Reflections on my conjecture, and several new ones

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1 Introduction

In a concluding talk at a December, 1999, conference, Jeff Xia [19] called attention to my easily stated, but unanswered 1969 conjecture [13] about the Newtonian N-body problem:

> *if the polar moment of inertia, I, is a constant, then the N-body motion is that of a rotating rigid body* (a "relative equilibrium").

Since then, significant progress has been made for the collinear problem (Diacu, Perez-Chavela, and Santoprete [2] and Saari [15]) and the general setting for $N = 3$ (e.g., Llibre and Pina [8] and McCord [9]). The current highlight is where Rick Moeckel [4] used notions from algebraic geometry to develop a computer assisted proof verifying the conjecture for $N = 3$. Then, during the conference associated with these proceedings, Moeckel [5] outlined his proof showing that the conjecture holds for the three body problem when physical space is $\mathbb{R}^d$ for all positive integer d values! I find this last result to be astonishing because, as explained later, there were reasons to doubt that the conjecture would hold for $\mathbb{R}^4$.

While this conjecture has received considerable attention, it is more interesting when viewed in a more general setting. Indeed, any conjecture should be considered in the context of a larger question; perhaps as part of an effort to understand basic properties of a system, or maybe to question current beliefs. The value of taking a broader perspective is obvious; in this manner we can appreciate how a conjecture relates to other issues and research efforts. So in the next section I outline what motivated my conjecture from the late 1960s. By doing so, several other questions emerge and it becomes possible to see, for instance, how the fascinating "figure-eight" orbit of Chenciner and Montgomery [1] may be related.

While this problem has proved to be intriguing, I must confess that had it been made later in my career, rather than at the beginning, I would have phrased it in a more general manner where the original version becomes a special case. To make amends, the extended version of the conjecture is described in Sect. 3. This extension differs from the nice generalizations made by others, such as examining the conjecture with different force laws (e.g., Santoprete [16]), or allowing some masses to be negative (Roberts [12]), or seeking to identify which dynamical systems satisfy the conjecture, or demonstrating that the conjecture holds for generic potentials (Schmah and Stoica [17]). Closely related is the interesting work of Hernández-Garduno, Lawson and Marsden [6] who extended the conjecture to other physical systems and restated it in a coordinate free setting. Hernández [7] described aspects of this research at this conference.

All of these generalizations emphasize a constant moment of inertia. My extension explores a different direction; it is to find necessary and sufficient conditions for N-body motion where the bodies describe a fixed configuration for all time. As such, my extended conjecture goes beyond the original one to include elliptic and even homothetic orbits.

I also discuss a way to address the original and extended conjectures. To explain, although Moeckel's proof for $N = 3$ is a beautiful tour de force, his computer assisted proof does not help us understand the associated dynamics or *why* the conjecture is true. Consequently others, such as Fujiwara [3], continue to search for an analytical or dynamical proof of the conjecture. But an obstacle has hindered all progress, and I will identify what it is. As I show, the problem is caused by a particular "system velocity" component. To prove this, I derive a necessary and sufficient condition for the original conjecture to be true (for any N) in terms of the precise constant value for I: this condition holds iff the troubling velocity term equals zero.

As the issue now is to analyze the troublesome velocity component, appropriate structure is introduced starting in Sect. 4.1. (A more complete description of this structure is developed in my recent book [15].) I conclude by making some comments about behavior in $\mathbb{R}^4$.

2 Source of the original conjecture and an extended Virial Theorem

The notation is standard: the mass, position vector, and velocity vector for the j^{th} particle are given, respectively, by $m_j, \mathbf{r}_j, \mathbf{v}_j$. In a standard way using the integrals of motion for the center of mass of the system, assume that it is fixed at the origin. Namely,

$$\sum_{j=1}^{N} m_j \mathbf{r}_j = \sum_{j=1}^{N} m_j \mathbf{v}_j = \mathbf{0}. \tag{1}$$

The four remaining integrals of motion are the integral of angular momentum,

$$\sum_{j=1}^{N} m_j \mathbf{r}_j \times \mathbf{v}_j = \mathbf{c} = (0, 0, c) \tag{2}$$

and the energy integral,

$$T = U + h, \tag{3}$$

where $T = \frac{1}{2}\sum_{j=1}^{N} m_j {\mathbf{v}_j}^2$ is the kinetic energy, $U = \sum_{j<k} \frac{m_j m_k}{r_{jk}}$ is the self-potential, c and h are constants of integration, and $r_{jk} = |\mathbf{r}_j - \mathbf{r}_k|$. Central to my discussion is $I = \frac{1}{2}\sum_{j=1}^{N} m_j r_j^2$, which can be viewed as a measure of the (square of the) diameter of the N-body system. An important relationship governing the growth of I is the Lagrange-Jacobi equation

$$I'' = 2T - U = U + 2h. \tag{4}$$

2.1 Source of the original conjecture, and a new one

My original conjecture was in response to the way astronomers used the Virial Theorem. (This theorem and extensions, which significantly relax the traditional assumptions, are described in [15].) The theorem asserts that if I and T are bounded for all time, then

$$2\lim_{t\to\infty}\frac{1}{t}\int_0^t T(s)\,ds = \lim_{t\to\infty}\frac{1}{t}\int_0^t U(s)\,ds. \tag{5}$$

Time averages, however, are difficult to use. On the other hand, maybe we could avoid this difficulty if I, for a galaxy or another object, is "nearly constant." In such situations, it might not hurt to further assume that I *is* constant valued. If so, then the time averages of Eq. 5 would disappear because a constant I requires $I'' \equiv 0$, or (from Eq. 4) that $2T = U$.

