ON THE THEORIES OF MCDUFF'S II$_1$ FACTORS

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ABSTRACT. Recently, Boutonnet, Chifan, and Ioana proved that McDuff's family of continuum many pairwise nonisomorphic separable II$_1$ factors are in fact pairwise non-elementarily equivalent by proving that any ultrapowers of two distinct members of the family are nonisomorphic. We use Ehrenfeucht-Fraisse games to provide an upper bound on the quantifier-depth of sentences which distinguish these theories.

1. INTRODUCTION

Constructing non-isomorphic separable II$_1$ factors has an interesting history. Murray and von Neumann [10] gave the first example of two non-isomorphic separable II$_1$ factors by proving that the hyperfinite II$_1$ factor $\mathcal{R}$ was not isomorphic to $L(\mathbb{F}_2)$, the group von Neumannn algebra associated to the free group on two generators. The way they proved this was by considering an isomorphism invariant, namely property Gamma, and proving that $\mathcal{R}$ has property Gamma whilst $L(\mathbb{F}_2)$ does not. Dixmier and Lance [4] produced a new isomorphism class by constructing a separable II$_1$ factor that does have property Gamma but does not have another property, nowadays called being McDuff, that $\mathcal{R}$ does have. Work of Zeller-Meier [14] and Sakai [11] led to several more isomorphism classes. The lingering question remained: are there infinitely many isomorphism classes of separable II$_1$ factors? In [8], McDuff constructed a countably infinite set of isomorphism classes of separable II$_1$ factors; in the sequel [9], she extends her technique to construct a family $(\mathcal{M}_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha} \in 2^\omega}$ of pairwise non-isomorphic separable II$_1$ factors. Throughout this paper, we will refer to this family as the family of *McDuff examples*. We will describe in detail the construction of the McDuff examples later in this paper.

The model-theoretic study of tracial von Neumann algebras began in earnest in [7], where it was shown that both property Gamma and being McDuff are axiomatizable properties (in the appropriate continuous first-order language for studying tracial von Neumann algebras). It follows that $\mathcal{R}$, $L(\mathbb{F}_2)$, and the Dixmier-Lance example are pairwise non-elementarily equivalent. However, it proved difficult to find new elementary equivalence classes of II$_1$ factors, although it was generally agreed upon by researchers in the model theory of operator algebras that there should be continuum many pairwise non-elementarily equivalent II$_1$ factors. The current authors recognized that one of the properties considered by Zeller-Meier in [14] was axiomatizable, thus providing a fourth elementary equivalence class; we include a proof of this observation in the last section.

In the recent paper [3], Boutonnet, Chifan, and Ioana prove that the McDuff examples are pairwise non-elementarily equivalent. They do not, however, exhibit sentences that distinguish these examples. Indeed, their main result is the following: if $\boldsymbol{\alpha}, \boldsymbol{\beta} \in 2^\omega$ are distinct, then for any nonprincipal

ultrafilters $\mathcal{U}$, $\mathcal{V}$ on arbitrary index sets, one has that $\mathcal{M}_{\boldsymbol{\alpha}}^{\mathcal{U}} \not\cong \mathcal{M}_{\boldsymbol{\beta}}^{\mathcal{V}}$. It is now routine to see that $\mathcal{M}_{\boldsymbol{\alpha}}$ and $\mathcal{M}_{\boldsymbol{\beta}}$ are not elementarily equivalent. Indeed, since the question of whether or not $\mathcal{M}_{\boldsymbol{\alpha}}$ and $\mathcal{M}_{\boldsymbol{\beta}}$ are elementarily equivalent is absolute, one can safely assume CH (see, for example, [5]), whence $\mathcal{M}_{\boldsymbol{\alpha}}$ elementarily equivalent to $\mathcal{M}_{\boldsymbol{\beta}}$ would imply that, for any nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$, one has that $\mathcal{M}_{\boldsymbol{\alpha}}^{\mathcal{U}}$ and $\mathcal{M}_{\boldsymbol{\beta}}^{\mathcal{U}}$ are saturated models of the same theory and a familiar back-and-forth argument shows that they are isomorphic.[1] (See [6, Section 4.4] for more details.)

To a model-theorist, it is interesting to know what sentences separate these examples. Indeed, to show that $\mathcal{M}_{\boldsymbol{\alpha}}$ and $\mathcal{M}_{\boldsymbol{\beta}}$ are not elementarily equivalent, it would be interesting to write down an explicit set of sentences T such that, for some $\sigma \in T$, we have $\sigma^{\mathcal{M}_{\alpha}} \neq \sigma^{\mathcal{M}_{\beta}}$. At the end of this paper, we show how to do this when $\boldsymbol{\alpha}(0) \neq \boldsymbol{\beta}(0)$; for the general case, we do not know how to do this.

The main result of this paper is instead quantitative in nature. For II_1 factors $\mathcal{M}$ and $\mathcal{N}$ and $k \geq 1$, we say that $\mathcal{M} \equiv_k \mathcal{N}$ if $\sigma^{\mathcal{M}} = \sigma^{\mathcal{N}}$ for any sentence σ of "complexity" at most k. (The precise notion of complexity will be defined in the next section.) Here is our main result:

Theorem. Suppose that $\boldsymbol{\alpha}, \boldsymbol{\beta} \in 2^\omega$ are distinct and $k \in \omega$ is least such that $\boldsymbol{\alpha}(k) \neq \boldsymbol{\beta}(k)$. Then $M_{\boldsymbol{\alpha}} \not\equiv_{5k+3} M_{\boldsymbol{\beta}}$.

In the next section, we describe the needed facts from logic as well as the parts of the paper [3] that we will use in our argument. In Section 3, we prove the main result; the proof uses Ehrenfeucht-Fraïssé games. In Section 4, we take care of some miscellaneous facts. First, we write down an explicit list of sentences that distinguish $M_{\boldsymbol{\alpha}}$ from $M_{\boldsymbol{\beta}}$ when $\boldsymbol{\alpha}(0) \neq \boldsymbol{\beta}(0)$. Next we discuss how the model-theoretic behavior of "good unitaries" underlies much of the argument in [3]. We then go on to show how Zeller-Meier's notion of *inner asymptotic commutativity* is axiomatizable and discuss another of Zeller-Meier's notions (which he does not name but we call "super McDuff"), giving some evidence as to why it might be axiomatizable. Finally, we bring up the notion of the *first-order fundamental group* of a II_1 factor and show how finding a II_1 factor with proper first-order fundamental group would give a different proof of the existence of continuum many theories of II_1 factors.

We list here some conventions used throughout the paper. First, we follow set theoretic notation and view $k \in \omega$ as the set of natural numbers less than k: $k = \{0, 1 \ldots, k-1\}$. In particular, 2^k denotes the set of functions $\{0, 1, \ldots, k-1\} \to \{0, 1\}$. If $\boldsymbol{\alpha} \in 2^k$, then we set $\alpha_i := \boldsymbol{\alpha}(i)$ for $i = 0, 1, \ldots, k-1$ and we let $\boldsymbol{\alpha}^{\#} \in 2^{k-1}$ be such that $\boldsymbol{\alpha}$ is the concatenation of (α_0) and $\boldsymbol{\alpha}^{\#}$. If $\boldsymbol{\alpha} \in 2^\omega$, then $\boldsymbol{\alpha}|k$ denotes the restriction of $\boldsymbol{\alpha}$ to $\{0, 1, \ldots, k-1\}$.

Whenever we write a tuple $\vec{x}$, it will be understood that the length of the tuple is countable (that is, finite or countably infinite).

We use $\subset$ (as opposed to $\subseteq$) to denote proper inclusion of sets.

Fix a von Neumann algebra $\mathcal{M}$. For $x, y \in \mathcal{M}$, the commutator of x and y is the element $[x, y] := xy - yx$. If A is a subalgebra of $\mathcal{M}$, then the relative commutant of A in $\mathcal{M}$ is the set

$$A' \cap \mathcal{M} := \{x \in \mathcal{M} \mid [x, a] = 0 \text{ for all } a \in A\}.$$

In particular, the center of $\mathcal{M}$ is $Z(\mathcal{M}) := \mathcal{M}' \cap \mathcal{M}$. For a tuple $\vec{a}$ from $\mathcal{M}$, we write $C(\vec{a})$ to denote $A' \cap \mathcal{M}$, where A is the subalgebra of $\mathcal{M}$ generated by the coordinates of $\vec{a}$. (Technically,

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[1]For those uncomfortable with the use of CH here, one can alternatively quote the Keisler-Shelah theorem as done in [3].

this notation should also mention $\mathcal{M}$, but the ambient algebra will always be clear from context, whence we omit any mention of it in the notation.)

The continuous logic used in this paper follows [6]; in particular, our formulae take values in $\mathbb{R}$.

2. Preliminaries

2.1. **Logic.**

Definition 2.1. If φ is a formula and k is a nonnegative integer, we define what it means for φ to have *quantifier depth at most* k, written $\mathrm{depth}(\varphi) \leq k$, by induction on the complexity of φ.

