ON POPA'S FACTORIAL COMMUTANT EMBEDDING PROBLEM

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ABSTRACT. An open question of Sorin Popa asks whether or not every $\mathcal{R}^{\mathcal{U}}$-embeddable factor admits an embedding into $\mathcal{R}^{\mathcal{U}}$ with factorial relative commutant. We show that there is a locally universal McDuff II_1 factor M such that every property (T) factor admits an embedding into $M^{\mathcal{U}}$ with factorial relative commutant. We also discuss how our strategy could be used to settle Popa's question for property (T) factors if a certain open question in the model theory of operator algebras has a positive solution.

1. INTRODUCTION

In this note, all II_1 factors are assumed to be separable unless they are ultrapowers. $\mathcal{R}$ denotes the hyperfinite II_1 factor. $\mathcal{U}$ denotes an arbitrary nonprincipal ultrafilter on $\mathbb{N}$. We say that a factor is **embeddable** if it embeds into $\mathcal{R}^{\mathcal{U}}$. In order to avoid any set-theoretic subtleties, we also assume that the Continuum Hypothesis (CH) holds.[1]

The starting point of this note is the following question of Popa:

Question 1.1 (The factorial commutant embedding problem (FCEP)). *Suppose that N is an embeddable factor. Is there an embedding $\pi : N \hookrightarrow \mathcal{R}^{\mathcal{U}}$ such that $\pi(N)' \cap \mathcal{R}^{\mathcal{U}}$ is a factor?*

The question is known to have a positive answer in some cases, e.g. $N = \mathcal{R}$ [5, Proposition 12] and $N = SL_3(\mathbb{Z})$ [14, Section 1.7], but seems to be wide-open in general. The question itself even seems to be open for the class of property (T) factors.

The main result of this note, proven in Section 2, is that there is a McDuff II_1 factor making the conclusion of the FCEP true for all property (T) factors:

Theorem. *There is a locally universal McDuff II_1 factor M such that, for any property (T) factor N, there is an embedding $\pi : N \hookrightarrow M^{\mathcal{U}}$ such that $\pi(N)' \cap M^{\mathcal{U}}$ is a factor.*

We recall that a **locally universal** factor is one whose ultrapower contains all (separable) II_1 factors. Locally universal factors were first shown to exist in [7, Example 6.4(2)], thus providing a "poor man's resolution" to the Connes Embedding Problem (CEP). Thus, in some sense, our theorem is a "poor man's resolution" to the FCEP for property (T) factors.

[1]It would be interesting to investigate if any of our results depend on set theory

Recently, a negative solution to the CEP was announced in [13]; if correct, it would imply that a locally universal factor is not embeddable. It thus makes sense to wonder whether or not M as in the previous theorem can be taken to be embeddable if one restricts attention to embeddable property (T) factors; we discuss a hurdle to this being true in Section 3, where we also discuss how the success of this approach to settle the FCEP for property (T) factors is connected to an open question about so-called infinitely generic embeddable factors.

Popa's question was given a geometric reformulation by Nate Brown in [4, Proposition 5.2], who showed that an embedding $\pi : N \hookrightarrow \mathcal{R}^{\mathcal{U}}$ has factorial commutant if and only if $[\pi]$ is an extreme point in the convex-like space $\mathrm{Hom}(N, \mathcal{R}^{\mathcal{U}})$ of embeddings of N into $\mathcal{R}^{\mathcal{U}}$ modulo unitary equivalence. Scott Atkinson [1, Theorem 5.4] showed a similar result when $\mathcal{R}$ is replaced by a McDuff factor. Consequently, our result shows that, for the M as in the above theorem, $\mathrm{Hom}(N, M^{\mathcal{U}})$ has an extreme point for any property (T) factor N.

Our proofs use ideas from model theory although we do our best to provide logic-free definitions of the main concepts. Crucial to our proof are some results from our earlier works [6] and [9].

We would like to thank Scott Atkinson and Srivatsav Kunnawalkam Elayavalli for bringing Popa's question to our attention and for useful conversations regarding this work. We would also like to thank Sorin Popa for providing historical context for his question and for providing us with some references.

