Progress on a sumset conjecture of Erdős

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van der Warden

Recall that an *arithmetic progression* is a finite sequence $a, a+r, a+2r, \ldots, a+kr$ for some $a, r, k \in \mathbb{N}$.

A *finite coloring of* $\mathbb{N}$ is just a partition $\mathbb{N} = C_1 \sqcup \cdots \sqcup C_k$ into finitely many sets. We refer to the C_i's as *colors*.

Theorem (van der Warden, 1927)

Given any finite coloring of $\mathbb{N}$, there is a color that contains arbitrarily long arithmetic progressions.

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Szemerédi's Theorem

Definition

For $A \subseteq \mathbb{N}$, the *upper density of A* is the quantity

$$\bar{d}(A) := \limsup_{n\to\infty} \frac{|A \cap [1,n]|}{n}.$$

Theorem (Szemerédi, 1975)

If $\bar{d}(A) > 0$, then A contains arbitrarily long arithmetic progressions.

Given a finite coloring of $\mathbb{N}$, some color must have positive density, so Szemerédi is a *drastic* generalization of van der Warden.

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Another coloring theorem-Hindman's Theorem

Given $A \subseteq \mathbb{N}$, set

$$\mathrm{FS}(A) := \{x_1 + \cdots + x_n \ : \ x_1, \ldots, x_n \text{ distinct elements of } A, n \in \mathbb{N}\}.$$

Theorem (Hindman, 1974)

Given any finite coloring of $\mathbb{N}$, there is an infinite monochromatic set A such that $\mathrm{FS}(A)$ is also monochromatic.

Question

Is the "density version" of Hindman's Theorem true? Namely, if $\bar{d}(A) > 0$, is there infinite $B \subseteq \mathbb{N}$ such that $\mathrm{FS}(B) \subseteq A$?

Answer

No! Just let A be the odd numbers!

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Erdős' conjectures

Seeing that arithmetic progressions are *translates* of (finite) FS-sets, Erdős asked the following:

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If $\bar{d}(A) > 0$, is there $t \in \mathbb{N}$ and infinite $B \subseteq \mathbb{N}$ such that $t + \mathrm{FS}(B) \subseteq A$?

Answer-Strauss

No! In fact, there are counterexamples with $\bar{d}(A)$ as close to 1 as you like.

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Erdős' conjectures (continued)

Given $A \subseteq \mathbb{N}$, set

$$\mathrm{PS}(A) := \{x + y \ : \ x, y \in A, \ x \neq y\}.$$

Erdős then changed his question.

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This question is still open. In fact, the following more specific conjecture is open:

Erdős' "B+C" conjecture

If $\underline{d}(A) > 0$, then there are infinite $B, C \subseteq \mathbb{N}$ such that $B + C \subseteq A$.

Here,

$$\underline{d}(A) := \liminf_{n \to \infty} \frac{|A \cap [1, n]|}{n}.$$

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Our results

For $A \subseteq \mathbb{N}$, the *Banach density of A* is the quantity

$$\mathrm{BD}(A) := \lim_{n\to\infty} \sup_{x\in\mathbb{N}} \frac{|A \cap [x, x+n-1]|}{n}.$$

It is possible to have $\mathrm{BD}(A) > 0$ while $\bar{d}(A) = 0$, so $\mathrm{BD}(A) > 0$ is a milder assumption.

Theorem (DGJLLM, 2013)

Let $A \subseteq \mathbb{N}$.

1. *If $\mathrm{BD}(A) > 1/2$, then A satisfies the conclusion of the B+C conjecture.*
2. *If $\mathrm{BD}(A) > 0$, then there are infinite $B, C \subseteq \mathbb{N}$ and $k \in \mathbb{N}$ such that $B + C \subseteq A \cup (A + k)$. Moreover, enumerating $B = (b_i)$ and $C = (c_i)$ in increasing order, which translate $b_i + c_j$ lands in depends only on whether $i < j$ or $i \geq j$.*

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(2) implies (1)

- For $A \subseteq \mathbb{N}$ and $n \in \mathbb{N}$, set

$$A_{[n]} := \{x \in \mathbb{N} \ : \ A \cap [nx, nx + n - 1] \neq \emptyset\}.$$

