Model theory and the Weak Expectation Property

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WEP

All C* algebras are unital and all inclusions are unital.

Definition

Let A be a C* algebra. We say that A has the *Weak Expectation Property* (WEP) if, whenever B is a C* algebra containing A, there is a u.c.p. map $\phi : B \to A^{**}$ that is the identity on A.

- A has WEP if and only if whenver $A \subseteq B$ and C is another C* algebra, then $A \otimes_{\max} C \subseteq B \otimes_{\max} C$.
- In particular, if A is nuclear, then A has WEP.
- A theorem of Kirchberg says that A has WEP if and only if there is a unique C*-norm on $A \odot C^*(\mathbb{F})$.

Goal of the talk

Investigate the model-theoretic content of this notion.

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Continuous logic-The case of C* algebras

- Atomic formulae: $\varphi(\vec{x}) := \|p(\vec{x})\|$, $p(\vec{x})$ a *polynomial (over $\mathbb{C}$).
- Quantifier-free formulae: $\varphi(\vec{x}) := f(\varphi_1(\vec{x}), \ldots, \varphi_n(\vec{x}))$, each φ_i atomic, $f : \mathbb{R}^n \to \mathbb{R}$ continuous.
- Quantifiers: If φ is a formula and $n \in \mathbb{N}$, then $\sup_{\|x\| \leq n} \varphi$ and $\inf_{\|x\| \leq n} \varphi$ are also formulae.
- If A is a C* algebra, $\varphi(\vec{x})$ is a formula, and $\vec{a}$ a tuple from A, then $\varphi(\vec{a})^A$ is a real number.
- For example, $\varphi(x) := \sup_y \|xy - yx\|$ is a formula. If A is a C* algebra and $a \in A$, then $\varphi(a)^A = 0$ if and only if a is in the *center of A*.
- A *sentence* is a formula with no free variables.
- C* algebras A and B are *elementarily equivalent*, written $A \equiv B$, if $\sigma^A = \sigma^B$ for all sentences σ.

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Axiomatizable classes

Definition

Let $\mathcal{K}$ be a class of (separable) C* algebras. We say that $\mathcal{K}$ is *axiomatizable* if there is a set T of sentences such that a (separable) C* algebra A belongs to $\mathcal{K}$ if and only if $\sigma^A = 0$ for all $\sigma \in T$.

Examples

Abelian, non-abelian, real-rank 0, n-subhomogeneous (fixed n), C* algebras that admit a trace, C* algebras that admit a character,...

Some of these examples are proven to be axiomatizable using an abstract test: $\mathcal{K}$ is axiomatizable if and only if it is closed under isomorphism, ultraproducts, and ultraroots.

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Let A be a C* algebra, E an operator system contained in A, and $\phi : E \to B(H)$ a u.c.p. map. Then there is a u.c.p. map $\psi : A \to \mathcal{B}(H)$ extending ϕ.

Question

Is there a *finitary version* of Arveson Extension? More precisely: Is it true that given an operator system $E \subseteq M_n$ and $k \in \mathbb{N}$, there exists $l \in \mathbb{N}$ and $\delta > 0$ such that, for any unital map $\phi : E \to \mathcal{B}(H)$ with $\|\phi\|_l < 1 + \delta$, there is a unital map $\psi : M_n \to \mathcal{B}(H)$ with $\|\psi\|_k < 1 + \frac{1}{k}$ and $\|\psi|E - \phi\| < \frac{1}{k}$?

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FAE is false

Let FAE denote the statement that the finitary version of Arveson Extension mentioned above is true.

Theorem (G.-Sinclair; Ozawa)

FAE is false.

Using work of Choi-Effros, it turns out that FAE is equivalent to $\mathcal{B}(H)^\omega$ having WEP. So:

Corollary

$\mathcal{B}(H)^\omega$ do not have WEP. In particular, WEP is not an axiomatizable property.

Corollary

A^ω has WEP if and only if A is subhomogeneous.

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1-exact operator systems

Definition

1. Suppose that E is a finite-dimensional operator system with basis $\vec{a}$. We say that E is *1-exact* if, for every $\epsilon > 0$, there is $n \in \mathbb{N}$ and c.b. maps $\phi : E \to M_n$ and $\psi : \phi(E) \to E$ such that $\|(\psi \circ \phi)(\vec{a}) - \vec{a}\| < \epsilon$.
2. An arbitrary operator system is 1-exact if and only if each of its finite-dimensional subsystems is 1-exact.