My research interests at the time, which included using the analytic nature of N-body solutions and searching for upper bounds on I, strongly indicated that a constant I carried the unintended consequences that the orbit must be a relative equilibrium solution where (if physical space is $\mathbb{R}^3$) the particles would rotate as a planar rigid body forming what is called a *central configuration* (see Eq. 6). This is my original conjecture.[1]

Knowing the context of the original conjecture leads to a natural, related question: *were the astronomers essentially correct?* To restate this new question in mathematical terms, if I remains "nearly constant valued," will $2T - U$ remain nearly zero? If so, then the spirit of what the astronomers want to do would be justified. Rephrasing this question in a manner that proposes a valuable extension of the Virial Theorem, we have:

> *characterize in terms of I all bounded N-body settings where, for a given $\epsilon > 0$, it follows for all $t > 0$ (or for all t in some specified interval) that $|2T(t) - U(t)| < \epsilon$.*

The astronomers' conversion of a "nearly constant I to a constant I" reveals their belief that if the variance of I values is very small, then $|2T - U|$ values are also small. This may be correct (and, for reasons described later, I believe it is), but there are mathematical concerns. For instance, for any choice of δ, the value assumed by $I = D + \frac{\delta}{t+1}\sin t^8$ remains within δ distance of D, but its second derivative (which, by Eq. 4, would characterize $2T - U$) is highly erratic and of order t^{13}. "Tauberian theorems" can exclude such behavior for certain N-body settings (see [15]), but the point is made: integrating preserves bounds but differentiating (because of oscillations) need not. While this new question, which replaces a constant I from my original conjecture with a variable I, appears to be more difficult than the original conjecture, it may be more tractable and an answer most surely would be more useful. Later I explain why a complete answer for this extended Virial Theorem question is blocked by the same obstacle that hinders solving the original conjecture.

This related question is *not* one of stability. After all, if my original conjecture is correct, then solutions include orbits that we know are unstable. (A collinear relative equilibrium solution, for instance, is unstable.) Rather than examining the stability of the orbit, we seek to determine whether and under what situations is the "stability" of the $2T - U$ value preserved.

[1] I suspect that somewhere in my notes of the time, or those of C. Simon or J. Palmore, is my proof for $N = 3$. But, it has yet to surface.

Encouragement that a positive answer might be found comes from the Chenciner-Montgomery figure-eight orbit and the fascinating extensions that Montgomery described at this conference: they may provide supporting examples. This is because it follows from their construction that their orbit has a "nearly constant" I, and, indeed, the orbit passes through central configurations. While I have not carried out any computations, the nature of the orbit suggests that it is accompanied by small $2T - U$ values. This raises the question whether the "extended Virial Theorem" question that I propose can be addressed via Chenciner and Montgomery's variational methods. Maybe, but a complete answer probably requires delving into the dynamics of the N-body problem.

3 Extended conjecture with configurational measures

If my original conjecture is correct, then (when physical space is $\mathbb{R}^3$) the configurations are central configurations. This is where a constant λ can be found so that for each j,

$$\mathbf{r}_j'' = \lambda \mathbf{r}_j. \tag{6}$$

It is well-known (e.g., see the classic book by Wintner [18] or my recent book [15]) that a central configuration occurs iff $\nabla IU^2 = \mathbf{0}$. To be compatible with what follows, let $R^2 = 2I$ so that R becomes a measure of the diameter of the system and $\nabla IU^2 = \mathbf{0}$ becomes $\nabla RU = \mathbf{0}$.

As the RU term is homogeneous of degree zero and invariant with respect to rotations, its value is determined by the shape of a configuration, but not its size or orientation. Admittedly, RU is an imperfect measure of the shape of the configuration formed by the particles, but it is one of the few we have. Moreover, as indicated later, this RU term arises so often in the study of the N-body problem that it deserves a name: I call RU (and IU^2) the *configurational measure* of the system. For the inverse q-force law, $q \neq 1$, the configurational measure is $R^{q-1}U$ or $RU^{1/(q-1)}$.