- If φ is atomic, then $\mathrm{depth}(\varphi) \leq 0$.
- If $\varphi_1, \ldots, \varphi_n$ are formulae, $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous function and $\varphi = f(\varphi_1, \ldots, \varphi_n)$, then $\mathrm{depth}(\varphi) \leq \max_{1 \leq i \leq n} \mathrm{depth}(\varphi_i)$.
- If $\varphi = \sup_{\vec{x}} \psi$ or $\varphi = \inf_{\vec{x}} \psi$, then $\mathrm{depth}(\varphi) \leq \mathrm{depth}(\psi) + 1$.

The main tool in this paper is the following variant of the usual Ehrenfeucht-Fraisse game.

Definition 2.2. Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures and let $k \in \mathbb{N}$. $\mathfrak{G}(\mathcal{M}, \mathcal{N}, k)$ denotes the following game played by two players. First, player I plays either a tuple $\vec{x_1} \in \mathcal{M}$ or a tuple $\vec{y_1} \in \mathcal{N}$. Player II then responds with a tuple $\vec{y_1} \in \mathcal{N}$ or $\vec{x_1} \in \mathcal{M}$. The play continues in this way for k rounds. We say that *Player II wins* $\mathfrak{G}(\mathcal{M}, \mathcal{N}, k)$ if there is an isomorphism between the substructures generated by $\{\vec{x_1}, \ldots, \vec{x_k}\}$ and $\{\vec{y_1}, \ldots, \vec{y_k}\}$ that maps $\vec{x_i}$ to $\vec{y_i}$.

Definition 2.3. Suppose that $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}$-structures.

(1) We write $\mathcal{M} \equiv_k \mathcal{N}$ if $\sigma^{\mathcal{M}} = \sigma^{\mathcal{N}}$ whenever $\mathrm{depth}(\sigma) \leq k$.
(2) We write $\mathcal{M} \equiv_k^{EF} \mathcal{N}$ if II has a winning strategy for $\mathfrak{G}(\mathcal{M}, \mathcal{N}, k)$.

It is a routine induction to show that $\mathcal{M} \equiv_k^{EF} \mathcal{N}$ implies $\mathcal{M} \equiv_k \mathcal{N}$. In the next lemma, the reader unfamiliar with the notion of an $\aleph_1$-saturated structure can simply think of an ultraproduct of structures with respect to a nonprincipal ultrafilter on $\mathbb{N}$.

Lemma 2.4. *Suppose that $\mathcal{M}$ and $\mathcal{N}$ are $\aleph_1$-saturated. Then $\mathcal{M} \equiv_k \mathcal{N}$ if and only if $\mathcal{M} \equiv_k^{EF} \mathcal{N}$.*

Proof. We prove the lemma by induction on k. Suppose first that $k = 0$ and that $\mathcal{M} \equiv_0 \mathcal{N}$. Let $\mathcal{M}_0$ and $\mathcal{N}_0$ be the substructures of $\mathcal{M}$ and $\mathcal{N}$ respectively generated by the emptyset. It follows immediately that there is an isomorphism between $\mathcal{M}_0$ and $\mathcal{N}_0$ that sends $c^{\mathcal{M}}$ to $c^{\mathcal{N}}$ for each constant symbol c, whence II always wins $\mathfrak{G}(\mathcal{M}, \mathcal{N}, 0)$.

Now suppose that $k > 0$ and inductively assume that the lemma holds for all integers smaller than k. We now describe a winning strategy for II in $\mathfrak{G}(\mathcal{M}, \mathcal{N}, k)$. Suppose that I first plays $\vec{a_1} \in \mathcal{M}$ (the case that I's first move is in $\mathcal{N}$ is analogous). Consider the set $\Gamma(\vec{x})$ given by

$$\Gamma(\vec{x}) := \{|\varphi(\vec{x}) - r| = 0 \ : \ \mathrm{depth}(\varphi) < k, \ \varphi^{\mathcal{M}}(\vec{a_1}) = r\}.$$

We claim that $\Gamma(\vec{x})$ is finitely satisfiable in $\mathcal{N}$. Towards this end, consider conditions "$|\varphi_i(\vec{x}) - r_i| = 0$" in Γ, $i = 1, \ldots, p$. Let $\sigma := \inf_{\vec{x}} \max_{1 \leq i \leq p} |\varphi_i(\vec{x}) - r_i|$. Note that $\sigma^{\mathcal{M}} = 0$ and that $\mathrm{depth}(\sigma) \leq k$. Since $\mathcal{M} \equiv_k \mathcal{N}$, we have that $\sigma^{\mathcal{N}} = 0$, witnessing that Γ is finitely satisfiable. Since $\mathcal{N}$ is $\aleph_1$-saturated, it follows that there is $\vec{b_1} \in \mathcal{N}$ satisfying Γ. The strategy for II in $\mathfrak{G}(\mathcal{M}, \mathcal{N}, k)$ starts by demanding that II play $\vec{b_1}$. Note now that $(\mathcal{M}, \vec{a_1}) \equiv_{k-1} (\mathcal{N}, \vec{b_1})$, so by induction we have that

$(\mathcal{M}, \vec{a_1}) \equiv_{k-1}^{EF} (\mathcal{N}, \vec{b_1})$. The rest of the strategy for II in $\mathfrak{G}(\mathcal{M}, \mathcal{N}, k)$ is to have II play according the winning strategy for II in $\mathfrak{G}((\mathcal{M}, \vec{a_1}), (\mathcal{N}, \vec{b_1}), k-1)$, where, for $p \geq 2$, round p in $\mathfrak{G}(\mathcal{M}, \mathcal{N}, k)$ is viewed as round $p-1$ in $\mathfrak{G}((\mathcal{M}, \vec{a_1}), (\mathcal{N}, \vec{b_1}), k-1)$. This strategy is clearly a winning strategy for II in $\mathfrak{G}(\mathcal{M}, \mathcal{N}, k)$, whence $\mathcal{M} \equiv_k^{EF} \mathcal{N}$. □

In the sequel, we will often assume that $\mathcal{M} \equiv_k^{EF} \mathcal{N}$ and that $\mathcal{M}$ is nonseparable. For reasons that will become clear in the next section, we actually want to know that $\mathcal{N}$ is also nonseparable.

Lemma 2.5. *Suppose that $\mathcal{M} \equiv_2^{EF} \mathcal{N}$ and $\mathcal{M}$ is nonseparable. Then $\mathcal{N}$ is nonseparable.*

Proof. Let $\vec{b}$ be any tuple from $\mathcal{N}$. Let I play $\vec{b}$ and have II respond with $\vec{a}$ from $\mathcal{M}$. Since $\mathcal{M}$ is nonseparable, there is $\epsilon > 0$ and $c \in \mathcal{M}$ such that $d(c, a_i) \geq \epsilon$ for all i. Have I play c and II responds with $d \in \mathcal{N}$. Since II wins $\mathfrak{G}(\mathcal{M}, \mathcal{N}, 2)$, we have that $d(d, b_i) \geq \epsilon$ for all i, whence $\vec{b}$ is not dense in $\mathcal{N}$. □

Note that the conclusion of the lemma need not hold if one merely assumes that $\mathcal{M} \equiv_1^{EF} \mathcal{N}$. Indeed, we invite the reader to verify that $H \equiv_1^{EF} H'$ for any two infinite-dimensional Hilbert spaces H and H'.

2.2. **McDuff's examples and property $\tilde{V}$.** First, we recall McDuff's examples. Let Γ be a countable group. For $i \geq 1$, let Γ_i denote an isomorphic copy of Γ and let Λ_i denote an isomorphic copy of $\mathbb{Z}$. Let $\tilde{\Gamma} := \bigoplus_{i \geq 1} \Gamma_i$. If S_∞ denotes the group of permutations of $\mathbb{N}$ with finite support, then there is a natural action of S_∞ on $\bigoplus_{i \geq 1} \Gamma$ (given by permutation of indices), whence we may consider the semidirect product $\tilde{\Gamma} \rtimes S_\infty$. Given these conventions, we can now define two new groups:

$$T_0(\Gamma) := \langle \tilde{\Gamma}, (\Lambda_i)_{i \geq 1} \mid [\Gamma_i, \Lambda_j] = 0 \text{ for } i \geq j \rangle$$

and

$$T_1(\Gamma) := \langle \tilde{\Gamma} \rtimes S_\infty, (\Lambda_i)_{i \geq 1} \mid [\Gamma_i, \Lambda_j] = 0 \text{ for } i \geq j \rangle.$$

Note that if Δ is a subgroup of Γ and $\alpha \in \{0, 1\}$, then $T_\alpha(\Delta)$ is a subgroup of $T_\alpha(\Gamma)$. Given a sequence $\boldsymbol{\alpha} \in 2^{\leq \omega}$, we define a group $K_{\boldsymbol{\alpha}}(\Gamma)$ as follows:

(1) $K_{\boldsymbol{\alpha}}(\Gamma) := \Gamma$ if $\boldsymbol{\alpha} = \emptyset$;
(2) $K_{\boldsymbol{\alpha}}(\Gamma) := (T_{\alpha_0} \circ T_{\alpha_1} \circ \cdots T_{\alpha_{n-1}})(\Gamma)$ if $\boldsymbol{\alpha} \in 2^n$;
(3) $K_{\boldsymbol{\alpha}}$ is the inductive limit of $(K_{\boldsymbol{\alpha}|n})_n$ if $\boldsymbol{\alpha} \in 2^\omega$.