Omitting types

Definition

Let $\mathcal{K}$ be a class of C* algebras (or operator spaces or operator systems...). We say that $\mathcal{K}$ is an *omitting types class* if there is a family of nonnegative formulae $\varphi_{m,n}(\vec{x}_n)$ such that a C* algebra A belongs to $\mathcal{K}$ if and only if, for all n, we have

$$\left(\sup_{\vec{x}_n} \inf_m \varphi_{m,n}(\vec{x}_n)\right)^A = 0.$$

One should think of this as a particularly nice kind of *infinitary* axiomatizability.

Sketch of the proof

- We first show that if FAE held, then the class of 1-exact operator systems is omitting types.
- We then consider two spaces: $\mathfrak{M}_n$ is the space of "codes" for n-dimensional operator systems, equipped with the logic topology; and $\mathcal{OS}_n$, the space of (isomorphism classes of) n-dimensional operator systems, equipped with its weak topology. Both are Polish spaces.
- There is a "forgetful" map $F : \mathfrak{M}_n \to \mathcal{OS}_n$, which is surjective, open, and continuous.
- Let $\mathcal{E}_n$ denote the 1-exact elements of $\mathcal{OS}_n$. We show that the class of 1-exact operator systems being omitting types implies that $F^{-1}(\mathcal{E}_n)$ is G_δ, so Polish.
- By the Open Mapping Theorem for Polish spaces, we get that $\mathcal{E}_n$ is weakly Polish. But this contradicts fundamental work of Junge and Pisier.

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Existentially closed C* algebras

Definition

A C* algebra A is said to be *existentially closed* (e.c.) if, for any quantifier-free formula $\varphi(\vec{x}, \vec{y})$, tuple $\vec{a}$ from A, and extension $B \supseteq A$, we have

$$\inf\{\varphi(\vec{b}, \vec{a})^A \ : \ \vec{b} \in A_1\} = \inf\{\varphi(\vec{c}, \vec{a})^B \ : \ \vec{c} \in B_1\}.$$

Theorem (G. and Sinclair)

If A is an existentially closed C algebra, then A has WEP. More generally, if $X \subseteq \mathcal{B}(H)$ is an operator system that is e.c. in $\mathcal{B}(H)$, then there is a u.c.p. $\mathcal{B}(H) \to \overline{X}$ (weak closure) restricting to id_X.*

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Idea of the proof

Theorem

if $X \subseteq \mathcal{B}(H)$ is an operator system that is e.c. in $\mathcal{B}(H)$, then there is a u.c.p. $\mathcal{B}(H) \to \overline{X}$ (weak closure) restricting to id_X.

Proof.

- It is enough to show (by weak compactness) that for every self-adjoint $b \in \mathcal{B}(H)^k$ there is u.c.p. $X + \mathbb{C}b_1 + \cdots + \mathbb{C}b_k \to \overline{X}$ extending id_X.
- Since X is e.c., any instance of positivity in $M_n(X + \mathbb{C}b_1 + \cdots + \mathbb{C}b_k)$ can be approximately witnessed by an element of $M_n(X)$.
- By weak compactness, this is enough.

□

No model companion

Theorem (Eagle, Farah, Kirchberg, Vignati)

The class of existentially closed C^* algebras is not axiomatizable (no "model companion.")

Proof (G.)

Suppose that A is separable and e.c. If being e.c. were axiomatizable, then A^ω has a cofinal family of separable subalgebras with WEP (namely the elementary substructures), so itself has WEP. Thus, A is subhomogeneous, in particular finite. But Sinclair and I showed e.c. C^* algebras are purely infinite. □

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Semi-p.e.c. as an operator system

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- A quantifier-free formula is *positive* if it is built using only increasing connectives.
- We say that a C* algebra A is *semi-p.e.c. as an operator system* if, whenever $A \subseteq B$ is an inclusion of C* algebras, $\varphi(\vec{x}, \vec{y})$ is a positive quantifier-free formula in the language of operator systems, and $\vec{a}$ is a tuple from A, we have

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WEP implies semi-p.e.c.

