On convergence of trajectory attractors of 3D Navier–Stokes-α model as α approaches 0

V.V.Chepyzhov*, E.S.Titi†, and M.I.Vishik*

*Institute for Information Transmission Problems
Russian Academy of Sciences, Bolshoy Karetniy 19
Moscow 127994, GSP-4, Russia
Email: chep@iitp.ru, vishik@iitp.ru

†Department of Mathematics and Department of
Mechanical and Aerospace Engineering,
University of California, Irvine, CA 92697-3875, USA,
Email: etiti@math.uci.edu
Also:
Department of Computer Science and Applied
Mathematics, Weizmann Institute of Science,
Rehovot 76100, Israel,
Email: edriss.titi@weizmann.ac.il

Abstract

We study the relations between the long-time dynamics of the Navier–Stokes-α model and the exact 3D Navier–Stokes system. We prove that bounded sets of solutions of the Navier–Stokes-α model converge to the trajectory attractor $\mathfrak{A}_0$ of the 3D Navier–Stokes system as time tends to infinity and α approaches zero. In particular, we show that the trajectory attractor $\mathfrak{A}_\alpha$ of the Navier–Stokes-α model converges to the trajectory attractor $\mathfrak{A}_0$ of the 3D Navier–Stokes system when $\alpha \to 0+$. We also construct the minimal limit $\mathfrak{A}_{\min}$ $(\subseteq \mathfrak{A}_0)$ of the trajectory attractor $\mathfrak{A}_\alpha$ as $\alpha \to 0+$ and we prove that the set $\mathfrak{A}_{\min}$ is connected and strictly invariant.

Introduction (Date: January 17, 2007)

In this paper, we study the connection between the long-time dynamics of solutions of the Lagrange averaged Navier–Stokes-α model (N.–S.-α model) and the exact 3D Navier–Stokes system (3D N.–S. system) with periodic boundary conditions. The Navier–Stokes-α model (also known as the viscous 3D Camassa–Holm system) under the consideration was introduced in the works [1] – [6] (see also [7] and the references therein). This model is a regularized approximation of the 3D Navier–Stokes system depending on a small parameter α, where, in some terms, the unknown velocity vector-function v is replaced by a smoother vector function u which are related by the elliptic system $v = u - \alpha^2 \Delta u$ (see Sect.2). For $\alpha = 0$, the model is reduced to the exact 3D N.–S. system.

Since the uniqueness theorem for the global weak solutions (or the global existence of strong solutions) of the initial-value problem of the 3D Navier–Stokes system is not proved yet, the known theory of global attractors of infinite dimensional dynamical systems (making a good showing in the study of the 2D N.–S. system and other important evolution equations of mathematical physics, see, e.g., [8] – [14]) is not applicable to the 3D N.–S. system.

It was demonstrated analytically and numerically in many works that the mentioned above N.–S.-α model gives a good approximation in the study of many problems related to the turbulent flows (see [1] – [4], [7, 15, 16]). In particular, it was found that the explicit steady analytical solutions of the N.–S.-α model compare successfully with empirical and numerical experimental data for a wide range of Reynolds numbers in turbulent channel and pipe flows (see [1] – [3]). Along the same lines it is worth mentioning that other approximate α-models for the 3D N.–S. system also demonstrate good fit with empirical data: Clark-α model [17], Leray-α model [18], Modified-Leray-α model [19], simplified Bardina-α model [20] and some other models. Closed problems related to the regularization of the 3D N.–S. system were also considered in the works of Lions [21] and Ladyzhenskaya [22].

In [6], the Cauchy problem for the 3D Navier–Stokes-α model was studied, the global existence and uniqueness of weak solutions were established, the smoothing property of solutions was proved, and the global attractor for this system was constructed. Besides, upper bounds for the dimension of the global attractor (the number of degree of freedom) were found in terms of the relevant physical parameters and some other turbulence related features and characteristics (such as spectra and boundary layer) were discussed (see also [23, 24]).

The theory of trajectory attractors for evolution partial differential equations was developed in [14, 25, 26, 27] with an emphasis on equations for which the uniqueness theorem of solutions of the corresponding initial-value problem is not proved yet, e.g. for the 3D N.–S. system (see also [13, 28]).

In the present paper, we study the connection between the solutions of the Navier–Stokes-α model and the exact 3D Navier–Stokes system as $\alpha \to 0+$. Our main theorem states that bounded (in the corresponding norm) families of solutions $\{u_\alpha(x,t)\}$ of the N.–S.-α models converge to the trajectory attractor $\mathfrak{A}_0$ of 3D Navier–Stokes system as $\alpha \to 0+$ and $t \to +\infty$. In particular, the trajectory attractors $\mathfrak{A}_\alpha$ of the N.–S.-α model converges to $\mathfrak{A}_0$ as $\alpha \to 0+$. In [29, 30], analogous theorems were proved for the Leray-α model.

This paper consists of introduction, five sections, and an appendix. In Sect. 1, we recall the definition of the trajectory attractor $\mathfrak{A}_0$ of the exact 3D N.–S. system. In Sect. 2, we consider the N.–S.-α model (the viscous Camassa–Holm equations). Following [6], we formulate the main properties of this model.

In Sect. 3 and 4, we prove the convergence of the trajectory attractor $\mathfrak{A}_\alpha$ of the N.–S.-α model to the trajectory attractor $\mathfrak{A}_0$ of the exact 3D N.–S. system as $\alpha \to 0+$. It turns out that in order to establish this convergence it is very fruitful to study the equation to which the function $w_\alpha(t) = (1-\alpha^2\Delta)^{1/2}u_\alpha(t)$ is satisfied. Here, $u_\alpha(t)$ is the smoother velocity field of the solution of the N.–S.-α model. The main theorem of Sect. 3 states that if a sequence of solutions $w_{\alpha_n}(t)$ of the mentioned above equation converges to the limit $w(t)$ as $\alpha_n \to 0+$ and $n \to \infty$ in the space $\Theta_+^{\rm loc}$ (see Sect. 1), then $w(t)$ is a Leray–Hopf weak solution of the exact 3D N.–S. system. Using mostly this theorem in Sect. 4, we prove the convergence of the trajectory attractors $\mathfrak{A}_\alpha$ to the trajectory attractor $\mathfrak{A}_0$ in the space $\Theta_+^{\rm loc}$ as $\alpha \to 0+$.

In Sect. 5, we establish the existence of the *minimal limit* $\mathfrak{A}_{\min}$ ($\subseteq \mathfrak{A}_0$) of the trajectory attractors $\mathfrak{A}_\alpha$ as $\alpha \to 0+$, i.e., $\mathfrak{A}_\alpha \to \mathfrak{A}_{\min}$ ($\alpha \to 0+$), where $\mathfrak{A}_{\min}$ is the smallest closed subset of $\mathfrak{A}_0$ satisfying this limit relation (See Sect. 5). We prove that the set $\mathfrak{A}_{\min}$ is connected and strictly invariant with respect to the translation semigroup. These properties of the minimal limit $\mathfrak{A}_{\min}$ make it a very useful object in the study of various models that approximate the 3D N.–S. system.

We note that the question of the connectedness of the trajectory attractor $\mathfrak{A}_0$ of the 3D N.–S. system remains open.

Now, the hypothesis also arises that, to different α-models of the 3D N.–S. system (Camassa–Holm, Leray-α, Clark-α, simplified Bardina-α, etc.), different minimal limits of their trajectory attractors $\mathfrak{A}_\alpha$ as $\alpha \to 0+$ may correspond.

This work was partly supported by the Russian Foundation of Basic Researches (project no. 05-01-00390 and 04-01-00735), Civilian Research & Development Foundation (Grant RUM1-2654-MO-05), and the Russian Science Support Foundation. E.S.T. was also supported in part by the NSF grant no. DMS–0204794, the ISF grant no.120/06, and the BSF grant no. 2004271.

1 3D Navier–Stokes system and its trajectory attractor

We consider the autonomous 3D N.–S. system with periodic boundary conditions

$$\begin{cases} \partial_t v - \nu\Delta v + \sum\limits_{j=1}^{3} v^j \partial_{x_j} v + \nabla p = g(x), \\ \nabla \cdot v = 0,\ x \in \mathbb{T}^3 := [\mathbb{R} \bmod 2\pi L]^3, t \geq 0, \end{cases}$$

which is equivalent to the nonlinear nonlocal functional differential equation

$$\partial_t v - \nu\Delta v + P\sum_{j=1}^{3} v^j \partial_{x_j} v = g(x),\ \nabla \cdot v = 0. \tag{1.1}$$

Here $v = v(x,t) = (v^1(x,t), v^2(x,t), v^3(x,t))$ is the unknown vector function describing the motion of the fluid in $\mathbb{T}^3$, P is the Leray–Helmholtz orthogonal projector, $g(x) = (g^1(x), g^2(x), g^3(x))$ is a given external force with zero mean in x-variable, i.e., $\int_{\mathbb{T}^3} g(x)dx = 0$, and $g = Pg$. We assume that $v(x,t)$ is a periodic function in $x = (x_1, x_2, x_3) \in \mathbb{T}^3$ with zero mean, i.e., $\int_{\mathbb{T}^3} v(x,t)dx = 0$, and $v = Pv$.

We set $\mathcal{V} = \{\phi(x) = (\phi^1(x), \phi^2(x), \phi^3(x)), x \in \mathbb{T}^3 \mid \phi^j(x)$ *are trigonometrical polynomial with period* $2\pi L$ *in each* $x_i, i,j = 1,2,3,$ *such that* $\nabla\cdot\phi = 0$ *and* $\int_{\mathbb{T}^3}\phi(x)dx = 0\}$. We denote by H and V the closure of the set $\mathcal{V}$ in the norms $\|\cdot\|_H =: |\cdot|$ and

$\|\cdot\|_{H^1} =: \|\cdot\|$ of the spaces $L_2(\mathbb{T}^3)^3$ and $H^1(\mathbb{T}^3)^3$, respectively (see, e.g., [11, 31]). Then the Leray–Helmholtz projector $P : L_2(\mathbb{T}^3)^3 \to H$.

We define also the space $D(A) = \{v \in H \mid \Delta v \in H\}$, where $A = -P\Delta$ is the Stokes operator with domain $D(A)$. Recall that, in the periodic case, $A = -\Delta$ and the norm $|Av| =: \|v\|_{D(A)}$ on $D(A)$ is equivalent to the norm induced by $H^2(\mathbb{T}^3)^3$. The operator A is self-adjoint, positive, and has a compact resolvent. We denote by

$$((u, v)) := (A^{1/2}u, A^{1/2}v) = (\nabla u, \nabla v), \ \|u\| := |A^{1/2}u|, \ u, v \in V,$$

the scalar product and the norm in V, respectively. The Poincaré inequality implies that

$$|v|^2 \le \lambda_1^{-1}\|v\|^2, \ \forall v \in V, \tag{1.2}$$

where λ_1 is the first eigenvalue of the Stokes operator A. Let $V' = H^{-1}$ be the dual space of V. For any $f \in V'$, we denote by $\langle f, v\rangle$ the action of the functional $f \in V'$ on any $v \in V$. The operator A is an isomorphism from V to V' and $((u, v)) = \langle Au, v\rangle$ for all $u, v \in V$.

We rewrite equation (1.1) in a standard short form:

$$\partial_t v + \nu Av + B(v, v) = g(x), \ t \ge 0. \tag{1.3}$$

Here, we denote

$$B(u, v) = P[(u \cdot \nabla)v] = P\sum_{j=1}^{3} u^j \partial_{x_j} v. \tag{1.4}$$

Recall that for u satisfying $\nabla \cdot u = 0$ we have:

$$B(u, v) = P\sum_{j=1}^{3} \partial_{x_j}\left(u^j v\right) \tag{1.5}$$

(see [21, 11, 31]). For all $w \in D(A)$ and $u, v \in V$, we have the estimate

$$|\langle B(u, v), w\rangle| \le C|u| \cdot \|v\| \cdot \|w\|_{L_\infty} \le C_1\lambda_1^{-1/4}|u| \cdot \|v\| \cdot \|w\|_{D(A)} \tag{1.6}$$

and therefore

$$\|B(u, v)\|_{D(A)'} \le C_1\lambda_1^{-1/4}|u| \cdot \|v\|, \tag{1.7}$$

where $D(A)'$ is the dual space of $D(A)$.

Let a function $v(\cdot) \in L_2(0, M; V) \cap L_\infty(0, M; H)$ be given. Therefore, $Av \in L_2(0, M; V')$ and due to (1.7) we have

$$B(v(\cdot), v(\cdot)) \in L_2(0, M; D(A)'). \tag{1.8}$$

Consider the space of distributions $\mathcal{D}'(0, M; D(A)')$ (see, e.g., [21]). Recall that a function $v(\cdot) \in L_2(0, M; V) \cap L_\infty(0, M; H)$ is said to be a *weak solution* of equation (1.3) if it satisfies this equation in the space $\mathcal{D}'(0, M; D(A)')$. Then it follows from (1.8) that $\partial_t v(\cdot) \in L_2(0, M; D(A)')$ for any weak solution $v(\cdot)$ of (1.3) and hence $v(\cdot) \in C([0, M]; D(A)')$. Recall that $v(\cdot) \in L_\infty(0, M; H)$. Then, by the known lemma from [33] (see also [31]), the function $v(\cdot) \in C_w([0, M]; H)$ and, consequently, the initial data

$$v|_{t=0} = v_0(x) \in H \tag{1.9}$$

for equation (1.3) has a sense in the class of weak solutions from the space $L_2(0,M;V)\cap L_\infty(0,M;H)$.

We now formulate the classical theorem on the existence of weak solutions of the Cauchy problem for the 3D N.–S. system in the form we need in the sequel (see, e.g., [11, 21, 32, 31]).

Theorem 1.1 *Let $g\in V'$ and $v_0\in H$. Then for every $M>0$, there exists a weak solution $v(t)$ of equation (1.3) from the space $L_2(0,M;V)\cap L_\infty(0,M;H)$ such that $v(0)=v_0$ and $v(t)$ satisfies the energy inequality*

$$\frac{1}{2}\frac{d}{dt}|v(t)|^2+\nu\|v(t)\|^2\le\langle g,v(t)\rangle\ ,\ t\in[0,M]. \tag{1.10}$$

Inequality (1.10) means that for any function $\psi(\cdot)\in C_0^\infty(]0,M[)$, $\psi(t)\ge 0$,

$$-\frac{1}{2}\int_0^M|v(t)|^2\psi'(t)dt+\nu\int_0^M\|v(t)\|^2\psi(t)dt\le\int_0^M\langle g,v(t)\rangle\,\psi(t)dt. \tag{1.11}$$

The proof of Theorem 1.1 is given, e.g., in [11, 21, 14].

Remark 1.1 *For the 3D Navier–Stokes system the question of the uniqueness of a weak solution of problem (1.3) and (1.9) remains open. It is also unknown, whether every weak solution satisfies the energy inequality (1.10). However, it is known that every weak solution resulting from the Faedo–Galerkin approximation method satisfies this energy inequality. The class of weak solutions which satisfy the energy inequality (1.10) or (1.11) is called Leray–Hopf weak solutions.*

In the sequel, we define the trajectory attractor for the N.–S. equation (1.3). (For more details, see [26, 14].)

To begin with, we define the trajectory space $\mathcal{K}^+$ of equation (1.3). We consider a set of weak solutions $v(t),t\ge 0$, belonging to the space $L_2^{\mathrm{loc}}(\mathbb{R}_+;V)\cap L_\infty^{\mathrm{loc}}(\mathbb{R}_+;H)$ that satisfy equation (1.3) in the space of distributions $\mathcal{D}'(0,M;D(A)')$ for any $M>0$.

