# THE ALMOST SURE THEORY OF FINITE METRIC SPACES

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ABSTRACT. We establish an approximate 0-1 law for finite metric spaces (of diameter at most 1) by establishing the existence of a complete theory $T_{AS}$ of metric spaces for which, given any sentence $\sigma$ in the language of pure metric spaces and any $\epsilon > 0$, almost surely in sufficiently large finite metric spaces, the value of $\sigma$ is within $\epsilon$ of the value $T_{AS}$ assigns to $\sigma$. We also establish some model-theoretic properties of the theory $T_{AS}$.

## 1. INTRODUCTION

Recall that the *Urysohn sphere* $\mathbb{U}$ is the unique Polish metric space of diameter 1 satisfying two properties: *universality:* all Polish metric spaces of diameter at most 1 embed into $\mathbb{U}$; and *ultrahomogeneity:* any isometry between finite subspaces of $\mathbb{U}$ extends to a self-isometry of $\mathbb{U}$. From the model-theoretic perspective, $\mathbb{U}$ is the Fraïssé limit of the class of all finite metric spaces of diameter at most 1 and its complete theory is the model completion of the pure theory of metric spaces.

A lingering question about $\mathbb{U}$ is whether or not it is *pseudofinite*, that is, elementarily equivalent to an ultraproduct of finite metric spaces, or, equivalently, whether or not, given a sentence $\sigma$ for which $\sigma^{\mathbb{U}} = 0$ and $\epsilon > 0$, there is a finite metric space $X$ of diameter at most 1 such that $\sigma^X < \epsilon$. In an earlier preprint, the first two authors claimed that not only is $\mathbb{U}$ pseudofinite, but indeed a stronger result is true, namely $\mathrm{Th}(\mathbb{U})$ is the almost-sure theory of finite metric spaces, which means, given any sentence $\sigma$ and any $\epsilon > 0$, almost all sufficiently large finite metric spaces $X$ of diameter at most 1 satisfy $|\sigma^X - \sigma^{\mathbb{U}}| < \epsilon$.

However, a serious flaw in the argument was discovered by the third author, and thus the pseudofiniteness of the Urysohn sphere is still in question. It is the purpose of this note to rescue the latter fact, namely that there is an almost-sure theory of finite metric spaces of diameter at most 1. The motivation for the definition of this theory comes from the fact that almost all sufficiently large metric spaces of diameter at most 1 have all nontrivial distances at least $\frac{1}{2} - O(n^{-c})$ for

some $c > 0$ (see [7] and [9]). This led us to consider spaces axiomatized using "extension axioms" just like $\mathbb{U}$ except with all nontrivial distances being at least $\frac{1}{2}$. Since any assignment of distances between distinct points taking values at least $\frac{1}{2}$ automatically satisfies the triangle inequality, this allowed us to salvage a version of our argument in this context.

In the next section, we define the aforementioned extension axioms and show that they hold approximately almost-surely in sufficiently large finite metric spaces. In Section 3, we show that these extension axioms axiomatize a complete theory and show that this theory is the almost-sure theory of all finite metric spaces. The final section establishes further model-theoretic properties of the almost-sure theory, including the fact that it has quantifier-elimination, has continuum many nonisomorphic separable models, and is unstable but is supersimple of U-rank 1.

In the first two sections, we assume minimal familiarity with continuous logic as established in [3, Sections 2-4]. The final section will assume familiarity with more sophisticated model-theoretic notions.

Throughout the paper, L denotes the "empty" metric language, that is, the metric language consisting solely of the metric symbol d. The words "formula" and "sentence" will be used as abbreviations for "L-formula" and "L-sentence" respectively. All metric spaces will be viewed also as L-structures.

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## 2. Extension axioms

Suppose that $\mathcal{C}$ is the class of finite metric spaces in which the distance between any two distinct points lies in the interval $[\frac{1}{2}, 1]$. Note that all such metric spaces are discrete as any ball of radius $\frac{1}{4}$ consists just of its center.

