D \succ A \succ B$.

5 Proofs

General proof of Thm. 1: three alternatives A general proof of Sen's result requires handling all possible choices for which agents are decisive, over what pairs, and whether the Pareto condition is, or is not, needed. Sen's assumptions for three alternatives and any finite number of agents require each of at least two agents to be decisive over separate pairs. (If they were decisive over the same pair, it would be trivial to create a contradiction.) Start with two agents each decisive over a pair. Using the geometry of the cube, for each assigned pair, the decisive agent selects one of two opposing cube faces. The societal outcome must be on each selected face, so it is one of the two vertices on the edge connecting the two faces selected by the decisive agents. The four possible pairs of faces define four edges that are parallel to each other. Each edge is defined by two vertices: each vertex represents the outcome depending on how the remaining pair of alternatives is ranked.

In any coordinate direction, the cube has four parallel edges. Each of the cube's eight vertices is on precisely one edge. Thus, each cyclic vertex is on one of the four described edges. As the cyclic vertices are diametrically opposite one another, both cannot be on the same edge. So, for each decisive agent, select a ranking of their assigned pair of alternatives (equivalently, for each agent, select the appropriate cube surface) so that the pair of selected faces defines an edge with a cyclic vertex.

The next step is to determine the societal ranking for the remaining pair; that is, we need to select one of the two vertices of the identified edge. If one of the two agents (or a third agent) is decisive over the remaining pair, select the agent's ranking for this pair so that it requires the cyclic vertex. If no agent is decisive over the remaining pair, then use the Pareto condition: select each agent's ranking for the remaining pair to force the cyclic vertex ranking.

It remains to find transitive preferences that support this societal outcome. If an agent is decisive, the societal outcome must be represented by one of the vertices on the cube face she selected. Likewise, if a ranking for the societal outcome is determined by the Pareto condition, then each agent's ranking must be on that face. An important point is that at most two conditions are imposed on any agent's preferences. For instance, if an agent is decisive over two pairs, then the above construction specifies his preferences for two pairs (i.e., for two faces). Thus his preferences must be one of the two vertices along the edge defined by these particular cube faces. Similarly, if an agent is decisive over one pair and shares a common (Pareto) ranking with another pair, these choices define two surfaces, so her preferences are represented by one of the two vertices on the defined cube edge. Finally, for a non-decisive agent, either there are no restrictions on his preferences (when others are decisive over all three pairs), or his preferences are restricted to the four vertices on the cube face mandated by the Pareto condition where everyone agrees on the ranking of a pair. These vertices catalogue all possible supporting profiles.

The societal outcome is the vertex surviving the intersection of the specified cube faces, so one of the choices for each agent is the cyclic vertex representing the societal outcome. But the choices for each agent includes an edge, and, according to the geometry of the cube, each edge emanating from a cyclic vertex has a transitive vertex. Select this vertex. This completes the proof. □

Notice from the above argument that of the four choices of profiles for the two decisive agents, only one is transitive.

Proof for $n \geq 4$ alternatives. The n alternatives define $\binom{n}{2} = \frac{n(n-1)}{2}$ pairs of alternatives. For each pair, (X, Y), represent its three rankings $X \succ Y, X \sim Y, Y \succ X$, by the points $0, \frac{1}{2}, 1$ on a particular $\mathbb{R}^{\binom{n}{2}}$ coordinate axis. Over all pairs, the integer points define the vertices of a unit hyper-cube in $\mathbb{R}^{\binom{n}{2}}$: each vertex defines the rankings over all pairs of alternatives. We will use a lower k-dimensional cube.