- It is relatively straightforward to check that, if $\mathrm{BD}(A) > 0$, then for any $\epsilon > 0$, there is $n \in \mathbb{N}$ such that $\mathrm{BD}(A_{[n]}) > 1 - \epsilon$.
- Take $n \in \mathbb{N}$ such that $\mathrm{BD}(A_{[n]}) > 1/2$ and take infinite B', C' such that $B' + C' \subseteq A_{[n]}$, that is, writing $B' = (b_i)$ and $C' = (c_i)$, we have $[nb_i + nc_j, nb_i + nc_j + n - 1] \cap A \neq \emptyset$ for each i, j.
- By Ramsey's Theorem, we may assume that there are $m_1, m_2 \in [0, n - 1]$ such that, for any $i < j$, we have $nb_i + nc_j + m_1 \in A$, $nb_j + nc_i + m_2 \in A$.
- Taking $B := \{nb_i + m_1 \ : i \text{ is even}\}$, $C := \{nc_j \ : \ j \text{ is odd}\}$, and $k := m_1 - m_2$, we have $B + C \subseteq A \cup (A + k)$.

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Nonstandard analysis

- Our proofs use techniques from *nonstandard analysis*.
- But why?
- Densities on natural numbers “feel like” measures but often lack many of the nice properties of measures.
- It is often useful to replace statements about densities by statements about measures.
- Case in point: Furstenberg’s correspondence principle
- It turns out that densities on sets of natural numbers are intimately related to certain measures on their nonstandard extensions, namely the *Loeb measures*.

An axiomatic approach to $\mathbb{R}^*$

We will work in a *nonstandard universe* $\mathbb{R}^*$ that has the following properties:

1. $(\mathbb{R}; +, \cdot, 0, 1, <)$ is an *ordered subfield* of $(\mathbb{R}^*; +, \cdot, 0, 1, <)$.
2. $\mathbb{R}^*$ has a *positive infinitesimal* element, that is, there is $\epsilon \in \mathbb{R}^*$ such that $\epsilon > 0$ but $\epsilon < r$ for every $r \in \mathbb{R}^{>0}$.
3. For every $n \in \mathbb{N}$ and every function $f : \mathbb{R}^n \to \mathbb{R}$, there is a "natural extension" $f : (\mathbb{R}^*)^n \to \mathbb{R}^*$. The natural extensions of the field operations $+, \cdot : \mathbb{R}^2 \to \mathbb{R}$ coincide with the field operations in $\mathbb{R}^*$. Similarly, for every $A \subseteq \mathbb{R}^n$, there is a subset $A^* \subseteq (\mathbb{R}^*)^n$ such that $A^* \cap \mathbb{R}^n = A$.
4. $\mathbb{R}^*$, equipped with the above assignment of extensions of functions and subsets, "behaves logically" like $\mathbb{R}$.

Standard parts

- Say that $x \in \mathbb{R}^*$ is *finite* if $|x| \leq n$ for some $n \in \mathbb{N}$.
- For example, for any $r \in \mathbb{R}$ and any (positive or negative) infinitesimal ϵ, $r + \epsilon$ is finite.
- Conversely:

Fact

If $x \in \mathbb{R}^*$ is finite, then there is a unique $r \in \mathbb{R}^{>0}$ such that $x - r$ is infinitesimal. We call r *the standard part of* x and denote it by $\mathrm{st}(x)$.

Proof.

WLOG, $x > 0$. Let $A := \{r \in \mathbb{R}^{>0} : r < x\}$. Then $0 \in A$ and A is bounded above (since x is finite). By the completeness of the reals, $\sup(A)$ exists. Check that $\mathrm{st}(x) = \sup(A)$. □

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Extending sequences

- Recall that every function $f : \mathbb{R} \to \mathbb{R}$ has a nonstandard extension $f : \mathbb{R}^* \to \mathbb{R}^*$.
- Partial functions $f : A \to \mathbb{R}$ have nonstandard extensions $f : A^* \to \mathbb{R}^*$ as well.
- In particular, if $(a_n \ : \ n \in \mathbb{N})$ is a sequence of reals, viewing (a_n) as the function $a : \mathbb{N} \to \mathbb{R}$, we get a nonstandard extension $a : \mathbb{N}^* \to \mathbb{R}^*$. We also write this in sequence notation $(a_n \ : \ n \in \mathbb{N}^*)$ and refer to a_ν for $\nu \in \mathbb{N}^* \setminus \mathbb{N}$ as an *extended term of the sequence*.

Subsequential limits

Lemma

If (a_n) is a sequence and $L \in \mathbb{R}$, then L is a subsequential limit of (a_n) if and only if there is $\nu \in \mathbb{N}^* \setminus \mathbb{N}$ such that a_ν is finite and $\mathrm{st}(a_\nu) = L$.

Proof of the “if” direction.

