Thorn-forking and Rosiness in Continuous Logic

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Outline

Geometric Definition

In this section, fix a complete (classical) theory T in a signature L and let $\mathfrak{M}$ denote a monster model for T.

Definition (Adler?)

Let A, B, C be small subsets of $\mathfrak{M}^{\mathrm{eq}}$.

- $A \mathop{\smile\!\!\!\!\!\!|}^{M}_{C} B$ if and only if for every C' with $C \subseteq C' \subseteq \operatorname{acl}(BC)$, we have
$$\operatorname{acl}(AC') \cap \operatorname{acl}(BC') = \operatorname{acl}(C').$$
- $A \mathop{\smile\!\!\!\!\!\!|}^{þ}_{C} B$ if and only if for every $D \supseteq BC$, there is $A' \equiv_{BC} A$ such that $A' \mathop{\smile\!\!\!\!\!\!|}^{M}_{C} D$.

Formula Definition

Definition (Scanlon, Onshuus)

Let $\varphi(x, b)$ be a formula and C a small set of parameters.

- $\varphi(x, b)$ **strongly k-divides over** C if $b \notin \mathrm{acl}(C)$ and whenever $b_1, \ldots, b_k \models \mathrm{tp}(b/C)$ are distinct, we have that $\bigwedge \varphi(x, b_i)$ is inconsistent.
- $\varphi(x, b)$ **strongly divides over** C if it strongly k-divides over C for some k.
- $\varphi(x, b)$ þ-divides (read: **thorn-divides**) over C if there is $D \supseteq C$ such that $\varphi(x, b)$ strongly divides over D.
- $\varphi(x, b)$ þ-forks over C if it implies a (finite) disjunction of formulae which thorn-divide over C.
- $\mathrm{tp}(A/B)$ þ-forks over C if it contains a formula which þ-forks over C.

Equivalence of the Definitions

Fact (Adler?)

$\mathrm{tp}(A/BC)$ þ-forks over C if and only if $A \mathop{\not\smile\!\!\!\!|}^{þ}_{C} B$.

Main Reason

For a finite tuple b, we have $b \in \mathrm{acl}(AC) \setminus \mathrm{acl}(C)$ if and only if there is a formula $\varphi(x, b)$ in $\mathrm{tp}(A/bC)$ such that $\varphi(x, b)$ strongly divides over C.

Rosy Theories

Definition

T is said to be **rosy** if $\mathop{\smile\!\!\!\!\!\!\;|}^{þ}$ is a strict independence relation for T^{eq}.

Facts (Adler, Ealy, Onshuus)

- T is rosy if and only if $\mathop{\smile\!\!\!\!\!\!\;|}^{þ}$ satisfies local character on small subsets of $\mathfrak{M}^{\mathrm{eq}}$.
- T is rosy if and only if there is a strict independence relation for T^{eq}.
- If T is rosy, then $\mathop{\smile\!\!\!\!\!\!\;|}^{þ}$ is the weakest strict independence relation for T^{eq}, that is, if $\mathop{\smile\!\!\!\!\!\!\;|}^{*}$ is a strict independence relation for T^{eq}, then $\mathop{\smile\!\!\!\!\!\!\;|}^{*} \Rightarrow \mathop{\smile\!\!\!\!\!\!\;|}^{þ}$.

Example

Stable theories, simple theories, and o-minimal theories are all rosy.

A Word About Imaginaries

In this section, fix a complete (continuous) theory T in a (bounded continuous) signature L and let $\mathfrak{M}$ be a monster model for T.

To construct $\mathfrak{M}^{eq}$, we add extra sorts for products (finite or countable) of sorts quotiented by 0*-definable pseudo-metrics*, where a 0-definable pseudo-metric is a formula or uniform limit of formulae which defines a pseudometric.

An imaginary which corresponds to a pseudometric defined by a formula is called a **finitary imaginary** and we let $\mathfrak{M}^{\mathrm{feq}}$ denote the reduct of $\mathfrak{M}^{\mathrm{eq}}$ which only considers finitary imaginaries.

Strong Dividing

Definition

Let $\varphi(x, b)$ be an formula and C a small set of parameters.