By measuring only the shape of the configuration, but not its size nor orientation, it follows immediately that RU is constant valued for an orbit where the configuration of the particles remains fixed: this is called a *homographic solution.* My extended conjecture, which if true would provide a useful representation for homographic motion, is

> *the configurational measure RU is a constant iff the N-body motion is homographic.*

To see how the extended conjecture subsumes the original one, recall from Eq. 4 that if $I = \frac{1}{2}R^2$ has a constant value along a trajectory, then $U + 2h \equiv 0$, or $h < 0$ and $U \equiv 2|h|$. The original conjecture, then, requires each term of RU to be a constant. Moreover, the relative equilibrium behavior asserted by the original conjecture is a special case of a homographic solution: it is where the distances between particles remain fixed. In the other direction, however, there are many solutions whereby RU is fixed, but R is not. As illustrations, start with any central configuration and assign zero velocities to each particle: the resulting *homothetic solution* tends toward a complete collapse retaining the same configuration. As the configuration does not change, the RU value remains fixed, but R most surely does not. As another example, velocities can be assigned to a coplanar central configuration so that the configuration remains fixed and each particle has an elliptic motion. Similarly, an appropriate choice of initial conditions for the configuration will send the particles off to infinity with either the parabolic or hyperbolic behavior of a two-body problem while retaining the configuration. The extended conjecture is more general.

3.1 Related conjectures by mimicking the two-body problem

Homographic orbits resemble two-body orbits. The reason is that if physical space is $\mathbb{R}^d$ for $d = 1, 2, 3$, then the fixed configuration accompanying a homographic solution must be a central configuration where each particle behaves as though it is a "two-body problem." (A simpler proof of this nineteenth century result is in [15].) This comment raises an interesting question. There are other N-body settings that resemble, in some manner, the two-body problem; which ones of these must be characterized by homographic motion?

Let me illustrate what I mean with an example. Recall that the equations of motion for the two-body problem are

$$\mathbf{r}'' = \nabla(\frac{\mu}{r}) = -\frac{\mu\mathbf{r}}{r^3} \tag{7}$$

with the associated scalar equation

$$r'' = -\frac{\mu}{r^2} + \frac{c^2}{r^3}. \tag{8}$$

By mimicking the form of Eq. 8, we have the following conjecture:

the scalar equation for R in the N-body system can be expressed as

$$R'' = -\frac{A}{R^2} + \frac{B}{R^3}, \tag{9}$$

for constants A and B, iff the solution is homographic.

The following result identifies this question with the extended conjecture while relating constant A in Eq. 9 with the configurational measure.

Theorem 3.1 *(Saari [15]) Equation 9 holds for an N-body solution iff the configurational measure RU is a constant. In this case, $A = (RU)$.*

3.2 System inner product

Other N-body systems that resemble the two-body problem, where the configurational measure plays an important role, emerge by using the system inner product (Saari [14, 15]) defined over $(\mathbb{R}^d)^N$, which is

$$< \mathbf{a}, \mathbf{b} >= \sum_{j=1}^{N} m_j \mathbf{a}_, \cdot \mathbf{b}_j, \quad \mathbf{a}, \mathbf{b} \in (\mathbb{R}^d)^N.$$

I introduced this inner product to facilitate my search for aspects of the *system* of N-bodies that might be hidden when using the traditional, individual $\mathbf{r}_j''$ equations. To suggest what is gained, notice that this inner product changes the gradient to

$$\nabla U = (\frac{1}{m_1}\frac{\partial U}{\partial \mathbf{r}_1}, \dots, \frac{1}{m_1}\frac{\partial U}{\partial \mathbf{r}_1}).$$

Consequently, if $\mathbf{R} = (\mathbf{r}_1 \dots, \mathbf{r}_N)$, then $R^2 =< \mathbf{R}, \mathbf{R} >= 2I$ (as given earlier), and the equations of motion for the N-body *system* mimic the simpler Eq. 7 form by becoming

$$\mathbf{R}'' = \nabla U. \tag{10}$$

Another advantage of emphasizing the system is that the central configuration equations (Eq. 6) take the simpler expression $\lambda \mathbf{R} = \nabla U$. To solve for λ, notice that

$$\lambda R^2 =< \mathbf{R}, \lambda \mathbf{R} >=< \mathbf{R}, \nabla U >= -U,$$

where the last equality follows from Euler's Theorem and the fact U is homogeneous of degree -1. This leads to the standard $\lambda = -U/R^2$. But instead, if we express λ as $-(RU)/R^3$, we have that if a solution keeps a central configuration for all time, then Eq. 10 can be expressed in terms of the configurational measure as

$$\mathbf{R}'' = -\frac{(RU)}{R^3}\mathbf{R}. \tag{11}$$

As the configurational measure RU is constant valued, we have another parallel with the two-body problem: Eq. 11 assume the form of the last term in Eq. 7 for homographic solutions (in $\mathbb{R}^d$ for $d = 1, 2, 3$).

We would like to establish the converse. To do so, we need the general equations of motion for the system. This suggests expressing Eq. 10 in a system form by first finding (for $(\mathbb{R}^d)^N$) the ∇U component in the $\mathbf{R}$ direction. The somewhat surprising result, which emerges from a system approach, is $-\frac{(RU)}{R^3}\mathbf{R}$; for all motion this component depends on the configurational measure! This leads to the following.

Theorem 3.2 *(Saari [15]) With respect to the system inner product, the equations of motion for the N-body problem are*

$$\mathbf{R}'' = \nabla U = -\frac{(RU)}{R^3}\mathbf{R} + \mathbf{D}(\mathbf{R}) \tag{12}$$

where $\mathbf{D}(\mathbf{R})$ is orthogonal to $\mathbf{R}$ as well as to the $SO(d)$ orbit of $\mathbf{R}$.