Given a finite metric space $X = \{x_1, \ldots, x_n\}$, we let $\mathrm{Conf}_X(v_1, \ldots, v_n)$ denote the formula

$$\max_{1 \leq i < j \leq n} |d(x_i, x_j) - d(v_i, v_j)|.$$

We use the notation $X \sqsubset Y$ when $X$ is a finite metric space and $Y$ is a one-point extension of $X$, in which case the extra point is denoted by $y$.

Given $X \sqsubset Y$ with $X, Y \in \mathcal{C}$, we let $\Psi^{\epsilon}_{X \sqsubset Y}$ denote the sentence

$$\sup_{\bar{v}} \min \left\{ \epsilon \,\dot{-}\, \mathrm{Conf}_X(\vec{v}), \inf_w \mathrm{Conf}_Y(\vec{v}, w) \,\dot{-}\, \epsilon \right\}.$$

We identify $\bar{d} = (d_{ij} : 1 \le i < j \le n) \in [0,1]^{\binom{n}{2}}$ with the metric space on $\{1, \ldots, n\}$ with $d(i,j) := d_{ij}$. In this manner, if $X \in \mathcal{C}$, we write $\mathrm{Conf}_X(\bar{d})$, with the interpretation that the appearance of $d(v_i, v_j)$ gets replaced with $d_{ij}$. We perform a similar identification with $\Psi^\epsilon_{X \sqsubset Y}(\bar{d})$.

Let $M_n \subseteq [0,1]^{\binom{n}{2}}$ denote the set of all metric spaces on $\{1, \ldots, n\}$ with values in $[0,1]$. We let $\lambda_n$ be Lebesgue measure on $[0,1]^{\binom{n}{2}}$, and we let $\nu_n$ be Lebesgue measure normalized to $M_n$, that is,

$$\nu_n(A) = \frac{\lambda_n(A)}{\lambda_n(M_n)}.$$

The following is a less precise version of [7, Theorem 1.3].

**Fact 2.1.** *There is a decreasing sequence $(\delta_n)$ from $(0, \frac{1}{2})$ which tends to 0 such that, setting*

$$D_n := \left\{ \bar{d} \in M_n \; : \; d_{ij} \ge \frac{1}{2} - \delta_n \text{ for all } 1 \le i < j \le n \right\},$$

*we have $\lim_{n \to \infty} \nu_n(D_n) = 1$.*

We let $\mu_n$ be Lebesgue measure normalized to $D_n$.

**Theorem 2.2.** *For any $X_1 \sqsubset Y_1, \ldots, X_m \sqsubset Y_m$ from $\mathcal{C}$ and any $\epsilon > 0$, we have*

$$\lim_{n \to \infty} \mu_n \left( \left\{ \bar{d} \in D_n \; : \; \max_{i=1,\ldots,m} \Psi^\epsilon_{X_i \sqsubset Y_i}(\bar{d}) = 0 \right\} \right) = 1.$$

*Proof.* Fix $i \in \{1, \ldots, m\}$ and set $X := X_i$ and $Y := Y_i$. Let $k = |X|$. We decompose elements $\bar{d}$ from $[0,1]^{\binom{n}{2}}$ as $\bar{d} = (\bar{d}', \bar{d}^{k+1}, \ldots, \bar{d}^n, \bar{d}'')$, where $\bar{d}' \in [0,1]^{\binom{k}{2}}$, $\bar{d}'' \in [0,1]^{\binom{n-k}{2}}$, and $\bar{d}^t \in [0,1]^k$ for $t = k+1, \ldots, n$. The intention is that $\bar{d}'$ represents $d_{ij}$ for $1 \le i < j \le k$, $\bar{d}^t$ represents $d_{it}$ for $i = 1, \ldots, k$ and $t = k+1, \ldots, n$, and $\bar{d}''$ represents $d_{ij}$ for $k < i < j \le n$.

Let $E_n$ denote the projection of $D_n$ onto the last $\binom{n-k}{2}$ coordinates, and set $S_n := [\frac{1}{2}, 1]^{\binom{k}{2}} \times [\frac{1}{2} + \delta_n, 1]^{k(n-k)} \times E_n$. Note that the values of $d$ specified by an element in $S_n$ cannot violate the triangle inequality, so we have $S_n \subseteq D_n$.