Definition 1.1 *The* trajectory space $\mathcal{K}^+$ *is the set of all Leray–Hopf weak solutions $v(\cdot)$ of equation (1.3) in the space $L_2^{\mathrm{loc}}(\mathbb{R}_+;V)\cap L_\infty^{\mathrm{loc}}(\mathbb{R}_+;H)$ that satisfy the energy inequality (1.10) for $t\ge 0$, that is,*

$$-\frac{1}{2}\int_0^\infty|v(t)|^2\psi'(t)dt+\nu\int_0^\infty\|v(t)\|^2\psi(t)dt\le\int_0^\infty\langle g,v(t)\rangle\,\psi(t)dt \tag{1.12}$$

for all $\psi\in C_0^\infty(\mathbb{R}_+),\psi\ge 0$.

It follows from Theorem 1.1 that, for any $v_0\in H$, there is a trajectory $v(\cdot)\in\mathcal{K}^+$ such that $v(0)=v_0$.

We need the Banach space

$$\mathcal{F}_+^{\mathrm{b}}=\{z\mid z(\cdot)\in L_2^{\mathrm{b}}(\mathbb{R}_+;V)\cap L_\infty(\mathbb{R}_+;H),\ \partial_t z(\cdot)\in L_2^{\mathrm{b}}(\mathbb{R}_+;D(A)')\}$$

with norm

$$\|z\|_{\mathcal{F}_+^{\mathrm{b}}}=\|z\|_{L_2^{\mathrm{b}}(\mathbb{R}_+;V)}+\|z\|_{L_\infty(\mathbb{R}_+;H)}+\|\partial_t z\|_{L_2^{\mathrm{b}}(\mathbb{R}_+;D(A)')}, \tag{1.13}$$

where $\|z\|^2_{L_2^{\rm b}(\mathbb{R}_+;V)} = \sup_{t\ge 0}\int_t^{t+1}\|z(s)\|^2 ds$, $\|z\|_{L_\infty(\mathbb{R}_+;H)} = \operatorname{ess\,sup}_{t\ge 0}|z(t)|$, and $\|\partial_t z\|^2_{L_2^{\rm b}(\mathbb{R}_+;D(A)')} = \sup_{t\ge 0}\int_t^{t+1}\|z(s)\|^2_{D(A)'}ds$.

We denote by $\{T(h)\} := \{T(h), h \ge 0\}$ the translation semigroup acting on a function $\{z(t), t \ge 0\}$ by the formula

$$T(h)z(t) = z(t+h),\ t \ge 0.$$

Clearly, the semigroup $\{T(h)\}$ acts on $\mathcal{F}_+^{\rm b}$. We consider the action of the semigroup $\{T(h)\}$ on the trajectory space $\mathcal{K}^+$ of equation (1.3). It follows from the definition of $\mathcal{K}^+$ that if $v(\cdot) \in \mathcal{K}^+$, then $v_h(\cdot) = T(h)v(\cdot) = v(\cdot + h) \in \mathcal{K}^+$ for all $h \ge 0$. That is,

$$T(h)\mathcal{K}^+ \subseteq \mathcal{K}^+,\ \forall h \ge 0. \tag{1.14}$$

We are going to construct the global attractor of the translation semigroup $\{T(h)\}$ on $\mathcal{K}^+$. We call this attractor the *trajectory attractor* since the semigroup $\{T(h)\}$ acts on the trajectory space $\mathcal{K}^+$. The following key proposition is proved in [14].

Proposition 1.1 *Let $g \in V'$. Then*

1. *The trajectory space $\mathcal{K}^+ \subseteq \mathcal{F}_+^{\rm b}$;*
2. *for any function $v(\cdot) \in \mathcal{K}^+$,*

$$\|T(h)v(\cdot)\|_{\mathcal{F}_+^{\rm b}} \le C_0\|v(\cdot)\|^2_{C_\infty(0,1;H)}e^{-\nu\lambda_1 h} + R_0^2,\ \forall h \ge 1, \tag{1.15}$$

where the constant C_0 depends on ν, λ_1 and R_0 depends on ν, λ_1, $\|g\|_{V'}$.

We need a topology in the space $\mathcal{K}^+$. Similarly to $\mathcal{F}_+^{\rm b}$, we consider the space

$$\mathcal{F}_+^{\rm loc} = \{z \mid z(\cdot) \in L_2^{\rm loc}(\mathbb{R}_+;V) \cap L_\infty^{\rm loc}(\mathbb{R}_+;H),\ \partial_t z(\cdot) \in L_2^{\rm loc}(\mathbb{R}_+;D(A)')\}.$$

We define on $\mathcal{F}_+^{\rm loc}$ the following sequential topology which we denote $\Theta_+^{\rm loc}$. By definition, a sequence of functions $\{z_n\} \subseteq \mathcal{F}_+^{\rm loc}$ converges to a function $z \in \mathcal{F}_+^{\rm loc}$ in the topology $\Theta_+^{\rm loc}$ as $n \to +\infty$ if, for any $M > 0$,

$$\begin{aligned} z_n(\cdot) &\rightharpoonup z(\cdot)\ (n \to \infty) \text{ weakly in } L_2(0,M;V),\\ z_n(\cdot) &\rightharpoonup z(\cdot)\ (n \to \infty) \text{ weakly-}* \text{ in } L_\infty(0,M;H), \end{aligned}$$

and

$$\partial_t z_n(\cdot) \rightharpoonup \partial_t z(\cdot)\ (n \to \infty) \text{ weakly in } L_2(0,M;D(A)').$$

Note that the topology $\Theta_+^{\rm loc}$ can be described in terms of open neighborhoods. $\Theta_+^{\rm loc}$ is a Hausdorff topological space with a countable base of its topology (however, the topology $\Theta_+^{\rm loc}$ is not metrizable). Recall that $\mathcal{F}_+^{\rm b} \subseteq \Theta_+^{\rm loc}$. Besides, any ball $B_R = \{z \in \mathcal{F}_+^{\rm b} \mid \|z\|_{\mathcal{F}_+^{\rm b}} \le R\}$ is compact in $\Theta_+^{\rm loc}$. Hence, the set B_R with topology induced by $\Theta_+^{\rm loc}$ is metrizable and the corresponding metric space is complete (see the details in [26, 14]). This property simplifies the construction of the trajectory attractor (in the topology $\Theta_+^{\rm loc}$) of the semigroup $\{T(h)\}$ acting on $\mathcal{K}^+$. It follows from the definition of the topology $\Theta_+^{\rm loc}$ that the translation semigroup $\{T(h)\}$ is continuous in $\Theta_+^{\rm loc}$. The following assertion is important for us (see the proof in [14]) .

Proposition 1.2 *The trajectory space $\mathcal{K}^+$ is closed in the space Θ_+^{loc}.*

In a standard manner, we define an attracting set in $\mathcal{K}^+$ (see [8, 9, 10, 28]). A set $P \subseteq \mathcal{F}_+^{\mathrm{b}}$ is called *attracting* for the space $\mathcal{K}^+$ in the topology Θ_+^{loc} if, for any bounded (in the norm of $\mathcal{F}_+^{\mathrm{b}}$) set $B \subset \mathcal{K}^+$, the set P attracts $T(h)B$ in the topology Θ_+^{loc} as $h \to +\infty$, that is, for any neighborhood $\mathcal{O}(P)$ (in Θ_+^{loc}), there is a number $h_1 = h_1(B, \mathcal{O})$ such that $T(h)B \subseteq \mathcal{O}(P)$ for all $h \geq h_1$.

We now define the trajectory attractor.

Definition 1.2 *A set $\mathfrak{A} \subset \mathcal{K}^+$ is called the trajectory attractor of the semigroup $\{T(h)\}$ in the topology Θ_+^{loc} if*

1. *$\mathfrak{A}$ is bounded in $\mathcal{F}_+^{\mathrm{b}}$ and compact in Θ_+^{loc};*
2. *$\mathfrak{A}$ is strictly invariant with respect to $\{T(h)\} : T(h)\mathfrak{A} = \mathfrak{A}$, $\forall h \geq 0$;*
3. *$\mathfrak{A}$ is an attracting set in the topology Θ_+^{loc} for $\{T(h)\}$ on $\mathcal{K}^+$.*

Following the terminology from [9], the set $\mathfrak{A}$ is also called the $(\mathcal{F}_+^{\mathrm{b}}, \Theta_+^{\mathrm{loc}})$-*attractor* of the semigroup $\{T(h)\}|_{\mathcal{K}^+}$.

The main inequality (1.15) implies that the ball B_{2R_0} in $\mathcal{F}_+^{\mathrm{b}}$ is an attracting (and even absorbing) set of the semigroup $\{T(h)\}$ in $\mathcal{K}^+$. The ball B_{2R_0} is clearly compact in Θ_+^{loc} and $T(h)B_{2R_0} \subseteq B_{2R_0}$ for all $h \geq 0$. Therefore, the continuous semigroup $\{T(h)\}$ has a compact attracting set. Consequently, the translation semigroup $\{T(h)\}$ has the trajectory attractor $\mathfrak{A} \subset \mathcal{K}^+ \cap B_{2R_0}$ and moreover

$$\mathfrak{A} = \bigcap_{s>0} \left[\bigcup_{h \geq s} T(h)(\mathcal{K}^+ \cap B_{2R_0}) \right]_{\Theta_+^{\mathrm{loc}}},$$

where $[\cdot]_{\Theta_+^{\mathrm{loc}}}$ denotes the closure in Θ_+^{loc} (see [14]).

Notice that the following embeddings are continuous:

$$\Theta_+^{\mathrm{loc}} \subset L_2^{\mathrm{loc}}(\mathbb{R}_+; H^{1-\delta}), \tag{1.16}$$
$$\Theta_+^{\mathrm{loc}} \subset C^{\mathrm{loc}}(\mathbb{R}_+; H^{-\delta}), \text{ for } 0 < \delta \leq 1, \tag{1.17}$$

(see [14, 21, 34]). Hence, the trajectory attractor $\mathfrak{A}$ satisfies the following properties: for any bounded (in $\mathcal{F}_+^{\mathrm{b}}$) set $B \subset \mathcal{K}^+$,

$$\begin{aligned} \mathrm{dist}_{L_2(0,M;H^{1-\delta})}(T(h)B, \mathfrak{A}) &\to 0 \ (h \to +\infty), \\ \mathrm{dist}_{C([0,M];H^{-\delta})}(T(h)B, \mathfrak{A}) &\to 0 \ (h \to +\infty), \end{aligned}$$

where M is an arbitrary positive number.

To describe the structure of the trajectory attractor $\mathfrak{A}$ we need the notion of the kernel of equation (1.3). The *kernel* $\mathcal{K}$ is the set of all weak solutions $v(t), t \in \mathbb{R}$, bounded in the space

$$\mathcal{F}^{\mathrm{b}} = \{z \mid z(\cdot) \in L_2^{\mathrm{b}}(\mathbb{R}; V) \cap L_\infty(\mathbb{R}; H),\ \partial_t z(\cdot) \in L_2^{\mathrm{b}}(\mathbb{R}; D(A)')\}$$

that satisfies an inequality similar to (1.12): for all $\psi \in C_0^\infty(\mathbb{R}), \psi \geq 0$,

$$-\frac{1}{2}\int_{-\infty}^{\infty} |v(t)|^2\psi'(t)dt + \nu\int_{-\infty}^{\infty} \|v(t)\|^2\psi(t)dt \leq \int_{-\infty}^{\infty} \langle g, v(t)\rangle\,\psi(t)dt. \tag{1.18}$$

(The norm in $\mathcal{F}^{\mathrm{b}}$ is defined in a similar way that the norm in $\mathcal{F}_+^{\mathrm{b}}$ (see (1.13)) replacing $\mathbb{R}_+$ by $\mathbb{R}$).

We denote by Π_+ the restriction operator onto $\mathbb{R}_+$. It is proved in [14] that the trajectory attractor $\mathfrak{A}$ of the 3D Navier–Stokes system coincides with the restriction of the kernel $\mathcal{K}$ of equation (1.3) onto $\mathbb{R}_+$:

$$\mathfrak{A} = \Pi_+\mathcal{K}. \tag{1.19}$$

The set $\mathcal{K}$ is bounded in $\mathcal{F}^{\mathrm{b}}$ and compact in Θ^{loc}. The topology Θ^{loc} is defined similar to Θ_+^{loc} where the intervals $(0, M)$ are replaced by $(-M, M)$.

2 Navier–Stokes-α model and its attractor

2.1 Some properties of the Navier–Stokes-α model

We consider the following system with periodic boundary conditions:

$$\partial_t v - \nu\Delta v - P(u \times (\nabla \times v)) = g(x), \tag{2.1}$$

$$v = u - \alpha^2\Delta u,\ \nabla \cdot v = 0, \nabla \cdot u = 0,\ x \in \mathbb{T}^3. \tag{2.2}$$

This system is an approximation of the 3D N.–S. system (1.1) discussed in the previous section. The unknown vector function is $u = u(x,t) = (u^1, u^2, u^3)$. The function $v = v(x,t) = (v^1, v^2, v^3)$ is auxiliary. We assume that functions $u(x,t)$, $v(x,t)$, and the (known) external force $g(x)$ are periodic in $x \in \mathbb{T}^3$ and have zero spatial mean. In equation (2.2), α is a fixed positive parameter called "the sub-grid (filter) length scale" of the model (see the motivations in [6] and the references therein). As in (1.1), P denotes the Leray–Helmholtz projector and $a \times b$ is the vector product in $\mathbb{R}^3$. We will see shortly that, for $\alpha = 0$, the function $v \equiv u$ and formally equations (2.1) and (2.2) coincides with the 3D N.–S. system (1.1). The system (2.1) and (2.2) is called sometimes as the 3D Camassa–Holm equations (it is also known as the Lagrange averaged Navier–Stokes-α model or just the Navier–Stokes-α model).

Recall that the nonlinear term in (2.1) satisfies the following identity

$$u \times (\nabla \times v) = \sum_{j=1}^{3}\left(u^j\partial_j v - u^j\nabla v^j\right) = -(u\cdot\nabla)v - \sum_{j=1}^{3} u^j\nabla v^j \tag{2.3}$$

assuming that $u, v \in C^1$ (see [6]). For $u = v$, we have

$$v \times (\nabla \times v) = -(v\cdot\nabla)v - \frac{1}{2}\nabla(\sum_{j=1}^{3} v^j v^j) \tag{2.4}$$

and hence, for $\alpha = 0$, the system (2.1) and (2.2) becomes (1.1) since P projects any gradient function to zero, so, $P\nabla(\sum_{j=1}^{3} v^j v^j) = 0$ (see [11, 31]).

We now rewrite system (2.1) and (2.2) in the short form

$$\partial_t v + \nu A v + \tilde{B}(u,v) = g(x), \tag{2.5}$$
$$v = u + \alpha^2 A u. \tag{2.6}$$

Here as in equation (1.3), A denotes the Stokes operator and the bilinear operator

$$\tilde{B}(u,v) = -P(u \times (\nabla \times v)). \tag{2.7}$$

Recall that

$$\tilde{B}(v,v) = B(v,v), \tag{2.8}$$

where $B(u,v) = (u \cdot \nabla)v$ (see (1.4)) and, for $\alpha = 0$, system (2.5), (2.6) coincides with the 3D N.–S. system (1.3).

