The Connes Embedding Problem and Model Theory

Isaac Goldbring

University of California, Irvine

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What are these tutorials about?

- The Connes Embedding Problem (or CEP) is a famous problem posed by Alain Connes in his landmark 1976 paper in the field of von Neumann algebras.
- The CEP is a model theory problem when viewed in the right light.
- In early 2020, a group of computer scientists proved a result in quantum complexity theory known as $\mathrm{MIP}^* = \mathrm{RE}$.
- Besides being intrinsically fascinating, it yielded a refutation of CEP.
- The "standard" path from $\mathrm{MIP}^* = \mathrm{RE}$ to $\neg\mathrm{CEP}$ uses a lot of heavy machinery.
- We will show how basic continuous model theory can give an alternate proof of this implication, bypassing many of the intermediate ingredients, as well as yielding further results.

The ubiquity of the CEP

- C^*algebras: CEP is equivalent to Kirchberg's QWEP problem, a problem stemming from the theory of C^*-algebra tensor products.
- Quantum information theory: CEP is equivalent to Tsirelson's problem about the equality of two different models for quantum correlations.
- Free probability: CEP is equivalent to microstate free entropy dimension being nonnegative (Voiculescu).
- Group theory: CEP for group von Neumann algebras is equivalent to every countable discrete group being hyperlinear. (Rădulescu)
- Noncommutative real algebraic geometry: CEP is equivalent to a certain tracial Positivstellenzats (Klep and Schweighofer).
- Model theory: Connections with decidability, e.c. models, and Henkin constructions...

$B(H)$

- Throughout, H is a complex Hilbert space.
- A linear operator $T : H \to H$ is called **bounded** if

$$\|T\| := \sup\{\|T\xi\| \ : \ \|\xi\| \leq 1\} < \infty.$$

- $B(H)$ denotes the set of bounded operators on H. It is a unital Banach $*$-algebra when equipped with the operator norm, operator addition, scalar multiplication, composition and **adjoint**: $\langle T\xi, \eta\rangle = \langle \xi, T^*\eta\rangle$ for all $\xi, \eta \in H$.
- The **weak operator topology** (WOT) on $B(H)$ is given by $T_i \xrightarrow{\text{WOT}} T$ if and only if $\langle T^i\xi, \eta\rangle \to \langle T\xi, \eta\rangle$ for all $\xi, \eta \in H$.
- The WOT is a weaker topology than the operator norm topology.

C^*-algebras and von Neumann algebras

Definition

1. A C^***-algebra** is a *-subalgebra of $B(H)$ closed in the operator norm topology.
2. A **von Neumann algebra** is a *unital* *-subalgebra of $B(H)$ closed in the WOT.

Example

$B(H)$ is a von Neumann algebra. In particular, when $\dim(H) = n$, we see that $M_n(\mathbb{C})$ is a von Neumann algebra.

Example

If (X, μ) is a measure space, then $L^\infty(X, \mu)$ is an abelian von Neumann subalgebra of $B(L^2(X, \mu))$.

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von Neumann's bicommutant theorem

- Given $X \subseteq B(H)$, we set

$$X' := \{T \in B(H) \ : \ TS = ST \text{ for all } S \in X\}.$$

- If X is closed under adjoint, it is easy to see that X' is a von Neumann algebra and $X \subseteq X'' := (X')'$.

Theorem (von Neumann's bicommutant theorem)

If M is a unital $*$-subalgebra of $B(H)$, then M is a von Neumann algebra if and only if $M = M''$.

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Group von Neumann algebras

- Let Γ denote a countable discrete group.
- $\ell^2(\Gamma)$ is the Hilbert space with orthonormal basis $(\delta_\gamma)_{\gamma\in\Gamma}$.
- The **left-regular representation** of Γ is the unitary representation $\lambda_\Gamma : \Gamma \to U(\ell^2(\Gamma)) \subseteq B(\ell^2(\Gamma))$ given by $\lambda_\Gamma(\gamma)(\delta_\eta) = \delta_{\gamma\eta}$.
- Note that $\operatorname{span}(\lambda_\Gamma(\Gamma)) \cong \mathbb{C}[\Gamma]$.
- The **group von Neumann algebra** of Γ is $L(\Gamma) := \lambda_\Gamma(\Gamma)''$.
- Example: $L(\mathbb{Z}) \cong L^\infty(\mathbb{T})$ (Fourier analysis).

Factors

Definition

If M is a von Neumann algebra, then its **center** is

$$Z(M) := M \cap M' = \{T \in B(H) \ : \ TS = ST \text{ for all } S \in M\}.$$

M is a **factor** if $Z(M) = \mathbb{C} \cdot 1$.

- Factors are the "building blocks" of von Neumann algebras (every von Neumann algebra is a *direct integral* of factors).

