The model-theoretic content of some conjectures about C* algebras

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1 C^* algebras and logic

2 Kirchberg's embedding problem

3 The QWEP conjecture

4 The weak QWEP conjecture

C^* algebras

- In this talk, H denotes a complex Hilbert space. A linear operator $T : H \to H$ is *bounded* if $\|T\| < \infty$, where
$$\|T\| := \sup\{\|Tx\| \ : \ \|x\| = 1\}.$$
- A *concrete C^* algebra* is a $*$-subalgebra of $B(H)$ that is closed in the operator norm topology.
- An *abstract C^* algebra* is a Banach $*$-algebra A satisfying the C^* equality: $\|T^*T\| = \|T\|^2$ for all $T \in A$.
- It is not hard to see that every concrete C^* algebra is an abstract C^* algebra. Conversely, it can be shown that every abstract C^* algebra admits a faithful representation as a norm closed subalgebra of some $B(H)$, so these really are the same notion.

Examples of C^* algebras

- $B(H)$ (boring!)
- If X is a compact topological space, then $C(X)$ is a *unital, commutative* C^* algebra when equipped with the sup-norm. By Gelfand theory, every unital commutative C^* algebra is isomorphic to $C(X)$ for some compact topological space X, whence C^* algebra theory is sometimes dubbed *noncommutative topology*.
- If Γ is a discrete group and one considers the unitary representation $l : \Gamma \to U(\ell^2(\Gamma))$ given by $(l(\gamma)(f))(\eta) := f(\gamma^{-1}\eta)$, the *left-regular representation*, then the C^* algebra generated by $l(\Gamma)$ inside of $B(\ell^2(\Gamma))$ is called the *reduced group C^* algebra of* Γ, denoted $C^*_r(\Gamma)$.
- There is also the *full group C^* algebra of* Γ, denoted $C^*(\Gamma)$, which is the completion of $\mathbb{C}[\Gamma]$ with respect to the norm

$$\|x\| = \sup_{\pi} \|\pi(x)\|$$

as π ranges over all unitary representations of Γ.

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$\mathcal{O}_2$

- Suppose that H is separable with orthonormal basis $(e_n : n \in \mathbb{N})$. Let $T_1, T_2 : H \to H$ be the bounded operators defined by $T_1(e_n) = e_{2n}$ and $T_2(e_n) = e_{2n+1}$.
- The C* subalgebra of $B(H)$ generated by T_1 and T_2 is called the *Cuntz algebra* $\mathcal{O}_2$.
- $\mathcal{O}_2$ is a very interesting C* algebra for many reasons. It has played a crucial role in the classification programme for (simple, separable, nuclear) C* algebras.

Continuous logic-The case of C^* algebras

- Atomic formulae: $\varphi(\vec{x}) := \|p(\vec{x})\|$, $p(\vec{x})$ a *polynomial (over $\mathbb{C}$).
- Quantifier-free formulae: $\varphi(\vec{x}) := f(\varphi_1(\vec{x}), \ldots, \varphi_n(\vec{x}))$, each φ_i atomic, $f : \mathbb{R}^n \to \mathbb{R}$ continuous.
- Quantifiers: If φ is a formula, then so are $\sup_x \varphi$ and $\inf_x \varphi$.
- If A is a C^* algebra, $\varphi(\vec{x})$ is a formula, and $\vec{a}$ a tuple from A, then $\varphi(\vec{a})^A$ is a real number.
- For example, $\varphi(x) := \sup_y \|xy - yx\|$ is a formula. If A is a C^* algebra and $a \in A$, then $\varphi(a)^A = 0$ if and only if a is in the *center of A*.
- A *sentence* is a formula with no free variables.
- C^* algebras A and B are *elementarily equivalent*, written $A \equiv B$, if $\sigma^A = \sigma^B$ for all sentences σ.