2. The main theorem

We recall the following definition:

Definition 2.1. *Suppose that M is a II$_1$ factor with a subfactor N. We say that N has **w-spectral gap in** M if $N' \cap M^{\mathcal{U}} = (N' \cap M)^{\mathcal{U}}$.*

We remind the reader that property (T) factors have w-spectral gap in any extension.

We will need the following notion from model theory, defined in ultra-power[2] terms.

Definition 2.2. *If M is a subfactor of the factor Q, we say that M is **existentially closed in** Q if there is an embedding $Q \hookrightarrow M^{\mathcal{U}}$ that restricts to the diagonal embedding $M \hookrightarrow M^{\mathcal{U}}$. We say that the II$_1$ factor M is **existentially closed** (**e.c.**) if it is existentially closed in all extensions.*

The following is [10, Section 2].

Fact 2.3. *An e.c. factor is locally universal and McDuff.*

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[2] Read: operator algebraist-friendly

Definition 2.4. *A class $\mathcal{C}$ of II_1 factors is said to be **extensive**[3] if every II_1 factor embeds in an element of $\mathcal{C}$.*

The following are well-known (see, e.g. [15, Fact 2.8]).

Fact 2.5. *The class of e.c. factors is extensive.*

Fact 2.6. *E.c. factors are locally universal and McDuff.*

The following appears in [9]:

Fact 2.7. *Suppose that N is a w-spectral gap subfactor of the e.c. factor M. Then $(N' \cap M)' \cap M = N$.*

Theorem 2.8. *Suppose that N is a w-spectral gap subfactor of the e.c. factor M. Then $N' \cap M$ is a factor.*

Proof. Set $P := N' \cap M$. We show that P is a factor. Take $x \in P$ such that $[x, y] = 0$ for all $y \in P$. Then $x \in (N' \cap M)' \cap M = N$, so $x \in N$. Now suppose that $z \in N$. Then since $x \in P$, we have $[x, z] = 0$. So $x \in Z(N) = \mathbb{C}$, as desired. $\qquad\square$

Corollary 2.9. *Suppose that N is a w-spectral gap subfactor of the e.c. factor M. Then $N' \cap M^{\mathcal{U}}$ is a factor.*

Proof. $N' \cap M^{\mathcal{U}} = (N' \cap M)^{\mathcal{U}}$ and the ultrapower of a factor is once again a factor. $\qquad\square$

Although we won't need the following result, it might be of independent interest:

Digression 2.10. *If N is a w-spectral gap of the e.c. factor M, then $N' \cap M$ is a locally universal McDuff II_1 factor.*

Proof. Once again, set $P := N' \cap M$. We first show that P is locally universal. Let Q be any II_1 factor. Since M is e.c. in $M \otimes Q$, we have an embedding $M \otimes Q \hookrightarrow M^{\mathcal{U}}$ that restircts to the diagonal embedding $M \hookrightarrow M^{\mathcal{U}}$. In particular, $Q \hookrightarrow M' \cap M^{\mathcal{U}} \subseteq N' \cap M^{\mathcal{U}} = (N' \cap M)^{\mathcal{U}} = P^{\mathcal{U}}$, whence P is locally universal.

Since P is locally universal, it follows that P is a II_1 factor.

Finally, we show that P is McDuff. It suffices to show that $M_2(\mathbb{C})$ embeds in $P' \cap P^{\mathcal{U}}$. Take an embedding $M \otimes M_2(\mathbb{C}) \hookrightarrow M^{\mathcal{U}}$ restricting to the diagonal embedding of $M \hookrightarrow M^{\mathcal{U}}$. As in the previous argument, this embedding sends $M_2(\mathbb{C})$ into $P^{\mathcal{U}}$. Moreover, since $M' \cap M^{\mathcal{U}} \subseteq P' \cap M^{\mathcal{U}}$, this embedding sends $M_2(\mathbb{C})$ into $P' \cap P^{\mathcal{U}}$, as desired. $\qquad\square$

Returning to the main thread, at this moment, we simply have that every property (T) factor N embeds in a II_1 factor M such that the diagonal embedding $N \hookrightarrow M^{\mathcal{U}}$ has factorial relative commutant. We would like a single M that works for all property (T) factors. This leads us to the following:

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[3]In the model-theoretic literature, one would say that $\mathcal{C}$ is model-consistent with the class of II_1 factors. We prefer the above terminology.

