$\mathrm{MIP}^* = \mathrm{RE} \Rightarrow \neg\,\mathrm{CEP}$

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Group C^*-algebras

- Recall the left-regular representation $\lambda_\Gamma : \Gamma \to U(\ell^2(\Gamma))$.
- The **reduced group C^*-algebra** of Γ is $C_r^*(\Gamma) := \overline{\mathrm{span}(\lambda_\Gamma(\Gamma))}^{\|\cdot\|}$.
- There is another C^*-algebra associated to Γ, called the **universal group C^*-algebra** of Γ, denoted $C^*(\Gamma)$, characterized by the universal property: any unitary representation $\Gamma \to U(H)$ extends to a *-homomorphism $C^*(\Gamma) \to B(H)$.
- Always have a surjective *-homomorphism $C^*(\Gamma) \to C_r^*(\Gamma)$. It is an isomorphism if and only if Γ is amenable.

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The problem with C^*-algebra tensor products

- Given two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, their vector space tensor product $A \odot B$ carries a natural *-algebra operation.
- One would like to equip $A \odot B$ with a C^**-norm*, that is, a norm such that the completion with respect to that norm is a C^*-algebra.
- Issue: in general, there many such norms yielding nonisomorphic completions.
- For example, there are continuum many C^*-norms on $B(H) \odot B(H)$. (Ozawa-Pisier)
- There is always a smallest and largest such norm on $A \odot B$, whose completions are denoted $A \otimes_{\min} B$ and $A \otimes_{\max} B$.

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Nuclear pairs

Definition

- (A, B) are a **nuclear pair** if there is a unique C^*-norm on $A \odot B$ (equivalently $A \otimes_{\min} B = A \otimes_{\max} B$).
- A is **nuclear** if (A, B) is a nuclear pair for all B.

Example

If Γ is a group, then $C_r^*(\Gamma)$ is nuclear if and only if $C^*(\Gamma)$ is nuclear if and only if Γ is amenable (in which case $C_r^*(\Gamma) \cong C^*(\Gamma)$).

Example

$(C_r^*(\mathbb{F}), C_r^*(\mathbb{F}))$ is not a nuclear pair.

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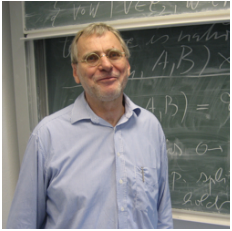
Kirchberg's QWEP Problem

Theorem (Kirchberg)

$(C^*(\mathbb{F}), B(H))$ is a nuclear pair.



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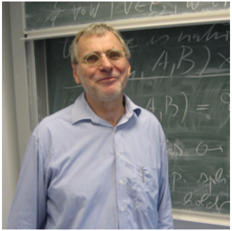

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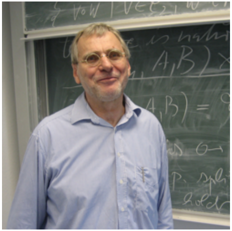

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Another kind of quantum correlation

Definition

The set $C_{qc}(k,n) \subseteq [0,1]^{k^2n^2}$ of **quantum commuting strategies** is the set of all p for which there are:

- a *single* (possibly infinite-dimensional) Hilbert space H,
- a unit vector $\xi \in H$, and
- POVMs A^x and B^y on H of length n (for each $x, y \in [k]$) satisfying $A^x_a B^y_b = B^y_b A^x_a$ for all $x, y \in [k]$ and $a, b \in [n]$,

such that $p(a,b|x,y) = \langle A^x_a B^y_b \xi, \xi \rangle$.

- Note $C_q(k,n) \subseteq C_{qc}(k,n)$.
- It can be shown that $C_{qc}(k,n)$ is closed, so $C_{qa}(k,n) \subseteq C_{qc}(k,n)$.

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Tsirelson's Problem

- Boris Tsirelson claimed in a paper, without proof, that $C_q(k,n) = C_{qc}(k,n)$.
- It soon became clear that the question of equality for both inclusions $C_q(k,n) \subseteq C_{qa}(k,n) \subseteq C_{qc}(k,n)$ was nontrivial.
- In 2018, Slofstra showed that $C_q(k,n) \subsetneq C_{qa}(k,n)$ for sufficiently large (k,n).