Assign each of $2 \leq k \leq \binom{n}{2}$ pairs to a decisive agent; there are at least two decisive agents. Take two of the pairs, $\{(X_i, Y_i)\}_{i=1}^{j}$ that are assigned to two different decisive agents, say (X_1, Y_1) and (X_2, Y_2). The pairs cannot be the same, so either at most one alternative is in both pairs, or they are disjoint. In the former case, assume that $Y_1 = X_2$ and add the pair (Y_2, X_1). As this setting reduces to the three alternative setting already analyzed, the proof is completed. (For pairs outside of this triplet, let everyone agree on the ranking.) In the latter case, add the pairs (Y_1, X_2) and (Y_2, X_1). The four pairs define a hyper-cube in the four dimensional space R^4 with $2^4 = 16$ vertices. There are two cyclic vertices: one is given by $X_1 \succ Y_1, Y_1 \succ X_2, X_2 \succ Y_2, Y_2 \succ X_1$ and the other reverses the rankings. As a 0 in a vertex coordinate (i.e., representing the ranking of one pair) for one cycle is replaced by a 1 for the other cycle, these vertices are diametrically opposite each other. By being four-dimensional, four edges emanate from each vertex of this cube. As each edge coming from a cyclic preference reverses the ranking of precisely one pair, this change will will make the ranking transitive (with respect to the specified pairs). Thus, as in the three-dimensional setting, the cyclic rankings have a special geometric relationship with transitive rankings.

The rest of the construction mimics the three-alternative case. Select a cycle. Let the rankings for the two pairs determined by decisive agents coincide with the selected cycle. For the remaining two pairs, if assigned to a decisive agent, select the ranking according to the chosen cycle. If not, use the Pareto condition by requiring all voters to have the specified ranking. Each selection corresponds to selecting a cube face. By construction, the specified cyclic vertex is on all cube faces, so it defines the societal outcome for these pairs.

To find transitive preferences for the voters, notice that with two (or more) decisive agents, the above construction mandates for each agent the choice of preferences for at most three pairs (at most three cube faces). Thus the selection of an agent's preference is one of the vertices on the agent's selected faces. By construction, the choice of vertices always includes the two vertices on at least one edge coming out of the cyclic vertex. One of the vertices must be transitive: transitive preferences can always be found.

While the above proves the theorem, notice that the argument generalizes to any number of pairs involving any number of cycles. Note that a cycle must involve at least two decisive agents. If this were not true, then some agent's choice must agree with each ranking in the cycle (through the Pareto condition and by being decisive), so this agent must have cyclic preferences. (This is the only place the transitivity assumption occurs: it shows the need for at least two decisive agents.) Second, a cycle involving k pairs can be represented by a vertex on a k-dimensional cube: each vertex on an edge with the cyclic vertex is a transitive ranking. □

Proof of Thm. 2. The proof follows immediately from the comments made prior to the statement of the theorem. The societal outcome is the intersection of all faces of a hypercube determined by the decisive agents and the Pareto condition. Each agent's preferences are given by the vertices of the object that result from the intersection of *some* (but not all) faces. Consequently, the societal outcome must be one of these vertices. □

Proof of Thm. 3. With the assumptions on preferences, this is a counting exercise. When the voters can rank each pair independently, then the condition that all voters rank one pair in the same manner means that there are four remaining choices of rankings for each voter. The ranking of the other two pairs, however, are determined by the two decisive voters, so the choices of the $n-2$ voters do not matter. Their choice define four parallel edges; each is equally likely. On each edge, the Pareto condition mandates that only one vertex can be selected. Thus, the outcomes are one of the four vertices on a cube face; each is equally likely. But on each cube face, three of the vertices are transitive and one is cyclic. This completes the proof for the first part.

Now assume that the voters can select only transitive preferences, and that they agree on the ranking for one pair. Thus each voter has three choices for the transitive ranking as determined by whether the remaining alternative is ranked above, below, or between the specified pair. There are nine possible rankings for the two decisive voters. As demonstrated in the proof for three alternatives, for each way the societal ranking of a pair can be determined by the Pareto condition, there is precisely one transitive profile for the two decisive agents that gives rise to the cyclic outcome. Consequently, the likelihood of a cyclic outcome is $\frac{1}{9}$ and of a transitive outcome is $\frac{8}{9}$.

With three decisive agents, each agent must select the societal ranking for one pair. It is immaterial how they rank the other pairs. Therefore, there are eight choices for the selected rankings (cube faces); each is equally likely. Of the eight choices, two define cyclic vertices and six define transitive ones. Thus, the likelihood of a cyclic outcome is $\frac{1}{4}$. □

Proof of Prop. 1. According to the above construction of any example and the hypothesis, each agent's preferences are given by the vertices of the edge that is on both of the agent's specified faces of the cube. Moreover, the edges for the two agents must meet in the vertex that describes the societal outcome, so the edges are orthogonal.