- $\mathrm{Ind}(b/C)$ denotes the set of C-indiscernible sequences of realizations of $\mathrm{tp}(b/C)$.
- $\chi(b/C) := \max\{d(b', b'') \mid b, b'' \in I,\ I \in \mathrm{Ind}(b/C)\}$, so $b \in \mathrm{acl}(C)$ if and only if $\chi(b/C) = 0$.
- $\varphi(x, b)$ **strongly ϵ-k-divides over** C if:
  - $\epsilon \leq \chi(b/C)$, and
  - for every $b_1, \ldots, b_k \models \mathrm{tp}(b/C)$ satisfying $d(b_i, b_j) \geq \epsilon$ for all $1 \leq i < j \leq k$, we have

$$\inf_x \max_{1 \leq i \leq k} \varphi(x, b_i) = 1.$$

Connection with Algebraic Closure

Theorem (Ealy, G.)

Let A and C be small parameter sets an b a countable tuple. Then the following are equivalent:

- $b \in \operatorname{acl}(AC) \setminus \operatorname{acl}(C)$;
- $b \notin \operatorname{acl}(C)$ *and for every* ϵ *with* $0 < \epsilon \leq \chi(b/C)$, *there is a formula* $\varphi_\epsilon(x, b)$ *such that "*$\varphi_\epsilon(x, b) = 0$*" is in* $\operatorname{tp}(A/bC)$ *and* $\varphi_\epsilon(x, b)$ *strongly* ϵ*-divides over* C.

Formula Definition of Thorn-Forking

Let A, B, C be small parameter sets.

- Suppose b is a countable tuple from B. Then $\text{tp}(A/bC)$ **thorn-divides over** C if there is $D \supseteq C$ such that $b \notin \text{acl}(D)$ and for every $0 < \epsilon \leq \chi(b/D)$, there is a formula $\varphi_\epsilon(x, b)$ such that "$\varphi_\epsilon(x, b) = 0$" is in $\text{tp}(A/bC)$ and $\varphi_\epsilon(x, b)$ strongly ϵ-divides over D.
- $\text{tp}(A/BC)$ **thorn-divides over** C if there is a countable $b \subseteq B$ such that $\text{tp}(A/bC)$ thorn-divides over C.
- $\text{tp}(A/BC)$ **thorn-forks over** C if there is $D \supseteq BC$ such that every extension of $\text{tp}(A/BC)$ to D thorn-divides over C.
- One can then show that $\text{tp}(A/BC)$ thorn-forks over C if and only if $A \underset{C}{\not\!\downarrow^{þ}} B$ (in the geometric sense).

Countable Character

In classical logic, $\mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}$ always satisfied **finite character**:

$$A \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_{C} B \text{ if and only if } A_0 \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_{C} B \text{ for all finite } A_0 \subseteq A.$$

In continuous logic, $\mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}$ satisfies **countable character**:

$$A \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_{C} B \text{ if and only if } A_0 \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_{C} B \text{ for all countable } A_0 \subseteq A.$$

The reason for this is that, in continuous logic, $a \in \mathrm{acl}(B)$ implies $a \in \mathrm{acl}(B_0)$ for some *countable* $B_0 \subseteq B$.

A **countable independence relation** is defined just like an independence relation except that finite character is replaced by countable character.

Rosy Continuous Theories

T is **rosy** if $\mathop{\smile\hskip-0.9em^{|}\,}^{\text{þ}}$ satisfies local character for small subsets of $\mathfrak{M}^{\mathrm{eq}}$.

Theorem

T is rosy if and only if $\mathop{\smile\hskip-0.9em^{|}\,}^{\text{þ}}$ is a strict countable independence relation for T^{eq} if and only if there is a strict countable independence relation for T^{eq}. If T is rosy, then $\mathop{\smile\hskip-0.9em^{|}\,}^{\text{þ}}$ is the weakest strict countable independence relation for T^{eq}.

Example

All stable and simple continuous theories are rosy.

Theorem (Ealy, G.)

If T is a classical theory, then T is rosy as a classical theory if and only if T is rosy with respect to finitary imaginaries when considered as a continuous theory.

Thorn-forking in Classical Logic

Thorn-forking in Continuous Logic

An Example: Urysohn space

Another Notion of Rosiness

A Question of Ben-Yaacov

Question (Ben-Yaacov)

Is there an *essentially continuous* simple unstable theory?