We then set $\mathcal{M}_{\boldsymbol{\alpha}}(\Gamma) := L(T_{\boldsymbol{\alpha}}(\Gamma))$. When $\Gamma = \mathbb{F}_2$, we simply write $\mathcal{M}_{\boldsymbol{\alpha}}$ instead of $\mathcal{M}_{\boldsymbol{\alpha}}(\mathbb{F}_2)$; these are the McDuff examples referred to the introduction.

Given $n \geq 1$, we let $\tilde{\Gamma}_{\boldsymbol{\alpha}, n}$ denote the subgroup of $T_{\alpha_0}(K_{\boldsymbol{\alpha}\#}(\Gamma))$ given by the direct sum of the copies of $K_{\boldsymbol{\alpha}\#}(\Gamma)$ indexed by those $i \geq n$ and we let $P_{\boldsymbol{\alpha}, n} := L(\tilde{\Gamma}_{\boldsymbol{\alpha}, n})$. We define a *generalized McDuff ultraproduct corresponding to $\boldsymbol{\alpha}$ and Γ* to be an ultraproduct of the form $\prod_{\mathcal{U}} \mathcal{M}_{\boldsymbol{\alpha}}(\Gamma)^{\otimes t_s}$, where (t_s) is a sequence of natural numbers and $\mathcal{U}$ is a nonprincipal ultrafilter on $\mathbb{N}$, and we refer to subalgebras of the form $\prod_{\mathcal{U}} P_{\boldsymbol{\alpha}, n_s}^{\otimes t_s}$ as *special*.

We will need the following key facts; for (1) and (2), we ask the reader to recall the notations $\boldsymbol{\alpha}^{\#}$ and $C(\vec{a})$ from the introduction.

Facts 2.6. *Suppose that $\boldsymbol{\alpha} \in 2^{<\omega}$ is nonempty, Γ is a countable group, and (t_s) is a sequence of natural numbers.*

(1) *Suppose that (m_s) and (n_s) are two sequences of natural numbers such that $n_s < m_s$ for all s. Further suppose that Γ is an ICC group. Then $(\prod_{\mathcal{U}} P_{\boldsymbol{\alpha}, m_s}^{\otimes t_s})' \cap (\prod_{\mathcal{U}} P_{\boldsymbol{\alpha}, n_s}^{\otimes t_s})$ is a generalized McDuff ultraproduct corresponding to $\boldsymbol{\alpha}^{\#}$ and Γ.*

(2) *For any sequence (n_s), there is a pair of unitaries $\vec{a}$ from $\prod_{\mathcal{U}} \mathcal{M}_{\boldsymbol{\alpha}}(\Gamma)^{\otimes t_s}$ such that $\prod_{\mathcal{U}} P_{\boldsymbol{\alpha}, n_s}^{\otimes t_s} = C(\vec{a})$.*

(3) *Given any separable subalgebra A of $\prod_{\mathcal{U}} \mathcal{M}_{\boldsymbol{\alpha}}(\Gamma)^{\otimes t_s}$, there is a sequence (n_s) such that $\prod_{\mathcal{U}} P_{\boldsymbol{\alpha}, n_s}^{\otimes t_s} \subset A' \cap \prod_{\mathcal{U}} \mathcal{M}_{\boldsymbol{\alpha}}(\Gamma)^{\otimes t_s}$.*

Proof. (1) is [3, Lemma 3.3] and (3) is [3, Corollary 2.4]. We now mention how (2) follows from various parts of [3]. For simplicty, let us suppose $\alpha = (0)$; the general case is no harder, only notationally more cumbersome. Fix n and note that

$$T_0(\Gamma) = \langle \tilde{\Gamma}, \Lambda_1, \ldots, \Lambda_{n-1} \rangle *_{\tilde{\Gamma}_n} \langle \tilde{\Gamma}_n, \Lambda_n, \Lambda_{n+1}, \ldots \rangle.$$

Let a and b be generators of Λ_1 and Λ_n, respectively. For any $t \geq 1$, consider the elements $a_t := (a, \ldots, a)$ and $b_t := (b, \ldots, b)$ of $T_0(\Gamma)^{\oplus t}$. Note that a_t and b_t commute with $\tilde{\Gamma}_n^t$. Let $u_{t,n}$ and $v_{t,n}$ denote the corresponding unitaries of $L(T_0(\Gamma))^{\otimes t}$ and note that $u_{t,n}$ and $v_{t,n}$ commute with $L(\tilde{\Gamma}_n)^{\otimes t}$. Moreover, the proofs of [3, Lemmas 2.8-2.10] show that $u_{t,n}$ and $v_{t,n}$ witness that $L(\tilde{\Gamma}_n)^{\otimes t}$ is a $(2, 100)$-residual subalgebra of $L(T_0(\Gamma))^{\otimes t}$. Thus, given any sequence (n_s), we have unitaries $(u_{t_s, n_s})^{\bullet}, (v_{t_s, n_s})^{\bullet} \in \prod_{\mathcal{U}} \mathcal{M}_{\boldsymbol{\alpha}}(\Gamma)^{\otimes t_s}$ that commute with $\prod_{\mathcal{U}} P_{\boldsymbol{\alpha}, n_s}^{\otimes t_s}$. Moreover, the proof of [3, Lemma 2.6] shows that $\prod_{\mathcal{U}} P_{\boldsymbol{\alpha}, n_s}^{\otimes t_s}$ lies in $C((u_{t_s, n_s})^{\bullet}, (v_{t_s, n_s})^{\bullet})$. $\square$

We recall the definition of property $\tilde{V}$.

Definition 2.7. Let $\mathcal{M}$ be a nonseparable von Neumann algebra. We say that $\mathcal{M}$ has property $\tilde{V}$ if there is a separable subalgebra $A \subseteq \mathcal{M}$ such that, for all separable $B \subseteq A' \cap \mathcal{M}$ and all separable $C \subseteq \mathcal{M}$, there is $u \in U(\mathcal{M})$ such that $uBu^* \subseteq C' \cap \mathcal{M}$.

The following is [3, Lemma 4.3].

Fact 2.8. *If Γ is any countable group, then $\prod_{\mathcal{U}} L(T_1(\Gamma))^{\otimes t_s}$ has $\tilde{V}$.*

Notation. If $\vec{a}$ and $\vec{b}$ are tuples from $\mathcal{M}$, we set $\vec{a} \leq \vec{b}$ if and only if $C(\vec{b}) \subseteq C(\vec{a})$. As with any preorder, we write $\vec{a} < \vec{b}$ to indicate that $\vec{a} \leq \vec{b}$ but $\vec{b} \not\leq \vec{a}$.

Definition 2.9. Let k be a natural number. We define what it means for a nonseparable von Neumann algebra $\mathcal{M}$ to have $\tilde{V}$ at depth k:

- $\mathcal{M}$ has $\tilde{V}$ at depth 0 if it has $\tilde{V}$;
- If $k > 0$, then $\mathcal{M}$ has $\tilde{V}$ at depth k if for any $\vec{a}$, there is $\vec{b} > \vec{a}$ such that, for all $\vec{c} > \vec{b}$, there is $\vec{d} > \vec{c}$ for which there is a von Neumann algebra $\mathcal{N}$ with $C(\vec{c})' \cap C(\vec{b}) \subseteq \mathcal{N} \subseteq C(\vec{d})' \cap C(\vec{a})$ and such that $\mathcal{N}$ has $\tilde{V}$ at depth $k - 1$.

In connection with this definition, let us set up some further notation.

Notation. Let $\mathcal{M}$ be a nonseparable von Neumann algebra and let $\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{d}$ range over tuples from $\mathcal{M}$. Furthermore, let $k \geq 1$.

(1) For $\vec{a} < \vec{b} < \vec{c} < \vec{d}$, $\Phi(\vec{a}, \vec{b}, \vec{c}, \vec{d}; k)$ denotes the statement "there is a von Neumann algebra $\mathcal{N}$ with $C(\vec{c})' \cap C(\vec{b}) \subseteq \mathcal{N} \subseteq C(\vec{d})' \cap C(\vec{a})$ and such that $\mathcal{N}$ has $\tilde{V}$ at depth $k-1$."
(2) For $\vec{a} < \vec{b} < \vec{c}$, $\Phi(\vec{a}, \vec{b}, \vec{c}; k)$ denotes the statement "there is $\vec{d} > \vec{c}$ such that $\Phi(\vec{a}, \vec{b}, \vec{c}, \vec{d}; k)$ holds."
(3) For $\vec{a} < \vec{b}$, $\Phi(\vec{a}, \vec{b}; k)$ denotes the statement "for all $\vec{c} > \vec{b}$, $\Phi(\vec{a}, \vec{b}, \vec{c}; k)$ holds."
(4) $\Phi(\vec{a}; k)$ denotes the statement "there is $\vec{b} > \vec{a}$ such that $\Phi(\vec{a}, \vec{b}; k)$ holds."