As a special case, if the equations of motion can be expressed as $\mathbf{R}'' = -\frac{A}{R^3}\mathbf{R}$ for some constant A, then $A = (RU)$, $\mathbf{D} = \mathbf{0}$, and the resulting motion is homographic. Similarly, if $\mathbf{D} \equiv \mathbf{0}$, it is not difficult to prove that the motion is homographic, so RU is a constant. But notice that Eq. 12 holds in general, so even when RU is not constant valued, the configurational measure still plays a central role in the dynamics of the N-body problem. This adds importance to the extended conjecture. (For other applications of the configurational measure, see [15].) Indeed, according to Thm. 3.2, if my extended conjecture is correct, then the configurational measure is a constant iff $\mathbf{D} = \mathbf{0}$ in Eq. 12 when physical space is $\mathbb{R}^d$ where $d = 1, 2, 3$. As described later, interesting differences arise for $d \geq 4$.

4 Dynamics

In his proof of my original conjecture for three-bodies, Moeckel reduces the question to a large number of equations, such as $U = 2|h|, U' = U'' = U''' = U'''' = 0$, etc., and then uses techniques from algebraic geometry to prove that the mutual distances must be constant valued. A flavor of his proof is captured by the simpler three-body setting where the particles always lie on a line. This situation, where particle 2 is in the middle, can be reduced to the three equations $I = \frac{1}{2M}\sum_{j<k} m_j m_k r_{jk}^2 = D, U = 2|h|$, and $r_{1,2} + r_{2,3} = r_{1,3}$. Here, M is the total mass, and the expression for I uses the fact the center of mass is at the origin. Differentiating leads to three equations, in the three variables $r'_{1,2}, r'_{2,3}, r'_{1,3}$, that equal zero. With the possible exception of the particles forming a central configuration (which is what we want to prove), the equations are independent, so the variables all must equal zero: this proves the original conjecture in this case.

The difficulty with this proof and Moeckel's is that the conjecture is verified without providing insight into the dynamics. A proof for the collinear case that does depend on the dynamics, which holds for all N, is in [15]. (The same proof verifies the extended conjecture for the collinear case, but an extra argument is required for the collinear homothetic case where $\mathbf{c} = \mathbf{0}$.) To handle the more general setting, however, we need more information about the dynamics. To put these comments in perspective, by using higher order derivatives, Moeckel's approach implicitly captures relationships among changes in velocities: the approach described next explicitly identifies different velocity terms.

4.1 System velocity

Above I describe the system position (or configuration) vector of $\mathbf{R} = (\mathbf{r}_1, \ldots, \mathbf{r}_N)$; the *system velocity vector* is $\mathbf{V} = (\mathbf{v}_1, \ldots, \mathbf{v}_N)$. With this notation and the system inner product, we have the following expression that plays a major role in our discussion:

$$2T =< \mathbf{V}, \mathbf{V} >= V^2. \tag{13}$$

As developed in [15], the idea is to decompose $\mathbf{V}$ into terms that describe the rotation, expansion, and change in configuration of the system. The scalar expansion of the system is $\mathbf{W}_{scale} = \frac{R'}{R}\mathbf{R}$. To find the rotational velocity, denoted by $\mathbf{W}_{rot}$, start with the rigid body rotations of $\mathbf{R}$; in $(\mathbb{R}^d)^N$ this is given by

$$\mathcal{M}_{\mathbf{R}} = \{\Omega(\mathbf{R}) = (\Omega(\mathbf{r}_1), \ldots, \Omega(\mathbf{r}_N)) \,|\, \Omega \in SO(d)\}.$$

The $\mathbf{V}$ component in the tangent space $T_{\mathbf{R}}\mathcal{M}_{\mathbf{R}}$ defines the system rotational velocity $\mathbf{W}_{rot}$.

By construction, $\mathbf{W}_{rot}$ describes the portion of the system velocity that rotates the configuration as a rigid body, $\mathbf{W}_{scale}$ describes the portion of $\mathbf{V}$ that changes the scale of the configuration, so $\mathbf{W}_{config} = \mathbf{V} - (\mathbf{W}_{rot} + \mathbf{W}_{scale})$ is the *configurational velocity*: this portion of the system velocity changes the shape of the configuration. By construction, these three velocity components are mutually orthogonal (with respect to the system inner product $< -, - >$), so (by use of Eq. 13) the conservation of energy integral (Eq. 3) becomes the equality

$$\mathbf{W}_{rot}^2 + \mathbf{W}_{scale}^2 + \mathbf{W}_{config}^2 = V^2 = 2T = 2(U + h). \tag{14}$$

Equation 14 immediately provides information about the original conjecture.