Let $B_n$ be the set of $\bar{d} \in D_n$ such that:
- $\mathrm{Conf}_X(\bar{d}') \le \epsilon$, and
- $\mathrm{Conf}_Y(\bar{d}', \bar{d}^t) > \epsilon$ for all $t = k+1, \ldots, n$.

If we let $A = \{\bar{d}' \in M_k : \mathrm{Conf}_X(\bar{d}') \le \epsilon\}$, then we have $\lambda_n(B_n) \le \lambda_k(A) \cdot ((\frac{1}{2} + \delta_n)^k - \epsilon^k)^{n-k} \cdot \lambda_{n-k}(E_n)$, so

$$\begin{aligned}
\mu_n(B_n) = \frac{\lambda_n(B_n)}{\lambda_n(D_n)} &\le \frac{\lambda_n(B_n)}{\lambda_n(S_n)} \\
&\le \frac{\lambda_k(A) \cdot ((\frac{1}{2} + \delta_n)^k - \epsilon^k)^{n-k} \cdot \lambda_{n-k}(E_n)}{(\frac{1}{2})^{\binom{k}{2}} \cdot (\frac{1}{2} - \delta_n)^{k(n-k)} \cdot \lambda_{n-k}(E_n)} \\
&= 2^{\binom{k}{2}} \lambda_k(A) \left( \frac{(\frac{1}{2} + \delta_n)^k - \epsilon^k}{(\frac{1}{2} - \delta_n)^k} \right)^{n-k}
\end{aligned}$$

We have that

$$\frac{(\frac{1}{2} + \delta_n)^k - \epsilon^k}{(\frac{1}{2} - \delta_n)^k} \le \left( \frac{\frac{1}{2} + \delta_n}{\frac{1}{2} - \delta_n} \right)^k - (2\epsilon)^k$$

and so, since $\lim_{n \to \infty} \delta_n = 0$, it follows that there exists a constant $C$ and a constant $p < 1$ such that $\mu_n(A_n) \le Cp^n$ for all large enough $n$.

The previous calculation yielded an upper bound on the probability that a random element of $D_n$ failed the extension axiom $\Psi^\epsilon_{X_i \sqsubset Y_i}$ as witnessed by the first $k$ elements. The calculation is identical if one focuses on any other $k$ element subset instead of the first $k$ coordinates. Moreover, if we were considering $m$ extension axioms instead of just one, we would obtain a similar expression, possibly with different constants $C$ and $p$. It follows that for some constants $K$ and $q < 1$,

$$\mu_n \left( \left\{ \bar{d} \in D_n : \max_{i=1,\dots,m} \Psi^\epsilon_{X_i \sqsubset Y_i}(\bar{d}) > 0 \right\} \right) \le mn^k K q^n$$

for sufficiently large $n$. As $n$ tends to infinity, this quantity goes to zero, yielding the desired result. $\square$

**Corollary 2.3.** *For any $X_1 \sqsubset Y_1, \dots, X_m \sqsubset Y_m$ from $\mathcal{C}$ and any $\epsilon > 0$, we have*

$$\lim_{n \to \infty} \nu_n \left( \left\{ \bar{d} \in M_n : \max_{i=1,\dots,n} \Psi^\epsilon_{X_i \sqsubset Y_i}(\bar{d}) = 0 \right\} \right) = 1.$$

*Proof.* Let $A := \left\{ \bar{d} \in M_n : \max_{i=1,\dots,n} \Psi^\epsilon_{X_i \sqsubset Y_i}(\bar{d}) = 0 \right\}$. We then have

$$\nu_n(A) = \mu_n(A \cap D_n) \cdot \nu_n(D_n) + \nu_n(A \setminus D_n).$$

By Theorem 2.2, $\lim_{n \to \infty} \mu_n(A \cap D_n) = 1$. Since $\lim_{n \to \infty} \nu_n(D_n) = 1$, it follows that $\lim_{n \to \infty} \nu_n(A) = 1$, as desired. $\square$

## 3. The approximate 0 - 1 law

We first recall the Compactness Theorem for continuous logic. Given a theory $T$ (in some language), we define the theory $T^+$ to consist of all sentences $\sigma \mathbin{\dot-} \epsilon$, where $\sigma \in T$ and $\epsilon > 0$. We say that $T$ is *approximately finitely satisfiable* if $T^+$ is finitely satisfiable.