Examples

1. $B(H)$ is a factor.
2. $L(\Gamma)$ is a factor if and only if Γ is an **ICC group**, i.e. all nontrivial conjugacy classes are infinite, e.g. $\Gamma = S_\infty$ or $\mathbb{F}_n$.

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Traces

Definition

1. A **trace** on M is a normal, positive linear functional $\tau : M \to \mathbb{C}$ with $\tau(1) = 1$ and $\tau(xy) = \tau(yx)$. (E.g. integration on $M = L^\infty(X, \mu)$.)
2. A **tracial von Neumann algebra** is a pair (M, τ), where M is a von Neumann algebra and τ is a trace on M.
3. A II_1 **factor** is an infinite-dimensional factor that admits a trace (which is then necessarily unique).

Examples

1. $M_n(C)$ is a tracial factor, but not II_1. If $\dim(H) = \infty$, then $B(H)$ admits no trace.
2. $L(\Gamma)$ admits the trace $x \mapsto \langle x\delta_e, \delta_e\rangle$. $\therefore$ If Γ is a countably infinite ICC group, then $L(\Gamma)$ is a II_1 factor.

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The hyperfinite II_1 factor

- Consider the map $A \mapsto \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$ from $M_{2^n}(\mathbb{C})$ to $M_{2^{n+1}}(\mathbb{C})$.
- This map is a $*$-homomorphism that preserves the normalized trace on $M_{2^n}(\mathbb{C})$.
- The limit of this chain, denoted M, possesses a natural trace τ for which we can apply the GNS procedure, obtaining a faithful representation $\pi_\tau : M \hookrightarrow B(H)$.
- The **hyperfinite II_1 factor** is the von Neumann algebra $\mathcal{R} := \pi_\tau(M)''$.
- By a major theorem of Connes, $\mathcal{R} \cong L(\Gamma)$ for any infinite ICC **amenable** group Γ.
- $\mathcal{R}$ is contained in any II_1 factor.

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Ultrapowers of II_1 factors

- Fix a family $(M_i)_{i\in I}$ of II_1 factors and an **ultrafilter** $\mathcal{U}$ on I.
- The **(tracial) ultraproduct of the family** $(M_i)_{i\in I}$ **with respect to the ultrafilter** $\mathcal{U}$ is the II_1 factor $\prod_{\mathcal{U}} M_i := \ell^\infty(M_i)/c_{\mathcal{U}}$, where:
  - $\ell^\infty(M_i) := \{(a_i) \in \prod_{i\in I} M_i \ : \ \sup_{i\in I} \|a_i\| < \infty\}$ (operator norm bounded)
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- It carries the ultraproduct trace $\tau((a_i)^\bullet) := \lim_{\mathcal{U}} \tau_i(a_i)$.
- When each $M_i = M$, speak of **ultrapowers** of M, denoted $M^{\mathcal{U}}$. Have the **diagonal embedding** $M \hookrightarrow M^{\mathcal{U}}$, $a \mapsto (a, a, a, \ldots)^\bullet$.
- If $\mathcal{U}$ is principal, say supported on $j \in I$, then $\prod_{\mathcal{U}} M_i \cong M_j$. Otherwise, $\prod_{\mathcal{U}} M_i$ is nonseparable.

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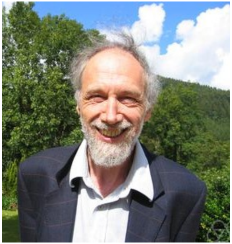
Connes' Embedding Problem



Quote (Connes, 1976)

"We now construct an approximate imbedding of N in $\mathcal{R}$. Apparently such an imbedding ought to exist for all II_1 factors because it does for the regular representation of free groups. However, the construction below relies on condition 6."

The Connes Embedding Problem

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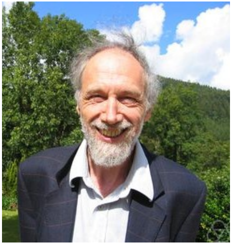

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The language for tracial von Neumann algebras

  

- Developed by Ilijas Farah, Bradd Hart, and David Sherman
- *Domains of quantification*: operator norm balls of integer radii
- Function symbols for the *-algebra operations
- Real-valued predicate symbols for the (real and imaginary parts of the) trace
- Distinguished predicate symbol for the metric arising from the trace

An example of a continuous formula

Example

Let $\varphi(x, y)$ be the formula $\|xy - yx\|_2$ (with x and y ranging over the unit ball), let M be a tracial von Neumann algebra, and let $a, b \in M_1$. We then have:

- $\varphi(a, b)^M = 0$ if and only if a and b commute.
- $(\sup_y \varphi(a, y))^M = 0$ if and only if $a \in Z(M)$.
- $(\sup_x \sup_y \varphi(x, y))^M = 0$ if and only if M is abelian.

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- $(\sup_y \varphi(a, y))^M = 0$ if and only if $a \in Z(M)$.
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The formula appearing in the third bullet has no free variables (so is a **sentence**) and is in fact a **universal** sentence.