The definition of $\mathcal{M}$ having $\tilde{V}$ at depth k can thus be recast as: for every $\vec{a}$, $\Phi(\vec{a}; k)$ holds. The following is the main result of [3] and appears there as Theorem 4.2 (really, Remark 4.3).

Fact 2.10. *Suppose that $\boldsymbol{\alpha} \in 2^\omega$. Then $\mathcal{M}_{\boldsymbol{\alpha}}^{\mathcal{U}}$ has $\tilde{V}$ at depth k if and only if $\alpha_k = 1$.*

3. The main result

Proposition 3.1. *Suppose that $\mathcal{M}$ and $\mathcal{N}$ are nonseparable with $\mathcal{M} \equiv_3^{EF} \mathcal{N}$ and $\mathcal{M}$ has $\tilde{V}$. Then $\mathcal{N}$ has $\tilde{V}$.*

Proof. Let $A \subseteq \mathcal{M}$ witness that $\mathcal{M}$ has $\tilde{V}$ and fix a winning strategy for Player II in $\mathfrak{G}(\mathcal{M}, \mathcal{N}, 3)$; we will show that $\mathcal{N}$ has $\tilde{V}$. Let $\vec{a}$ enumerate a countable dense subset of A and let I play $\vec{a}$. II then plays $\vec{a_1} \in \mathcal{N}$. Let A_1 denote the subalgebra of $\mathcal{N}$ generated by $\vec{a_1}$. We claim that A_1 witnesses that $\mathcal{N}$ has $\tilde{V}$. Towards this end, take separable $B_1 \subseteq A_1' \cap \mathcal{N}$ and $C_1 \subseteq \mathcal{N}$. Let $\vec{b_1}$ and $\vec{c_1}$ enumerate countable dense subsets of B_1 and C_1 respectively. I then plays $\vec{b_1}$ and $\vec{c_1}$. II then responds with $\vec{b}, \vec{c} \in \mathcal{M}$. Since II wins, it follows that $B \subseteq A' \cap \mathcal{M}$, so there is $u \in U(M)$ such that $uBu^* \subseteq C' \cap \mathcal{M}$. I finally plays u and II responds with $u_1 \in \mathcal{N}$. It remains to observe that $u_1 \in U(\mathcal{N})$ and $u_1 B_1 u_1^* \subseteq C_1' \cap \mathcal{N}$. $\square$

Theorem 3.2. *Suppose that $\boldsymbol{\alpha} \in 2^{k+1}$ with $\boldsymbol{\alpha}(k) = 1$. Further suppose that Γ is any countable ICC group and that $\mathcal{M}$ is a generalized McDuff ultraproduct corresponding to $\boldsymbol{\alpha}$ and Γ. Finally suppose that $\mathcal{M} \equiv_{5k+3}^{EF} \mathcal{N}$. Then $\mathcal{N}$ has property $\tilde{V}$ at depth k.*

Proof. We proceed by induction on k. Fact 2.8 and the previous proposition establishes the case $k = 0$. So suppose that $k > 0$ and the result holds for all smaller k. Choose a $\vec{b_0}$ from $\mathcal{N}$ and we would like to show that $\Phi(\vec{b_0}; k)$ holds. We obtain this by having player I play cooperatively in $\mathfrak{G}(\mathcal{M}, \mathcal{N}, 5k+3)$. View $\vec{b_0}$ as the first play for player I; II responds with $\vec{a_0}$ from $\mathcal{M}$ according to her winning strategy. Let P be a special subalgebra of $\mathcal{M}$ such that $P \subset C(\vec{a_0})$; this is possible by Fact 2.6(3). By Fact 2.6(2), at the next round, I plays $\vec{a_1}$ from $\mathcal{M}$ such that $P = C(\vec{a_1})$ (so $\vec{a_0} < \vec{a_1}$) and II responds with $\vec{b_1}$ from $\mathcal{N}$.

Claim 1 $\vec{b_0} < \vec{b_1}$.

Proof of Claim 1: We show that otherwise, player I could win the game. First suppose that there is $y \in C(\vec{b_1}) \setminus C(\vec{b_0})$; since $5k+3 \geq 3$, we can have I play y and II responds with $x \in \mathcal{M}$ according to her winning strategy. We have that $x \in C(\vec{a_1}) \setminus C(\vec{a_0})$, a contradiction. This shows that $\vec{b_0} \leq \vec{b_1}$. Now suppose that $z \in C(\vec{a_0}) \setminus C(\vec{a_1})$ and have I play z, II responding with $w \in \mathcal{N}$; since II wins, it follows that $w \in C(\vec{b_0}) \setminus C(\vec{b_1})$, whence $\vec{b_0} < \vec{b_1}$.

Now we would like to show that $\Phi(\vec{b_0}, \vec{b_1}; k)$ holds. Choose any $\vec{b_2} > \vec{b_1}$ and we will show $\Phi(\vec{b_0}, \vec{b_1}, \vec{b_2}; k)$ holds. Agreeably I plays $\vec{b_2}$, II responding with $\vec{a_2}$ from $\mathcal{M}$. Since $5k + 3 \geq 4$, the proof of Claim 1 shows that $\vec{a_2} > \vec{a_1}$. Now choose a special subalgebra Q such that $Q \subset C(\vec{a_2})$ and $Q = C(\vec{a_3})$. Player I now plays $\vec{a_3}$ and II responds with $\vec{b_3} \in \mathcal{N}$. Since $5k + 3 \geq 5$, repeating Claim 1 shows that $\vec{b_3} > \vec{b_2}$. To finish, we show that $\Phi(\vec{b_0}, \vec{b_1}, \vec{b_2}, \vec{b_3}; k)$ holds.

Set $\mathcal{M}_1 := C(\vec{a_3})' \cap C(\vec{a_1})$ and $\mathcal{N}_1 := C(\vec{b_3})' \cap C(\vec{b_1})$. By Fact 2.6(1), $\mathcal{M}_1$ is a generalized McDuff ultraproduct corresponding to $\boldsymbol{\alpha}^{\#} \in 2^k$ and Γ and that $\boldsymbol{\alpha}^{\#}(k-1) = 1$.

Claim 2: $\mathcal{M}_1 \equiv^{EF}_{5k-2} \mathcal{N}_1$.

Proof of Claim 2: We view any round p in $\mathfrak{G}(\mathcal{M}_1, \mathcal{N}_1, 5k-2)$ as round $p+4$ in $\mathfrak{G}(\mathcal{M}, \mathcal{N}, 5k+3)$ where the first four rounds are played out as above II plays according to the winning strategy for that game. A priori II's moves come from $\mathcal{M}$ or $\mathcal{N}$ but if they do not land in $\mathcal{M}_1$ or $\mathcal{N}_1$ then I can win the game in 1 more step since $p + 4 + 1 \leq 5k - 2 + 5 = 5k + 3$ and this would be a contradiction.

Since $5k - 2 = 5(k-1) + 3$, by induction we see that $\mathcal{N}_1$ has $\tilde{V}$ at depth $k - 1$. Since we have $C(\vec{b_2})' \cap C(\vec{b_1}) \subset \mathcal{N}_1 \subset C(\vec{b_3})' \cap C(\vec{b_0})$, it follows that $\Phi(\vec{b_0}, \vec{b_1}, \vec{b_2}, \vec{b_3}; k)$. $\qquad\square$

Corollary 3.3. *Suppose that* $\boldsymbol{\alpha}, \boldsymbol{\beta} \in 2^{\omega}$ *and* k *are such that* $\boldsymbol{\alpha}|k = \boldsymbol{\beta}|k$, $\boldsymbol{\alpha}(k) = 1$, $\boldsymbol{\beta}(k) = 0$. *Then* $\mathcal{M}_{\boldsymbol{\alpha}} \not\equiv_{5k+3} \mathcal{M}_{\boldsymbol{\beta}}$.