According to the geometry of the cube (Fig. 1b), three edges emanate from each vertex; thus, for each vertex representing a societal outcome, there are three pairs of edges that satisfy the "two-edge" condition. Also according to the geometry of the cube, if the common vertex represents a transitive ranking, then there is one pair of edges where all four vertices represent transitive rankings. For the other two pairs of edges, one vertex on one edge corresponds to a cyclic ranking, while the other three vertices represent transitive rankings. (For example, if 3 is the common vertex (so this vertex represents the societal outcome of $C \succ A \succ B$), then the set of all possible pairs of edges are $\{3-2, 3-4\}$, $\{3-2, 3-7\}$, and $\{3-4, 3-7\}$. The first choice involves only transitive rankings; the other two involve the cyclic vertex 7.) Each pair of edges gives rise to four possible profiles; e.g., the $\{3-2, 3-4\}$ edges define the profiles $(3,3), (3,2), (2,3), (2,4)$. If none of the edges has a cyclic vertex, then all profiles are transitive. If one edge has one cyclic vertex, then half of the possible profiles have a cyclic preference, but no profile consists strictly of cyclic preferences.

If the outcome is cyclic, then each edge for each of the three pairs of edges, has one cyclic vertex. When computing the four resulting profiles, one choice has both rankings cyclic, two others have one cyclic ranking, and the fourth has both transitive rankings. This completes the proof. □

Proof of Thm. 4. The vertices that can be assigned to each agent consist of all vertices on an edge, or a face, or a two-dimensional cube, or ... . Each of these geometric objects have the societal outcome as one of the vertices. As such, each person's averaged outcome is a vertex of the subcube with the societal outcome as a vertex. By the properties of decisive agents with choice over different alternatives, at least two of these averaged outcomes are not on the same edge of the subcube. Thus, the average of the averaged ranking must be in the interior of the cube. This proves the theorem. □

Proof of Thm. 5. To handle the first part of the theorem, we know from Thm. 4 that one choice of preferences is where the preferences of all agents agree with the transitive societal outcome. With agreement, the choices made by the decisive agents do not even register as a negative externality. Then there are $\binom{n}{2}$ edges coming out of this vertex: $n-1$ of them connect to another transitive ranking—these edges represent a pair of alternatives that are adjacently ranked in a ranking representing the vertex. (For instance, for $A \succ B \succ C \succ D$, these would be the edges for $\{A, B\}, \{B, C\}, \{C, D\}$.) By being adjacent, reversing the ranking of a pair does not affect transitivity and, while a change in these rankings would create a negative externality, it would not be a strong negative externality. What remains are $\binom{n}{2} - (n-1) = \binom{(n-1)}{2}$ edges: here a change in the indicated pair leads to a cycle. This is where the pairs are not adjacently ranked in the original ranking. (With $A \succ B \succ C \succ D$, reversing the $\{A, B\}$ ranking leads to a cycle.)

For a cyclic ranking, changing the ranking of any pair defines a transitive ranking. The changed alternatives cannot be adjacently ranked in the transitive ranking—if they were, then reversing them again to return to the original setting would create a transitive rather than a cyclic ranking. Thus, each voter suffers a strong negative externality. □

Proof of Thm. 6. With a cycle, only one ranking needs to be changed to convert it into a transitive ranking. Changing one of these rankings into indifference converts the ranking into a quasi-transitive ranking because, by removing an offending strict ranking, the remaining strict rankings are transitive. □

Proof of Thm. 7. If a cycle created by decisive agents and the Pareto condition involved only one decisive agent, this agent's preferences would have to agree with the cyclic outcome. This violates the assumption of transitive preferences. (Again, this is about the only place the assumption of transitivity is used.) The remainder of the theorem follows because a rule is specified to reverse the ranking of one pair, and that will create a transitive ranking. □

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