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MIP* = RE and Tsirelson

- Recall $\mathrm{val}^*(\mathfrak{G})$ is effectively approximable from below.
- Set $\mathrm{val}^{co}(\mathfrak{G}) := \sup_{p \in C_{qc}(k,n)} \mathrm{val}(\mathfrak{G}, p)$.
- Note that $\mathrm{val}^{co}(\mathfrak{G}) \geq \mathrm{val}^*(\mathfrak{G})$.
- It can be shown that $\mathrm{val}^{co}(\mathfrak{G})$ can be effectively approximated from above! (Semidefinite programming or model theory)
- Thus, if Tsirelson has a positive solution, then one can effectively approximate $\mathrm{val}^*(\mathfrak{G}) = \mathrm{val}^{co}(\mathfrak{G})$, contradicting MIP* = RE!

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Tsirelson and QWEP

Theorem (Fritz and Junge, et. al. (independently); Ozawa)

The QWEP problem is equivalent to Tsirelson's problem.

- Set $\mathbb{F}(k,n)$ to be the group freely generated by k elements of order n.
- The key point in the backwards direction is the existence of an element $\eta_{\mathfrak{G}} \in C^*(\mathbb{F}(k,n)) \odot C^*(\mathbb{F}(k,n))$ such that:
  - $\mathrm{val}^*(\mathfrak{G}) = \|\eta_{\mathfrak{G}}\|_{\min}$
  - $\mathrm{val}^{co}(\mathfrak{G}) = \|\eta_{\mathfrak{G}}\|_{\max}$.
- The last bullet explains why $\mathrm{val}^{co}(\mathfrak{G})$ is effectively approximable from above: for any finitely presented group Γ, the norm on $C^*(\Gamma)$ is effectively approximable from above (Fritz, Netzer and Thom) and $C^*(\mathbb{F}(k,n)) \otimes_{\max} C^*(\mathbb{F}(k,n)) \cong C^*(\mathbb{F}(k,n) \times \mathbb{F}(k,n))$.

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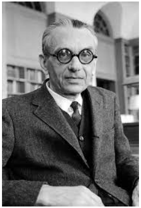
Model theory to the rescue



- After seeing the initial derivation of $\neg$ CEP from MIP$^* =$ RE, our initial reaction was: ???
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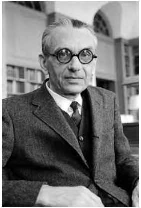

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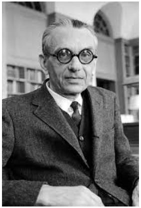


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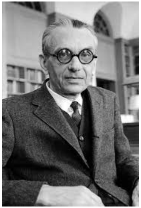

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CEP and computability

Theorem (G. and Hart (2016))

If CEP holds, then the universal theory of $\mathcal{R}$ is computable.

- This means that there is an algorithm such that, upon input a universal sentence σ, returns an interval $I \subseteq \mathbb{R}$ with $|I| < \epsilon$ and $\sigma^{\mathcal{R}} \in I$.
- Lower bounds: brute force.
- The Completeness theorem for continuous logic says that $\sup\{\sigma^M : M \models T_{II_1}\} = \inf\{r \in \mathbb{Q}^{>0} : T_{II_1} \vdash \sigma \dot{-} r\}$.
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On the other hand...

Theorem (G. and Hart (2020))

The universal theory of $\mathcal{R}$ is not computable.

Of course, we are going to use $\text{MIP}^* = \text{RE}$, but how?

Synchronous correlations and synchronous games

Definition

1. A correlation $p(a,b|x,y)$ is **synchronous** if $p(a,b|x,x)=0$ whenever $a\neq b$.
2. $C_{qa}^{s}(n,k)$ denotes the synchronous elements of $C_{qa}(n,k)$.
3. $\text{s-val}^*(\mathfrak{G})=\sup_{p\in C_{qa}^{s}(n,k)}\text{val}^*(\mathfrak{G},p)$.

- Clearly $\text{s-val}^*(\mathfrak{G})\leq\text{val}^*(\mathfrak{G})$.