Proof. Fix $\mathcal{U} \in \beta\mathbb{N} \setminus \mathbb{N}$. If $\mathcal{M}_{\boldsymbol{\alpha}} \equiv_{5k+3} \mathcal{M}_{\boldsymbol{\beta}}$, then $\mathcal{M}_{\boldsymbol{\alpha}}^{\mathcal{U}} \equiv^{EF}_{5k+3} \mathcal{M}_{\boldsymbol{\beta}}^{\mathcal{U}}$. Since $\mathcal{M}_{\boldsymbol{\alpha}} = L(K_{\boldsymbol{\alpha}|(k+1)}(\Gamma))$ for some group Γ, the previous theorem implies that $\mathcal{M}_{\boldsymbol{\beta}}^{\mathcal{U}}$ has $\tilde{V}$ at depth k, contradicting Fact 2.10. $\qquad\square$

4. Miscellanea

4.1. Distinguishing $\tilde{V}$ with a sentence.
As mentioned in the introduction, it would be interesting to find concrete sentences that are actually distinguishing the McDuff examples. In this subsection, we show how we can find a set of sentences to distinguish $\mathcal{M}_{\boldsymbol{\alpha}}$ from $\mathcal{M}_{\boldsymbol{\beta}}$ when $\boldsymbol{\alpha}(0) = 1$ and $\boldsymbol{\beta}(0) = 0$.

Suppose that $\mathcal{M}$ is a separable McDuff II$_1$ factor for which $\mathcal{M}^{\mathcal{U}}$ has $\tilde{V}$ as witnessed by separable $A \subseteq \mathcal{M}^{\mathcal{U}}$. Since any separable subalgebra of $\mathcal{M}^{\mathcal{U}}$ containing A also witnesses that $\mathcal{M}^{\mathcal{U}}$ has $\tilde{V}$, by considering a separable elementary substructure of $\mathcal{M}^{\mathcal{U}}$ containing A (see, for example, [6, Theorem 4.6]), we may assume that A is a separable McDuff II$_1$ factor, whence singly generated (see [1, Theorem 1]), say by $a \in A$. Fix $n \in \mathbb{N}$ and let $\theta_n(w)$ be the meta-statement

$$\forall \vec{x}, \vec{y} \left(\max_{1 \leq i \leq n} \|[w, x_i]\|_2 = 0 \rightarrow \inf_{u \in U} \max_{1 \leq i, j \leq n} \|[ux_i u^*, y_j]\|_2 = 0 \right).$$

Now $\theta_n(w)$ is not an official statement of continuous logic, but [2, Proposition 7.14] together with the fact that $(\mathcal{M}^{\mathcal{U}}, a)$ is $\aleph_1$-saturated and $\theta_n(a)$ holds in $(\mathcal{M}^{\mathcal{U}}, a)$ implies that there are nondecreasing, continuous functions $\gamma_n : [0, 1] \rightarrow [0, 1]$ with $\gamma_n(0) = 0$ such that $\psi_n(a)^{(\mathcal{M}^{\mathcal{U}}, a)} = 0$ for each n, where $\psi_n(w)$ is the formula

$$\sup_{\vec{x}, \vec{y}} \left(\left(\inf_u \max_{1 \leq i, j \leq n} \|ux_i u^*, y_j]\|_2 \right) \dotminus \gamma_n \left(\max_{1 \leq i \leq n} \|[w, x_i]\|_2 \right) \right).$$

In the above formula, $\dotminus$ denotes truncated subtraction, that is, for $r, s \in \mathbb{R}$, $r \dotminus s := \max(r - s, 0)$.

Proposition 4.1. *Suppose that $\boldsymbol{\alpha}, \boldsymbol{\beta} \in 2^\omega$ are such that $\boldsymbol{\alpha}(0) = 1$ and $\boldsymbol{\beta}(0) = 0$. Then there are γ_n, ψ_n as above such that:*

(1) For each $n \geq 1$, $(\inf_w \psi_n(w))^{\mathcal{M}_{\boldsymbol{\alpha}}} = 0$.
(2) There is $n \geq 1$ such that $(\inf_w \psi_n(w))^{\mathcal{M}_{\boldsymbol{\beta}}} \neq 0$.

Proof. Let $\mathcal{M} := \mathcal{M}_{\boldsymbol{\alpha}}^{\mathcal{U}}$, $\mathcal{N} := \mathcal{M}_{\boldsymbol{\beta}}^{\mathcal{U}}$. Then $\mathcal{M}$ has $\tilde{V}$, whence the discussion preceding the current proposition holds and we have γ_n, ψ_n satisfying (1). Suppose, towards a contradiction, that (2) fails, namely that $(\inf_w \psi_n(w))^{\mathcal{M}_{\boldsymbol{\beta}}} = 0$ for all n. We claim that $\mathcal{N}$ has $\tilde{V}$, a contradiction. By saturation (together with the fact that the ψ_n's get successively stronger), there is $a_1 \in \mathcal{N}$ such that $\psi_n(a_1) = 0$ for all n. Let A_1 be the subalgebra of $\mathcal{N}$ generated by a_1. We claim that A_1 witnesses that $\mathcal{N}$ has $\tilde{V}$. Towards this end, fix separable $B \subseteq A_1' \cap \mathcal{N}$ and separable $C \subseteq \mathcal{N}$. Let $\vec{b}$ and $\vec{c}$ enumerate countable dense subsets of B and C respectively. Set

$$\Omega(u) := \{u \in U\} \cup \{\|[ub_i u^*, c_j]\|_2 = 0 \ : \ i, j \in \mathbb{N}\}.$$

By choice of a_1, $\Omega(u)$ is finitely satisfiable in $\mathcal{N}$, whence satisfiable in $\mathcal{N}$; if u satisfies Ω, then $uBu^* \subseteq C' \cap \mathcal{N}$, yielding the desired contradiction. $\square$

Notice that each $\inf_w \psi_n(w)$ has depth 3 which agrees with the 3 appearing in Proposition 3.1. Also note that the above discussion goes through with $\mathcal{M}_{\boldsymbol{\alpha}}^{\mathcal{U}}$ replaced with any generalized McDuff ultraproduct corresponding to $\boldsymbol{\alpha}$ and any countable group Γ and likewise for $\mathcal{M}_{\boldsymbol{\beta}}^{\mathcal{U}}$.

4.2. **Good unitaries and definable sets.** We would like to draw the reader's attention to some of the underlying model theory in [3] and recast Theorem 3.2. We highlight and give a name to the following concept that played a critical role in [3].

Definition 4.2. We say that a pair of unitaries u, v in a II_1 factor $\mathcal{M}$ are *good unitaries* if $C(u, v)$ is a (2,100)-residual subalgebra of $\mathcal{M}$ (in the terminology of [3]) with respect to the unitaries u and v, that is, for all $\zeta \in \mathcal{M}$,

$$\inf_{\eta \in C(u,v)} \|\zeta - \eta\|_2 \leq 100(\|[\zeta, u]\|_2^2 + \|[\zeta, v]\|_2^2).$$

We will call $C(u, v)$ a *good subalgebra* with respect to u and v.

If u and v are good unitaries, then $C(u, v)$ is a $\{u, v\}$-definable set, which follows immediately from [2, Proposition 9.19]. Moreover, we claim that if u_1, v_1 are another pair of good unitaries for which $C(u_1, v_1) \subseteq C(u, v)$, then $C(u_1, v_1)' \cap C(u, v)$ is $\{u, v, u_1, v_1\}$-definable. To see this, we first recall the following fact, due to Christensen and communicated to us by David Sherman; see [13, Lemma 3.6.5(ii)].

Fact 4.3. *Suppose that $\mathcal{M}$ is a tracial von Neumann algebra with subalgebra $\mathcal{N}$. Let $E : \mathcal{M} \to \mathcal{N}' \cap \mathcal{M}$ denote the conditional expectation map. Then for any $x \in \mathcal{M}$, we have*

$$\|E(x) - x\|_2 \leq \sup_{y \in \mathcal{N}_{\leq 1}} \|[x, y]\|_2.$$

Note already that this fact shows $C(u, v)' \cap \mathcal{M}$ is $\{u, v\}$-definable for any pair of good unitaries u, v. In general, intersections of definable subsets of metric structures need not be definable, so to show that $C(u_1, v_1)' \cap C(u, v)$ is definable, we need to do a bit more.