**Fact 3.1** (Compactness Theorem). *Given a theory $T$, we have that $T$ is satisfiable if and only if it is approximately finitely satisfiable.*

We let $T_{AS}$ denote the L-theory consisting of the set of all extension axioms $\psi^\epsilon_{X \sqsubset Y}$ together with the sentence $\varphi_{\geq \frac{1}{2}}$:

$$\sup_x \sup_y (\min\{d(x, y), \frac{1}{2} \mathbin{\dot-} d(x, y)\})$$

which when satisfied says that $d(x, y) \geq \frac{1}{2}$ when $x \neq y$. A familiar amalgamation construction shows that $T_{AS}$ is satisfiable. By Corollary 2.3, we can say more.

**Proposition 3.2.** $T_{AS}$ *has the* finite model property: *every finite subset of $T_{AS}$ is approximately satisfied in a finite metric space. Equivalently, there is an ultraproduct of finite metric spaces which satisfies $T_{AS}$.*

*Proof.* By Fact 2.1 and Corollary 2.3, any finite number of extension axioms, together with any sentence $\varphi_{\geq \frac{1}{2}} \mathbin{\dot-} \epsilon$ for $\epsilon > 0$, are satisfied in a sufficiently large finite metric space, whence the proposition follows. The ultraproduct equivalence is a standard reformulation of the finite model property. (See, for example, [6].) $\qquad\square$

In order to prove our 0-1 law for finite metric spaces, we show that $T_{AS}$ is a complete theory, that is, for all models $X$ and $Y$ of $T_{AS}$, we have that $X$ and $Y$ are elementarily equivalent. We establish this fact using *Ehrenfeucht-Fraïssé games*.

**Definition 3.3.** *Given metric spaces $X$ and $Y$, $n \in \mathbb{N}$, and $\epsilon > 0$, we define $\mathfrak{G}(X, Y, n, \epsilon)$ to be the two-player, $n$-round game, where at round $i$, player I chooses $a_i \in X$ or $b_i \in Y$ and then player II chooses $b_i \in Y$ or $a_i \in X$ accordingly. We say that player II wins a run of $\mathfrak{G}(X, Y, n, \epsilon)$ if*

$$|d_X(a_i, a_j) - d_Y(b_i, b_j)| < \epsilon$$

*for all $1 \leq i < j \leq n$; otherwise, player I wins. We write $X \equiv_{n,\epsilon} Y$ if player II has a winning strategy in $\mathfrak{G}(X, Y, n, \epsilon)$.*

The classical version of the following fact is well-known (see, e.g. [8, Theorem 2.4.6]). For a proof (of a slight variant) in the continuous setting, one can consult [5, Lemma 2.4].

**Fact 3.4.** *If $X$ and $Y$ are metric spaces, then $X$ and $Y$ are elementarily equivalent if and only if $X \equiv_{n,\epsilon} Y$ for all $n \in \mathbb{N}$ and $\epsilon > 0$.*

**Theorem 3.5.** $T_{AS}$ *is complete.*

*Proof.* Fix $X, Y \models T_{AS}$. It suffices to show that $X \equiv_{n,\epsilon} Y$ for all $n \in \mathbb{N}$ and $\epsilon > 0$. We prove this by induction on $n$, the base case $n = 1$ being trivial. Now suppose that $n > 1$ and $X \equiv_{n-1,\epsilon} Y$ for all $\epsilon > 0$. Fix $\epsilon > 0$ and let player II play the first $n - 1$ rounds of $\mathfrak{G}(X, Y, n, \epsilon)$ according to a winning strategy for $\mathfrak{G}(X, Y, n - 1, \frac{\epsilon}{2})$, yielding $a_1, \ldots, a_{n-1} \in X$ and $b_1, \ldots, b_{n-1} \in Y$ with