We now formulate some properties of the bilinear operator $\tilde{B}$ that are analogous to the properties of the operator B. The operator $\tilde{B}$ maps $V \times V$ to V' and the following inequalities holds

$$|\left\langle \tilde{B}(u,v), w \right\rangle| \;\leq\; c|u|^{1/4}\|u\|^{3/4}\|v\| \cdot |w|^{1/4}\|w\|^{3/4}, \tag{2.9}$$
$$|\left\langle \tilde{B}(u,v), w \right\rangle| \;\leq\; c|u|^{1/2}\|u\|^{1/2}\|v\| \cdot \|w\| \;\forall u,v,w \in V. \tag{2.10}$$

(For the proof, see [6].) We have also the identity

$$\left\langle \tilde{B}(u,v), w \right\rangle = -\left\langle \tilde{B}(w,v), u \right\rangle, \;\forall u,v,w \in V, \tag{2.11}$$

which follows from the vector calculus formulas

$$(a \times b) \cdot c = \det[a,b,c] = -\det[c,b,a] = -(c \times b) \cdot a, \;\forall a,b,c \in \mathbb{R}^3,$$

where we set $a = u,\; b = \nabla \times v$, and $c = w$. From (2.11), we conclude that

$$\left\langle \tilde{B}(u,v), u \right\rangle = 0 \;\forall u,v \in V. \tag{2.12}$$

We need also the following inequality proved in [6]:

$$\begin{aligned} |\left\langle \tilde{B}(u,v), w \right\rangle| \;&\leq\; c\left(|u|^{1/2}\|u\|^{1/2}|v||Aw| + \|u\||v|\|w\|^{1/2}|Aw|^{1/2}\right), \\ \forall u,v \;&\in\; V \text{ and } w \in D(A). \end{aligned} \tag{2.13}$$

To prove (2.13) one uses the following identity

$$\left\langle \tilde{B}(u,v), w \right\rangle = \langle B(w,u) - B(u,w), v \rangle \tag{2.14}$$

and the known properties of the operator B (see (1.6) and [6]). The identity (2.14) can be verified by the direct calculation.

It follows from (2.13) that

$$|\left\langle \tilde{B}(u,v), w \right\rangle| \leq c\|u\| \cdot |v| \cdot |Aw|, \;\forall u,v \in V,\; w \in D(A). \tag{2.15}$$

This means that $\tilde{B}$ maps $V \times H$ into $D(A)'$ and

$$\|\tilde{B}(u,v)\|_{D(A)'} \leq c\|u\| \cdot |v| \tag{2.16}$$

(compare with (1.7)).

2.2 Cauchy problem and attractor for the N.–S.-α model

Let now a function $u(\cdot) \in L_\infty(0, M; V) \cap L_2(0, M; D(A))$ be given. Then the function $v(\cdot) = u(\cdot) + \alpha^2 Au(\cdot) \in L_\infty(0, M; V') \cap L_2(0, M; H)$ and $Av(\cdot) \in L_2(0, M; D(A)')$. Consequently, from inequality (2.16) we conclude that the corresponding function $\tilde{B}(u(\cdot), v(\cdot)) \in L_2(0, M; D(A)')$. Therefore, all the terms of equation (2.5) (except the time derivative) belongs to the space $L_2(0, M; D(A)')$ and the equation itself is meaningful in the distribution space $\mathcal{D}'(0, M; D(A)')$. We supplement system (2.5) and (2.6) with initial data

$$u|_{t=0} = u_0 \in V. \tag{2.17}$$

(Compare with (1.9), where $v_0 \in H$.)

Definition 2.1 *Let $g \in H$, $u_0 \in V$, and $M > 0$. A function $u(\cdot) \in L_\infty(0, M; V) \cap L_2(0, M; D(A))$ is called a solution of system (2.5), (2.6), and (2.17) if*

(i) *$u(t)$ satisfies the equation in the space of distributions $\mathcal{D}'(0, M; D(A)')$, i.e., for every $\omega \in D(A)$*

$$\begin{aligned} \frac{d}{dt}\left\langle u + \alpha^2 Au, \omega\right\rangle &+ \left\langle A(u + \alpha^2 Au), \omega\right\rangle \\ &+ \left\langle \tilde{B}(u, u + \alpha^2 Au), \omega\right\rangle = \langle g, \omega\rangle, \end{aligned} \tag{2.18}$$

where (2.18) is understood in the scalar distribution sense of the space $\mathcal{D}'(0, M)$, that is, for every $\varphi \in C_0^\infty(]0, M[)$,

$$\begin{aligned} &- \int_0^M \langle v(t), \omega\rangle \varphi'(t)dt + \int_0^M \langle Av(t), \omega\rangle \varphi(t)dt \\ &+ \int_0^M \left\langle \tilde{B}(u(t), v(t)), \omega\right\rangle \varphi(t)dt = \int_0^M \langle g, \omega\rangle \varphi(t)dt, \end{aligned} \tag{2.19}$$

where $v(t) = u(t) + \alpha^2 Au(t)$.

(ii) *$u(0) = u_0$. Since $u(t)$ is a solution of (2.5) and (2.6), $dv/dt \in L_2(0, M; D(A)')$. We note that $u = (1 + \alpha^2 A)^{-1} v$, so $du/dt \in L_2(0, M; H)$, therefore, $u \in C([0, M]; H)$, and the initial condition (2.17) is meaningful.*

Remark 2.1 *Sometimes, the function $v(t) = (1+\alpha^2 A)u(t)$ is also called the solution of system (2.5) and (2.6). This terminology forms a correspondence between the solutions of (2.5), (2.6) and the solutions of the exact Navier–Stokes system (1.3).*

In the work [6], the following theorem was proved.

Theorem 2.1 *Let $g \in H$ and $u_0 \in V$. Then, for every $M > 0$, the Cauchy problem (2.5), (2.6), and (2.17) has a unique solution $u(t)$ that belongs to the space*

$$C([0, M]; V) \cap L_\infty(0, M; D(A)).$$

Here, we formulate and prove some corollaries from this theorem we need in the sequel. First of all, we are interested in solution estimates that are independent of α as $\alpha \to 0+$.

Corollary 2.1 ***(The energy equality)*** *Let $u(t)$ be a solution of (2.5), (2.6), and (2.17) then the following identity holds:*

$$\frac{1}{2}\frac{d}{dt}\left\{|u(t)|^2+\alpha^2\|u(t)\|^2\right\} \\ +\nu\left\{\|u(t)\|^2+\alpha^2|Au(t)|^2\right\}=\langle g,u(t)\rangle\,,\ t\in[0,M], \tag{2.20}$$

the function $|u(t)|^2+\alpha^2\|u(t)\|^2$ is absolutely continuous, and its time derivative satisfies (2.20) in the usual sense for a.e. $t\in(0,M)$.

Proof. We take the scalar product in H of equation (2.5) with $u(t)$ and use the facts that $u\in L_2(0,M;D(A))$ and $\partial_t u\in L_2(0,M;H)$. Then due to the known theorem from [31]

$$\begin{aligned}\frac{d}{dt}|u(t)|^2 &= 2(u,\partial_t u),\\ \frac{d}{dt}\|u(t)\|^2 &= 2(A^{1/2}u,\partial_t A^{1/2}u)=2(Au,\partial_t u)\\ &= 2(u,\partial_t Au),\end{aligned}$$

(recall that $A^{1/2}u\in L_2(0,M;V)$ and $\partial_t A^{1/2}u\in L_2(0,M;V'))$. Besides,

$$(\tilde{B}(u,v),u)=0$$

(see (2.12)). To complete the proof we note that $(Av,u)=\|u(t)\|^2+\alpha^2|Au(t)|^2$. (Analogous approach is used to prove the uniqueness of a solution of (2.5), (2.6), and (2.17), see [6].) ∎

Corollary 2.2 ***(A priori estimates)*** *If $u(t)$ is a solution of (2.5), (2.6), and (2.17), then the following inequalities hold:*

$$|u(t)|^2+\alpha^2\|u(t)\|^2 \le \left(|u(0)|^2+\alpha^2\|u(0)\|^2\right)e^{-\nu\lambda_1 t}+\frac{|g|^2}{\lambda_1^2\nu^2}, \tag{2.21}$$

$$\nu\int_t^{t+1}\left\{\|u(s)\|^2+\alpha^2|Au(s)|^2\right\}ds \le \left(|u(0)|^2+\alpha^2\|u(0)\|^2\right)e^{-\nu\lambda_1 t} \\ +\frac{|g|^2}{\lambda_1^2\nu^2}+\frac{|g|^2}{\lambda_1\nu},\ \forall t\ge 0. \tag{2.22}$$

Proof. We use the energy equality (2.20) and estimate the right-hand side as follows:

$$\begin{aligned}|(g,u)| &\le \frac{\nu}{2}\|u\|^2+\frac{1}{2\nu}\|g\|_{V'}^2\\ &\le \frac{\nu}{2}\|u\|^2+\frac{1}{2\nu\lambda_1}|g|^2\le\frac{\nu}{2}\left\{\|u\|^2+\alpha^2|Au|^2\right\}+\frac{1}{2\nu\lambda_1}|g|^2.\end{aligned}$$

Hence,

$$\frac{d}{dt}\left\{|u(t)|^2+\alpha^2\|u(t)\|^2\right\}+\nu\left\{\|u(t)\|^2+\alpha^2|Au(t)|^2\right\}\le\frac{1}{\nu\lambda_1}|g|^2. \tag{2.23}$$

It follows from the Poincaré inequality that

$$|u(t)|^2 + \alpha^2 \|u(t)\|^2 \le \lambda_1^{-1} \left\{ \|u(t)\|^2 + \alpha^2 |Au(t)|^2 \right\}$$

(since $\lambda_1 |u|^2 \le \|u\|^2$ and $\lambda_1 \|u\|^2 \le |Au|^2$). Consequently, from (2.23) we have

$$\frac{d}{dt} \left\{ |u(t)|^2 + \alpha^2 \|u(t)\|^2 \right\} + \nu\lambda_1 \left\{ |u(t)|^2 + \alpha^2 \|u(t)\|^2 \right\} \le \frac{1}{\nu\lambda_1} |g|^2.$$

Using now the known assertion

$$\frac{d}{dt}\varphi + \nu\lambda_1 \varphi \le \frac{1}{\nu\lambda_1}|g|^2 \implies \varphi(t) \le \varphi(0) e^{-\nu\lambda_1 t} + \frac{1}{\nu^2\lambda_1^2}|g|^2,$$

where $\varphi(t) = |u(t)|^2 + \alpha^2 \|u(t)\|^2$, we obtain (2.21).

Integrating (2.23) over $[t, t+1]$ we find that

$$\begin{aligned} & |u(t+1)|^2 + \alpha^2 \|u(t+1)\|^2 + \nu \int_t^{t+1} \left\{ \|u(s)\|^2 + \alpha^2 |Au(s)|^2 \right\} ds \\ \le \ & |u(t)|^2 + \alpha^2 \|u(t)\|^2 + \frac{1}{\nu\lambda_1}|g|^2 \\ \le \ & \left(|u(0)|^2 + \alpha^2 \|u(0)\|^2 \right) e^{-\nu\lambda_1 t} + \frac{|g|^2}{\lambda_1^2 \nu^2} + \frac{1}{\nu\lambda_1}|g|^2, \end{aligned}$$

where we have applied (2.21). Thus, (2.22) is also proved. ■

Remark 2.2 ***(i)*** *Estimates (2.21) and (2.22) imply that, for* $\alpha > 0$, $u \in L_\infty(\mathbb{R}_+; V) \cap L_2^{loc}(\mathbb{R}_+; D(A))$. *This inclusion is essentially used in the proof of Theorem 2.1 (see [6]).* ***(ii)*** *It follows also that* $v \in L_2^b(\mathbb{R}_+; H)$.

Remark 2.3 *We note that the constants in the right-hand sides of estimates (2.21) and (2.22) are independent of* α *(for* $0 < \alpha \le 1$*). This fact plays the key role in the proof of the convergence of solutions of the Navier–Stokes-*α *model to the solutions of the real Navier–Stokes system as* $\alpha \to 0+$.

Remark 2.4 *In the work [6], the following smoothing property for solutions of (2.5), (2.6), and (2.17) is established:*

$$t|Au(t)|^2 + \nu \int_0^t s|A^{3/2}u(s)|^2 ds \le C(\alpha, t, \|u(0)\|, |g|), \tag{2.24}$$

where $C(\alpha, z, r_1, r_2)$ *is a monotone increasing function in each variable* z, r_1, r_2 *and* $C(\alpha, z, r_1, r_2) \to +\infty$ *as* $\alpha \to 0+$.

We now consider the semigroup $\{S_\alpha(t)\} = \{S(t)\}, \alpha > 0$, acting in the space V by the formula $S(t)u_0 = u(t)$, where $u(t)$ is a solution of problem (2.5), (2.6), and (2.17). It follows from (2.21) that the semigroup $\{S(t)\}$ has bounded (in V) absorbing set $P_0 = \left\{ u \mid \|u\| \le \frac{2|g|}{\alpha\lambda_1\nu} \right\}$. The set $P_1 = S(1)P_0$ is also absorbing and inequality (2.24) implies that P_1 is precompact in V. It can be verified that the semigroup $\{S(t)\}$ is continuous in V. These facts are sufficient to state that the semigroup $\{S(t)\}$ corresponding to the

Navier–Stokes-α model has the global attractor $\mathcal{A}_\alpha$, that is $\mathcal{A}_\alpha$ compact in V, strictly invariant with respect to $\{S(t)\}: S(t)\mathcal{A}_\alpha = \mathcal{A}_\alpha,\ for$ all $t \geq 0$, and, $\mathrm{dist}_V(S(t)B, \mathcal{A}_\alpha) \to 0+$ as $t \to +\infty$ for any bounded (in V) set of initial data $B = \{u_0\}$ (see [8, 9, 10, 14, 28]). Moreover, $\mathcal{A}_\alpha$ is bounded in $D(A) \cap H^3(\mathbb{T})^2$ for every fixed $\alpha > 0$ but not uniformly with respect to α (see [6]).

In the next section, we study the behaviour of the Navier–Stokes-α model as $\alpha \to 0+$. We establish its relation with solutions of the 3D Navier–Stokes system.

3 On the convergence of solutions of N.–S.-α model

First of all, we need an estimate for the derivative $\partial_t v$ in which constants are independent of α similar to that proved for u in Corollaries 2.1 and 2.2.

Proposition 3.1 *Let $g \in H$. Then any solution $u(t)$ of (2.5), (2.6), and (2.17) satisfies the inequality*

$$\left(\int_t^{t+1} \|\partial_t v(s)\|^2_{D(A)'} ds\right)^{1/2} \leq C \cdot \left(|u(0)|^2 + \alpha^2\|u(0)\|^2\right) e^{-\nu\lambda_1 t} + R^2, \tag{3.1}$$

where C depends on λ_1, ν; R depends on $\lambda_1, \nu, |g|$ and the values C and R are independent of α.