Theorem 4.1 *(Saari [15]) If physical space is $\mathbb{R}^d$ for $d = 1, 2, 3,$ a necessary and sufficient condition for the original conjecture to be true, which asserts that an orbit with constant I must be a relative equilibrium orbit, is that*

$$\frac{1}{2}R^2 = I \equiv \frac{c^2}{4|h|}. \tag{15}$$

Proof: If I has a fixed value, then $\mathbf{W}_{scale} \equiv \mathbf{0}$. Moreover, a computation given in [15] shows that for $\mathbb{R}^d$ for $d \leq 3$, $\frac{c^2}{R^2} \leq \mathbf{W}^2_{rot}$ where equality holds iff the motion is in the invariable plane (the plane with $\mathbf{c}$ as a normal vector). A similar computation holds for $d \geq 4$, except the invariable plane comment no longer holds. Returning to $d \leq 3$, Eq. 14 becomes

$$\frac{c^2}{R^2} + \mathbf{W}^2_{config} \leq 2(U + h) = 2|h|,$$

where equality holds iff I has a fixed value (to ensure that $\mathbf{W}_{scal} = \mathbf{0}$ and $U = 2|h|$) and the motion is in the invariable plane. So if I has a fixed value, then

$$I = \frac{1}{2}R^2 \geq \frac{c^2}{4|h| - 2\mathbf{W}^2_{config}} \tag{16}$$

where, for $d \leq 3$, equality holds iff the motion is in the invariable plane. By definition, a relative equilibrium orbit must be in the invariable plane and $\mathbf{W}_{scale} = \mathbf{W}_{config} = \mathbf{0}$. Thus, if the conjecture is true, it must be that $I \equiv \frac{c^2}{4|h|}$.

Conversely, if $I \equiv \frac{c^2}{4|h|}$, then, according to Eq. 16, the motion must be in the invariable plane and $\mathbf{W}_{config} \equiv \mathbf{0}$: the motion must be a relative equilibrium orbit. $\square$

For a physical space with dimension 4 or larger, the equivalence condition is that $\mathbf{W}^2_{rot} \equiv 2|h|$, which leads to the assertion that $\frac{1}{2}R^2 = I \geq \frac{c^2}{4|h|}$. Sharper results are possible, but I have not tried to find them. (For readers interested in doing so, find a bound on $\mathbf{W}^2_{rot}$ for dimensions $d \geq 4$.)

A similar result holds for the extended conjecture.

Theorem 4.2 *If physical space is $\mathbb{R}^d$ for $d = 1, 2, 3$, then RU being a constant implies the motion is homographic iff RU equals $\frac{R}{2}[\frac{c^2}{R^2} + R'^2 - 2h]$.*

The proof, which uses Eq. 14, is essentially the same. If a constant RU implies the motion is homographic, there are two possibilities. The first is that the motion is homothetic. Here $\mathbf{W}_{rot} = \mathbf{W}_{config} \equiv \mathbf{0}$, so $c = 0$ and Eq. 14 become $(R')^2 = 2(U + h)$. The remaining setting requires the motion to be in the invariable plane, so $\mathbf{W}^2_{rot} = \frac{c^2}{R^2}$ and $\mathbf{W}_{config} \equiv \mathbf{0}$. The assertion follows from Eq. 14.

In the opposite direction, if a constant RU equals $\frac{R}{2}[\frac{c^2}{R^2} + R'^2 - 2h]$ then, using the arguments above and Eq. 14, the motion must be in the invariable plane and $\mathbf{W}_{config} \equiv \mathbf{0}$.

4.2 Configurational velocity

The Eq. 15 condition is intriguigingly, maybe frustratingly, close to resolving the problem. This is because if $\mathbf{c} \neq \mathbf{0}$ and I (or R) is bounded for all time, then each local maximum provides a lower bound on the next local minimum for I. (See [15].) An interesting feature about these maxima and minima values is that they fall on either side of the magical $\frac{c^2}{4|h|}$ value from Thm. 4.1. In particular, the closer a local maximum is to $\frac{c^2}{4|h|}$, the closer the next local minima must be to this value. Indeed, the original conjecture (and the posed problem about finding a stronger Virial Theorem) could be finished if either $\mathbf{W}_{config} = \mathbf{0}$ at one of these local extreme points, or if it could be shown that any of these maxima or minima corresponds to a strict local extreme point.

In any case, to verify the original conjecture for any value of N, it suffices either to prove that $\mathbf{W}_{config} \equiv \mathbf{0}$, or that Eq. 15 holds. My earlier work emphasized Eq. 15, so I now discuss the configurational velocity. In doing so, notice that if I is a constant for the planar N-body problem, then, whether or not the conjecture is true, $\mathbf{W}^2_{config}$ must be a constant. (This is because $\mathbf{W}^2_{rot} + \mathbf{W}^2_{config}$ and $\mathbf{W}^2_{rot} = \frac{c^2}{R^2}$ are constants.) A way to show that this $\mathbf{W}^2_{config}$ constant value is zero might be to use the various relationships involving $\mathbf{W}_{config}$; several (as described in [15]) come from differentiating the orthogonality relationships $<\mathbf{W}_j, \mathbf{W}_{config}>= 0$, where $j =$ "scale" or "rot".