Lemma 4.4. *Suppose that $\mathcal{M}$ is a tracial von Neumann algebra with subalgebra $\mathcal{N}$. Let $E : \mathcal{M} \to \mathcal{N}$ denote the conditional expectation and let $P(x) := d(x, \mathcal{N})$ for all $x \in \mathcal{M}$. Then E is an A-definable function if and only if P is an A-definable predicate.*

Proof. If E is an A-definable function, then $P(x) := \|E(x) - x\|_2$ is an A-definable predicate. Conversely, if P is an A-definable predicate, then for any $x, y \in \mathcal{M}$, we have $\|E(x) - y\|_2^2 = \|x - y\|_2^2 - P(x - y)^2 + P(y)^2$, whence E is an A-definable function.   □

Lemma 4.5. *Suppose that $\mathcal{M}$ is a tracial von Neumann algebra with subalgebras $\mathcal{N}_1 \subseteq \mathcal{N}_2 \subseteq \mathcal{M}$. Furthermore suppose that $\mathcal{N}_1$ and $\mathcal{N}_2$ are A-definable subsets of $\mathcal{M}$. Then $\mathcal{N}_2' \cap \mathcal{N}_1$ is an A-definable subset of $\mathcal{M}$.*

Proof. Since $\mathcal{N}_2' \cap \mathcal{N}_1 = (\mathcal{N}_2' \cap \mathcal{M}) \cap \mathcal{N}_1$ and the intersection of two zerosets is again a zeroset, it suffices to show that the distance to $\mathcal{N}_2' \cap \mathcal{N}_1$ is a definable predicate. To keep things straight, let $E_1 : \mathcal{M} \to \mathcal{N}_1$ and $E_2 : \mathcal{M} \to \mathcal{N}_2' \cap \mathcal{N}_1$ denote the respective conditional expectations. By assumption, E_1 is A-definable. If $x \in \mathcal{M}$, we have

$$\|E_2(x) - x\|_2 \le \|E_2(x) - E_2(E_1(x))\|_2 + \|E_2(E_1(x)) - E_1(x)\|_2 + \|E_1(x) - x\|_2$$

$$\le 2\|x - E_1(x)\|_2 + \sqrt{\sup_{y \in (\mathcal{N}_2)_{\le 1}} \|[E_1(x), y]\|_2}.$$

Since $\mathcal{N}_2$ is an A-definable set and E_1 is an A-definable function, we see that $\mathcal{N}_2' \cap \mathcal{N}_1$ is an A-definable set.   □

In particular, if u, v, u_1, v_1 are as above, then $C(u_1, v_1)' \cap C(u, v)$ is an $\{u, v, u_1, v_1\}$-definable subset of $\mathcal{M}$.

We note that Fact 2.6 (and the proof of Lemma 2.9 of [3]) shows that a special subalgebra of a generalized McDuff ultraproduct is a good subalgebra with respect to some pair of good unitaries. In the definition of $\tilde{V}$ at depth k, one could modify the definition to only work with pairs of good unitaries instead of arbitrary countable tuples. It follows from the work in [3] that if $\mathcal{M}$ is a generalized McDuff ultraproduct, then $\mathcal{M}$ has $\tilde{V}$ at depth k if and only if $\mathcal{M}$ has $\tilde{V}$ at depth k in this augmented sense.

Returning now to the proof of Theorem 3.2, by the previous paragraph we see that at each play of the game, I could have chosen a pair of good unitaries instead of a countable sequence. Moreover, player I would also choose good unitaries corresponding to special subalgebras whenever they played a special subalgebra. Since II has a winning strategy by assumption, it follows that II always responds with pairs of good unitaries. Indeed, suppose that I plays good unitaries u,v (say in $\mathcal{M}$) and then II responds with $u_1, v_1 \in \mathcal{N}$. Since II wins, we have that u_1, v_1 are unitaries. To see that they are good, we need to play two more rounds of a side-game. Fix $\zeta_1 \in \mathcal{N}$ and $\epsilon > 0$. Have I play ζ_1 and have II reply with $\zeta \in \mathcal{M}$. Since u, v are good, there is $\eta \in C(u, v)$ such that

$$\|\zeta - \eta\|_2 < 100(\|[\zeta, u]\|_2^2 + \|[\zeta, v]\|_2^2) + \epsilon.$$

Have I play η and II responds with $\eta_1 \in \mathcal{N}$. It follows that $\eta_1 \in C(u_1, v_1)$ and

$$\|\zeta_1 - \eta_1\|_2 < 100(\|[\zeta_1, u_1]\|_2^2 + \|[\zeta_1, v_1]\|_2^2) + \epsilon.$$

Since $\zeta_1 \in \mathcal{N}$ and $\epsilon > 0$ were arbitrary, it follows that u_1, v_1 are good.

We see then that the subalgebras called $\mathcal{M}_1$ and $\mathcal{N}_1$ in the proof were in fact definable subalgebras defined over the parameters picked during the game.

We conclude this subsection with a discussion of what goes wrong when trying to distinguish $\mathcal{M}_{\boldsymbol{\alpha}}$ and $\mathcal{M}_{\boldsymbol{\beta}}$ with a sentence when $\boldsymbol{\alpha}(0) = \boldsymbol{\beta}(0)$ but $\boldsymbol{\alpha}(1) = 1$ and $\boldsymbol{\beta}(1) = 0$. Motivated by the game played in the previous section, it seems natural to try to use sentences of the form

$$\sup_{u_1,v_1} \inf_{u_2,v_2} \sup_{u_3,v_3} \inf_{u_4,v_4} \chi,$$

where at every stage we quantify only over good unitaries above the previous unitaries in the partial order on tuples and χ expresses that $C(u_4,v_4)' \cap C(u_2,v_2)$ has $\tilde{V}$. There is no issue in saying that the unitaries involved are good and get progressively stronger; moreover, if the unitaries "played" at the inf stages yield a special subalgebra, then $C(u_4,v_4)' \cap C(u_2,v_2)$ is definable and so one can relativize the sentences from the previous subsection to this definable set and indeed express that this commutant has $\tilde{V}$. The issue arises in that there were "mystery" connectives γ_n used in the sentences from the previous subsection and for different choices of good unitaries u_3, v_3, the generalized McDuff ultraproducts corresponding to $\boldsymbol{\alpha}^{\#}$, $C(u_4,v_4)' \cap C(u_2,v_2)$, may require different connectives to express that they have $\tilde{V}$. Of course, a positive answer to the following question alleviates this concern and shows how one can find sentences distinguishing $\mathcal{M}_{\boldsymbol{\alpha}}$ from $\mathcal{M}_{\boldsymbol{\beta}}$ when $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ differ for the first time at the second digit (and by induction one could in theory find sentences distinguishing all McDuff examples):

Question 4.6. *Given $\boldsymbol{\alpha} \in 2^{\omega}$ and a countable group Γ, are all generalized McDuff ultraproducts corresponding to $\boldsymbol{\alpha}$ and Γ elementarily equivalent?*

4.3. **Inner asymptotic commutativity and super McDuffness.** Motivated by Sakai's definition of asymptotically commutative II$_1$ factors from [11], Zeller-Meier introduced the following notion in [14]:

Definition 4.7. Suppose that $\mathcal{M}$ is a separable II$_1$ factor. We say that $\mathcal{M}$ is *inner asymptotically commutative* (IAC) if and only if there is a sequence of unitaries (u_n) such that, for all $x, y \in \mathcal{M}$, we have $\lim_n \|[u_n x u_n^*, y]\|_2 = 0$.

Proposition 4.8. *Inner asymptotic commutativity is an axiomatizable property amongst separable II$_1$ factors..*

Proof. For $n \geq 1$, consider the sentence

$$\sigma_n := \sup_{\vec{x},\vec{y}} \inf_u \max_{1 \leq i,j \leq n} \|[u x_i u^*, y_j]\|_2.$$

We claim that a separable II$_1$ factor $\mathcal{M}$ is IAC if and only if $\sigma_n^{\mathcal{M}} = 0$ for all n. The forward implication is clear. For the converse, suppose that $\sigma_n^{\mathcal{M}} = 0$ for all n. Let $\{a_i \ : \ i \in \mathcal{N}\}$ be a dense subset of $\mathcal{M}$. For each n, let $u_n \in U(\mathcal{M})$ be such that $\|[u_n a_i u_n^*, a_j]\|_2 < 1/n$ for all $i, j \leq n$. It then follows that (u_n) witnesses that $\mathcal{M}$ is IAC. $\square$

Before the appearance of [3], the previous proposition was interesting as it gave an example of a fourth theory of II$_1$ factors. Indeed, in the notation of [14], the hyperfinite II$_1$ factor, $L(S_{\infty})$, $L(K)$, and $L(\tilde{K})$ all have different theories.

Zeller-Meier also considers another property that may or may not hold for separable II$_1$ factors. Before we can define this property, we need some preparation:

Proposition 4.9. *Suppose that $\mathcal{M}$ is a separable McDuff II$_1$ factor and $\mathcal{M} \preceq \mathcal{C} \preceq \tilde{\mathcal{C}}$ with $\mathcal{C}$ and $\tilde{\mathcal{C}}$ both $\aleph_1$-saturated. Then the following are equivalent:*

(1) $Z(\mathcal{M}' \cap \mathcal{C}) = \mathbb{C}$
(2) $Z(\mathcal{M}' \cap \tilde{\mathcal{C}}) = \mathbb{C}$.

Proof. First suppose that (2) fails, so there is $a \in Z(\mathcal{M}' \cap \tilde{\mathcal{C}})$ such that $d(a, \mathrm{tr}(a) \cdot 1) \geq \epsilon$. Since $\mathcal{M}$ is McDuff, it is singly generated, say by $m \in \mathcal{M}$. Since $(\tilde{\mathcal{C}}, a, m)$ is $\aleph_1$-saturated, there is a continuous function $\gamma : [0, 1] \to [0, 1]$ with $\gamma(0) = 0$ such that

$$(\tilde{\mathcal{C}}, a, m) \models \sup_y (\|[a, y]\|_2 \mathbin{\dot{-}} \gamma(\|[y, m]\|_2) = 0.$$

It follows that

$$(\tilde{\mathcal{C}}, m) \models \inf_x \max(\|[x, m]\|_2, \sup_y (\|[x, y]\|_2 \mathbin{\dot{-}} \gamma(\|[y, m]\|_2), \epsilon \mathbin{\dot{-}} d(x, \mathrm{tr}(x) \cdot 1) = 0.$$

By elementarity, the same statement holds in $(\mathcal{C}, m)$; by saturation, the infimum is realized by $b \in \mathcal{C}$. It follows that $b \in Z(M' \cap \mathcal{C}) \setminus \mathbb{C}$, so (1) fails.