Proof. We use inequality (2.16):

$$\|\tilde{B}(u, v)\|_{D(A)'} \leq c\|u\| \cdot |v|, \ \forall u \in V, v \in H. \tag{3.2}$$

Replacing here a solution $u(t)$ of (2.5), (2.6), (2.17) and $v = u + \alpha^2 Au$, we obtain

$$\begin{aligned} & \|\tilde{B}(u(t), v(t))\|_{D(A)'} \leq c\|u(t)\| \left\{|u(t)| + \alpha^2|Au(t)|\right\} \\ = \ & c\left\{|u(t)|\|u(t)\| + \alpha\|u(t)\|\alpha|Au(t)|\right\} \\ \leq \ & c\left(|u(t)|^2 + \alpha^2\|u(t)\|^2\right)^{1/2}\left(\|u(t)\|^2 + \alpha^2|Au(t)|^2\right)^{1/2}, \end{aligned} \tag{3.3}$$

where we have used the simplest Cauchy inequality. Applying inequality (2.21), we have that

$$\|\tilde{B}(u(t), v(t))\|^2_{D(A)'} \leq c^2\left\{\varphi(0)e^{-\nu\lambda_1 t} + \frac{|g|^2}{\lambda_1^2\nu^2}\right\}\left(\|u(t)\|^2 + \alpha^2|Au(t)|^2\right),$$

where $\varphi(0) = |u(0)|^2 + \alpha^2\|u(0)\|^2$. Integrating this inequality over $[t, t+1]$, we find

$$\begin{aligned} & \int_t^{t+1} \|\tilde{B}(u(s), v(s))\|^2_{D(A)'} ds \\ \leq \ & c^2\left\{\varphi(0)e^{-\nu\lambda_1 t} + \frac{|g|^2}{\lambda_1^2\nu^2}\right\}\int_t^{t+1}\left(\|u(s)\|^2 + \alpha^2|Au(s)|^2\right) ds \end{aligned}$$

(The function in braces was just majorized on $[t, t+1]$ by its values at t.) We now use (2.22) and obtain

$$\begin{aligned} & \int_t^{t+1} \|\tilde{B}(u(s), v(s))\|^2_{D(A)'} ds \\ \leq \ & c^2\left\{\varphi(0)e^{-\nu\lambda_1 t} + \frac{|g|^2}{\lambda_1^2\nu^2}\right\}\frac{1}{\nu}\left\{\varphi(0)e^{-\nu\lambda_1 t} + \frac{|g|^2}{\lambda_1^2\nu^2} + \frac{|g|^2}{\lambda_1\nu}\right\} \end{aligned}$$

Hence,

$$\left(\int_t^{t+1} \|\tilde{B}(u(s),v(s))\|^2_{D(A)'}ds\right)^{1/2} \leq C_1\varphi(0)e^{-\nu\lambda_1 t} + R_1^2, \tag{3.4}$$

where $C_1 = c\nu^{-1/2}$ and $R_1^2 = \frac{|g|^2}{\lambda_1^2\nu^2} + \frac{|g|^2}{\lambda_1\nu}$.

¿From the estimate

$$\|Av\|_{D(A)'} = \|v\|_H = |v| \leq |u| + \alpha^2|Au|$$

we conclude that

$$\begin{aligned}
\int_t^{t+1} \|Av(s)\|^2_{D(A)'}ds &\leq 2\left(\int_t^{t+1} |u(s)|^2 ds + \alpha^2 \int_t^{t+1} \alpha^2|Au(s)|^2 ds\right) \\
&\leq 2\int_t^{t+1} \left\{|u(s)|^2 + \alpha^2|Au(s)|^2\right\} ds
\end{aligned}$$

(Recall that $\alpha \leq 1$.) Using once more inequality (2.22), we obtain that

$$\int_t^{t+1} \|Av(s)\|^2_{D(A)'}ds \leq C_2\varphi(0)e^{-\nu\lambda_1 t} + R_2^2 \tag{3.5}$$

for an appropriate C_2 and R_2 independent of α.

The functions u and v satisfy equation (2.5), i.e.,

$$\partial_t v = -\nu Av - \tilde{B}(u,v) + g \tag{3.6}$$

We apply to (3.6) the triangle inequality taking into account inequalities (3.4) and (3.5):

$$\begin{aligned}
&\left(\int_t^{t+1} \|\partial_t v(s)\|^2_{D(A)'}ds\right)^{1/2} \leq \nu\left(\int_t^{t+1} \|Av(s)\|^2_{D(A)'}ds\right)^{1/2} \\
&\quad + \left(\int_t^{t+1} \|\tilde{B}(u(s),v(s))\|^2_{D(A)'}ds\right)^{1/2} + \|g\|_{D(A)'} \\
&\leq \nu\left(C_2\varphi(0)e^{-\nu\lambda_1 t} + R_2^2\right)^{1/2} + C_1\varphi(0)e^{-\nu\lambda_1 t} + R_1^2 + \lambda_1^{-1}|g| \\
&\leq \nu\left(C_2\varphi(0)e^{-\nu\lambda_1 t} + R_2^2 + 1\right) + C_1\varphi(0)e^{-\nu\lambda_1 t} + R_1^2 + \lambda_1^{-1}|g| \\
&\leq C\varphi(0)e^{-\nu\lambda_1 t} + R^2 = C\cdot\left(|u(0)|^2 + \alpha^2\|u(0)\|^2\right)e^{-\nu\lambda_1 t} + R^2,
\end{aligned}$$

where $C = \nu C_2 + C_1$ and $R = \nu(R_2^2 + 1) + R_1^2 + \lambda_1^{-1}|g|$. The proof is completed. ∎

The following inequality holds:

$$\|f\|^2_{D(A)'} \leq \|f + \alpha^2 Af\|^2_{D(A)'},\ \forall f \in D(A)'.$$

Indeed, the operator A is self-adjoint and positive. Therefore,

$$\begin{aligned}
\|f\|^2_{D(A)'} &= \sum_{j=1}^{\infty} |f_j|^2\lambda_j^{-1} \leq \sum_{j=1}^{\infty} |f_j|^2(1+\alpha^2\lambda_j)\lambda_j^{-1} \\
&= \|(1+\alpha^2 A)f\|^2_{D(A)'} = \|f + \alpha^2 Af\|^2_{D(A)'},
\end{aligned}$$

where $f = \sum_{j=1}^{\infty} f_j e_j$, $Ae_j = \lambda_j e_j$, $j = 1, 2, \ldots$, $\{e_j\}$ are the eigenvectors of the operator A and $\{\lambda_j\}$ are the corresponding eigenvalues.

Thus we conclude that

$$\int_t^{t+1} \|\partial_t u(s)\|^2_{D(A)'} ds \leq \int_t^{t+1} \|\partial_t v(s)\|^2_{D(A)'} ds, \ t \geq 0, \tag{3.7}$$

where $v = u + \alpha^2 Au$.

Corollary 3.1 *The inequality (3.1) also holds for the function* $\partial_t u$:

$$\left(\int_t^{t+1} \|\partial_t u(s)\|^2_{D(A)'} ds \right)^{1/2} \leq C \cdot \left(|u(0)|^2 + \alpha^2 \|u(0)\|^2 \right) e^{-\nu\lambda_1 t} + R^2, \tag{3.8}$$

with the same constants C *and* R.

To construct the trajectory attractor for system (2.5) and (2.6) we have to pass to a new function variable w that occupies an intermediate position between the function u and v. We consider the function

$$w = (1 + \alpha^2 A)^{1/2} u. \tag{3.9}$$

Then we clearly have

$$v = (1 + \alpha^2 A)u = (1 + \alpha^2 A)^{1/2} w. \tag{3.10}$$

It is easy to verify the following identities:

$$\begin{aligned} |w|^2 &= |u|^2 + \alpha^2 \|u\|^2, && (3.11) \\ \|w\|^2 &= \|(1 + \alpha^2 A)^{1/2} u\|^2 = |A^{1/2}(1 + \alpha^2 A)^{1/2} u|^2 \\ &= (A(1 + \alpha^2 A)u, u) = ((1 + \alpha^2 A)u, Au) \\ &= \|u\|^2 + \alpha^2 |Au|^2. && (3.12) \end{aligned}$$

We note that the function $w = w(x, t)$ satisfies the following equation:

$$\begin{aligned} \partial_t w + \nu Aw + (1 + \alpha^2 A)^{-1/2} \tilde{B}((1 + \alpha^2 A)^{-1/2} w, (1 + \alpha^2 A)^{1/2} w) \\ = (1 + \alpha^2 A)^{-1/2} g \end{aligned} \tag{3.13}$$

that is a consequence of (2.5), (3.9), and (3.10).

Using the function w, we rewrite inequalities (2.21), (2.22), and (3.1).

Corollary 3.2 *The following inequalities hold:*

$$|w(t)|^2 \leq |w(0)|^2 e^{-\nu\lambda_1 t} + \frac{|g|^2}{\lambda_1^2 \nu^2}, \tag{3.14}$$

$$\nu \int_t^{t+1} \|w(s)\|^2 ds \leq |w(0)|^2 e^{-\nu\lambda_1 t} + \frac{|g|^2}{\lambda_1^2 \nu^2} + \frac{|g|^2}{\lambda_1 \nu}, \tag{3.15}$$

$$\left(\int_t^{t+1} \|\partial_t w(s)\|^2_{D(A)'} ds \right)^{1/2} \leq C \cdot |w(0)|^2 e^{-\nu\lambda_1 t} + R^2, \ \forall t > 0. \tag{3.16}$$

We note that (3.16) follows from (3.1) if one takes into account similarly to (3.7) that

$$\int_t^{t+1} \|\partial_t w(s)\|^2_{D(A)'} ds \leq \int_t^{t+1} \|\partial_t v(s)\|^2_{D(A)'} ds, \ t \geq 0. \tag{3.17}$$

We now consider the Banach space $\mathcal{F}_+^{\rm b}$ defined in Sect. 1. Recall that

$$\mathcal{F}_+^{\rm b} = \{z \mid z(\cdot) \in L_2^{\rm b}(\mathbb{R}_+; V) \cap L_\infty(\mathbb{R}_+; H),\ \partial_t z(\cdot) \in L_2^{\rm b}(\mathbb{R}_+; D(A)')\}$$

Inequalities (3.14) – (3.16) provide the following

Proposition 3.2 *If $g \in H$, then, for any solution $u(t)$ of problem (2.5), (2.6), and (2.17), the corresponding function $w(t) = (1 + \alpha^2 A)^{1/2} u(t)$ being a solution of (3.13) satisfies the inequality*

$$\|T(h)w(\cdot)\|_{\mathcal{F}_+^{\rm b}} \le C_3 |w(0)|^2 e^{-\nu\lambda_1 h} + R_3^2,\ \forall h \ge 0, \tag{3.18}$$

where the constant C_3 depends on ν, λ_1 and R_3 depends on ν, λ_1, $|g|$. (We stress that C_3 and R_3 are independent of α.)

We consider the trajectory space $\mathcal{K}_\alpha^+$ of system (2.5) and (2.6). By definition, the space $\mathcal{K}_\alpha^+$ is the union of all functions $w(t) = (1 + \alpha^2 A)^{1/2} u(t)$, where $u(t)$ is a solution of (2.5), (2.6), and (2.17) with an arbitrary $u_0 \in V$. Proposition 3.2 implies that $\mathcal{K}_\alpha^+ \subset \mathcal{F}_+^{\rm b}$ for $\alpha > 0$.

We rewrite the energy equality (2.20) in the integral form we need in the sequel.

Proposition 3.3 *For every $w \in \mathcal{K}_\alpha^+$,*

$$-\frac{1}{2}\int_0^\infty |w(t)|^2 \psi'(t) dt + \nu \int_0^\infty \|w(t)\|^2 \psi(t) dt = \int_0^\infty \langle g, u(t)\rangle\, \psi(t) dt \tag{3.19}$$

for all $\psi \in C_0^\infty(\mathbb{R}_+)$.

To prove (3.19) we rewrite the identity (2.20) in the form

$$\frac{1}{2}\frac{d}{dt}|w(t)|^2\ + \nu \|w(t)\|^2 = \langle g, u(t)\rangle\,,\ t \ge 0,$$

multiply by an arbitrary test function $\psi \in C_0^\infty(\mathbb{R}_+)$, and integrate in t from 0 to $+\infty$. Then, integrating by part in the first integral term (that is legitimate since the function $|w(t)|^2$ is absolutely continuous), we obtain the needed result (3.19).

We also consider the topological space $\Theta_+^{\rm loc}$ introduced in Sect. 1 in connection with the initial Navier–Stokes system. Recall that $\mathcal{F}_+^{\rm b} \subset \Theta_+^{\rm loc}$.

Lemma 3.1 *Let two sequences $\{u_n(t)\} \subset \mathcal{F}_+^{\rm b}$ and $\{\alpha_n\} \subset]0, 1]$ be given such that $\alpha_n \to 0+$ as $n \to \infty$. We denote $w_n = (1 + \alpha_n A)^{1/2} u_n$ for $n \in \mathbb{N}$. We assume that the sequence $\{w_n(t)\}$ is bounded in $\mathcal{F}_+^{\rm b}$ and $w_n(t) \to w(t)$ in $\Theta_+^{\rm loc}$ as $n \to \infty$. Then the sequence $\{u_n(t)\}$ is bounded in $\mathcal{F}_+^{\rm b}$ and $u_n(t) \to w(t)$ in $\Theta_+^{\rm loc}$ as $n \to \infty$.*

Proof. The first assertion follows from the apparent inequalities

$$\begin{aligned} |u_n|^2 &\le |u_n|^2 + \alpha^2 \|u_n\|^2 = |w_n|^2, & (3.20)\\ \|u_n\|^2 &\le \|u_n\|^2 + \alpha^2 |Au_n|^2 = \|w_n\|^2 & (3.21) \end{aligned}$$

(see (3.11 and (3.12)). Besides similar to (3.17), we prove that

$$\int_t^{t+1} \|\partial_t u_n(s)\|^2_{D(A)'} ds \le \int_t^{t+1} \|\partial_t w_n(s)\|^2_{D(A)'} ds. \tag{3.22}$$

Consequently,

$$\|u_n\|_{\mathcal{F}^{\mathrm{b}}_+} \le \|w_n\|_{\mathcal{F}^{\mathrm{b}}_+}, \ \forall n \in \mathbb{N}. \tag{3.23}$$

¿From (3.23), we conclude that $\{u_n(t)\}$ is bounded in $\mathcal{F}^{\mathrm{b}}_+$. Since a ball in $\mathcal{F}^{\mathrm{b}}_+$ is a weakly compact set in Θ^{loc}_+, we can extract from $\{u_n(t)\}$ a convergent subsequence and we denote the limit of this subsequence by $u(t)$. For simplicity, we denote this subsequence by $\{u_n(t)\}$. We also keep the corresponding subsequence of $\{w_n(t)\}$. Then we have

$$u_n(t) \to u(t), \ w_n(t) \to w(t) \text{ in } \Theta^{\mathrm{loc}}_+ \text{ as } n \to \infty.$$

We state that $u \equiv w$. Consider an arbitrary interval $[0, M]$. By our assumption, $w_n(t) \to w(t)$ $(n \to \infty)$ weakly in $L_2(0, M; V)$ and $\partial_t w_n(t) \to \partial_t w(t)$ $(n \to \infty)$ weakly in $L_2(0, M; D(A)')$. Then, by the Aubin theorem (see [34, 21, 35]), we obtain that $w_n(t) \to w(t)$ $(n \to \infty)$ strongly in $L_2(0, M; H)$. Arguing similarly, we have that $u_n(t) \to u(t)$ $(n \to \infty)$ strongly in $L_2(0, M; H)$.

We note that $\|(1 + \alpha_n A)^{-1/2}\|_{\mathcal{L}(H,H)} < 1$ and therefore

$$\begin{aligned} & \|(1 + \alpha_n A)^{-1/2} w_n - (1 + \alpha_n A)^{-1/2} w\|_{L_2(0,M;H)} \\ \le \ & \|w_n - w\|_{L_2(0,M;H)} \to 0 \ (n \to \infty). \end{aligned} \tag{3.24}$$

It follows from Lemma 3.2 (see below) that

$$\|(1 + \alpha_n A)^{-1/2} w - w\|_{L_2(0,M;H)} \to 0 \ (n \to \infty). \tag{3.25}$$

Combining (3.24) and (3.25), we observe that

$$\begin{aligned} \|u_n - w\|_{L_2(0,M;H)} &= \|(1 + \alpha_n A)^{-1/2} w_n - w\|_{L_2} \\ &\le \|(1 + \alpha_n A)^{-1/2} w_n - (1 + \alpha_n A)^{-1/2} w\|_{L_2} \\ &+ \|(1 + \alpha_n A)^{-1/2} w - w\|_{L_2} \to 0 \ (n \to \infty), \end{aligned}$$

that is,

$$u_n(t) \to w(t) \text{ strongly in } L_2(0, M; H),$$

consequently, $u(t) \equiv w(t)$ and Lemma 3.1 is completely proved. ■

Lemma 3.2 *Let $f(t) \in L_2(0, M; H)$ and let $\alpha_n \to 0+$ $(n \to \infty)$. Then*

$$(1 + \alpha_n A)^{-1/2} f(t) \to f(t) \textit{ strongly in } L_2(0, M; H).$$

The proof is given in Appendix.