- Suppose A has WEP and $A \subseteq B$, both separable.
- Let $E_1 \subseteq E_2 \subseteq \cdots$ be a filtration of B by finite-dimensional subspaces such that $A \cap \bigcup_i E_i$ is dense in A.
- WEP gives us linear maps $\phi_i : E_i \to E_i \cap A$ such that $\|\phi_i\|_i \leq 1$ and $\|\phi_i|_{E_i \cap A} - \mathrm{id}|_{E_i \cap A}\| \leq 1/i$.
- Get $\phi_\omega : \bigcup_i E_i \to A^\omega$ that extends to unital, completely contractive (hence u.c.p.) $\phi_\omega : B \to A^\omega$ that restricts to the identity on A.
- If $\psi(\vec{x}, \vec{y})$ is a positive quantifier-free formula in the language of operator systems and a is a tuple from A, and b is a tuple from B, then

$$\psi(b, a)^B \geq \psi(\phi_\omega(b), a)^{A^\omega} \geq \inf_x \psi(x, a)^{A^\omega} = \inf_x \psi(x, a)^A.$$

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An application

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An inclusion of C* algebras $A \subseteq B$ has the *complete tight Riesz interpolation property* if, for any n and any finite collection $(x_1, \ldots, x_m, y_1, \ldots, y_p) \in M_n(A)_{sa}$, if there is $z \in M_n(B)$ so that $x_1, \ldots, x_m < z < y_1, \ldots, y_p$, then there is $z' \in M_n(A)$ satisfying the same property.

Theorem (Kavruk)

If A is a unital separable C^* algebra, then A has WEP if and only if there is an inclusion $A \subseteq \mathcal{B}(H)$ with the complete tight Riesz interpolation property.

The proof becomes fairly straightforward using the fact that WEP is the same as semi-p.e.c. as an operator system.

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The proof becomes fairly straightforward using the fact that WEP is the same as semi-p.e.c. as an operator system.

QWEP

Definition

If A is a quotient of a C* algebra with WEP, then we say that A has QWEP.

Kirchberg's QWEP conjecture

Every separable C* algebra is QWEP.

Theorem (Kirchberg)

The following are equivalent:

1. *The QWEP conjecture.*
2. $C^*(\mathbb{F})$ *is QWEP.*
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QWEP is axiomatizable

Proposition (G.)

QWEP is axiomatizable. Consequently, the QWEP conjecture is equivalent to the existence of a QWEP C* algebra A such that $A \equiv C^*(\mathbb{F})$

Proof.

- Closure under isomorphism ✓
- Closure under ultraproduct: Kirchberg showed that QWEP is closed under direct product
- Closure under ultraroot: one can check that ultraroots are relatively weakly injective in the ultrapower; Kirchberg showed that QWEP is preserved under r.w.i. subalgebras □

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LLP and QWEP conjecture

Definition

A separable C*-algebra A has the *local lifting property* (LLP) if whenever $\phi : A \to C/J$ is a u.c.p. map and $E \subseteq A$ is a finite-dimensional operator system, then $\phi|E$ has a u.c.p. lift.

- Nuclear C* algebras have LLP (Choi-Effros).
- C$^*(\mathbb{F})$ has LLP. (Kirchberg)
- QWEP+LLP implies WEP. (Kirchberg)
- It follows that the QWEP conjecture is equivalent to the statement that LLP implies WEP.
- Weak QWEP conjecture There is a non-nuclear C* algebra with both WEP and LLP.

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LLP models

Observation (G.)

Suppose that there is an LLP C* algebra B such that $B \equiv C_r^*(\mathbb{F})$ or $\prod_\omega M_n$. Then B witnesses the truth of the weak QWEP conjecture.

Proof.

Both $C_r^*(\mathbb{F})$ and $\prod_\omega M_n$ are QWEP, whence so is B. Since B has LLP, it follows that B has WEP. B is not nuclear by a result of Farah, Hart, et. al. □

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Locally universal C* algebras

- Call a separable C* algebra S *locally universal* if every separable C* algebra embeds into some (any) ultrapower of S.
- Locally universal C* algebras exist.
- Kirchberg's Embedding Problem (KEP) Does there exist a nuclear locally universal C* algebra?
- LLPEP Does there exist an LLP locally universal C* algebra?
- Clearly KEP implies LLPEP.
- Sinclair and I proved that KEP is equivalent to the existence of a nuclear e.c. C* algebra (which is necessarily $\mathcal{O}_2$).

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Wrapping up

- Suppose that LLP is an omitting types property. (Sinclair and I came close to proving this.)
- Suppose that LLPEP is true. Then we can use the technique of *model-theoretic forcing* to build an e.c. LLP C^* algebra A.
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References

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