Now suppose that (2) holds and consider $a \in Z(M' \cap \mathcal{C})$. Then there is a continuous function $\eta : [0, 1] \to [0, 1]$ with $\eta(0) = 0$ such that

$$\mathcal{C} \models \sup_y (\|[y, a]\|_2 \mathbin{\dot{-}} \eta(\|[y, m]\|_2)) = 0.$$

By elementarity, the same statement holds in $\tilde{\mathcal{C}}$, that is, $a \in Z(M' \cap \tilde{\mathcal{C}}) = \mathbb{C}$, whence (1) holds. $\square$

Observe that the end of the above proof actually shows that, under the same hypotheses as in the proposition, we have $Z(M' \cap \mathcal{C}) \subseteq Z(M' \cap \tilde{\mathcal{C}})$.

Corollary 4.10. *Suppose that $\mathcal{M}$ is a separable McDuff II$_1$ factor. Then the following are equivalent:*

(1) $Z(\mathcal{M}' \cap \mathcal{C}) = \mathbb{C}$ *for every $\aleph_1$-saturated elementary extension $\mathcal{C}$ of $\mathcal{M}$.*
(2) $Z(\mathcal{M}' \cap \mathcal{C}) = \mathbb{C}$ *for some $\aleph_1$-saturated elementary extension $\mathcal{C}$ of $\mathcal{M}$.*

Definition 4.11. *If $\mathcal{M}$ is a separable McDuff II$_1$ factor, we say that $\mathcal{M}$ is *super McDuff* if either of the equivalent conditions of the previous corollary hold.*

It would be nice to know if being super McDuff is axiomatizable, for then, by [14], $L(\breve{\mathbb{F}}_2)$ has a different theory from the hyperfinite II$_1$ factor, $L(S_\infty)$, $L(K)$, and $L(\tilde{K})$. At the moment, the following proposition is the best that we can do.

Proposition 4.12. *Suppose that $\mathcal{M}, \mathcal{N}$ are separable McDuff II$_1$ factors with $\mathcal{M} \preceq \mathcal{N}$. If $\mathcal{N}$ is super McDuff, then so is $\mathcal{M}$.*

First, we need a little bit of preparation. Given $p \in S(\mathcal{M})$, we define $p^{\mathcal{U}} \in S(\mathcal{M}^{\mathcal{U}})$ by declaring, for every formula $\varphi(x, y)$ and every element $a := (a_i)^\bullet \in \mathcal{M}^{\mathcal{U}}$, $\varphi(x, a)^{p^{\mathcal{U}}} := \lim_{\mathcal{U}} \varphi(x, a_i)^p$. Recall also that $p \in S(\mathcal{M})$ is *algebraic* if all realizations of p are in $\mathcal{M}$.

Lemma 4.13. *If $p \in S(\mathcal{M})$ is not algebraic, then neither is $p^{\mathcal{U}} \in S(\mathcal{M}^{\mathcal{U}})$.*

Proof. Suppose that $p^{\mathcal{U}}$ is algebraic. Let $\mathcal{N}$ be an elementary extension of $\mathcal{M}$ containing a realization a of p. Then $a^\bullet \in \mathcal{N}^{\mathcal{U}}$ is a realization of $p^{\mathcal{U}}$, whence it belongs to $\mathcal{M}^{\mathcal{U}}$ by algebraicity of $p^{\mathcal{U}}$. It follows that a is the limit of a sequence from $\mathcal{M}$, whence it belongs to $\mathcal{M}$ as well. Since a was an arbitrary realization of p, we conclude that p is algebraic. $\square$

Proof of Proposition 4.12. Fix a nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$. By [6, Lemma 4.12], without loss of generality, we may assume that $\mathcal{M} \preceq \mathcal{N} \preceq \mathcal{M}^{\mathcal{U}}$. Suppose that $\mathcal{M}$ is not super McDuff as witnessed by $a \in Z(\mathcal{M}' \cap \mathcal{M}^{\mathcal{U}}) \setminus \mathbb{C}$. Let $p := \operatorname{tp}(a/\mathcal{M})$; since $\mathcal{M}$ is a II_1 factor, p is not algebraic, whence neither is $p^{\mathcal{U}}$. Let $\mathcal{C}$ be a $(2^{\aleph_0})^+$-saturated elementary extension of $\mathcal{M}^{\mathcal{U}}$.

Claim 1: $p^{\mathcal{U}}(\mathcal{C}) \subseteq (\mathcal{M}^{\mathcal{U}})' \cap \mathcal{C}$.

Proof of Claim 1: Let $\varphi(x, y)$ denote the formula $\|[x, y]\|_2$. Then for any $b \in \mathcal{M}$, we have that $\varphi(x, b)^p = 0$, whence it follows that for any element $b \in \mathcal{M}^{\mathcal{U}}$ we have $\varphi(x, b)^{p^{\mathcal{U}}} = 0$, verifying the claim.

Claim 2: $p(\mathcal{M}^{\mathcal{U}}) \subseteq Z(\mathcal{M}' \cap \mathcal{M}^{\mathcal{U}})$.

Proof of Claim 2: Fix $\epsilon > 0$. Then the following set of conditions is unsatisfiable in $\mathcal{M}^{\mathcal{U}}$:

$$\{\|[x, b]\|_2 = 0 \ : \ b \in \mathcal{M}\} \cup \{\|[x, a]\|_2 \geq \epsilon\}.$$

By saturation, there are $b_1, \ldots, b_n \in \mathcal{M}$ such that the following meta-statement is true in $\mathcal{M}^{\mathcal{U}}$:

$$\mathcal{M}^{\mathcal{U}} \models \forall x \left(\max_{1 \leq i \leq n} \|[x, b_i]\|_2 = 0 \rightarrow (\|[x, a]\|_2 \dotminus \epsilon) = 0 \right).$$

As above, by saturation this meta-statement can be made into an actual first-order formula with parameters from $\mathcal{M}$ that holds of a, whence it holds of any other realization of p in $\mathcal{M}^{\mathcal{U}}$. This shows that if $a' \in p(\mathcal{M}^{\mathcal{U}})$ and $c \in \mathcal{M}' \cap \mathcal{M}^{\mathcal{U}}$, then $\|[a', c]\|_2 \leq \epsilon$; since $\epsilon > 0$ is arbitrary, this proves the claim.

Claim 3: $p^{\mathcal{U}}(\mathcal{C}) \subseteq Z((\mathcal{M}^{\mathcal{U}})' \cap \mathcal{C})$.

Proof of Claim 3: Suppose that $a' \in \mathcal{C}$ realizes $p^{\mathcal{U}}$. Fix $b' \in (\mathcal{M}^{\mathcal{U}})' \cap \mathcal{C}$. Take $a'', b'' \in \mathcal{M}^{\mathcal{U}}$ such that $\operatorname{tp}(a', b'/\mathcal{M}) = \operatorname{tp}(a'', b''/\mathcal{M})$. By Claim 2, $a'' \in Z(\mathcal{M}' \cap \mathcal{M}^{\mathcal{U}})$. Note also that $b'' \in \mathcal{M}' \cap \mathcal{M}^{\mathcal{U}}$. It follows that $\|[a', b']\|_2 = \|[a'', b'']\|_2 = 0$, yielding the desired conclusion.

In order to establish that $\mathcal{N}$ is not super McDuff, by Lemma 4.13, it suffices to establish the following claim:

Claim 4: $p^{\mathcal{U}}|\mathcal{N}(\mathcal{M}^{\mathcal{U}}) \subseteq Z(\mathcal{N}' \cap \mathcal{M}^{\mathcal{U}})$.

Proof of Claim 4: Arguing as in the proof of Claim 1, we see that $p^{\mathcal{U}}|\mathcal{N}(\mathcal{M}^{\mathcal{U}}) \subseteq \mathcal{N}' \cap \mathcal{M}^{\mathcal{U}}$. Now suppose that $a' \in p^{\mathcal{U}}|\mathcal{N}(\mathcal{M}^{\mathcal{U}})$ and $b' \in \mathcal{N}' \cap \mathcal{M}^{\mathcal{U}}$. Then $a' \in Z(\mathcal{M}' \cap \mathcal{M}^{\mathcal{U}})$ by Claim 2 and $b' \in \mathcal{M}' \cap \mathcal{M}^{\mathcal{U}}$, so $[a, b] = 0$ as desired. $\qquad\square$

4.4. The first-order fundamental group.

For a II_1 factor $\mathcal{M}$ and $t \in \mathbb{R}_+$, we let $\mathcal{M}_t$ denote the amplification of $\mathcal{M}$ by t. Note that if $\mathcal{U}$ is an ultrafilter, then $(\mathcal{M}^{\mathcal{U}})_t$ is canonically isomorphic to $(\mathcal{M}_t)^{\mathcal{U}}$, whence we can unambiguously write $\mathcal{M}_t^{\mathcal{U}}$.