We now formulate and prove the main theorem of this section.

Theorem 3.1 *Let a sequence $\{w_n\} \subset \mathcal{K}^+_{\alpha_n}$ be given such that $\{w_n\}$ is bounded in $\mathcal{F}^{\mathrm{b}}_+$, $\alpha_n \to 0+$ $(n \to \infty)$, and $w_n(t) \to w(t)$ in Θ^{loc}_+ as $n \to \infty$. Then $w(t)$ is a weak solution of the 3D Navier–Stokes system such that w satisfies the energy inequality (1.12), i.e., $w \in \mathcal{K}^+$, where $\mathcal{K}^+$ is the trajectory space of the 3D Navier–Stokes system.*

Proof. By the assumption, we have

$$\|w_n\|_{\mathcal{F}_+^{\mathrm{b}}} \leq C, \ \forall n \in \mathbb{N} \tag{3.26}$$

and since $w_n(t) \to w(t)$ in Θ_+^{loc} as $n \to \infty$ we conclude that

$$\|w\|_{\mathcal{F}_+^{\mathrm{b}}} \leq C. \tag{3.27}$$

We set $u_n = (1 + \alpha_n^2 A)^{-1/2} w_n$. It is clear that u_n is a solution of the original system (2.5) and (2.6). Inequality (3.26) implies that

$$\operatorname*{ess\,sup}_{t\geq 0} \left\{|u_n(t)|^2 + \alpha_n^2 \|u_n(t)\|^2\right\} \ \leq \ C, \tag{3.28}$$

$$\sup_{t\geq 0} \int_t^{t+1} \left\{\|u_n(s)\|^2 + \alpha_n^2 |Au_n(s)|^2\right\} ds \ \leq \ C, \tag{3.29}$$

$$\sup_{t\geq 0} \int_t^{t+1} \|\partial_t u_n(s)\|_{D(A)'}^2 ds \leq \sup_{t\geq 0} \int_t^{t+1} \|\partial_t w_n(s)\|_{D(A)'}^2 ds \ \leq \ C. \tag{3.30}$$

We now prove that $w(t)$ is a weak solution of the 3D Navier–Stokes system on any interval $(0, M)$.

The function $w_n(t)$ satisfies the equation

$$\partial_t w_n + \nu A w_n + (1 + \alpha_n^2 A)^{-1/2} \tilde{B}(u_n, v_n) = (1 + \alpha_n^2 A)^{-1/2} g \tag{3.31}$$

in the space $\mathcal{D}'(0, M; D(A)')$. Here $v_n = u_n + \alpha_n^2 A u_n$.

By the assumption of the theorem,

$$w_n(t) \to w(t) \ (n \to \infty) \tag{3.32}$$

weakly in $L_2(0, M; V)$, weak-$*$ in $L_\infty(0, M; H)$, and

$$\partial_t w_n(t) \to \partial_t w(t) \ (n \to \infty) \tag{3.33}$$

weakly in $L_2(0, M; D(A)')$. Then, these convergencies take place in the space of distributions $\mathcal{D}'(0, M; D(A)')$. Moreover, it follows from (3.32) that

$$A w_n(t) \to A w(t) \ (n \to \infty) \tag{3.34}$$

weakly in $L_2(0, M; V')$ and, hence, in the topology of $\mathcal{D}'(0, M; D(A)')$ as well. Applying Lemma 3.2 in a particular case, where the function $f(t) \equiv g$ is time independent, we find that

$$(1 + \alpha_n^2 A)^{-1/2} g \to g \ (n \to \infty) \tag{3.35}$$

strongly in $L_2(0, M; H)$ and, clearly, in $\mathcal{D}'(0, M; D(A)')$ as well.

Thus having (3.33) – (3.35), to prove that w satisfy the equation

$$\partial_t w + \nu A w + B(w, w) = g \tag{3.36}$$

we must establish that

$$(1 + \alpha_n^2 A)^{-1/2} \tilde{B}(u_n, v_n) \to B(w, w) \text{ as } n \to \infty \tag{3.37}$$

in the space $\mathcal{D}'(0, M; D(A)')$.

Firstly, we prove that

$$\tilde{B}(u_n, v_n) \to B(w, w)\ (n \to \infty) \tag{3.38}$$

weakly in the space $L_q(0, M; D(A)')$ for some q, $1 < q < 2$.

It follows from Lemma 3.1 that

$$u_n(t) \to w(t)\ (n \to \infty) \text{ in } \Theta_+^{\mathrm{loc}}. \tag{3.39}$$

We note that

$$\begin{aligned} \tilde{B}(u_n, v_n) &= \tilde{B}(u_n, u_n + \alpha_n^2 A u_n) \\ &= \tilde{B}(u_n, u_n) + \alpha_n^2 \tilde{B}(u_n, A u_n) \\ &= B(u_n, u_n) + \alpha_n^2 \tilde{B}(u_n, A u_n) \end{aligned} \tag{3.40}$$

(Here, we have used the identity (2.8).) Consider both terms of (3.40) separately. We start with the second. By (2.16), we have

$$\|\alpha_n^2 \tilde{B}(u_n, A u_n)\|_{D(A)'} \le c\alpha_n^2 \|u_n\| \cdot |A u_n|. \tag{3.41}$$

Fixing an arbitrary β, $1 < \beta < 2$, we obtain the following chain of inequalities

$$\begin{aligned} &\int_0^M \|\alpha_n^2 \tilde{B}(u_n(t), A u_n(t))\|_{D(A)'}^\beta dt \le c^\beta \alpha_n^{2\beta} \int_0^M \|u_n(t)\|^\beta \cdot |A u_n(t)|^\beta dt \\ \le\ & c^\beta \alpha_n^{2\beta} \left(\sup_{t \in [0,M]} \|u_n(t)\|^\gamma \right) \int_0^M \|u_n(t)\|^{\beta-\gamma} \cdot |A u_n(t)|^\beta dt \\ \le\ & c^\beta \alpha_n^{2\beta} \left(\sup_{t \in [0,M]} \|u_n\|^\gamma \right) \left[\int_0^M \|u_n\|^{q(\beta-\gamma)} dt \right]^{\frac{1}{q}} \left[\int_0^M |A u_n|^{p\beta} dt \right]^{\frac{1}{p}}, \end{aligned} \tag{3.42}$$

where γ is an arbitrary number such that $0 < \gamma < \beta$, and, in (3.42), we have applied the Hölder inequality with $1/p + 1/q = 1$ (these numbers will be determined later on). Continuing the chain of inequalities after (3.42), we have

$$\begin{aligned} &\int_0^M \|\alpha_n^2 \tilde{B}(u_n, A u_n)\|_{D(A)'}^\beta dt \\ \le\ & c^\beta \alpha_n^{2\beta} \left(\sup_{t \in [0,M]} \|u_n\|^2 \right)^{\frac{\gamma}{2}} \left[\int_0^M \|u_n\|^{q(\beta-\gamma)} dt \right]^{\frac{1}{q}} \left[\int_0^M |A u_n|^{p\beta} dt \right]^{\frac{1}{p}}. \end{aligned} \tag{3.43}$$

We now set $p = 2/\beta$, $q = 2/(2-\beta)$, and find the number γ from the equation $q(\beta-\gamma) = 2$, that is,

$$\frac{2}{2-\beta}(\beta - \gamma) = 2 \iff \gamma = 2(\beta - 1).$$

We see that such γ satisfies the inequality $0 < \gamma < \beta$, since

$$\gamma = 2(\beta - 1) < \beta \iff \beta < 2.$$

Replacing such $p,\ q$, and γ into (3.43), we obtain the following estimate:

$$\int_0^M \|\alpha_n^2 \tilde{B}(u_n, Au_n)\|^\beta_{D(A)'} dt$$
$$\leq \ c^\beta \alpha_n^{2-\beta} \left(\sup_{t\in[0,M]} \alpha_n^2 \|u_n\|^2 \right)^{\beta-1} \left[\int_0^M \|u_n\|^2 dt \right]^{\frac{2-\beta}{2}} \left[\int_0^M \alpha_n^2 |Au_n|^2 dt \right]^{\frac{\beta}{2}}. \quad (3.44)$$

We now use estimates (3.28) and (3.29) and find that the right-hand side of (3.44) is less or equal than $C_1\alpha_n^{2-\beta}$:

$$\int_0^M \|\alpha_n^2 \tilde{B}(u_n, Au_n)\|^\beta_{D(A)'} dt \leq C_1 \alpha_n^{2-\beta},\ 1 < \beta < 2. \quad (3.45)$$

Therefore, the term

$$\alpha_n^2 \tilde{B}(u_n, Au_n) \to 0 \ (n \to \infty) \quad (3.46)$$

strongly in $L_\beta(0, M; D(A)')$ for any $\beta,\ 1 < \beta < 2$.

We now study the behavior of the term $B(u_n, u_n)$ from (3.40). It follows from (3.39) that

$$u_n(t) \to w(t) \ (n \to \infty)$$

weakly in $L_2(0, M; V)$ and $\{u_n(t)\}$ is bounded in this space. Besides,

$$\partial_t u_n(t) \to \partial_t w(t) \ (n \to \infty)$$

weakly in $L_2(0, M; D(A)')$ and, thereby, $\{\partial_t u_n(t)\}$ is bounded in this space. Thus, applying the Aubin compactness theorem (see [34, 21, 35]), we obtain that

$$u_n(t) \to w(t) \ (n \to \infty) \quad (3.47)$$

strongly in $L_2(0, M; H)$. Recall that $L_2(0, M; H) \subset L_2(\mathbb{T}^3 \times [0, M])^3$ and therefore we may assume that

$$u_n(x, t) \to w(x, t) \ (n \to \infty) \text{ for a.e. } (x, t) \in \mathbb{T}^3 \times [0, M]. \quad (3.48)$$

The identity (1.5) implies that

$$B(u_n, u_n) = P \sum_{j=1}^3 \partial_{x_j} \left(u_n^j u_n \right). \quad (3.49)$$

It follows from (3.48) that

$$u_n^j(x, t) u_n(x, t) \to w^j(x, t) w(x, t) \ (n \to \infty) \text{ for a.e. } (x, t) \in \mathbb{T}^3 \times [0, M]. \quad (3.50)$$

Recall that $\{u_n\}$ is bounded in $L_2(0, M; V)$ and in $L_\infty(0, M; H)$. Hence, the well-known inequality

$$\|B(u, u)\|_{V'} \leq c|u|^{1/2} \|u\|^{3/2},\ \forall u \in V,$$

implies that

$$\{u_n^j u_n\} \text{ is bounded in } L_{4/3}(0, M; H) \quad (3.51)$$

and in $L_{4/3}(\mathbb{T}^3 \times [0, M])^3$. Applying the known lemma on weak convergence from [21], we conclude from (3.50) and (3.51) that

$$u_n^j(t)u_n(t) \to w^j(t)w(t) \ (n \to \infty)$$

weakly in $L_{4/3}(\mathbb{T}^3 \times [0, M])^3$ and weakly in $L_{4/3}(0, M; H)$. Then, due to (3.49),

$$B(u_n(t), u_n(t)) \to B(w(t), w(t)) \ (n \to \infty) \tag{3.52}$$

weakly in $L_{4/3}(0, M; V')$.

Combining (3.46) and (3.52), we find that

$$\tilde{B}(u_n, v_n) \to B(w, w) \ (n \to \infty)$$

weakly in $L_{4/3}(0, M; D(A)')$.

We now state that

$$(1 + \alpha_n^2 A)^{-1/2} \tilde{B}(u_n, v_n) \to B(w, w) \ (n \to \infty) \tag{3.53}$$

weakly in $L_{4/3}(0, M; D(A)')$. Here, we need the following lemma that is analogous to Lemma 3.1.

Lemma 3.3 *Let $f_n(t) \in L_q(0, M; D(A)')$ and $f_n \rightharpoonup f$ $(n \to \infty)$ weakly in the space $L_q(0, M; D(A)')$, $q > 1$. Let also $\alpha_n \to 0+$ $(n \to \infty)$. Then*

$$(1 + \alpha_n A)^{-1/2} f_n(t) \to f(t) \textit{ weakly in } L_q(0, M; D(A)').$$

Proof. By the assumption, for all $\varphi \in L_p(0, M; D(A))$ $(1/p + 1/q = 1)$

$$\int_0^M \langle f_n(t), \varphi(t) \rangle \, dt \to \int_0^M \langle f(t), \varphi(t) \rangle \, dt \ (n \to \infty). \tag{3.54}$$

We have

$$\int_0^M \left\langle (1 + \alpha_n A)^{-1/2} f_n(t), \varphi(t) \right\rangle dt = \int_0^M \left\langle f_n, (1 + \alpha_n A)^{-1/2} \varphi \right\rangle dt$$
$$= \int_0^M \left\langle f_n, (1 + \alpha_n A)^{-1/2} \varphi - \varphi \right\rangle dt + \int_0^M \langle f_n, \varphi \rangle \, dt. \tag{3.55}$$

By Lemma 3.4 (see below), $(1 + \alpha_n A)^{-1/2} \varphi \to \varphi$ $(n \to \infty)$ strongly in $L_p(0, M; D(A))$. Then

$$\begin{aligned} \left| \int_0^M \left\langle f_n, (1 + \alpha_n A)^{-1/2} \varphi - \varphi \right\rangle dt \right| &\leq \|f_n\|_{L_q(0,M;D(A)')} \times \|(1 + \alpha_n A)^{-1/2} \varphi - \varphi\|_{L_p(0,M;D(A))} \\ &\leq C \|(1 + \alpha_n A)^{-1/2} \varphi - \varphi\|_{L_p} \to 0 \ (n \to \infty) \end{aligned}$$

and the right-hand side of (3.55) tends to $\int_0^M \langle f(t), \varphi(t) \rangle \, dt$ as $n \to \infty$ (see (3.54)). Lemma 3.3 is proved. ■

Lemma 3.4 *Let $\varphi(t) \in L_p(0, M; D(A))$ and $\alpha_n \to 0+$ $(n \to \infty)$. Then*

$$(1 + \alpha_n A)^{-1/2} \varphi(t) \to \varphi(t) \textit{ strongly in } L_p(0, M; D(A)).$$

The proof is given in Appendix.