Definition 2.11. *II_1 factors M_1 and M_2 are said to be **elementarily equivalent**, denoted $M_1 \equiv M_2$, if $M_1^{\mathcal{U}} \cong M_2^{\mathcal{U}}$.*[4]

The following observation is obvious but crucial:

Lemma 2.12. *If $M_1 \equiv M_2$ and N is a II_1 factor, then N admits an embedding into $M_1^{\mathcal{U}}$ with factorial relative commutant if and only if N admits an embedding into $M_2^{\mathcal{U}}$ with factorial relative commutant.*

Consequently, **if** all e.c. factors were elementarily equivalent, then our main theorem would follow from Corollary 2.9 by taking any e.c. fcactor.

However, while still an open problem, it is highly unlikely (at least in this author's opinion) that all e.c. factors are elementarily equivalent. Instead, we look to an important subclass of these factors for which all members are elementarily equivalent. First, we need:

Definition 2.13. *Suppose that M_1 is a subfactor of the II_1 factor M_2. We say that M_1 is an **elementary subfactor** of M_2, denoted $M_1 \preceq M_2$, if there is an isomorphism $M_1^{\mathcal{U}} \cong M_2^{\mathcal{U}}$ that fixes the diagonal images of M_1.*[5]

Definition 2.14. *If $\mathcal{C}$ is a class of II_1 factors, we say that $\mathcal{C}$ is **model-complete** if, whenever M_1 and M_2 are elements of $\mathcal{C}$ with $M_1 \subseteq M_2$, then $M_1 \preceq M_2$.*

The following facts follow easily from the definitions:

Lemma 2.15. *Suppose that $\mathcal{C}$ is an extensive, model-complete class of II_1 factors. Then:*

    (1) *Every element of $\mathcal{C}$ is an e.c. factor.*
    (2) *If M_1 and M_2 belong to $\mathcal{C}$, then $M_1 \equiv M_2$.*

The following is a combination of [6, Proposition 5.7, Proposition 5.10, and Proposition 5.14]:

Fact 2.16. *There is an extensive, model-complete class of II_1 factors. In fact, there is a maximum such class $\mathcal{G}$.*

Definition 2.17. *Elements of $\mathcal{G}$ are called **infinitely generic** II_1 factors.*

We now have the main result:

Theorem 2.18. *Suppose that M is an infinitely generic II_1 factor. Then if N is a property (T) factor, then N admits an embedding into $M^{\mathcal{U}}$ with factorial relative commutant.*

Proof. Take an infinitely generic II_1 factor M_1 with $N \subseteq M_1$. By Lemma 2.15(1), M_1 is e.c. whence $N' \cap M_1^{\mathcal{U}}$ is a factor by Corollary 2.9. By Lemma 2.15(2), $M \equiv M_1$, whence we are done by Lemma 2.12. $\square$

Remark 2.19. *All that we used about property (T) factors is that they automatically have w-spectral gap in any extension. Are there other II_1 factors with this property?*

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[4]This is an evil, logic-free, definition, and makes heavy use of our standing CH assumption.

[5]Again, an evil logic-free definition taking full advantage of our standing CH assumption.

3. The case of embeddable factors

We now consider what happens when we restrict to embeddable factors. All notions from the last section relativize to this setting. For example, by an e.c. embeddable factor we mean an embeddable factor that is e.c. in all embeddable extensions. Similarly, one can define the class of infinitely generic embeddable factors, which forms a subclass of the class of e.c. embeddable factors. The class of e.c. embeddable factors and the subclass of infinitely generic embeddable factors are both extensive in the class of embeddable factors. See [6] for more details on this.

Conjecture 3.1. *Suppose that* N *is a w-spectral gap subfactor of the e.c. embeddable factor* M. *Then* $(N' \cap M)' \cap M = N$.