All known examples of continuous theories were either stable (e.g. Hilbert space, probability algebras, L^p-Banach lattices, $\mathbb{R}$-trees) or not simple (e.g. the Keisler randomization of a theory with the independence property).

Theorem (Ealy, G.)

There is an essentially continuous rosy theory, namely the theory of the Urysohn sphere.

The Urysohn Sphere

Definition

Urysohn sphere is the unique (up to isometry) Polish metric space of diameter ≤ 1 which is

1. *universal*: every Polish metric space of diameter ≤ 1 can be isometrically embedded into it, and
2. *ultrahomogeneous*: any isometry between finite subsets of it can be extended to an isometry of the whole space.

- Let L denotes the continuous signature consisting solely of the metric symbol d, which is assumed to have diameter bounded by 1.
- Let $\mathfrak{U}$ denote the Urysohn sphere, considered as an L-structure.
- Let $T_{\mathfrak{U}}$ denote the L-theory of $\mathfrak{U}$ and we let $\mathbb{U}$ denote a monster model for $T_{\mathfrak{U}}$.

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Model Theoretic Properties of $T_{\mathfrak{U}}$

Facts (Henson)

- $T_{\mathfrak{U}}$ is $\aleph_0$-categorical.
- $T_{\mathfrak{U}}$ admits quantifier elimination.
- $T_{\mathfrak{U}}$ is the model completion of the empty L-theory and is the theory of existentially closed metric spaces of diameter bounded by 1.
- For every small set of real parameters A from $\mathbb{U}$, the real algebraic closure of A in $\mathbb{U}$ equals the topological closure of A in $\mathbb{U}$, i.e. algebraic closure is trivial.

So in many ways, $T_{\mathfrak{U}}$ is the continuous analog of the theory of the infinite set in classical logic. However,...

Model Theoretic Properties of $T_{\mathfrak{U}}$ (cont'd)

Fact (Pillay)

$T_{\mathfrak{U}}$ is not simple.

Proof.

- Let A be a small set of real elements from $\mathbb{U}$ which are mutually $\frac{1}{2}$ apart.
- Let $p(x)$ be the unique 1-type over A determined by the conditions $\{d(x,a) = \frac{1}{4} \mid a \in A\}$.
- It suffices to show that p divides over any proper closed subset B of A. Fix such a B and let $a \in A \setminus B$. We can find a B-indiscernible sequence $(a_i \mid i < \omega)$ of realizations of $\mathrm{tp}(a/B)$ such that $d(a_i, a_j) = 1$ for all $i < j < \omega$.
- Then $d(x,a) = \frac{1}{4}$ divides over B.

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- Then $d(x,a) = \frac{1}{4}$ divides over B.

$\square$

$T_{\mathfrak{U}}$ is real rosy

Theorem (Ealy, G.)

$T_{\mathfrak{U}}$ is real rosy.

Proof (Sketch)

- First observe that $A \mathop{\smile\!\!\!\!\!\!|}^M_C B$ (in the real sense) if and only if $\bar{A} \cap \bar{B} \subseteq \bar{C}$; this follows from the triviality of algebraic closure in $T_{\mathfrak{U}}$.
- Using universality and ultrahomogeneity, one can show that $\mathop{\smile\!\!\!\!\!\!|}^M = \mathop{\smile\!\!\!\!\!\!|}^þ$, i.e. that $\mathop{\smile\!\!\!\!\!\!|}^M$ already satisfies extension.
- Let A and B be small subsets of $\mathbb{U}$. For each $x \in \bar{A} \cap \bar{B}$, let $B_x \subseteq B$ be countable so that $x \in \bar{B}_x$.
- Let $C := \bigcup\{B_x \mid x \in \bar{A} \cap \bar{B}\}$. Then $\bar{A} \cap \bar{B} \subseteq \bar{C}$, i.e. $A \mathop{\smile\!\!\!\!\!\!|}^þ_C B$.
- Since $|C| \leq |\bar{A}| \cdot \aleph_0$, this shows that $\mathop{\smile\!\!\!\!\!\!|}^þ$ has local character when restricted to the real sorts.

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- Since $|C| \leq |\bar{A}| \cdot \aleph_0$, this shows that $\mathop{\smile\!\!\!\!\!\!\mid}^{þ}$ has local character when restricted to the real sorts.