We now continue the proof of Theorem 3.1. To this moment, we have established relation (3.53) implying that the function $w(t)$ satisfies equation (3.36). It is left to prove that $w(t)$ satisfies the energy inequality (1.11) on every interval $(0, M)$. Indeed the functions $w_n(t)$ satisfies the energy equality (see (3.19))

$$-\frac{1}{2}\int_0^M |w_n(t)|^2\psi'(t)dt + \nu\int_0^M \|w_n(t)\|^2\psi(t)dt = \int_0^M \langle g, u_n(t)\rangle\,\psi(t)dt \tag{3.56}$$

for all $\psi \in C_0^\infty(0, M)$. Let now $\psi \geq 0$ for $t \in]0, M[$. We have already proved that $u_n(t) \to w(t)$ $(n \to \infty)$ strongly in $L_2(0, M; H)$ (see (3.47)). Similarly we prove that

$$w_n(t) \to w(t)\ (n \to \infty) \text{ strongly in } L_2(0, M; H). \tag{3.57}$$

Then the real functions $|w_n(t)|$ converge to $|w(t)|$ as $n \to \infty$ strongly in $L_2(0, M)$. In particular, passing to a subsequence, we may assume that

$$|w_n(t)|^2 \to |w(t)|^2\ (n \to \infty) \text{ for a.e. } t \in [0, M]. \tag{3.58}$$

Consider a sequence of functions $\{|w_n(t)|^2\psi'(t)\}$ in the space $L_1(0, M)$. It follows from the assumption of Theorem 3.1 that this sequence is essentially bounded and, hence, it has an integrable majorant. Then, by the Lebesque dominant convergence theorem, we obtain from (3.58) that

$$\int_0^M |w_n(t)|^2\psi'(t)dt \to \int_0^M |w(t)|^2\psi'(t)dt\ (n \to \infty). \tag{3.59}$$

We note that $w_n(t)\sqrt{\psi(t)} \to w(t)\sqrt{\psi(t)}$ $(n \to \infty)$ weakly in $L_2(0, M; V)$ (the assumption of Theorem 3.1). Consequently,

$$\int_0^M \|w(t)\|^2\psi(t)dt \leq \liminf_{n\to\infty}\int_0^M \|w_n(t)\|^2\psi(t)dt. \tag{3.60}$$

We have already notice that $u_n(t) \to w(t)$ $(n \to \infty)$ strongly in $L_2(0, M; H)$. Therefore,

$$\int_0^M \langle g, u_n(t)\rangle\,\psi(t)dt \to \int_0^M \langle g, w(t)\rangle\,\psi(t)dt\ (n \to \infty). \tag{3.61}$$

Using (3.59) – (3.61) and passing to the limit in (3.56), we obtain that

$$-\frac{1}{2}\int_0^M |w(t)|^2\psi'(t)dt + \nu\int_0^M \|w(t)\|^2\psi(t)dt \leq \int_0^M \langle g, w(t)\rangle\,\psi(t)dt \tag{3.62}$$

for all $\psi \in C_0^\infty(0, M)$, $\psi \geq 0$.

Thus, we have proved that the limit function $w(t)$ in Theorem 3.1 is a weak solution of the 3D Navier–Stokes system and satisfies the energy inequality, that is, $w \in \mathcal{K}^+$. ■

We use Theorem 3.1 in the next section, where we study the convergence of the trajectory attractors of the Navier–Stokes-α model to the trajectory attractor of the 3D N.–S. system.

4 Convergence of trajectories of the N.–S.-α model to the trajectory attractor of the 3D N.–S. system

We denote by $\mathfrak{A}_0$ the trajectory attractor of the 3D Navier–Stokes system

$$\partial_t v + \nu A v + B(v,v) = g(x),\ t \geq 0 \tag{4.1}$$

($\mathfrak{A}_0 \equiv \mathfrak{A}$, see Sect. 1). Recall that the set $\mathfrak{A}_0$ is bounded in $\mathcal{F}_+^{\mathrm{b}}$, compact in Θ_+^{loc}, and $\mathfrak{A}_0 \subset \mathcal{K}^+$.

We denote by $B_\alpha = \{w_\alpha(x,t),\ t \geq 0\}, 0 < \alpha \leq 1$, a family of functions $w_\alpha(t) = (1+\alpha^2 A)^{1/2} u_\alpha(t)$, where $u_\alpha(t)$ is a solution of system (2.5), (2.6), and the norms of $w_\alpha(t)$ in $\mathcal{F}_+^{\mathrm{b}}$ are uniformly bounded

$$\|w_\alpha\|_{\mathcal{F}_+^{\mathrm{b}}} = \|w_\alpha\|_{L_2^{\mathrm{b}}(\mathbb{R}_+;V)} + \|w_\alpha\|_{L_\infty^{\mathrm{b}}(\mathbb{R}_+;H)} + \|\partial_t w_\alpha\|_{L_2^{\mathrm{b}}(\mathbb{R}_+;D(A)')} \leq R,\ \forall w_\alpha \in B_\alpha,$$

where R is an arbitrary number. Recall that every $w_\alpha(t)$ satisfies the equation

$$\partial_t w_\alpha + \nu A w_\alpha + (1+\alpha^2 A)^{-1/2} \tilde{B}(u_\alpha, v_\alpha) = (1+\alpha^2 A)^{-1/2} g, \tag{4.2}$$

where $v_\alpha = (1+\alpha^2 A)^{1/2} w_\alpha(t)$ and $u_\alpha = (1+\alpha^2 A)^{-1/2} w_\alpha(t)$

We also set

$$\hat{B}_\alpha = (1+\alpha^2 A)^{-1/2} B_\alpha = \left\{u_\alpha \in \mathcal{F}_+^{\mathrm{b}} \mid (1+\alpha^2 A)^{1/2} u_\alpha = w_\alpha \in B_\alpha\right\}.$$

where u_α satisfies equations (2.5) and (2.6).

We denote by $\mathcal{K}_0$ the kernel of equation (4.1). Recall that $\mathcal{K}_0$ is the union of all bounded (in the nom $\mathcal{F}^{\mathrm{b}}$) complete weak solutions $\{v(t),\ t \in \mathbb{R}\}$ of the Navier–Stokes system (4.1) that satisfy the energy inequality (1.18). We saw in Sect. 1 that $\mathfrak{A}_0 = \Pi_+ \mathcal{K}_0$.

We now formulate the main theorem of this section.

Theorem 4.1 *Let $B_\alpha = \{w_\alpha(x,t),\ t \geq 0\}, 0 < \alpha \leq 1$, be bounded sets of solutions of equation (4.2) that satisfy the inequality*

$$\|w_\alpha\|_{\mathcal{F}_+^{\mathrm{b}}} \leq R, \forall \alpha,\ 0 < \alpha \leq 1. \tag{4.3}$$

Then the sets of shifted solutions $\{T(h)B_\alpha\}$ (recall that $T(h)w(t) = w(t+h)$) converge to the trajectory attractor $\mathfrak{A}_0 = \Pi_+\mathcal{K}_0$ of the 3D N.–S. system (4.1) in the topology Θ_+^{loc} as $h \to +\infty$ and $\alpha \to 0+$:

$$T(h)B_\alpha \to \mathfrak{A}_0 \text{ in } \Theta_+^{\mathrm{loc}} \text{ as } h \to +\infty \text{ and } \alpha \to 0+\,. \tag{4.4}$$

Moreover, the same convergence holds for the corresponding sets $\hat{B}_\alpha = (1+\alpha^2 A)^{-1} B_\alpha$:

$$T(h)\hat{B}_\alpha \to \mathfrak{A}_0 \text{ in } \Theta_+^{\mathrm{loc}} \text{ as } h \to +\infty \text{ and } \alpha \to 0+\,. \tag{4.5}$$

Proof. Assume that relation (4.4) does not hold, i.e., there exist a neighborhood $\mathcal{O}(\mathfrak{A}_0)$ in Θ_+^{loc} and two sequences $\alpha_n \to 0+, h_n \to +\infty$ $(n \to \infty)$ such that

$$T(h_n)B_{\alpha_n} \not\subset \mathcal{O}(\mathfrak{A}_0). \tag{4.6}$$

So, there are solutions $w_{\alpha_n}(\cdot) \in B_{\alpha_n}$ such that the functions $W_{\alpha_n}(t) = T(h_n)w_{\alpha_n}(t) = w_{\alpha_n}(t+h_n)$ do not belong to $\mathcal{O}(\mathfrak{A}_0)$:

$$W_{\alpha_n}(\cdot) \notin \mathcal{O}(\mathfrak{A}_0). \tag{4.7}$$

Notice that the function $W_{\alpha_n}(t)$ is a solution of equation (4.2) on the interval $[-h_n, +\infty)$ with $\alpha = \alpha_n$, since $W_{\alpha_n}(t)$ is a backward time shift of $w_{\alpha_n}(t)$ on h_n. Recall that the equation (4.2) is autonomous. Moreover, it follows from (4.3) that

$$\sup_{t \geq -h_n} |W_{\alpha_n}(t)| + \left(\sup_{t \geq -h_n} \int_t^{t+1} \|W_{\alpha_n}(s)\|^2 ds \right)^{1/2} + \sup_{t \geq -h_n} \left(\int_t^{t+1} \|\partial_t W_{\alpha_n}(s)\|^2_{D(A)'} ds \right)^{1/2} \leq R. \tag{4.8}$$

This inequality implies that the sequence $\{W_{\alpha_n}(\cdot)\}$ is weakly compact in the space $\Theta_{-M,M} = L_2(-M,M;V) \cap L_\infty(-M,M;H) \cap \{v \mid \partial_t v \in L_2(-M,M;D(A)')\}$ for every M, if we consider α_n with indices n such that $h_n \geq M$. Therefore, for every fixed $M > 0$, we can choose a subsequence $\{\alpha_{n'}\} \subset \{\alpha_n\}$ such that $\{W_{\alpha_{n'}}(\cdot)\}$ converges weakly in $\Theta_{-M,M}$. Then, using the standard Cantor diagonal procedure, we can construct a function $W(t), t \in \mathbb{R}$, and a subsequence $\{\alpha_{n''}\} \subset \{\alpha_n\}$ such that

$$W_{\alpha_{n''}} \to W \text{ weakly in } \Theta_{-M,M} \text{ as } n'' \to \infty \text{ for any } M > 0. \tag{4.9}$$

From (4.10), we obtain the inequality for the limit function $W(t), t \in \mathbb{R}$

$$\sup_{t \in \mathbb{R}} |W(t)| + \left(\sup_{t \in \mathbb{R}} \int_t^{t+1} \|W(s)\|^2 ds \right)^{1/2} + \sup_{t \in \mathbb{R}} \left(\int_t^{t+1} \|\partial_t W(s)\|^2_{D(A)'} ds \right)^{1/2} \leq R. \tag{4.10}$$

In particular, we have that $W \in \mathcal{F}^{\mathrm{b}} = L_2^{\mathrm{b}}(\mathbb{R};V) \cap L_\infty(\mathbb{R};H) \cap \{u \mid \partial_t u \in L_2^{\mathrm{b}}(\mathbb{R};D(A)')\}$.

We now apply Theorem 3.1, where we can assume that all the functions are defined on the semiaxis $[-M, +\infty)$ instead of $[0, +\infty)$ (equations are autonomous). Then, from (4.9) and (4.10), we conclude that $W(x,t)$ is a weak solution of the 3D N.–S. system for all $t \in \mathbb{R}$ and $W(x,t)$ satisfies the energy inequality, that is $W \in \mathcal{K}_0$, where $\mathcal{K}_0$ is the kernel of equation (4.1). But $\Pi_+\mathcal{K}_0 = \mathfrak{A}_0$ and we have $\Pi_+ W \in \mathfrak{A}_0$. At the same time, we have established that

$$\Pi_+ W_{\alpha_{n''}} \to \Pi_+ W \text{ in } \Theta_+^{\mathrm{loc}} \text{ as } n \to \infty \tag{4.11}$$

(see (4.9)). In particular for a large n''

$$\Pi_+ W_{\alpha_{n''}} \in \mathcal{O}(\Pi_+ W) \subseteq \mathcal{O}(\mathfrak{A}_0). \tag{4.12}$$

This contradicts (4.6). Therefore, (4.4) is true. To prove (4.5), we combine (4.4) and Lemma 3.1. The proof is completed. ■

We now use Theorem 3.1 in order to study the behaviour of trajectory attractors of the Navier–Stokes-α model as $\alpha \to 0+$.

As before, we consider the trajectory space $\mathcal{K}_\alpha^+$ for $\alpha > 0$, of the Navier–Stokes-α model (2.5) and (2.6) that was constructed in Sect. 3. Recall that $\mathcal{K}_\alpha^+$ consists of all the functions of the form $w_\alpha(t) = (1 + \alpha^2 A)^{1/2} u_\alpha(t), t \geq 0$, where $u_\alpha(t)$ is a solution of (2.5) and (2.6), or equivalently $w_\alpha(t)$ is a solution of (3.13). The space $\mathcal{K}_\alpha^+ \subset \mathcal{F}_+^{\mathrm{b}}$ for all $\alpha > 0$. We consider the topology Θ_+^{loc} on $\mathcal{K}_\alpha^+$. It is easy prove that the space $\mathcal{K}_\alpha^+$ is closed in Θ_+^{loc}. The translation semigroup $\{T(h)\}$ acts on $\mathcal{K}_\alpha^+$ by the formula $T(h)w_\alpha(t) = w_\alpha(t+h),\ h \geq 0$. From the definition of $\mathcal{K}_\alpha^+$, it follows that $T(h)\mathcal{K}_\alpha^+ \subseteq \mathcal{K}_\alpha^+$ for all $h \geq 0$. Finally, Proposition 3.2 implies (see (3.18)) that there exists an absorbing set of the semigroup $\{T(h)\}$ in $\mathcal{K}_\alpha^+$, bounded in $\mathcal{F}_+^{\mathrm{b}}$ and compact in Θ_+^{loc}. (We note, that this absorbing set does not depend on α, since the constants C_3 and R_3 in (3.18) are independent of α.) Then, similar to Sect. 1, we prove the existence of the trajectory attractor $\mathfrak{A}_\alpha \subset \mathcal{K}_\alpha^+$ such that $\mathfrak{A}_\alpha$ is bounded in $\mathcal{F}_+^{\mathrm{b}}$, compact in Θ_+^{loc},

$$\|\mathfrak{A}_\alpha\|_{\mathcal{F}_+^{\mathrm{b}}} \leq R', \forall \alpha,\ 0 < \alpha \leq 1 \tag{4.13}$$

for some $R' > 0$ (independent of α). Recall that $T(h)\mathfrak{A}_\alpha = \mathfrak{A}_\alpha$ for all $h \geq 0$; and $T(h)B_\alpha \to \mathfrak{A}_\alpha$ in Θ_+^{loc} as $h \to +\infty$ for any bounded set $B_\alpha \subset \mathcal{K}_\alpha^+$. Moreover, $\mathfrak{A}_\alpha = \Pi_+ \mathcal{K}_{\alpha,}$where $\mathcal{K}_\alpha$ is the kernel of equation (3.13).

Since the trajectory attractors $\mathfrak{A}_\alpha$ satisfy (4.13) Theorem 4.1 is applicable to these sets and we obtain

Corollary 4.1 *The following limit relations hold:*

$$\begin{aligned} \mathfrak{A}_\alpha &\to \mathfrak{A}_0 \textit{ in } \Theta_+^{\mathrm{loc}} \textit{ as } \alpha \to 0+, & (4.14)\\ (1+\alpha^2 A)^{-1/2}\mathfrak{A}_\alpha &\to \mathfrak{A}_0 \textit{ in } \Theta_+^{\mathrm{loc}} \textit{ as } \alpha \to 0+\,. & (4.15) \end{aligned}$$

Indeed, the family $\{\mathfrak{A}_\alpha,\ 0 < \alpha \leq 1\}$ is uniformly bounded with respect to $\alpha \in]0,1]$. Then, in (4.4), we set $B_\alpha = \mathfrak{A}_\alpha$ and obtain (4.14) because $T(h)\mathfrak{A}_\alpha = \mathfrak{A}_\alpha$ for all $h \geq 0$. The relation (4.15) is straightforward.