Actually, all of the questions raised in this note revolve around the behavior of $\mathbf{W}_{config}$. For instance, a constant R or a constant RU requires $\mathbf{W}_{config} \equiv \mathbf{0}$ iff, respectively, the original or extended conjecture is true. Now consider my "Virial Theorem" question whether small variations in the value of I require small values for $2T-U$. According to the energy integral, small values for $2T - U$ are equivalent to having small values for $U + 2h$, or U staying close to the $2|h|$ value. As stated earlier, if I is bounded, then its values oscillate above and below the $\frac{c^2}{4|h|}$ value of Thm. 4.1. So the search for an extended Virial Theorem reduces to analyzing the behavior of $\mathbf{W}^2_{config}$. This is because with coplanar motion, small variations in the value of I force only small variations in $\mathbf{W}^2_{rot}$, and (by use of the approach in [15]) it is not difficult to obtain bounds on $\mathbf{W}^2_{scal}$. According to Eq. 14, all that remains to establish an extended Virial Theorem—showing that I bounded slightly above $\frac{c^2}{4|h|}$ forces $2T - U$ to have small values— is to prove that $\mathbf{W}^2_{config}$ has small values. (Indeed, with I values near $\frac{c^2}{4|h|}$, and $\mathbf{W}_{scale} = \mathbf{W}_{config} \approx \mathbf{0}$, we have from Eq. 14 that U varies around the value of $2|h|$.) I believe this is true; I expect that if an upper bound for I is near $\frac{c^2}{4|h|}$, the small variations in the value of I may allow dramatic changes in configurations but with a small $\mathbf{W}^2_{config}$ value.

4.3 Coordinates for ∇U and $\mathbf{W}_{config}$

To analyze the configurational velocity, it would help to have a basis for $\mathbf{W}_{config}$. A convenient way to find one is with Jacobi coordinates where $\boldsymbol{\rho}_1 = \mathbf{r}_2 - \mathbf{r}_1$ and $\boldsymbol{\rho}_2$ is the vector from the center of mass of $\mathbf{r}_1$, $\mathbf{r}_2$ to $\mathbf{r}_3$. Using the usual generalized masses μ_1, μ_2, the system inner product for $(\mathbb{R}^3)^2$ becomes $[\mathbf{a}, \mathbf{b}] = \sum_{j=1}^{2} \mu_j(\mathbf{a}_j, \mathbf{b}_j)$.

If $\boldsymbol{\rho} = (\boldsymbol{\rho}_1, \boldsymbol{\rho}_2)$, then for the coplanar three-body problem (see [15]) the system equations are $\boldsymbol{\rho}'' = \nabla U$ (where the gradient is with respect to the inner product $[-,-]$) and

$$\mathbf{W}_{scal} = \rho' \frac{\boldsymbol{\rho}}{\rho}, \quad \mathbf{W}_{rot} = \frac{c}{\rho^2}(\mathbf{e}_3 \times \boldsymbol{\rho}_1, \mathbf{e}_3 \times \boldsymbol{\rho}_2) := \frac{c}{\rho}\mathbf{E}_3 \times \frac{\boldsymbol{\rho}}{\rho}. \tag{17}$$

The first term expresses the system velocity component that expands the configuration, the second is the system velocity component that rotates the configuration. These two vectors represent two of the four dimensions for the velocity space of the coplanar three-body problem expressed in Jacobi coordinates.

The other two dimensions in velocity space represent the individual scale and rotational changes of $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ that do not contribute to the Eq. 17 scaling. Thus these are the terms that change the shape of the configuration. For instance, the scalar change for the two $\boldsymbol{\rho}_j$ components is given by (ρ_1', ρ_2'), which is a vector in a two dimensional space. One axis for this space can be described in terms of the amount of this scalar change that retains the configuration; these are the $(\rho' \frac{\rho_1}{\rho}, \rho' \frac{\rho_2}{\rho})$ terms from the components of $\mathbf{W}_{scal}$. What remains of the scalar change for each $\boldsymbol{\rho}_j$ changes the shape of the configuration; this is given by $\mathbf{U}_{scal}$ in the following Thm. 4.3. A similar description holds for $\mathbf{U}_{rot}$.

Theorem 4.3 *(Saari [15]) For the coplanar three-body problem, the following is a basis for the configurational velocity.*[2]

$$\mathbf{U}_{scal} = (\frac{\rho_2}{\mu_1}\frac{\boldsymbol{\rho}_1}{\rho_1}, -\frac{\rho_1}{\mu_2}\frac{\boldsymbol{\rho}_2}{\rho_2}), \qquad \mathbf{U}_{rot} = (\frac{\rho_2}{\mu_1}\mathbf{e}_3 \times \frac{\boldsymbol{\rho}_1}{\rho_1}, -\frac{\rho_1}{\mu_2}\mathbf{e}_3 \times \frac{\boldsymbol{\rho}_2}{\rho_2}) \tag{18}$$

According to Thm. 4.3, $\mathbf{W}_{config} = a\mathbf{U}_{scal} + b\mathbf{U}_{rot}$. Because $\mathbf{U}_{scal}$ and $\mathbf{U}_{rot}$ are orthogonal, and a computations shows that $\mathbf{U}^2_{scal} = \mathbf{U}^2_{rot} = \frac{\boldsymbol{\rho}^2}{\mu_1\mu_2}$, the resulting expression

$$\mathbf{W}^2_{config} = a^2\mathbf{U}^2_{scal} + b^2\mathbf{U}^2_{rot} = (a^2+b^2)\frac{\boldsymbol{\rho}^2}{\mu_1\mu_2} \tag{19}$$

intimately connects the behavior of $\mathbf{W}^2_{config}$ with the growth of the system $\boldsymbol{\rho}^2$. This expression is the kind that may help us explore the different conjectures described in this paper. For instance, with the original conjecture where R and ρ are constant, I showed above that $\mathbf{W}^2_{comfig}$ is a constant. Thus Eq. 19 requires $a^2 + b^2$ to also be a constant. (Equations of motion for a and b in this setting are derived in [15].)