$$|d_X(a_i, a_j) - d_Y(b_i, b_j)| < \frac{\epsilon}{2}$$

for all $1 \leq i < j \leq n - 1$. Now suppose that player I plays $a_n \in X$ in the final round of $\mathfrak{G}(X, Y, n, \epsilon)$ (the case that they play $b_n \in Y$ is handled in a symmetric fashion). Let $X_0 := \{a_1, \ldots, a_{n-1}\}$ and let $Y_0 := X_0 \cup \{a_n\}$. Then $X_0$ and $Y_0$ are in $\mathcal{C}$, since $Y \models \varphi_{\geq \frac{1}{2}}$. Then $Y \models \psi_{X_0 \sqsubset Y_0}^{\frac{\epsilon}{2}}$ and $\mathrm{Conf}_{X_0}(b_1, \ldots, b_{n-1}) < \frac{\epsilon}{2}$, so we have that

$$Y \models \inf_w \left( \mathrm{Conf}(b_1, \ldots, b_{n-1}, w) \;\dot{-}\; \frac{\epsilon}{2} \right),$$

whence there is $b_n \in Y$ such that $\mathrm{Conf}_{Y_0}(b_1, \ldots, b_n) < \epsilon$. It follows that this strategy is winning for player II in $\mathfrak{G}(X, Y, n, \epsilon)$. $\square$

Given a sentence $\sigma$ in the language of metric spaces, let $\sigma^{AS}$ denote the unique real number $r$ such that $\sigma^X = r$ for all $X \models T_{AS}$. It follows that, given any $\epsilon > 0$, the theory $T_{AS} \cup \{\epsilon \;\dot{-}\; |\sigma - \sigma^{AS}|\}$ is not satisfiable. By Fact 3.1, there are extension axioms $\psi_{X_i \sqsubset Y_i}^{\epsilon_i}$ ($1 \leq i \leq m$) and $\eta > 0$ such that the theory

$$\left\{ \varphi_{\geq \frac{1}{2}} \;\dot{-}\; \eta, \left( \max_{1 \leq i \leq m} \psi_{X_i \sqsubset Y_i}^{\epsilon_i} \right) \;\dot{-}\; \eta, \epsilon \;\dot{-}\; |\sigma - \sigma^{AS}| \right\}$$

is not satisfiable. Combining this observation with Fact 2.1 and Corollary 2.3 immediately yields:

**Theorem 3.6** (Approximate 0-1 law). *For any sentence $\sigma$ in the language of metric spaces and any $\epsilon > 0$, we have*

$$\lim_{n \to \infty} \nu_n(\{X \in M_n \;:\; |\sigma^X - \sigma^{AS}| < \epsilon\}) = 1.$$

## 4. Further model-theoretic properties of $T_{AS}$

In this last section, we gather some further model-theoretic facts about $T_{AS}$. We assume that the reader is familiar with the model-theoretic terms defined in this section.

**Theorem 4.1.** $T_{AS}$ *has quantifier-elimination and is the model-completion of the theory* $T_0 := \{\varphi_{\geq \frac{1}{2}}\}$.

*Proof.* By a standard model-theoretic test for quantifier-elimination (see [3, Proposition 13.6]), it is enough to prove the following: given $X, Y \models T_{AS}$, $A \subseteq X$, an isometric embedding $f : A \hookrightarrow Y$, and $a \in X \setminus A$, there is an elementary extension $Y \preceq Y'$ and an isometric embedding $g : A \cup \{a\} \hookrightarrow Y'$ extending $f$. However, this follows easily from Fact 3.1 and the fact that $Y \models T_{AS}$.