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The elementary class of tracial von Neumann algebras

Theorem (Farah-Hart-Sherman)

1. *The class of tracial von Neumann algebras forms a universally axiomatizable elementary class in the language just described.*
2. *The model-theoretic ultraproduct coincides with the tracial ultraproduct.*
3. *The class of tracial factors and the class of II_1 factors form $\forall\exists$-axiomatizable subclasses.*

- This theorem is not super obvious.
- It uses the GNS construction to take a model of the theory and construct a *-algebra of operators on a Hilbert space.
- To see that the model forms a von Neumann algebra, one needs to use the fact that continuous structures are complete and the Kaplansky density theorem.

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CEP and Model theory: Part I

Definition

Given a continuous structure M, its **universal theory** is the function $\mathrm{Th}_\forall(M) : \{\text{universal sentences}\} \to \mathbb{R}$ given by $\mathrm{Th}_\forall(M)(\sigma) := \sigma^M$.

Model theory 101 (continuous version)

If M and N are structures in the same language, then $\mathrm{Th}_\forall(M) \leq \mathrm{Th}_\forall(N)$ (as functions) if and only if M embeds into an ultrapower of N.

Corollary

CEP is equivalent to:

- $\mathrm{Th}_\forall(M) \leq \mathrm{Th}_\forall(\mathcal{R})$ *for all tracial von Neumann algebras M.*
- $\mathrm{Th}_\forall(M) = \mathrm{Th}_\forall(\mathcal{R})$ *for all II_1 factors M.*

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Existentially closed structures

Definition

If $M \subseteq N$, then M is **existentially closed (e.c.) in** N if: for every quantifier-free formula $\varphi(x, y)$ and tuple a from M, we have

$$(\inf_y \varphi(a, y))^M = (\inf_y \varphi(a, y))^N.$$

More model theory 101

M is e.c. in N if and only if there is an embedding $\iota : N \hookrightarrow M^{\mathcal{U}}$ such that $\iota|M : M \hookrightarrow M^{\mathcal{U}}$ is the diagonal embedding.

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A model M of T is **locally universal** if every model of T embeds into an ultrapower of M.

So CEP asks: is $\mathcal{R}$ is locally universal?

Fact

If T has the joint embedding property (JEP), then an e.c. model M of T is a locally universal model of T, that is, all models of T embed into an ultrapower of M.

Since tracial von Neumann algebras have JEP (e.g. tensor products, free products,...), we get:

Theorem (Poor Man's CEP (FHS))

There is a locally universal tracial von Neumann algebra.

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Theorem (FGHS)

CEP is equivalent to $\mathcal{R}$ being an e.c. tracial von Neumann algebra.

Proof.

- Assume CEP holds and $\mathcal{R} \subseteq M$.
- By CEP we have $\iota : M \hookrightarrow \mathcal{R}^{\mathcal{U}}$.
- Issue: the composite map $\mathcal{R} \subseteq M \hookrightarrow \mathcal{R}^{\mathcal{U}}$ is not the diagonal.
- Folklore: the composite is *unitarily conjugate* to the diagonal. That is good enough.

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Building models by games (a la Hodges)

- We fix a countably infinite set C of distinct symbols (*witnesses*) that are to represent generators of a separable tracial vNa that two players (traditionally named $\forall$ and $\exists$) are going to build together (albeit adversarially).
- The two players take turns playing finite sets of expressions of the form $|\varphi(c) - r| < \epsilon$, where c is a tuple of variables, $\varphi(x)$ is a quantifier-free formula, and each player's move is required to extend the previous player's move. These sets are called (open) *conditions*.
- Moreover, these conditions are required to be *satisfiable*, meaning that there should be some vNa A and some tuple a from A such that $|\varphi(a) - r| < \epsilon$ for each such expression in the condition.

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Introducing the game (cont'd)

- We play this game for ω many steps.
- At the end of this game, we have enumerated some countable, satisfiable set of expressions.
- Player II can also ensure that the play is *definitive*, meaning that the final set of expressions yields complete information about all $*$-polynomials over the variables C (that is, for each $*$-polynomial $p(c)$, there should be a unique r such that the play of the game implies that $\|p(c)\|_2 = r$) and that this data describes a countable, dense $*$-subalgebra of a unique vNa, which is often called the *compiled structure*.
- With extra care, player II can also ensure that the compiled structure is actually a II_1 factor!
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Theorem

The following are equivalent:

1. *CEP has a positive solution.*
2. $\mathcal{R}$ *is the enforceable* II_1 *factor.*
3. $\mathcal{R}^{\mathcal{U}}$*-embeddability is enforceable.*

By the negative solution of CEP (and a little extra reasoning), we actually see that being a counterexample to CEP is enforceable (and thus model-theoretically generic).

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Does the enforceable II_1 factor $\mathcal{E}$ exist?

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