ON THE AXIOMATIZABILITY OF C*-ALGEBRAS AS OPERATOR SYSTEMS

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ABSTRACT. We show that the class of unital C*-algebras is an elementary class in the language of operator systems and that the algebra multiplication is a definable function in this language. Moreover, we prove that the aforementioned class is $\forall\exists\forall$-axiomatizable but not $\forall\exists$-axiomatizable nor $\exists\forall$-axiomatizable.

Recall that a C*-algebra is a *-subalgebra of $\mathcal{B}(H)$, the *-algebra of bounded operators on a complex Hilbert space, that is closed in the operator norm topology. In this note, we assume that all C*-algebras are unital, namely that they contain the identity operator. As shown in [5, Proposition 3.3], there is a natural (continuous) first-order language $\mathcal{L}_{\mathrm{C}^*}$ in which $\mathcal{K}_{\mathrm{C}^*}$, the class of $\mathcal{L}_{\mathrm{C}^*}$-structures that are unital C*-algebras, is an elementary class, meaning that there is a (universal) $\mathcal{L}_{\mathrm{C}^*}$-theory T_{C^*} for which $\mathcal{K}_{\mathrm{C}^*}$ is the class of models of T_{C^*}; in symbols, $\mathcal{K}_{\mathrm{C}^*} = \mathrm{Mod}(T_{\mathrm{C}^*})$. (The authors only treat not necessarily unital C*-algebras, but one just adds a constant to name the identity with no additional complications.)

An operator system is a *-closed subspace of $\mathcal{B}(H)$ that is closed in the operator norm topology. The appropriate morphisms between operator systems are the unital completely positive linear maps. There is a natural first-order language $\mathcal{L}_{\mathrm{os}}$ in which the class of operator systems is universally axiomatizable; see [3, Subsection 3.3] and [7, Appendix B]. Since the operator system structure on a C*-algebra is uniformly quantifier-free definable, we may assume that $\mathcal{L}_{\mathrm{os}} \subseteq \mathcal{L}_{\mathrm{C}^*}$. For a C*-algebra A, we let $A|\mathcal{L}_{\mathrm{os}}$ denote the reduct of A to $\mathcal{L}_{\mathrm{os}}$, which simply means that we view A merely as an operator system rather than as a C*-algebra. Set $\mathcal{K}_{\mathrm{C}^*}|\mathcal{L}_{\mathrm{os}} := \{A|\mathcal{L}_{\mathrm{os}} \ : \ A \in \mathcal{K}_{\mathrm{C}^*}\}$. In [6], the following question was raised: is $\mathcal{K}_{\mathrm{C}^*}|\mathcal{L}_{\mathrm{os}}$ an elementary class? The main result of this note is to give an affirmative answer to this question.

We first need a lemma, which is nearly identical to [2, Theorem 6.1]. Some notation: for a C*-algebra B and $x, y, z, b \in B$, let $\varphi(x, y, z, b)$ be the $\mathcal{L}_{\mathrm{os}}$-formula

$$\left| \left\| \begin{matrix} 0 & y & 1 & 0 \\ 2 \cdot 1 & x & z & b \end{matrix} \right\|^2 - \left\| \begin{matrix} 2 \cdot 1 & x & z & b \end{matrix} \right\|^2 \right|$$

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Lemma 1. *Suppose that A is a subsystem of the unital* C**-algebra B. Then A is closed under products if and only if we have*

$$\sup_{x,y\in A_1}\inf_{z\in A_1}\sup_{b\in B_2}\varphi(x,y,z,b)=0.$$

Proof. Fix $\epsilon\in(0,1)$ and $x,y\in A_1$. Choose $z\in A_1$ such that

$$\sup_{b\in B_2}\varphi(x,y,z,b)<\epsilon$$

and let b be the square root of $\|xx^*+zz^*\|\cdot 1-xx^*-zz^*\in B$ so that $\|b\|^2=\|b^2\|\leq\|xx^*+zz^*\|\leq 2$. Multiplying $[2\cdot 1\quad x\quad z\quad b]$ by its adjoint have that

$$\left\|2\cdot 1\quad x\quad z\quad b\right\|^2=4\cdot 1+xx^*+zz^*+bb^*=(4+\|xx^*+zz^*\|)\cdot 1.$$

Similarly, it holds that

$$\left\|\begin{matrix}0 & y & 1 & 0\\ 2\cdot 1 & x & z & b\end{matrix}\right\|^2=\left\|\begin{matrix}1+yy^* & yx^*+z^*\\ xy^*+z & (4+\|xx^*+zz^*\|)\cdot 1\end{matrix}\right\|.$$

Examining the second row, it follows that the norm of the right side is at least $(\|xy^*+z\|^2+(4+\|xx^*+zz^*\|)^2)^{1/2}$, whence $\|xy^*+z\|\leq 4\sqrt{\epsilon}$. As A is complete, it follows that $xy^*\in A$.