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The ultraproduct of C* algebras

- Suppose that $(A_i \ : \ i \in I)$ is a family of C* algebras and that $\mathcal{U}$ is an ultrafilter on I.
- The *ultraproduct* of the family is the C* algebra

$$\prod_{\mathcal{U}} A_i := \ell^\infty(A_i)/c_{\mathcal{U}}(A_i),$$

where

$$\ell^\infty(A_i) := \{(x_i) \in \prod_i A_i \ : \ \sup_i \|x_i\| < \infty\}$$

and

$$c_{\mathcal{U}}(A_i) := \{(x_i) \in \ell^\infty(A_i) \ : \ \lim_{\mathcal{U}} \|x_i\| = 0\}.$$

- When $A_i = A$ for all i, we write $A^{\mathcal{U}}$ and call it an *ultrapower of A*.

u.c.p. maps

- If A is a C* algebra, then elements of the form x^*x are called *positive*.
- If B is also a C* algebra, then a linear map $\phi : A \to B$ is called *positive* if it maps positive elements to positive elements.
- $*$ homomorphisms are positive: $\phi(x^*x) = \phi(x)^*\phi(x)$.
- A linear map $\phi : A \to B$ induces linear maps $\phi_n : M_n(A) \to M_n(B)$ (pointwise application).
- Since $M_n(A)$ and $M_n(B)$ are also C* algebras, it makes sense to speak of each ϕ_n being positive. If this happens, ϕ is said to be *completely positive*. If in addition $\phi(1) = 1$, we say that ϕ is *unital, completely positive*, or u.c.p.

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Nuclearity

- Suppose that A is a C* algebra and $m, n \in \mathbb{N}$.
- Define a function $P^A_{m,n} : A^n \to \mathbb{R}$ by

$$P^A_{m,n}(\vec{a}) = \inf_{\phi,\psi} \|(\psi \circ \phi)(\vec{a}) - \vec{a}\|,$$

  where $\phi : A \to M_m(\mathbb{C})$ and $\psi : M_m(\mathbb{C}) \to A$ are u.c.p. maps.
- We also define $\Delta^A_{\mathrm{nuc},n} := \inf_m P^A_{m,n}$.
- We say that A is *nuclear* if $\Delta^A_{\mathrm{nuc},n} \equiv 0$ for each n.
- Abelian C* algebras are nuclear. (Partitions of unity.)
- $C^*_r(\Gamma)$ is nuclear if and only if Γ is amenable. (Lance)
- $\mathcal{O}_2$ is nuclear (Cuntz).

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KEP

Theorem (Kirchberg-Phillips)

Every separable nuclear (even exact) C^* algebra embeds into $\mathcal{O}_2$.

Kirchberg's Embedding Problem (KEP)

Does every C* algebra embed into an ultrapower of $\mathcal{O}_2$?

In model-theoretic terms, if A is a C* algebra, does $A \models \mathrm{Th}_\forall(\mathcal{O}_2)$?

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Existentially closed C* algebras

Definition

A C* algebra A is said to be *existentially closed* (e.c.) if, for any quantifier-free formula $\varphi(\vec{x}, \vec{y})$, tuple $\vec{a}$ from A, and extension $B \supseteq A$, we have

$$\inf\{\varphi(\vec{b}, \vec{a})^A \ : \ \vec{b} \in A_1\} = \inf\{\varphi(\vec{c}, \vec{a})^B \ : \ \vec{c} \in B_1\}.$$

Proposition (G. and Sinclair)

1. $\mathcal{O}_2$ is the only possible nuclear C* algebra that is also e.c.
2. KEP is equivalent to the statement that $\mathcal{O}_2$ is e.c.

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Good nuclear witnesses

- A *condition* is a finite set $p(\vec{x})$ of expressions of the form $\varphi(\vec{x}) < r$, where $\varphi(\vec{x})$ is a quantifier-free formula and $r \in \mathbb{R}^{>0}$.
- We say that a tuple $\vec{a}$ from a C* algebra A *satisfies* $p(\vec{x})$ if $\varphi^A(\vec{a}) < r$ holds for all expressions in $p(\vec{x})$.
- We say that $p(\vec{x})$ is *satisfiable* if there is a tuple in a C* algebra satisfying it.
- We say that $p(\vec{x})$ has *good nuclear witnesses* if, for any $\epsilon > 0$, there is a C* algebra A and a tuple $\vec{a}$ from A satisying p with $\Delta^A_{\text{nuc}}(\vec{a}) < \epsilon$.

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KEP is equivalent to the statement that every satisfiable condition has good nuclear witnesses.