We notice that the following embeddings are continuous (see [14]):

$$\begin{aligned} \Theta_{-M,M} &\subseteq L_2(-M,M;H^{1-\delta}), & (4.16)\\ \Theta_{-M,M} &\subseteq C([-M,M];H^{-\delta}),\ \forall 0 < \delta \leq 1. & (4.17) \end{aligned}$$

We recall that the following quantity is called the Hausdorff (non-symmetric) semi-distance from a set X to a set Y in a Banach space E

$$\mathrm{dist}_E(X,Y) := \sup_{x\in X} \mathrm{dist}_E(x,Y) = \sup_{x\in X}\inf_{y\in Y} \|x-y\|_E. \tag{4.18}$$

¿From (4.14) and (4.15), we deduce

Corollary 4.2 *For any fixed $M > 0$, the following limit relations hold:*

$$\begin{aligned} \mathrm{dist}_{L_2(0,M;H^{1-\delta})}(\mathfrak{A}_\alpha,\mathfrak{A}_0) &\to 0+,\\ \mathrm{dist}_{C([0,M];H^{-\delta})}(\mathfrak{A}_\alpha,\mathfrak{A}_0) &\to 0+ \textit{ as } \alpha \to 0+\,. \end{aligned}$$

In conclusion of this section, we establish the relation between the trajectory attractor $\mathfrak{A}_\alpha$ and the global attractor $\mathcal{A}_\alpha$ of the Navier–Stokes-α model for a fixed $\alpha > 0$ (see [6] and Sect. 2).

Proposition 4.1 *The following relation holds:*

$$\mathfrak{A}_\alpha = \{w(t) = (1+\alpha^2 A)^{1/2}u(t) = (1+\alpha^2 A)^{1/2}S_\alpha(t)u_0, t \geq 0 \mid u_0 \in \mathcal{A}_\alpha\}, \qquad (4.19)$$

where $\{S_\alpha(t)\}$ is the semigroup corresponding to the α-model (2.5), (2.6) and acting in the space V.

To prove (4.19) we recall that the trajectory attractor $\mathfrak{A}_\alpha$ is described using the kernel $\mathcal{K}_\alpha$ of system (3.13), while the global attractor $\mathcal{A}_\alpha$ has the similar presentation in terms of the kernel of system (2.5) and (2.6). These kernels can be transformed to each other by mean of the operator $(1+\alpha^2 A)^{-1/2}$.

Finally, we formulate two more propositions that follow from the results of [6] on well-posedness of the N.–S.-α model.

Proposition 4.2 *For every $\alpha > 0$ the trajectory attractor $\mathfrak{A}_\alpha$ is connected in the topological space Θ_+^{loc}.*

Proposition 4.3 *The family of sets $\{\mathfrak{A}_\alpha, 0 < \alpha \leq 1\}$ is upper semicontinuous in Θ_+^{loc}, i.e., for every $\alpha, 0 < \alpha \leq 1$, and for any neighborhood $\mathcal{O}(\mathfrak{A}_\alpha)$ there is a $\delta = \delta(\alpha, \mathcal{O}) > 0$ such that*

$$\mathfrak{A}_{\alpha'} \subseteq \mathcal{O}(\mathfrak{A}_\alpha),\ \forall \alpha' > 0,\ |\alpha' - \alpha| < \delta. \qquad (4.20)$$

We omit the proofs of Propositions 4.2 and 4.3 because they use the standard reasoning known for well-posed problems (see, e.g., [8, 9]).

Remark 4.1 *It follows from inequality (2.24) that the trajectory attractor $\mathfrak{A}_\alpha$ is a set of more regular functions, i.e. it is bounded in the space $\mathcal{F}_+^{\mathrm{b,s}} = L_2^{\mathrm{b}}(\mathbb{R}_+; D(A)) \cap L_\infty(\mathbb{R}_+; V) \cap \{\partial_t w(\cdot) \in L_2^{\mathrm{b}}(\mathbb{R}_+; H)\}$ and, moreover, $\mathfrak{A}_\alpha$ attracts bounded set of trajectories from $\mathcal{K}_\alpha^+$ in the strong local topology of the space $\Theta_+^{\mathrm{loc,s}} = L_2^{\mathrm{loc}}(\mathbb{R}_+; D(A)) \cap L_\infty^{\mathrm{loc}}(\mathbb{R}_+; V) \cap \{\partial_t w(\cdot) \in L_2^{\mathrm{loc}}(\mathbb{R}_+; H)\}$. However, these properties are not uniform in α and they do not persist passing to the limit as $\alpha \to 0+$.*

5 Minimal limit of trajectory attractors $\mathfrak{A}_\alpha$ as $\alpha \to 0$

Let $\mathfrak{A}_\alpha$ be the trajectory attractor of the N.–S.-α model, $0 < \alpha \leq 1$. As it has been proved above $\mathfrak{A}_\alpha \subset B$, where B is the ball in $\mathcal{F}_+^{\mathrm{b}}$ (see (4.13)) with radius R' independent of α:

$$\|\mathfrak{A}_\alpha\|_{\mathcal{F}_+^{\mathrm{b}}} \leq \|B\|_{\mathcal{F}_+^{\mathrm{b}}} = R', \forall \alpha,\ 0 < \alpha \leq 1. \qquad (5.1)$$

It is clear that the trajectory attractor $\mathfrak{A}_0$ of the exact N.–S. system also belongs to B (see Sect. 1). Recall that the ball B is compact in the topology Θ_+^{loc}. It follows from the Uryson compactness theorem that the subspace $B \cap \Theta_+^{\mathrm{loc}}$ equipped with topology Θ_+^{loc} is metrizable (see [14] for more details). We denote the corresponding metric in $B \cap \Theta_+^{\mathrm{loc}}$ by $\rho(\,\cdot\,,\,\cdot\,)$. The metric space itself, we denote by B_ρ. This metric space is compact and complete. Using new notation, the result of the previous section can be written in the form

$$\mathrm{dist}_{B_\rho}(\mathfrak{A}_\alpha, \mathfrak{A}_0) \to 0+, \qquad (5.2)$$

where $\mathrm{dist}_{B_\rho}(\cdot\,,\,\cdot)$ denotes the Hausdorff distance from one set to another in B_ρ (see (4.18)). We note that, in fact, the limit relation (5.2) is stronger than the results of Corollary 4.2.

Recall that the set $\mathfrak{A}_0 \subset B_\rho$ is closed in B_ρ. Let $\mathfrak{A}_{\min}$ be the minimal closed subset of $\mathfrak{A}_0$ that satisfies the attracting property (5.2), i.e. $\lim_{\alpha\to 0+} \mathrm{dist}_{B_\rho}(\mathfrak{A}_\alpha, \mathfrak{A}_{\min}) = 0$ and $\mathfrak{A}_{\min}$ belongs to every closed subset $\mathfrak{A}' \subseteq \mathfrak{A}_0$ for which $\lim_{\alpha\to 0+} \mathrm{dist}_{B_\rho}(\mathfrak{A}_\alpha, \mathfrak{A}') = 0$. We call the set $\mathfrak{A}_{\min}$ *the minimal limit of the trajectory attractors* $\mathfrak{A}_\alpha$ *as* $\alpha \to 0+$.

To prove that such a set $\mathfrak{A}_{\min}$ exists we just show that

$$\mathfrak{A}_{\min} = \bigcap_{0<\delta\le 1} \left[\bigcup_{0<\alpha\le\delta} \mathfrak{A}_\alpha \right]_{B_\rho}. \tag{5.3}$$

It is easy to prove that a point w belongs to the right hand side of (5.3) if and only if there exist $w_{\alpha_n} \in \mathfrak{A}_{\alpha_n}$, $n = 1, 2, \ldots$ such that $\rho(w_{\alpha_n}, w) \to 0$ and $\alpha_n \to 0+$ as $n \to \infty$. Due to (5.2), such a limit point w always belongs to $\mathfrak{A}_0$ and, moreover, to *every* closed attracting set $\mathfrak{A}'$. We state that the set (5.3) is attracting for $\mathfrak{A}_\alpha$ as $\alpha \to 0+$. Assuming the converse, we have that there is a sequence $w_{\alpha_n} \in \mathfrak{A}_{\alpha_n}$, such that $\alpha_n \to 0+$ and

$$\mathrm{dist}_{B_\rho}(w_{\alpha_n}, \mathfrak{A}_{\min}) \ge \varepsilon \tag{5.4}$$

for some fixed $\varepsilon > 0$. Recall that $w_{\alpha_n} \in B_\rho$ and B_ρ is a compact metric space. Then, passing to a subsequence $\{w_{\alpha_{n'}}\} \subset \{w_{\alpha_n}\}$, we may assume that $\rho(w_{\alpha_{n'}}, w') \to 0$ as $\alpha_{n'} \to 0$ for some $w' \in B_\rho$. Therefore by definition, $w' \in \mathfrak{A}_{\min}$, that contradicts to (5.4). We have proved that the set $\mathfrak{A}_{\min}$ is a minimal closed attracting subset of $\mathfrak{A}_0$.

Proposition 5.1 *The minimal limit* $\mathfrak{A}_{\min}$ *of trajectory attractors* $\mathfrak{A}_\alpha$ *as* $\alpha \to 0+$ *is a connected subset of* $\mathfrak{A}_0$ *in* B_ρ.

Proof. Assume the converse. Then the set $\mathfrak{A}_{\min}$ is union of two closed non-intersecting subsets $\mathfrak{A}^1_{\min}$ and $\mathfrak{A}^2_{\min}$, i.e.,

$$\mathfrak{A}_{\min} = \mathfrak{A}^1_{\min} \cup \mathfrak{A}^2_{\min} \text{ and } \mathfrak{A}^1_{\min} \cap \mathfrak{A}^2_{\min} = \emptyset.$$

Since the space B_ρ is compact, there are two open sets $\mathcal{O}_1$ and $\mathcal{O}_2$ in B_ρ such that $\mathfrak{A}^1_{\min} \subset \mathcal{O}_1$, $\mathfrak{A}^2_{\min} \subset \mathcal{O}_2$, and $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$. Clearly, $\mathfrak{A}_{\min} \subset \mathcal{O}_1 \cup \mathcal{O}_2$. Therefore, by (5.2), there is a number $\alpha_0 > 0$ such that

$$\mathfrak{A}_\alpha \subset \mathcal{O}_1 \cup \mathcal{O}_2,\ \forall \alpha,\ 0 < \alpha \le \alpha_0. \tag{5.5}$$

We note that every set $\mathfrak{A}_\alpha$ is connected (see Proposition 4.2), that is, $\mathfrak{A}_\alpha \subset \mathcal{O}_1$ or $\mathfrak{A}_\alpha \subset \mathcal{O}_2$ for all $\alpha < \alpha_0$. At the same time, since $\mathfrak{A}_{\min}$ is the *minimal* limit of $\mathfrak{A}_\alpha$, we can find α_1 and α_2 such that

$$\mathfrak{A}_{\alpha_1} \subset \mathcal{O}_1 \textit{ and } \mathfrak{A}_{\alpha_2} \subset \mathcal{O}_2. \tag{5.6}$$

(otherwise, we can diminish $\mathfrak{A}_{\min}$). Let, for definiteness, $0 < \alpha_2 < \alpha_1 < \alpha_0$. We set

$$\delta^* = \sup\{\delta : \mathfrak{A}_\alpha \subset \mathcal{O}_2,\ \alpha_2 \le \alpha < \alpha_2 + \delta\}. \tag{5.7}$$

We note that $\alpha_2 + \delta^* \leq \alpha_1 < \alpha_0$, (see (5.6)) and $\mathfrak{A}_{\alpha_2+\delta^*} \subset \mathcal{O}_1 \cup \mathcal{O}_2$ since $\alpha_2 + \delta^* < \alpha_0$ (see (5.5)).

We now state that $\mathfrak{A}_{\alpha_2+\delta^*}$ can not belong to $\mathcal{O}_2$. Indeed, if $\mathfrak{A}_{\alpha_2+\delta^*} \subset \mathcal{O}_2$ then, by Proposition 4.3, there is a small $\delta_2 > 0$ such that $\mathfrak{A}_{\alpha_2+\delta^*+\delta_2} \subset \mathcal{O}_2$, which contradicts to the definition of δ^* in (5.7). At the same time, $\mathfrak{A}_{\alpha_2+\delta^*}$ can not belong to $\mathcal{O}_1$ neither. Indeed, if $\mathfrak{A}_{\alpha_2+\delta^*} \subset \mathcal{O}_1$ then again, by Proposition 4.3, there is a small $\delta_1 > 0$ such that $\mathfrak{A}_{\alpha_2+\delta^*-\delta_1} \subset \mathcal{O}_1$, which contradicts to the definition of δ^* as well. However, all these contradict to the inclusion $\mathfrak{A}_{\alpha_2+\delta^*} \subset \mathcal{O}_1 \cup \mathcal{O}_2$. The proof is completed. ■

Recall that the set $\mathfrak{A}_{\min}$ is compact. Finally we prove

Proposition 5.2 *The minimal limit $\mathfrak{A}_{\min}$ of trajectory attractors $\mathfrak{A}_\alpha$ as $\alpha \to 0+$ is strictly invariant with respect to the translation semigroup $\{T(h)\}$, that is*

$$T(h)\mathfrak{A}_{\min} = \mathfrak{A}_{\min}, \ \forall h \geq 0. \tag{5.8}$$

Proof. Consider an arbitrary $w \in \mathfrak{A}_{\min}$. By definition, there is a sequence $w_{\alpha_n} \in \mathfrak{A}_{\alpha_n}$ such that $\rho(w_{\alpha_n}, w) \to 0$ as $\alpha_n \to 0+$. The translation semigroup $\{T(h)\}$ is continuous in Θ_+^{loc} and, therefore, $\rho(T(h)w_{\alpha_n}, T(h)w) \to 0$ as $\alpha_n \to 0+$. Since every $\mathfrak{A}_{\alpha_n}$ is strictly invariant, $T(h)w_{\alpha_n} \in \mathfrak{A}_{\alpha_n}$. Thus, $T(h)w \in \mathfrak{A}_{\min}$ and we have proved that

$$T(h)\mathfrak{A}_{\min} \subseteq \mathfrak{A}_{\min}, \ \forall\, h \geq 0.$$

Let us prove the inverse inclusion. For any $h \geq 0$ and an arbitrary $w \in \mathfrak{A}_{\min}$ with corresponding $w_{\alpha_n} \in \mathfrak{A}_{\alpha_n}$, $\rho(w_{\alpha_n}, w) \to 0$ $(\alpha_n \to 0+)$, we have to find $W \in \mathfrak{A}_{\min}$ such that $T(h)W = w$. Since $\mathfrak{A}_{\alpha_n}$ is strictly invariant, there is an element $W_{\alpha_n} \in \mathfrak{A}_{\alpha_n}$ such that $T(h)W_{\alpha_n} = w_{\alpha_n}$. The sequence $\{W_{\alpha_n}\}$ belongs to the compact set B_ρ. Passing to a subsequence $\{\alpha_{n'}\}$, we have that $W_{\alpha_{n'}} \to W$ $(n' \to \infty)$ for some $W \in B_\rho$. Then $W \in \mathfrak{A}_{\min}$. Since $\{T(h)\}$ is continuous $T(h)W_{\alpha_{n'}} \to T(h)W$ $(n' \to \infty)$. However $T(h)W_{\alpha_{n'}} = w_{\alpha_{n'}}$, so, $w_{\alpha_{n'}} \to T(h)W$ $(n' \to \infty)$ but $w_{\alpha_n} \to w$ $(n \to \infty)$. Hence, $T(h)W = w$ and we have proved that

$$\mathfrak{A}_{\min} \subseteq T(h)\mathfrak{A}_{\min}, \ \forall h \geq 0.$$

Consequently, we obtain (5.8). ■

A Appendix

Proof of Lemma 3.2. Let $\{e_n\}$ be eigenvectors of the operator A, i.e. $Ae_k = \lambda_k e_k, \lambda_k > 0$ and $\lambda_k \to +\infty$ $(k \to \infty)$. Then $f(t) = \sum_{k=1}^\infty f_k(t)e_k$, where $f_k(t) = (f(t), e_k)$, and

$$\|f\|^2_{L_2(0,M;H)} = \sum_{k=1}^\infty \int_0^M |f_k(t)|^2 dt. \tag{A.1}$$

We have

$$(1 + \alpha_n^2 A)^{-1/2} f(t) = \sum_{k=1}^\infty \frac{f_k(t)}{(1 + \alpha_n^2 \lambda_k)^{1/2}} e_k$$

and due to (A.1)

$$\|(1+\alpha_n^2 A)^{-1/2} f(\cdot) - f(\cdot)\|^2_{L_2(0,M;H)} = \sum_{k=1}^{\infty} \left[1 - \frac{1}{(1+\alpha_n^2\lambda_k)^{1/2}}\right]^2 \int_0^M |f_k(t)|^2 dt. \quad \text{(A.2)}$$