Conversely, the above calculations show that if A is multiplicatively closed, then setting $z=-xy^*$ suffices. □

Theorem 2. *Let B be a* C**-algebra. If $A\subset B$ is a subsystem of B which is an elementary substructure in the language of operator systems, then A is a* C**-subalgebra of B.*

Proof. Since A inherits the unit of B and A is self-adjoint, we need only check that A is closed under products, that is, we need only verify the condition of the previous lemma. Fixing $x,y\in A_1$, we have

$$\inf_{z\in B_1}\sup_{b\in B_2}\varphi(x,y,z,b)=0.$$

Since A is an elementary substructure of B, we have

$$\inf_{z\in A_1}\sup_{b\in A_2}\varphi(x,y,z,b)=0.$$

Fix $\epsilon>0$ and take $z\in A_1$ such that

$$\sup_{b\in A_2}\varphi(x,y,z,b)\leq\epsilon.$$

By elementarity again, we have

$$\sup_{b\in B_2}\varphi(x,y,z,b)\leq\epsilon,$$

whence we have

$$\inf_{z\in A_1}\sup_{b\in B_2}\varphi(x,y,z,b)=0,$$

which is what we desired. □

Corollary 3. *$\mathcal{K}_{\mathrm{C}^*}|\mathcal{L}_{\mathrm{os}}$ is an elementary class.*

Proof. We use the semantic test for axiomatizability, that is, we show that $\mathcal{K}_{\mathrm{C}^*}|\mathcal{L}_{\mathrm{os}}$ is closed under isomorphism, ultraproduct, and ultraroot. (See [1, Proposition 5.14].) Closure under isomorphism and ultraproducts is clear. We thus only need to show that $\mathcal{K}_{\mathrm{C}^*}|\mathcal{L}_{\mathrm{os}}$ is closed under ultraroots. So suppose that A is an $\mathcal{L}_{\mathrm{os}}$-structure for which $A^{\mathcal{U}} \in \mathcal{K}_{\mathrm{C}^*}|\mathcal{L}_{\mathrm{os}}$. Note that this implies that A is an operator subsystem of $A^{\mathcal{U}}$, whence A is an elementary substructure by Łos' theorem. Thus A is a C*-subalgebra of $A^{\mathcal{U}}$. □

Recall that a formula $\psi(\vec{x})$ is *weakly stable* (relative to some elementary class $\mathcal{K}$) if, for every $\epsilon > 0$, there is $\delta > 0$ such that for every $A \in \mathcal{K}$ and every $\vec{a} \in A$ with $|\psi(\vec{a})|^A < \delta$, there is $\vec{b} \in A$ with $\max_i \|a_i - b_i\| < \epsilon$ and $\psi(\vec{b})^A = 0$. (See [4, Section 3.2].) The *proof* of Lemma 1 shows the following:

Corollary 4. *Set $\psi_{\mathrm{mult}}(x, y, z) := \sup_{\|b\| \leq 2} \varphi(-x, y^*, z, b)$. Then ψ_{mult} is a weakly stable formula relative to $\mathcal{K}_{\mathrm{C}^*}|\mathcal{L}_{\mathrm{os}}$.*

Now suppose that for every $A \in \mathcal{K}$ we fix a function $f^A : A^n \to A$ (with n independent of A). We say that f is a *definable function* (once again relative to $\mathcal{K}$) if there is a weakly stable formula $\psi(\vec{x}, y)$ (relative to $\mathcal{K}$) such that for all $A \in \mathcal{K}$ and all $\vec{a}, b \in A$, we have $f^A(\vec{a}) = b$ if and only if $\psi(\vec{a}, b)^A = 0$. Once again, the proof of Lemma 1 shows:

Corollary 5. *For $A \in \mathcal{K}_{\mathrm{C}^*}|\mathcal{L}_{\mathrm{os}}$, we let $\mathrm{mult}^A : A^2 \to A$ be the function given by $\mathrm{mult}^A(x, y) := x \cdot y$. Then ψ_{mult} witnesses that mult is a definable function relative to $\mathcal{K}_{\mathrm{C}^*}|\mathcal{L}_{\mathrm{os}}$.*

Let $\mathcal{L}_{\mathrm{os}}^{\#}$ be the language obtained from $\mathcal{L}_{\mathrm{os}}$ obtained by adding a new ternary predicate Q. Let $T^{\#}_{\mathrm{C}^*,\mathrm{os}}$ be the $\mathcal{L}^{\#}$-theory obtained from $T_{\mathrm{C}^*,\mathrm{os}}$ by adding the axiom

$$\sup_{x,y,z} |Q(x, y, z) - \psi_{\mathrm{mult}}(x, y, z)| = 0.$$

Proposition 6. *$T^{\#}_{\mathrm{C}^*,\mathrm{os}}$ is $\forall\exists$-axiomatizable.*

Proof. It suffices to show that a union of a chain of models of $T^{\#}_{\mathrm{C}^*,\mathrm{os}}$ is once again a model of $T^{\#}_{\mathrm{C}^*,\mathrm{os}}$ (see [4, Section 2.4]). If $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ is an ascending sequence of models of $T^{\#}_{\mathrm{C}^*,\mathrm{os}}$, then each A_m is a C*-algebra and the inclusions are in fact $*$-homomorphisms. It follows that the union is a C*-algebra and hence a model of $T_{\mathrm{C}^*,\mathrm{os}}$. It is readily verified that the extra axiom of $T^{\#}_{\mathrm{C}^*,\mathrm{os}}$ also holds in the union, whence the union is a model of $T^{\#}_{\mathrm{C}^*,\mathrm{os}}$. □

In classical logic, the next corollary would be immediate. In the continuous case, a few words are in order.

Corollary 7. *$T_{\mathrm{C}^*,\mathrm{os}}$ is $\forall\exists\forall$-axiomatizable.*

Proof. Let $\sigma := \sup_x \inf_y \varphi(x,y) = 0$ be an $\mathcal{L}_{\rm os}^{\#}$-axiom for $T^{\#}_{\rm C^*,os}$. We can view φ as an $\mathcal{L}_{\rm os}$-formula by replacing any occurrence of the new predicates Q_n by the $\mathcal{L}_{\rm os}$-formula ψ_n. Fix $\epsilon > 0$ and take a *restricted* $\mathcal{L}_{\rm os}$-formula $\theta(x,y)$ that approximates φ to within ϵ. (See [1, Section 6] for the definition of restricted formula.) The key property of restricted formulae is that they are monotone in their arguments. Thus, an easy induction shows that any formula formed from restricted connectives and atomic and existential formulae is logically equivalent to both a $\forall\exists$- and a $\exists\forall$-formulae. It follows that, without loss of generality, we may assume that θ is an $\exists\forall$-formula of $\mathcal{L}_{\rm os}$. The axiom $\sup_x \inf_y(\theta(x,y) \dot- \epsilon) = 0$ is an $\forall\exists\forall$-axiom and the totality of these axioms has the same expressive power as the original axiom. □

There is an alternative way of establishing the previous corollary. Indeed, one can take the universal axiomatization for C*-algebras given in [5] and "substitute" all mention of the multiplication by the formula $\psi_{\rm mult}$. For example, the axiom expressing the C*-identity could instead be written as the following $\mathcal{L}_{\rm os}^{\#}$-sentence:

$$\sup_x \inf_y \max(\psi_{\rm mult}(x, x^*, y), |\|y\| - \|x\|^2|) = 0.$$

Since $\psi_{\rm mult}$ is a universal formula in the language $\mathcal{L}_{\rm os}$, this sentence really is an $\forall\exists\forall$-sentence in the language $\mathcal{L}_{\rm os}$. Of course, in addition one needs to express that the zeroset of $\psi_{\rm mult}$ really is a function. The way one does this in continuous logic is via the bounds given in the proof of the main lemma. In other words, one would also need to add the following two axioms (the second of which is actually an axiom *scheme*, one such axiom for each rational ϵ):

- $\sup_{x,y} \inf_z \psi_{\rm mult}(x,y,z) = 0$;
- $\sup_{x,y,z_1,z_2} \min(\min_i \epsilon \dot- \psi_{\rm mult}(x,y,z_i), d(z_1,z_2) \dot- 8\sqrt{\epsilon}) = 0$.