Set $L := \mathrm{st}(a_\nu)$. Then for every $m \in \mathbb{N}$ and $\epsilon \in \mathbb{R}^{>0}$, $\mathbb{R}^*$ believes the statement “there is $n \in \mathbb{N}^*$ such that $n > m$ and $|a_n - L| < \epsilon$.” Consequently, $\mathbb{R}$ believes the statement “there is $n \in \mathbb{N}$ such that $n > m$ and $|a_n - L| < \epsilon$.” □

Corollary (Bolzano-Weierstrauss)

Every bounded sequence has a convergent subsequence.

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Nonstandard characterization of densities

- If (a_n) is a bounded sequence, we see that

$$\liminf a_n = \min\{\mathrm{st}(a_\nu) \ : \ \nu \in \mathbb{N}^* \setminus \mathbb{N}\}$$

and

$$\limsup a_n = \max\{\mathrm{st}(a_\nu) \ : \ \nu \in \mathbb{N}^* \setminus \mathbb{N}\}.$$

- Consequently, for $A \subseteq \mathbb{N}$, we have

$$\underline{d}(A) = \min\left\{\mathrm{st}\left(\frac{|A^* \cap [1,\nu]|}{\nu}\right) \ : \ \nu \in \mathbb{N}^* \setminus \mathbb{N}\right\}$$

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Internal sets

- The point of passing to the nonstandard framework is that the quantities $\text{st}\left(\frac{|A^* \cap [1,\nu]|}{\nu}\right)$ appearing in the nonstandard characterizations of the densities are actually certain measures on A^*, called *Loeb measures*. To define Loeb measure, we first need the concept of internal sets and hyperfinite sets.
- *Internal* subsets of $\mathbb{R}^*$ are the "definable" subsets of $\mathbb{R}^*$ in some precise way that we won't define. They "logically behave" like ordinary subsets of $\mathbb{R}$. For example, $A^* \cap [1,\nu]$ is an internal set.
- The set of all infinitesimals is not internal. Indeed, nonempty internal subsets of $\mathbb{R}^*$ bounded above have a sup. But what would the sup of the infinitesimals be?
- An internal set is *hyperfinite* if there is an internal bijection between it and an interval of the form $[1,\nu]$ from $\mathbb{N}^*$. Internal subsets of hyperfinite sets are hyperfinite, so, e.g., $A^* \cap [1,\nu]$ is hyperfinite.

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Loeb measure

- Suppose that $E \subseteq \mathbb{R}^*$ is hyperfinite. Then there is a unique $\nu \in \mathbb{N}^*$ such that there is an internal bijection $E \to [1, \nu]$; we call ν the *internal cardinality of E* and denote it by $|E|$.
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- Then μ_E is a finitely additive measure. Under a very mild assumption on the nonstandard extension, it can be shown that μ_E satisfies the conditions of the Caratheodory extension theorem, so extends to a countably additive measure on a certain σ-algebra containing the internal subsets of E; this measure is called the *Loeb measure*.
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Nonstandard characterization of densities again

- For $\nu \in \mathbb{N}^* \setminus \mathbb{N}$, let μ_ν be the Loeb measure on $[1, \nu]$.
- For $A \subseteq \mathbb{N}$, we have

$$\underline{d}(A) = \min\{\mu_\nu(A^* \cap [1, \nu]) \ : \ \nu \in \mathbb{N}^* \setminus \mathbb{N}\}$$

and

$$\bar{d}(A) = \max\{\mu_\nu(A^* \cap [1, \nu]) \ : \ \nu \in \mathbb{N}^* \setminus \mathbb{N}\}.$$

Reminder of the Main Theorem

Theorem

Suppose that $\mathrm{BD}(A) > \frac{1}{2}$*. Then there exists infinite* $B, C \subseteq \mathbb{N}$ *such that* $B + C \subseteq A$*.*

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Suppose that $\mathrm{BD}(A) := \alpha > 0$ and that (I_n) is a sequence of intervals with $|I_n| \to \infty$ such that $\lim_{n\to\infty} \frac{|A\cap I_n|}{|I_n|} = \alpha$. Then there is $L \subseteq \mathbb{N}$ such that:

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Bergelson's Theorem

The key measure-theoretic result that we will use is the following theorem of Bergelson:

Fact

Suppose that $(X, \mathcal{B}, \mu)$ is a probability space and (A_n) is a sequence of measurable sets for which there is $a \in (0, 1]$ such that $\mu(A_n) \geq a$ for all n. Then there is infinite $P \subseteq \mathbb{N}$ such that, for every finite $F \subseteq P$, we have

$$\mu(\bigcap_{n \in F} A_n) > 0.$$

Proof of main theorem

- Fix (I_n) witnessing that $\mathrm{BD}(A) = \alpha > 1/2$. Fix $L = (\ell_n)$ satisfying the conclusion of the key technical lemma.
- Recursively define $D := (d_n) \subseteq A$ such that $\ell_i + d_n \in A$ for $i \leq n$.
- Fix $\nu \in \mathbb{N}^* \setminus \mathbb{N}$ such that $\mu(L^* \cap I_\nu) = \mathrm{st}\left(\frac{|L^* \cap I_\nu|}{|I_\nu|}\right) \geq \alpha$.
- Then, for every $n \in \mathbb{N}$, we have

$$\mu(L^* \cap (A^* - d_n) \cap I_\nu) \geq 2\alpha - 1 > 0.$$

- By Bergelson, after passing to a subsequence, we may assume that, for all $n \in \mathbb{N}$, we have

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Proof of main theorem (cont'd)

- The takeaway: for every $n \in \mathbb{N}$, we have $L \cap \bigcap_{i \leq n}(A - d_i)$ is infinite.
- We are now home free. Pick $b_1 \in L$ arbitrary and take $c_1 \in D$ such that $b_1 + c_1 \in A$.
- Now take $b_2 \in (L \cap (A - c_1)) \setminus \{b_1\}$ and take $c_2 \in D$ such that $b_1 + c_2, b_2 + c_2 \in A$.
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Proof of the key technical lemma

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- Suppose we have constructed internal sets $X_1, X_2, ..., X_{i-1} \subseteq I_\nu$ and standard natural numbers $n_1 < n_2 < n_3 < \cdots < n_{i-1}$ with the desired properties.
- Fix $K \in \mathbb{N}^* \setminus \mathbb{N}$ and set Z to be the set of all $M \in \mathbb{N}^*$ such that:
  - $n_{i-1} < M \leq K$;
  - $\delta_\nu(\{x \in I_\nu : \delta_M(A^* \cap (x + I_M)) \geq \alpha - \frac{1}{i}\}) > 1 - \epsilon^i$.
- Then Z is internal. An appropriate choice of K and a calculation (to be done on the next slide) shows that Z contains all elements of $\mathbb{N}^* \setminus \mathbb{N}$ below K.
- By *underflow*, we must have $Z \cap \mathbb{N} \neq \emptyset$. Take $n_i \in Z \cap \mathbb{N}$ and define X_i as it should be.

Proof of the key technical lemma (cont'd)

- Suppose we have constructed internal sets $X_1, X_2, ..., X_{i-1} \subseteq I_\nu$ and standard natural numbers $n_1 < n_2 < n_3 < \cdots < n_{i-1}$ with the desired properties.
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Proof of the key technical lemma (conclusion)

Fix $\mathbb{N} < M \leq K$. If K is "small enough", we have that

$$\begin{aligned}\frac{1}{|I_\nu|}\sum_{x\in I_\nu}\delta_M(A^*\cap(x+I_M)) &= \frac{1}{|I_M|}\sum_{y\in I_M}\frac{1}{|I_\nu|}\sum_{x\in I_\nu}\chi_{A^*}(x+y)\\ &\approx \frac{1}{|I_M|}\sum_{y\in I_M}\delta_\nu(A^*\cap I_\nu)\\ &\approx \alpha.\end{aligned}$$

Since $\mathrm{BD}(A)=\alpha$, we have that $\mathrm{st}(\delta_M(A^*\cap(x+I_M)))\leq\alpha$, so μ_ν-almost all $x\in I_\nu$ are such that $\delta_M(A^*\cap(x+I_M))\approx\alpha$. □

Proof of the key technical lemma (conclusion)

Fix $\mathbb{N} < M \leq K$. If K is "small enough", we have that

$$\frac{1}{|I_\nu|} \sum_{x \in I_\nu} \delta_M(A^* \cap (x + I_M)) = \frac{1}{|I_M|} \sum_{y \in I_M} \frac{1}{|I_\nu|} \sum_{x \in I_\nu} \chi_{A^*}(x + y)$$

$$\approx \frac{1}{|I_M|} \sum_{y \in I_M} \delta_\nu(A^* \cap I_\nu)$$

$$\approx \alpha.$$

Since $\mathrm{BD}(A) = \alpha$, we have that $\mathrm{st}(\delta_M(A^* \cap (x + I_M))) \leq \alpha$, so μ_ν-almost all $x \in I_\nu$ are such that $\delta_M(A^* \cap (x + I_M)) \approx \alpha$. □

References

- M. DiNasso, I. Goldbring, R. Jin, S. Leth, M. Lupini, and K. Mahlburg, *Progress on a sumset conjecture of Erdos*, submitted. arXiv 1307.0767