This computation suggests that if the original conjecture were false, then the change in configurations generated by the particles would form an elliptical curve. This statement is consistent with what Fujiwara [3] found and with what we observe by plotting a constant $I = \frac{1}{2M}\sum_{j<k} m_j m_k r^2_{jk}$ and a constant U in a space where the coordinates are the mutual distances. The constant I forms an ellipsoid that the constant U intersects either in a curve or a point. The point corresponds to the conjecture, the curve, which represents what happens if the conjecture is false, imposes a constraint on the a, b variables.

To the best of my knowledge, prior to [15], no natural coordinates have been advanced for ∇U. With a system approach to the N-body problem, such coordinates follow immediately,

[2]At each instant of time, the three particles of the three-body problem define the *osculating plane.* The motion of this plane is governed by $\mathbf{W}_{rot}$, so $\mathbf{W}_{config}$ acts in the plane. Consequently, to extend $\mathbf{U}_{rot}$ to $\mathbb{R}^3$, replace $\mathbf{e}_3$ with a vector orthogonal to the osculating plane.

and they are consistent with what happens with the system velocity. To explain, as noted in Eq. 12, ∇U has a component in the $\boldsymbol{\rho}$ direction, which involves the configurational measure, and the other $\mathbf{D}(\boldsymbol{\rho})$ components that are orthogonal to $\boldsymbol{\rho}$ and rigid body rotations. This means that

$$\nabla U = -\frac{(\rho U)}{\rho^3}\boldsymbol{\rho} + \alpha \mathbf{U}_{scal} + \beta \mathbf{U}_{rot}. \tag{20}$$

With a computation (carried out in [15]), $\boldsymbol{\rho}''$ can be expressed in terms of $\boldsymbol{\rho}, \mathbf{W}_{rot}, \mathbf{W}_{scal}$, $\mathbf{U}_{scal}$ and $\mathbf{U}_{rot}$. The advantage of doing so is that both sides of the $\boldsymbol{\rho}'' = \nabla U$ equation are now expressed in the same system coordinates, where the ∇U side describes how the shape and size of the configuration of the particles changes the force in these different system directions, while the $\boldsymbol{\rho}'' = \mathbf{W}'_{scal} + \mathbf{W}'_{rot} + \mathbf{W}'_{config}$ side describes how the different system velocity terms are changed by the configuration. We now have more tools to handle the kinds of conjectures raised in this paper, but I am not sure if this is enough.

4.4 Higher dimensional physical spaces

At the conference associated with these proceeding, I probably was the most surprised with Moeckel's announcement that my original conjecture is true for the three-body problem in $\mathbb{R}^4$. To explain why, notice that Eq. 20 suggests how the current shape of a configuration changes the dynamics. For instance, if α or β are not zero, then the ∇U force in these system directions would appear to affect $\mathbf{W}'_{config}$. But if $\mathbf{W}'_{config} \neq \mathbf{0}$, then the shape of the configuration must change—and the conjecture would be false. In other words, only when $\alpha = \beta \equiv 0$ should we expect homographic motion. As Eq. 20 demonstrates, this condition requires the configuration to be a central configuration.

For $\mathbb{R}^2$ and $\mathbb{R}^3$, the "homographic motion requires a central configuration" assertion is true. But, already in the early 1980s, I knew from the work of Wintner [18] and Palmore [10, 11] that a relative equilibria motion in $\mathbb{R}^4$ does not require a central configuration. This makes sense; in my terminology, homographic motion is governed primarily by $\mathbf{W}_{rot}$, which, for $\mathbb{R}^d$, resides in a $\binom{d}{2}$ dimensional space. Increasing the value of d increases the $\binom{d}{2}$ dimension, which suggests there is more flexibility in the choice of the configurations associated with homographic motion.

More precisely, at each instant of time when physical space is $\mathbb{R}^2$ or $\mathbb{R}^3$, the rotational effects of $\mathbf{W}_{rot}$ can be described as a rotation about a fixed axis. Compare this statement with what happens in $\mathbb{R}^d$, $d \geq 4$ where, generically, the rotation involves more than one axis. This added rotational behavior is what allows homographic and relative equilibria motion to occur with a non-central configuration, also, with an extra axis, a relative equilibrium orbit in $\mathbb{R}^4$ need not be constrained to a plane.