Remark

The games in $\text{MIP}^*=\text{RE}$ are such that, if $\text{val}^*(\mathfrak{G}_{\mathcal{M}})=1$, then $\text{s-val}^*(\mathfrak{G}_{\mathcal{M}})=1$.

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The games in $\text{MIP}^*=\text{RE}$ are such that, if $\text{val}^*(\mathfrak{G}_{\mathcal{M}})=1$, then $\text{s-val}^*(\mathfrak{G}_{\mathcal{M}})=1$.

Synchronous correlations and synchronous games

Definition

1. A correlation $p(a, b|x, y)$ is **synchronous** if $p(a, b|x, x) = 0$ whenever $a \neq b$.
2. $C_{qa}^s(n, k)$ denotes the synchronous elements of $C_{qa}(n, k)$.
3. $\text{s-val}^*(\mathfrak{G}) = \sup_{p \in C_{qa}^s(n,k)} \text{val}^*(\mathfrak{G}, p)$.

- Clearly $\text{s-val}^*(\mathfrak{G}) \leq \text{val}^*(\mathfrak{G})$.

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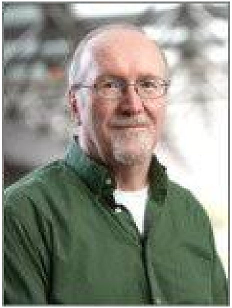
Synchronous strategies and operator algebras



Theorem (Kim-Paulsen-Shaufhauser)

$p \in C^s_{qa}(k,n)$ if and only if there are PVMs $e^1, \ldots, e^k$ of length n in $\mathcal{R}^{\mathcal{U}}$ such that $p(a,b|x,y) = \tau(e^x_a e^y_b)$, where τ is the unique trace on $\mathcal{R}^{\mathcal{U}}$.

We're getting closer...

Corollary

For any nonlocal game $\mathfrak{G}$*,*

$$\text{s-val}^*(\mathfrak{G}) = \left(\sup_{e^1,\ldots,e^k} \sum_{x,y} \lambda(x,y) \sum_{a,b} D(a,b,x,y)\,\mathrm{tr}(e^x_a e^y_b) \right)^{\mathcal{R}^{\mathcal{U}}},$$

where the supremum is being taken over PVMs of length n.

- This looks a lot more like a universal sentence being evaluated in $\mathcal{R}^{\mathcal{U}}$.
- If it were and the universal theory of $\mathcal{R}$ were computable, then we could effectively approximate $\text{s-val}^*(\mathfrak{G})$ for any nonlocal game $\mathfrak{G}$ and thus decide the halting problem!
- Issue: The supremum over PVMs is not officially part of the language!

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Definable sets in continuous logic

Theorem/Definition

Given a formula $\varphi(x)$ relative to some theory T, TFAE:

1. For any formula $\psi(x,y)$ and $\epsilon > 0$, there is a formula $\theta(y)$ such that
$$T \models \sup_y \left| \left(\sup_{\{x:\varphi(x)=0\}} \psi(x,y) \right) - \theta(y) \right| \leq \epsilon$$
and ditto for infimum.
2. For every $\epsilon > 0$, there is $\delta > 0$ such that, for all $M \models T$ and $a \in M$, if $\varphi(a) < \delta$, then there is $b \in M$ such that $\varphi(b) = 0$ and $d(a,b) < \epsilon$.
3. For any family of models $(M_i)_{i\in I}$ of T, any ultrafilter $\mathcal{U}$ on I, and any $a \in M := \prod_{\mathcal{U}} M_i$, if $\varphi^M(a) = 0$, then there are $a_i \in M_i$ such that $\varphi(a_i)^{M_i} = 0$ and $a = (a_i)_{\mathcal{U}}$.

We then call $\varphi(x)$ a **definable set relative to** T.

Some technical wrinkles

- It remains to check then that the set of PVMs in $\mathcal{R}$ of length n form a definable set relative to the theory of $\mathcal{R}$.
- Fortunately for us, this is the case, and Kim, Paulsen, and Schaufhauser themselves proved it!
- Then the translation from the expression using the definable set to an approximating family of "legitimate" sentences needs to be done effectively and the resulting sentences need to be universal...
- But it all works out just fine!