We fix an arbitrary $\varepsilon > 0$. It follows from (A.1) that there is a number $K > 0$ such that

$$\sum_{k=K+1}^{\infty} \int_0^M |f_k(t)|^2 dt \le \frac{\varepsilon}{2} \quad \text{(A.3)}$$

(since the series is convergent). We note that

$$0 < 1 - \frac{1}{(1+\alpha_n^2\lambda_k)^{1/2}} < 1. \quad \text{(A.4)}$$

Therefore, owing to (A.2) and (A.3),

$$\|(1+\alpha_n^2 A)^{-1/2} f - f\|^2_{L_2(0,M;H)} = \sum_{k=1}^{K} + \sum_{k=K+1}^{\infty}$$

$$\le \sum_{k=1}^{K} \left[1 - \frac{1}{(1+\alpha_n^2\lambda_k)^{1/2}}\right]^2 \int_0^M |f_k(t)|^2 dt + \frac{\varepsilon}{2}. \quad \text{(A.5)}$$

We now select a number N such that, for all $n \ge N$,

$$\sum_{k=1}^{K} \left[1 - \frac{1}{(1+\alpha_n^2\lambda_k)^{1/2}}\right]^2 \int_0^M |f_k(t)|^2 dt \le \frac{\alpha_n^4}{4} \sum_{k=1}^{K} \lambda_k^2 \int_0^M |f_k(t)|^2 dt \le \frac{\varepsilon}{2}.$$

This can be done since $\alpha_n \to 0+ \ (n \to \infty)$ and the number K is fixed. Here, we use the elementary inequality

$$0 < 1 - \frac{1}{(1+\alpha_n^2\lambda_k)^{1/2}} = \frac{(1+\alpha_n^2\lambda_k)^{1/2} - 1}{(1+\alpha_n^2\lambda_k)^{1/2}} < \frac{\alpha_n^2\lambda_k}{2}. \quad \text{(A.6)}$$

Consequently,

$$\|(1+\alpha_n^2 A)^{-1/2} f(\cdot) - f(\cdot)\|^2_{L_2(0,M;H)} \le \varepsilon, \ \forall n \ge N$$

and therefore

$$(1+\alpha_n^2 A)^{-1/2} f(t) \to f(t) \ (n \to \infty)$$

strongly in $L_2(0,M;H)$. The lemma is proved. ■

Proof of Lemma 3.4. Since $\varphi \in L_p(0,M;D(A))$, then $\varphi(t) = \sum_{k=1}^{\infty} \varphi_k(t) e_k$ and similarly to (A.1)

$$\|\varphi\|^p_{L_p(0,M;D(A))} = \int_0^M \left(\sum_{k=1}^{\infty} \lambda_k^2 |\varphi_k(t)|^2\right)^{\frac{p}{2}} dt.$$

Besides, the partial sum of the series

$$\sum_{k=1}^{K} \varphi_k(t) e_k \to \varphi(t) \ (K \to \infty)$$

strongly in $L_p(0,M;D(A))$, that is,

$$\left\|\sum_{k=1}^{K}\varphi_k(\cdot)e_k-\varphi(\cdot)\right\|^p_{L_p(0,M;D(A))}=\int_0^M\left(\sum_{k=K+1}^{\infty}\lambda_k^2|\varphi_k(t)|^2\right)^{\frac{p}{2}}dt\to 0\ (K\to\infty). \quad (A.7)$$

We have

$$(1+\alpha_n^2A)^{-1/2}\varphi(t)-\varphi(t)=\sum_{k=1}^{\infty}\left(\frac{1}{(1+\alpha_n^2\lambda_k)^{1/2}}-1\right)\varphi_k(t)e_k=\sum_{k=1}^{K}+\sum_{k=K+1}^{\infty}$$

and

$$\begin{aligned}&\left\|(1+\alpha_n^2A)^{-1/2}\varphi(t)-\varphi(t)\right\|^p_{L_p(0,M;D(A))}\\ =&\int_0^M\left(\sum_{k=1}^{\infty}\lambda_k^2\left[1-\frac{1}{(1+\alpha_n^2\lambda_k)^{1/2}}\right]^2|\varphi_k(t)|^2\right)^{\frac{p}{2}}dt. \quad (A.8)\end{aligned}$$

Therefore

$$\begin{aligned}&\|(1+\alpha_n^2A)^{-1/2}\varphi(\cdot)-\varphi(\cdot)\|_{L_p(0,M;D(A))}=\left\|\sum_{k=1}^{\infty}\left(\frac{1}{(1+\alpha_n^2\lambda_k)^{1/2}}-1\right)\varphi_k(\cdot)e_k\right\|_{L_p}\le\\ \le&\left\|\sum_{k=1}^{K}\right\|_{L_p}+\left\|\sum_{k=K+1}^{\infty}\right\|_{L_p}=\left\|\sum_{k=1}^{K}\left(\frac{1}{(1+\alpha_n^2\lambda_k)^{1/2}}-1\right)\varphi_k(\cdot)e_k\right\|_{L_p}+\left\|\sum_{k=K+1}^{\infty}\right\|_{L_p}\le\\ \le&\sum_{k=1}^{K}\left(1-\frac{1}{(1+\alpha_n^2\lambda_k)^{1/2}}\right)\|\varphi_k(\cdot)e_k\|_{L_p(0,M;D(A))}+\left\|\sum_{k=K+1}^{\infty}\right\|_{L_p}=\\ =&\sum_{k=1}^{K}\lambda_k\left(1-\frac{1}{(1+\alpha_n^2\lambda_k)^{1/2}}\right)\left(\int_0^M|\varphi_k(t)|^pdt\right)^{\frac{1}{p}}+\left\|\sum_{k=K+1}^{\infty}\right\|_{L_p} \quad (A.9)\end{aligned}$$

We notice that

$$\begin{aligned}\left\|\sum_{k=K+1}^{\infty}\right\|^p_{L_p}&=\left\|\sum_{k=K+1}^{\infty}\left(\frac{1}{(1+\alpha_n^2\lambda_k)^{1/2}}-1\right)\varphi_k(\cdot)e_k\right\|^p_{L_p}=\\ &=\int_0^M\left(\sum_{k=K+1}^{\infty}\lambda_k^2\left[1-\frac{1}{(1+\alpha_n^2\lambda_k)^{1/2}}\right]^2|\varphi_k(t)|^2\right)^{\frac{p}{2}}dt\le\\ &\le\int_0^M\left(\sum_{k=K+1}^{\infty}\lambda_k^2|\varphi_k(t)|^2\right)^{\frac{p}{2}}dt, \quad (A.10)\end{aligned}$$

since $0<1-(1+\alpha_n^2\lambda_k)^{-1/2}<1$.

It follows from (A.7) that the integral (A.10) can be made arbitrary small for a large K, that is, for an arbitrary $\varepsilon>0$, there is a number $K=K(\varepsilon)$ such that the last term in (A.9) does not exceed $\varepsilon/2$ for this fixed K.

It is left to estimate the first sum in (A.9). Using the inequality

$$0 < 1 - (1 + \alpha_n^2 \lambda_k)^{-1/2} < \alpha_n^2 \lambda_k / 2$$

(see (A.6)), we obtain that

$$\sum_{k=1}^{K} \lambda_k \left(1 - \frac{1}{(1 + \alpha_n^2 \lambda_k)^{1/2}}\right) \left(\int_0^M |\varphi_k(t)|^p dt\right)^{\frac{1}{p}} \le \frac{\alpha_n^2}{2} \sum_{k=1}^{K} \lambda_k \left(\int_0^M |\varphi_k(t)|^p dt\right)^{\frac{1}{p}} \tag{A.11}$$

Since K is fixed and $\alpha_n \to 0+$ as $n \to \infty$, we can choose N sufficiently large such that, for $n \ge N$, the right-hand side of (A.11) does not exceed $\varepsilon/2$ as well. Then we have the result:

$$\|(1 + \alpha_n^2 A)^{-1/2} \varphi(\cdot) - \varphi(\cdot)\|_{L_p} \le \varepsilon, \ \forall n \ge N.$$

that is

$$(1 + \alpha_n^2 A)^{-1/2} \varphi \to \varphi \ (n \to \infty) \text{ strongly in } L_p(0, M; D(A)). \ \blacksquare$$

References

[1] S.Chen, C.Foias, D.D.Holm, E.Olson, E.S.Titi, and S.Wynne, The Camassa–Holm equations as a closure model for turbulent channel and pipe flow, *Phys. Rev. Lett.* 81 (1998), 5338–5341.

[2] S.Chen, C.Foias, D.D.Holm, E.Olson, E.S.Titi, and S.Wynne, A connection between the Camassa–Holm equations and turbulent flows in pipes and channels, *Phys. Fluids*, 11 (1999), 2343–2353.

[3] S.Chen, C.Foias, D.D.Holm, E.Olson, E.S.Titi, and S.Wynne, The Camassa–Holm equations and turbulence, *Physica D*, 133 (1999), 49–65.

[4] S.Chen, D.D.Holm, L.G.Margolin, and R.Zhang, Direct numerical simulations of the Navier–Stokes-alpha model, *Physica D*, 133 (1999), 66–83.

[5] C.Foias, D.D.Holm, and E.S.Titi, The Navier–Stokes-alpha model of fluid turbulence, *Physica D*, 152/153 (2001), 505–519.

[6] C.Foias, D.D.Holm, and E.S.Titi, The three dimensional viscous Camassa–Holm equations, and their relation to the Navier–Stokes equations and turbulence theory, *Journal of Dynamics and Differential Equations*, 14 (2002), N. 1, 1–35.

[7] K.Mohseni, B.Kosović, S.Shkoller, and J.Marsden, Numerical simulations of the Lagrangian averaged Navier-Stokes equations for homogeneous isotropic turbulence, *Phys. Fluids*, 15 (2003), No.2, 524–544.

[8] R.Temam, *Infinite-dimensional Dynamical Systems in Mechanics and Physics, Applied Mathematics Series*, 68, New York, Springer-Verlag, 1988, 2nd ed. 1997.

[9] A.V.Babin and M.I.Vishik, *Attractors of Evolution Equations*, Nauka, Moscow, 1989. English transl.: North Holland, Amsterdam, 1992.

[10] J.K.Hale, *Asymptotic Behaviour of Dissipative Systems, Math. Surveys and Mon.*, 25, Amer. Math. Soc., Providence, RI, 1988.

[11] P.Constantin and C.Foias, *Navier–Stokes Equations*, The University of Chicago Press, Chicago and London, 1989.

[12] C.Foias, O.Manley, R.Rosa, and R.Temam, *Navier–Stokes Equations and Turbulence*, Cambridge University Press, Cambridge, 2001.

[13] G.Sell and Y.You, *Dynamics of Evolutionary Equations*, Springer-Verlag, New York, 2002.

[14] V.V.Chepyzhov and M.I.Vishik, *Attractors for Equations of Mathematical Physics*. AMS Colloquium Publications. V. 49. Providence: AMS, 2002.

[15] B.J.Geurts and D.D.Holm, Fluctuation effects on 3D Lagrangian mean and Eulerian mean fluid motion, *Physica D*, 133 (1999), 215–269.

[16] B.J.Geurts and D.D.Holm, Regularization modeling for large-eddy simulation, *Phys. Fluids*, 15 (2003), L13–L16.

[17] C.Cao, D.Holm, and E.S.Titi, On the Clark-α model of turbulence: global regularity and long-time dynamics, *Journal of Turbulence*, 6 (2005), 1–11.

[18] A.Cheskidov, D.D.Holm, E.Olson, and E.S.Titi, On Leray-α model of turbulence, *Royal Society London, Proc., Series A, Mathematical, Physical & Engineering Sciences*, 461 (2005), 629–649.

[19] A.A.Ilyin, E.M.Lunasin, and E.S.Titi, A modified-Leray-α subgrid scale model of turbulence, *Nonlinearity*, 19 (2006), 879–897.

[20] Y. Cao, E.M.Lunasin, and E.S.Titi, Global well-posedness of the three-dimensional viscous and inviscid simplified Bardina turbulence models, *Comm. Math. Sci.*, 19 (2006), N 4, 823–848.

[21] J.-L.Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*, Dunod Gauthier-Villars, Paris, 1969.

[22] O. A. Ladyzhenskaya, *The Boundary Value Problems of Mathematical Physics, Applied Mathematical Sciences*, vol.49, Springer-Verlag, 1985.

[23] A.Cheskidov, Boundary layer for the Navier-Stokes-alpha model of fluid turbulence, *Arch. Rational Mech. & Anal.*, 172 (2004), 333–362.

[24] J.Gibbon and D.Holm, Length-scale estimates for the LANS-α equations in terms of the Reynolds number, *Physica D*, 220 (2006), 69–78.

[25] V.V.Chepyzhov and M.I.Vishik, Trajectory attractors for evolution equations, *C. R. Acad. Sci.* Paris, Series I, 321 (1995), 1309-1314.

[26] V.V.Chepyzhov. and M.I.Vishik, Evolution equations and their trajectory attractors, *J.Math. Pures Appl.*, 76 (1997), 913–964.

[27] M.I.Vishik and V.V.Chepyzhov, Trajectory and global attractors of three-dimensional Navier–Stokes systems, *Math. Notes*, 71 (2002), 177–193.

[28] G.Sell, Global attractors for the three-dimensional Navier–Stokes equations. *J.Dyn. Diff. Eq.*, 8 (1996), 1–33.

[29] M.I.Vishik, E.S.Titi, and V.V.Chepyzhov, Trajectory attractor approximation of the 3D Navier–Stokes system by a Leray-α model, *Doklady Mathematics*, 71 (2005), N. 1, 92–95.

[30] V.V.Chepyzhov, E.S.Titi, and M.I.Vishik, On the convergence of solutions of the Leray-α model to the trajectory attractor of the 3D Navier–Stokes system, *Dicrete and Continuous Dyn. Sys.*, 17 (2007), N. 3, 33–52.

[31] R.Temam, *Navier–Stokes Equations, Theory and Numerical Analysis*, 3rd. rev. ed., North-Holland, Amsterdam, 1984.

[32] O.A.Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow*, Moscow, Nauka, 1970. English transl.: Gordon and Breach, New York, 1969.

[33] J.-L.Lions and E.Magenes, *Problèmes aux Limites Non Homogènes et Applications*, volume 1, Paris, Dunod, 1968.

[34] J.P.Aubin, Un théorème de compacité, *C.R.Acad.Sci.* Paris, 256 (1963), 5042-5044.

[35] Yu.A.Dubinskiĭ, Weak convergence in nonlinear elliptic and parabolic equations, *Mat. Sbornik*, 67, 4 (1965), pp.609–642. English transl.: *Sbornik: Mathematics*, 67, 4 (1965).