In order to prove that $T_{AS}$ is the model-completion of $T_0$, it remains to show that every model of $T_0$ embeds in a model of $T_{AS}$. By Fact 3.1, it suffices to show that, for any $X_0 = \{a_1, \ldots, a_n\}$ from $\mathcal{C}$ and $\epsilon > 0$, there is $X \models T_{AS}$ and $b_1, \ldots, b_n \in X$ such that $\mathrm{Conf}_{X_0}(b_1, \ldots, b_n) < \epsilon$. We do this by induction on $n$, the case $n = 1$ being trivial. Suppose that $n > 1$ and the claim is true for $n - 1$. Fix $X_0$ as above and $\epsilon > 0$. By induction, there is $X \models T_{AS}$ and $b_1, \ldots, b_{n-1} \in X$ such that, setting $X_0' := \{a_1, \ldots, a_{n-1}\}$, we have $\mathrm{Conf}_{X_0'}(b_1, \ldots, b_{n-1}) < \frac{\epsilon}{2}$. Since $X \models \psi^{\frac{\epsilon}{2}}_{X_0' \sqsubset X_0}$, it follows that there is $b_n \in X$ such that $\mathrm{Conf}_{X_0}(b_1, \ldots, b_n)$, as desired. $\square$

**Corollary 4.2.** $T_{AS}$ *has continuum many nonisomorphic separable models.*

*Proof.* Any model of $T_{AS}$ is topologically discrete, so a separable model of $T_{AS}$ is countable. In any such model, the metric $d$ only takes on countably many values in $[0, 1]$. By Theorem 4.1, given any separable model $X_0$ of $T_0$, there is a separable model $X$ of $T_{AS}$ such that $X_0$ embeds into $X$, so for every value $r \in [\frac{1}{2}, 1]$, there is a separable model $X$ of $T_{AS}$ such that the metric takes on the value $r$ in $X$. It follows that $T_{AS}$ has continuum many nonisomorphic separable models. $\square$

**Theorem 4.3.** $T_{AS}$ *is not stable but is supersimple of* U-*rank* 1. *Moreover, forking independence is characterized by*

$$A \underset{C}{\downarrow} B \Leftrightarrow A \cap B \subseteq C,$$

*where* $A, B, C$ *are small subsets of some monster model* $\mathbb{X}$ *of* $T_{AS}$.

*Proof.* It is straightforward to verify that the independence relation in the above display satisfies all of the axioms of forking independence in simple theories, whence we can conclude that $T_{AS}$ is simple and the above independence relation is forking independence.[1] We verify only the Independence Theorem over Models. Suppose that $X$ is a model, $X \subseteq A$, $X \subseteq B$, $A \underset{X}{\downarrow} B$, and $p(x) \in S(A)$ and $q(x) \in S(B)$

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[1] See [1, Theorem 5.4.5].

are independent extensions of their common restriction $p_0 \in S(X)$. Since $p$ and $q$ are independent extensions of $p_0$, all distances specified by $p$ and $q$ between $x$ and elements of $(A \cup B) \setminus X$ are at least $\frac{1}{2}$, whence we may find an abstract extension $Y := A \cup B \cup \{a\}$ of $A \cup B$ such that $Y \models T_0$ and $a$ is a tuple realizing the quantifier-free parts of both $p$ and $q$. Since $T_{AS}$ is the model completion of $T_0$ and $\mathbb{X}$ is saturated and strongly homogeneous, we may embed $Y$ in $\mathbb{X}$ over $A \cup B$. By quantifier elimination, the image of $a$ in $\mathbb{X}$ satisfies both $p$ and $q$. Consequently, the Independence Theorem over Models holds.

To see that the $U$-rank of the theory is 1, suppose that $p \in S_1(A)$ is a type with $U(p) \geq 1$. Take a forking extension $q \in S_1(B)$ of $p$ and let $a \models p$. Then $a \in B \setminus A$ and thus the condition $d(x, a) = 0$ belongs to $q$. It follows that $q$ is algebraic, whence $U(p) = 1$. Recall also that theories of $U$-rank 1 are supersimple. [2]

To see that $T_{AS}$ is not stable, let $p(x)$ be any 1-type over a model $X$, let $a \in \mathbb{X}$ realize $p$ and take $b \in \mathbb{X} \setminus Xa$. Then we can assign $d(x, b)$ to be any number in $[\frac{1}{2}, 1]$ and obtain an extension of $p$ to $Xb$ in this manner. Thus, there are continuum many different nonforking extensions of $p$ to $Xb$, whence $T_{AS}$ is not stable by [3, Theorem 14.12]. $\qquad\square$

The reader should contrast the previous result with the case of the Urysohn sphere, which is not simple (see [4, Theorem 5.4]).

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