Although we have established that $\mathcal{K}_{\rm C^*}|\mathcal{L}_{\rm os}$ is an $\forall\exists\forall$-axiomatizable $\mathcal{L}_{\rm os}$-theory, it is a priori possible that it is in fact two quantifier axiomatizable. Our last two results show that this is not the case. Recall that if X is an operator system and $u \in X$, then u is called a unitary of X if u is a unitary of the C*-envelope $\mathrm{C}^*_e(X)$.

Proposition 8. *$\mathcal{K}_{\rm C^*}|\mathcal{L}_{\rm os}$ is* not *$\forall\exists$-axiomatizable.*

Proof. If $\mathcal{K}_{\rm C^*}|\mathcal{L}_{\rm os}$ were $\forall\exists$-axiomatizable, then there would be $A \in \mathcal{K}_{\rm C^*}|\mathcal{L}_{\rm os}$ that is existentially closed for $\mathcal{K}_{\rm C^*}|\mathcal{L}_{\rm os}$. However, in [6, Section 5], it was observed that if $\phi : X \to Y$ is a complete order embedding that is also existential, then ϕ maps unitaries to unitaries. Take a complete order embedding of A into a C*-algebra B that is not a *-homomorphism (see, for example, [6, Section 5]); since this embedding maps unitaries to unitaries (since A is existentially closed for $\mathcal{K}$), this contradicts a well-known consequence of Pisier's linearization trick. (For the convenience of the reader, we include a proof of this fact in the appedix.) □

Proposition 9. *$\mathcal{K}_{\mathrm{C}^*}|\mathcal{L}_{\mathrm{os}}$ is* not *$\exists\forall$-axiomatizable.*

Proof. Fix a C*-algebra A and an operator system X that is **not** a C*-algebra with $A \subseteq X \subseteq A^{\mathcal{U}}$. Suppose, towards a contradiction, that $\mathcal{K}_{\mathrm{C}^*}|\mathcal{L}_{\mathrm{os}}$ is $\exists\forall$-axiomatizable and let $\sigma := \inf_x \sup_y \varphi(x,y)$ be such an axiom. Fix $\epsilon > 0$ and take $a \in A$ such that $(\sup_y \varphi(a,y))^A \leq \epsilon$. It follows that $(\sup_y \varphi(a,y))^{A^{\mathcal{U}}} \leq \epsilon$, whence $(\sup_y \varphi(a,y))^X \leq \epsilon$. Since $a \in X$ and $\epsilon > 0$ was arbitrary, we see that $\sigma^X = 0$. Since σ was an arbitrary axiom, we see that X is a C*-algebra, yielding a contradiction. □

Appendix on Pisier's Linearization Trick

The following facts follow immediately from Stinespring's Dilation Theorem; see, for example, [8, Theorem 18].

Fact 10. *Suppose that $\phi : A \to B$ is a u.c.p. map between C*-algebras. Then for all $x, y \in A$, we have*

$$\phi(x)^*\phi(x) \leq \phi(x^*x)$$

and

$$\|\phi(y^*x) - \phi(y)^*\phi(x)\| \leq \|\phi(y^*y) - \phi(y)^*\phi(y)\|^{1/2}\|\phi(x^*x) - \phi(x)^*\phi(x)\|^{1/2}.$$

Corollary 11. *Suppose that $\phi : A \to B$ is u.c.p. map between C*-algebras that maps unitaries to unitaries. Then ϕ is a *-homomorphism.*

Proof. The previous fact shows that the set

$$M_\phi := \{a \in A : \phi(a^*)\phi(a) = \phi(a^*a),\ \phi(a)\phi(a^*) = \phi(aa^*)\}$$

is a C*-subalgebra of A on which ϕ is a *-homomorphism. By assumption we have that $\mathcal{U}(A) \subset M_\phi$ whence $M_\phi = A$. □

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