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Weak Elimination of Finitary Imaginaries

Definition

A continuous theory T has **weak elimination of finitary imaginaries** (abbreviated: T has **WEFI**) if for any $a \in \mathfrak{M}^{feq}$, there is a finite real tuple b such that $a \in \operatorname{dcl}(b)$ and $b \in \operatorname{acl}(a)$.

Theorem (Ealy, G.)

If T is real rosy and has WEFI, then T is rosy with respect to finitary imaginaries.

$T_{\mathfrak{U}}$ has WEFI

Lemma (Lascar)

Suppose that T satisfies the following two conditions:

1. *There is no strictly decreasing sequence $A_0 \supsetneq A_1 \supsetneq A_2 \supsetneq \ldots$, where each A_n is the algebraic closure of a finite set.*
2. *If $\varphi(x)$ is a formula which is defined over A and defined over B, then $\varphi(x)$ is defined over $A \cap B$.*

Then T has WEFI.

$T_{\mathfrak{U}}$ clearly satisfies (1). That $T_{\mathfrak{U}}$ satisfies (2) follows from the following unpublished result of Julien Melleray.

Theorem

Let A and B be finite subsets of $\mathbb{U}$. Let $G := \mathrm{Aut}(\mathbb{U}|A \cap B)$ and let H be the subgroup of G generated by $\mathrm{Aut}(\mathbb{U}|A) \cup \mathrm{Aut}(\mathbb{U}|B)$. Then H is dense in G with respect to the topology of pointwise of convergence.

An Application

By the universality property of $\mathfrak{U}$, we know that $\mathfrak{U} \times \mathfrak{U}$ isometrically embeds in $\mathfrak{U}$. However,

Theorem (Ealy, G.)

There is no definable (in $\mathfrak{U}^{\mathrm{feq}}$) injection $f : \mathfrak{U} \times \mathfrak{U} \to \mathfrak{U}$.

Proof.

If there was such a definable map, then it would extend to a definable injective map $f : \mathbb{U} \times \mathbb{U} \to \mathbb{U}$. Using properties of $U^{\text{þ}}$-rank, one can show that $U^{\text{þ}}(\mathbb{U} \times \mathbb{U}) \leq U^{\text{þ}}(\mathbb{U})$. However, it is not too hard to show that $U^{\text{þ}}(\mathbb{U}^n) = n$ for all n. $\square$

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Maximal þ-forking

Definition (Ben-Yaacov)

Suppose $\varphi(x, b)$ is a formula and C is a small set of parameters.

- $\varphi(x, b)$ **strongly divides over** C if it $\chi(b/C)$-divides over C.
- $\varphi(x, b)$ **maximally þ-divides over** C if it strongly divides over D for some $D \supseteq C$.
- $\varphi(x, b)$ **maximally þ-forks over** C if $\mathrm{Zero}(\varphi) \subseteq \bigcup_{i=1}^{n} \mathrm{Zero}(\varphi_i)$, where each φ_i maximally þ-divides over C.
- $\mathrm{tp}(A/B)$ **maximally þ-forks over** C if it contains a condition "$\varphi = 0$", where φ þ-forks over C.
- We write $A \mathop{\smile\!\!\!\!\!\!|}_{C}^{m\text{þ}} B$ if $\mathrm{tp}(A/BC)$ does not maximally þ-fork over C.

Maximal Rosiness

Definition

T is **maximally rosy** if $\mathop{\smile\!\!\!\!\!\mid}^{m\text{þ}}$ satisfies local character.

Downsides

- Simple continuous theories are maximally rosy and maximally rosy theories are rosy. However, we don't know of any maximally rosy unstable theory. (It appears the same argument that shows that $T_{\mathfrak{U}}$ is not simple also shows that $T_{\mathfrak{U}}$ is not maximally rosy.)
- If T is a classical theory, then T is real rosy as a classical theory if and only if it is maximally real rosy as a continuous theory. However, it does not appear that this remains true when one considers finitary imaginaries.

Maximal Rosiness (cont'd)

Upsides

- In a maximally rosy theory, $\downfree^{m\text{þ}}$ is a strict independence relation; in particular it satisfies finite character.
- If T is a classical theory and T^R, the Keisler randomization of T, is maximally rosy with respect to finitary imaginaries, then T is rosy. We are unable to prove this when "maximally rosy" is replaced by "rosy".