It was also in the early 1980s when I developed the decomposition of the system velocity. Using considerations such as described above with Eq. 20, I suspected (but never investigated) that examples with a constant I behavior and a non-central configuration was an anomaly. After all, if the configuration is not a central configuration, then $\alpha^2 + \beta^2 \neq 0$, so the configuration introduces a force that, presumably, affects $\mathbf{W}'_{config}$ forcing the configuration to change shape. This suggested that the non-central configuration relative equilibria orbits were special cases that threaded their way though the $\binom{d}{2}$ dimensions of $\mathbf{W}_{rot}$ without influencing $\mathbf{W}_{config}$. As much more can happen with homographic behavior in $\mathbb{R}^4$, I had doubts whether my original conjecture would hold in $\mathbb{R}^d$ for $d \geq 4$.

For these reasons, I was surprised with Moeckel's announcement that my original conjecture, at least for $N = 3$, is true for $\mathbf{R}^d$ for any $d \geq 2$! Beyond verifying my conjecture, what I find exciting about Moeckel's result is the associated message that there exists unexplored, interesting N-body dynamics associated with $\mathbf{R}^4$. Let me hint at what some of it is.

The key term is $\mathbf{W}_{rot}$ as it captures differences in the system coordinates between different dimensional physical spaces. In $\mathbb{R}^3$, this component resides in a three-dimensional space, but in $\mathbb{R}^4$ it resides in a $\binom{4}{2} = 6$ dimensional space. In $\mathbb{R}^3$, as shown in [15], at each instant of time there is an $\mathbf{s} \in \mathbb{R}^3$ so that

$$\mathbf{W}_{rot} = \mathbf{S} \times \mathbf{R} := (\mathbf{s} \times \mathbf{r}_1, \mathbf{s} \times \mathbf{r}_2, \mathbf{s} \times \mathbf{r}_3). \tag{21}$$

But this expression does not hold for $\mathbb{R}^d$, $d \geq 4$, because of the added axes of rotation. In $\mathbb{R}^3$, the associated acceleration term is

$$\begin{aligned} \mathbf{W}'_{rot} &= \mathbf{S}' \times \mathbf{R} + \mathbf{S} \times (\mathbf{W}_{rot} + \mathbf{W}_{scal} + \mathbf{W}_{config}) \\ &= (\mathbf{S}' + \tfrac{R'}{R}\mathbf{S}) \times \mathbf{R} + \mathbf{S} \times (\mathbf{S} \times \mathbf{R}) + \mathbf{S} \times \mathbf{W}_{config}. \end{aligned} \tag{22}$$

In $\mathbb{R}^4$, the more complicated acceleration term has extra terms introduced by another axis: this is what allows extra dynamics with the more general homographic configurations.

As an example of a doable kind of question, consider the classical issue that homographic motion requires a central configuration. Homographic motion requires $\mathbf{W}_{config} \equiv \mathbf{0}$, so the Eq. 22 expression for $\mathbf{W}'_{rot}$ reduces to a rotational term (which we can ignore as it cannot be in ∇U) and the $\mathbf{S} \times (\mathbf{S} \times \mathbf{R})$ term. (The multiple of $\mathbf{R}$ needed to prove that the configuration is a central configuration comes from $\mathbf{W}'_{scal}$.) To prove the result ([15]), most effort is to show that $\mathbf{S} \times (\mathbf{S} \times \mathbf{R})$ is a multiple of $\mathbf{R}$—the analysis depends on the single axis of rotation property of $\mathbb{R}^3$ and it leads to the invariable plane assertion. As this analysis does not hold for $\mathbb{R}^d$ for $d \geq 4$ with the different $\mathbf{W}_{rot}$ representation, what does happen?

For instance, a way to characterize coplanar central configurations is that they are the only configurations for which the initial conditions $\mathbf{W}_{config} = \mathbf{W}_{scal} = \mathbf{0}$ always define a relative equilibrium solution. A natural question, then, is to *characterize all configurations in higher dimensional physical spaces for which the initial conditions* $\mathbf{W}_{scal} = \mathbf{W}_{config} = \mathbf{0}$ *define a relative equilibrium solution.*[3] The above description suggests how to do this: determine what happens with $\mathbf{W}_{rot}$ and each axis of rotation.

The above intuition involving Eq. 20 remains, but what probably will change with $\mathbb{R}^d$ is the $\mathbf{D}(\mathbf{R})$ term; e.g., an answer for the three body problem should determine what choices of α and β in Eq. 20 will not unleash $\mathbf{W}_{config}$ and why this is so. In other words, Eq. 12 describes ∇U in terms of the $\frac{RU}{R^3}\mathbf{R}$ component that keeps the configuration fixed and the $\mathbf{D}(\mathbf{R})$ terms that change the configuration. This expression holds in $\mathbb{R}^4$ for homothetic motion, but what is the related ∇U expression for homographic motion in $\mathbb{R}^4$?

Is homographic motion in $\mathbb{R}^d$ characterized by a constant configurational measure? I fully believe it is for $d = 1, 2, 3$. Emboldened by Moeckel's proof [5] of my original conjecture for $\mathbb{R}^d$, I now suspect that this always is true.

[3] Expect changes in the characterization with changes in N.

As the above demonstrates and as I have learned over years of enjoyable exploration of this topic, the N-body problem continues to be a source of fascinating questions with all sorts of delightful surprises!

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