Recall that the fundamental group of $\mathcal{M}$ is the set $\mathcal{F}(\mathcal{M}) := \{t \in \mathbb{R}_+ \ : \ \mathcal{M}_t \cong \mathcal{M}\}$. $\mathcal{F}(\mathcal{M})$ is a (not necessarily closed) subgroup of $\mathbb{R}_+$. We now consider the *first-order fundamental group of* $\mathcal{M}$, $\mathcal{F}_{\mathrm{fo}}(\mathcal{M}) := \{t \in \mathbb{R}_+ \ : \ \mathcal{M}_t \equiv \mathcal{M}\}$. Clearly $\mathcal{F}(\mathcal{M}) \subseteq \mathcal{F}_{\mathrm{fo}}(\mathcal{M})$. As the name indicates, $\mathcal{F}_{\mathrm{fo}}(\mathcal{M})$ is actually a group. The easiest way to see this is to recognize that $\mathcal{F}_{\mathrm{fo}}(\mathcal{M})$ is absolute, whence, assuming CH, we have $\mathcal{F}_{\mathrm{fo}}(\mathcal{M}) = \mathcal{F}(\mathcal{M}^{\mathcal{U}})$ for a fixed ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Alternatively, one can use Keisler-Shelah as follows. Suppose that $s, t \in \mathcal{F}_{\mathrm{fo}}(\mathcal{M})$. By Keisler-Shelah, there is $\mathcal{U}$ such that $\mathcal{M}^{\mathcal{U}} \cong \mathcal{M}_s^{\mathcal{U}}$. Note now that $\mathcal{M}^{\mathcal{U}} \equiv \mathcal{M}_t^{\mathcal{U}}$, whence there is $\mathcal{V}$ such that $(\mathcal{M}^{\mathcal{U}})^{\mathcal{V}} \cong (\mathcal{M}_t^{\mathcal{U}})^{\mathcal{V}}$. We then have

$$(\mathcal{M}^{\mathcal{U}})^{\mathcal{V}} \cong (\mathcal{M}_t^{\mathcal{U}})^{\mathcal{V}} \cong ((\mathcal{M}_s^{\mathcal{U}})_t)^{\mathcal{V}} = (\mathcal{M}_{st}^{\mathcal{U}})^{\mathcal{V}} \cong ((\mathcal{M}_{st})^{\mathcal{U}})^{\mathcal{V}},$$

whence it follows that $\mathcal{M} \equiv \mathcal{M}_{st}$.

Unlike the ordinary fundamental group, the first-order fundamental group is a closed subgroup of $\mathbb{R}_+$. Indeed, if (r_k) is a sequence from $\mathbb{R}_+$ with limit $r \in \mathbb{R}_+$, it is easy to verify that $\prod_{\mathcal{U}} \mathcal{M}_{r_k} \cong \mathcal{M}_r^{\mathcal{U}}$ for any nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$; if each $r_k \in \mathcal{F}_{\mathrm{fo}}(\mathcal{M})$, then $\prod_{\mathcal{U}} \mathcal{M}_{r_k} \equiv \mathcal{M}$, whence it follows that $r \in \mathcal{F}_{\mathrm{fo}}(\mathcal{M})$.

In summary:

Proposition 4.14. *$\mathcal{F}_{\mathrm{fo}}(\mathcal{M})$ is a closed subgroup of $\mathbb{R}_+$ containing $\mathcal{F}(\mathcal{M})$.*

Question 4.15. *Does there exist a separable II$_1$ factor $\mathcal{M}$ for which $\mathcal{F}_{\mathrm{fo}}(\mathcal{M}) \neq \mathbb{R}_+$?*

Recall that II$_1$ factors $\mathcal{M}$ and $\mathcal{N}$ are said to be *stably isomorphic* if $\mathcal{M} \cong \mathcal{N}_t$ for some $t \in \mathbb{R}_+$. So the above question is equivalent to the question: does stable isomorphism imply elementary equivalence? Since all of the free group factors are stably isomorphic, a special case of the above question is whether or not all of the free group factors are elementarily equivalent, a question Thomas Sinclair has called the *noncommutative Tarski problem*. (The original Tarski problem asked whether or not all the nonabelian free groups were elementarily equivalent; this was answered positively by Sela [12].)

In connection with the number of theories of II$_1$ factors, we have:

Proposition 4.16. *Suppose that $\mathcal{M}$ is a separable II$_1$ factor with $\mathcal{F}_{\mathrm{fo}}(\mathcal{M}) \neq \mathbb{R}_+$. Then*

$$|\{\mathrm{Th}(\mathcal{M}_t) \ : \ t \in \mathbb{R}_+\}| = 2^{\aleph_0}.$$

Proof. Since the map $t\mathcal{F}_{\mathrm{fo}}(\mathcal{M}) \mapsto \mathrm{Th}(\mathcal{M}_t)$ is injective, the result follows from the fact that closed subgroups of $\mathbb{R}_+$ are countable. $\square$

It seems very unlikely that $\mathcal{F}_{\mathrm{fo}}(\mathcal{M}) = \mathbb{R}_+$ for all separable II$_1$ factors $\mathcal{M}$. In fact, it seems very unlikely that $\mathcal{M} \equiv \mathcal{M}_2(\mathcal{M})$ for all separable II$_1$ factors $\mathcal{M}$. Let $\mathcal{F}_{\forall}(\mathcal{M}) := \{t \in \mathbb{R}_+ \ : \ \mathcal{M} \equiv_{\forall} \mathcal{M}_t\}$. Of course, if CEP holds, then $\mathcal{F}_{\forall}(\mathcal{M}) = \mathbb{R}_+$ for any II$_1$ factor $\mathcal{M}$, so what follows is only interesting if CEP fails.

Proposition 4.17. *The following statements are equivalent:*

  (1) If $\mathcal{M}$ is existentially closed, then $\mathcal{M}$ is McDuff.
  (2) If $\mathcal{M}$ is existentially closed, then $2 \in \mathcal{F}_{\forall}(\mathcal{M})$.
  (3) For any II$_1$ factor $\mathcal{M}$, $2 \in \mathcal{F}_{\forall}(\mathcal{M})$.
  (4) For any II$_1$ factor $\mathcal{M}$, $\mathcal{M} \equiv_{\forall} \mathcal{M} \otimes \mathcal{R}$.
  (5) For any II$_1$ factor $\mathcal{M}$, $\mathcal{F}_{\forall}(\mathcal{M}) = \mathbb{R}_+$.

In the statement of the proposition, when we say that $\mathcal{M}$ is existentially closed, we mean that $\mathcal{M}$ is an existentially closed model of its theory.

Proof of Proposition 4.17. Since McDuff II$_1$ factors have full fundamental group, (1) implies (2) is trivial. (2) implies (3) follows from the fact that $\mathcal{M} \equiv_{\forall} \mathcal{N}$ implies $\mathcal{F}_{\forall}(\mathcal{M}) = \mathcal{F}_{\forall}(\mathcal{N})$. (3) implies (4) follows from the fact that $\mathcal{M} \otimes \mathcal{R}$ embeds into $\prod_{\mathcal{U}} M_{2^n}(\mathcal{M})$. Now suppose that (4) holds and fix an arbitrary II$_1$ factor $\mathcal{M}$. Since $\mathcal{M} \otimes \mathcal{R}$ is McDuff, for any $t \in \mathbb{R}_+$ we have that

$$\mathcal{M} \equiv_{\forall} \mathcal{M} \otimes \mathcal{R} \cong (\mathcal{M} \otimes \mathcal{R})_t \equiv_{\forall} \mathcal{M}_t,$$

whence (5) holds. Finally assume that (5) holds and assume that $\mathcal{M}$ is existentially closed. By considering the chain

$$\mathcal{M} \subseteq M_2(\mathcal{M}) \subseteq M_4(\mathcal{M}) \subseteq M_8(\mathcal{M}) \subseteq \cdots$$

and noting that each element of the chain has the same universal theory as $\mathcal{M}$ by (5), we see that $\mathcal{M}$ is existentially closed in the union $\mathcal{M} \otimes \mathcal{R}$. Since $\mathcal{M} \otimes \mathcal{R}$ is McDuff and being McDuff is $\forall\exists$-axiomatizable, we have that $\mathcal{M}$ is McDuff as well. $\square$

Note that it is not always true that $\mathcal{M} \equiv_{\forall\exists} \mathcal{M} \otimes \mathcal{R}$ (e.g. when $\mathcal{M}$ is not McDuff).

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