Why is it not the case that Conjecture 3.1 is simply a theorem? Well, the proof of Fact 2.7 uses the fact that if N is a w-spectral gap subfactor of the e.c. factor M, then M is e.c. in the amalgamated free product $M *_N (N \otimes L(\mathbb{Z}))$. If M is an e.c. embeddable factor, then we could only conclude that M is e.c. in $M *_N (N \otimes L(\mathbb{Z}))$ *if* we knew that $M *_N (N \otimes L(\mathbb{Z}))$ is also embeddable. However, it is unknown at the moment whether or not this is the case.

Question 3.2. *Does taking amalgamated free products of embeddable factors with property* (T) *base preserve embeddability?*

Thus, we just argued that a positive answer to Question 3.2 yields a positive solution to Conjecture 3.1.

Suppose we have a positive solution to Conjecture 3.1. Since $\mathcal{R}$ is an e.c. embeddable factor (see [6, Lemma 2.1]), once again, **if** all e.c. embeddable factors were elementarily equivalent, we would actually arrive at a positive solution to the FCEP for property (T) factors. Once again, we believe this to be highly doubtful. Passing to infinitely generic embeddable factors and noting that the rest of the arguments of the previous section go through, we get:

Theorem 3.3. *Suppose that Conjecture 3.1 has a positive answer and that* M *is an infinitely generic embeddable factor. Then every embeddable property* (T) *factor admits a factorial embedding into* $M^{\mathcal{U}}$.

In light of the previous theorem and recalling Popa's original question, we arrive at the obvious question:

Question 3.4. *Is* $\mathcal{R}$ *an infinitely generic embeddable factor?*

Corollary 3.5. *If Conjecture 3.1 is true and Question 3.4 has a positive answer, then the FCEP for property* (T) *factors has a positive solution.*

In [6, Proposition 5.21], it was claimed that $\mathcal{R}$ is an infinitely generic embeddable factor. However, the proof there is horribly flawed and the question is still open at this time. Let us point out:

Lemma 3.6. *The following statements are equivalent:*

(1) $\mathcal{R}$ *is infinitely generic embeddable factor.*

(2) *There is an infinitely generic embeddable factor* M *such that* $\mathcal{R} \equiv$ M.

Proof. To prove the nontrivial direction, suppose that M is an infinitely generic embeddable factor such that $\mathcal{R} \equiv$ M. Fixing an embedding $\mathcal{R} \hookrightarrow$ M, we have that this embedding is automatically elementary.[6] We now quote [6, Proposition 5.17], which implies that an elementary subfactor of an infinitely generic embeddable factor is an infinitely generic embeddable factor. $\square$

There is another class of e.c. (embeddable) factors with the property that any two members are elementarily equivalent, namely the so-called **finitely generic (embeddable) factors** (see [6, Section 6] or [8, Section 3] for a precise definition). This class is also model-complete; in fact, by [8, Corollary 3.12], if a factor is e.c. in a finitely generic (embeddable) factor, then it is also a finitely generic (embeddable) factor. Consequently, $\mathcal{R}$ is a finitely generic embeddable factor.[7] Thus, at first glance, it might seem promising to look at this class instead. Unfortunately, this class is far from extensive:

Fact 3.7. ([2]) $\mathcal{R}$ *is the unique finitely generic embeddable factor.*

Remark 3.8. *In the case of groups, the finitely generic and the infinitely generic groups are not elementarily equivalent (see* [12, Theorem 11]). *Perhaps similar proofs could be used to negatively answer Question 3.4.*

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[6]This is well-known to those working in the model theory of operator algebras, but we sketch a quick proof for the sake of the reader. Fix an isomorphism $M^{\mathcal{U}} \cong \mathcal{R}^{\mathcal{U}}$ and note that the induced embedding $\mathcal{R} \hookrightarrow \mathcal{R}^{\mathcal{U}}$ is conjugate to the diagonal embedding by the easy direction of the main result of [11]. Consequently, there is an isomorphism $M^{\mathcal{U}} \cong \mathcal{R}^{\mathcal{U}}$ fixing the diagonal image of $\mathcal{R}$.

[7]With some revisionist history, one can use this fact to give an alternate definition of finitely generic embeddable factors, namely an embeddable factor M is a finitely generic embeddable factor if and only if: (i) $M \equiv \mathcal{R}$, and (ii) whenever $M \subseteq N$ and $N \equiv \mathcal{R}$, then M is an elementary subfactor of N.

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