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A Gödelian style refutation of CEP

- Perhaps it is too arrogant to simply expect all tracial von Neumann algebras to embed into $\mathcal{R}^{\mathcal{U}}$, but maybe by adding some "reasonable" set of extra conditions, we can ensure $\mathcal{R}^{\mathcal{U}}$-embeddability.
- Nope!

Theorem (G. and Hart)

Suppose that T is any "effective" satisfiable set of (first-order) conditions extending the axioms for being a II_1 factor. Then there is a II_1 factor satisfying T that does not embed in $\mathcal{R}^{\mathcal{U}}$.

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“Many” counterexamples to CEP

Corollary

There is a sequence $M_1, M_2, \ldots,$ of separable II_1 factors, none of which embed into an ultrapower of $\mathcal{R}$, and such that, for all $i < j$, M_i does not embed into an ultrapower of M_j.

Proof.

Suppose now that $M_1, \ldots, M_n$ have been constructed. Let σ_i be a sentence such that $\sigma_i^{\mathcal{R}} = 0$ but $\sigma_i^{M_i} > 0$. Fix a rational number $\delta_i \in (0, \sigma_i^{M_i})$. Let $T \subseteq \mathrm{Th}(\mathcal{R})$ be the effective theory of II_1 factors together with the single condition $\max_{i=1,\ldots,n}(\sigma_i \dot{-} \delta_i) = 0$. Thus there is a separable model M_{n+1} of T such that M_{n+1} does not embed into an ultrapower of $\mathcal{R}$. Since $\sigma_i^{M_i} > \delta_i$ while $\sigma_i^{M_{n+1}} \leq \delta_i$, it follows that M_i does not embed into an ultrapower of M_{n+1}. □

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$\mathrm{Th}_\forall(\mathcal{R})$ is not computable for operator algebraists

- Let $m_1, \ldots, m_L$ enumerate all *-monomials in the variables $x_1, \ldots, x_n$ of total degree at most d.
- We consider the map $\mu_{n,d} : \mathcal{R}_1^n \to \mathbb{D}^L$ given by $\mu_{n,d}(\vec{a}) = (\mathrm{tr}(m_i(\vec{a})) \ : \ i = 1, \ldots, L)$.
- We let $X(n,d)$ denote the range of $\mu_{n,d}$ and $X(n,d,p)$ be the image of $(M_p(\mathbb{C}))_1$ under $\mu_{n,d}$.
- Notice that $\bigcup_{p\in\mathbb{N}} X(n,d,p)$ is dense in $X(n,d)$.

Theorem (G. and Hart)

The following statements are equivalent:

1. *The universal theory of $\mathcal{R}$ is computable.*
2. *There is a computable function $F : \mathbb{N}^3 \to \mathbb{N}$ such that, for every $n, d, k \in \mathbb{N}$, $X(n,d,F(n,d,k))$ is $\frac{1}{k}$-dense in $X(n,d)$.*

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Effective computability of s-val$^{co}(\mathfrak{G})$ from above

Theorem (Paulsen, Severini, Stahlke, Todorov, Winter)

$p \in C_{qc}^s(k,n)$ if and only if there is a tracial C^*-algebra (A,τ) and PVMs $e^1,\ldots,e^k$ of length n in A such that $p(a,b|x,y) = \tau(e_a^x e_b^y)$

Proposition

For any nonlocal game $\mathfrak{G}$, we have s-val$^{co}(\mathfrak{G}) \geq r$ if and only if the theory $T \cup \{\theta_{\mathfrak{G},r} = 0\}$ is satisfiable, where $\theta_{\mathfrak{G},r}$ is the sentence $r \dot{-} \left(\sup_{e^1,\ldots,e^k} \sum_{x,y} \lambda(x,y) \sum_{a,b} D(a,b,x,y)\,\mathrm{tr}(e_a^x e_b^y)\right)$.

- For any continuous theory U, we have that U is satisfiable if and only if $U \not\models 1 \dot{-} \frac{1}{2}$.
- Combined with the previous proposition, we get that s-val$^{co}(\mathfrak{G})$ is effectively approximable from above (uniformly in $\mathfrak{G}$).

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See you next year!

2023 North American Annual Meeting of the Association for Symbolic Logic

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General Information

The 2023 North American Annual Meeting of the ASL will be held March 25-29 at the University of California, Irvine.