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Idea behind the proof

- Recall that A is nuclear if and only if, for each n, we have

$$\sup_{\vec{x}} \inf_m P^A_{m,n}(\vec{x}) = 0.$$

- Now the predicates $P_{m,n}$ are not officially part of the language we mentioned for studying C^* algebras, but the *Beth definability theorem* shows that they can be expressed in our language, and in fact by existential formulae.
- This shows that nuclearity is an *omitting types property*.
- The *Omitting Types Theorem* states that under a certain hypothesis, we can find a C^* algebra that omits these types (so is nuclear) and is simultaneously e.c. By our previous proposition, this shows that KEP holds.
- The hypothesis needed in this situation is exactly the statement that every satisfiable condition has good nuclear witnesses.

QWEP

Definition

Suppose that A is a C* algebra.

1. If B is a C* algebra containing A, we call a u.c.p. map $\phi : B \to A^{**}$ that restricts to the identity on A a *weak conditional expectation*.
2. We say that A has the *weak expectation property* of *WEP* if whenever $A \subseteq B$, then there is a weak expectation $B \to A^{**}$.
3. We say that A is *QWEP* if A is a quotient of a C* algebra with WEP.

Kirchberg's QWEP Conjecture

Every separable C* algebra is QWEP.

In deep work, Kirchberg proved that the QWEP conjecture is equivalent to one of the most famous open problems in operator algebras, the *Connes Embedding Problem*.

QWEP is axiomatizable

Observation (G.)

The class of QWEP algebras is axiomatizable. (Sinclair and I showed that this is not the case for WEP algebras.)

Proof.

- Closure under isomorphisms ✓
- Closure under ultraproducts: Kirchberg proved that QWEP is closed under products and it is clearly closed under quotients; an ultrapower is a quotient of a product
- Closure under ultraroot: One can show that there is a weak expectation $A^{\mathcal{U}} \to A^{**}$ and Kirchberg proved that if there is a weak conditional expectation $B \to A^{**}$ and B is QWEP then so is A.

□

Models with special properties

- Farah, Hart, et. al. asked whether every C* algebra was elementarily equivalent to a nuclear C* algebra.
- They soon showed that this was not the case. But perhaps they weren't aware of what a positive answer might mean:

Corollary (G.)

The QWEP conjecture is equivalent to the statement that $C^*(\mathbb{F}_\infty)$ is elementarily equivalent to a QWEP C^* algebra.

Proof.

Kirchberg proved that the QWEP conjecture is equivalent to the statement that $C^*(\mathbb{F}_\infty)$ is QWEP. □

The local lifting property

Definition

Let A be a C* algebra.

1. Suppose that $\phi : A \to C/J$ is a u.c.p. map and that $E \subseteq A$ is a finite-dimensional vector subspace that is closed under $*$. A *lift of* $\phi|E$ is a u.c.p. map $\psi : E \to C$ such that $\pi \circ \psi = \phi$, where $\pi : C \to C/J$ is the quotient map.
2. A has the *local lifting property* (or LLP) if whenever ϕ and E are as above, then $\phi|E$ has a lift.

Theorem (Kirchberg)

The QWEP conjecture is equivalent to the statement that LLP implies WEP.

The weak QWEP conjecture

The weak QWEP conjecture

There is a non-nuclear C* algebra with both WEP and LLP.

Proposition (G.)

If either $C_r^*(\mathbb{F})$ or $\prod_{\mathcal{U}} M_n$ are elementarily equivalent to an LLP C* algebra B, then B witnesses the truth of the weak QWEP conjecture.

Proof.

Both of the above algebras are QWEP, whence so is B. Kirchberg proved that QWEP+LLP implies WEP. Finally, Farah, Hart et. al. proved that the above algebras have no nuclear models. □

The LLPEP

- Call a C^* algebra S *locally universal* if every C^* algebra embeds into an ultrapower of S.
- It is easy to see that separable locally universal C^* algebras exist.
- KEP is equivalent to the statement that there is a separable nuclear locally universal C^* algebra.

Definition

We refer to the statement that there is a separable locally universal LLP C^* algebra the LLPEP.

Clearly KEP implies LLPEP.

Putting everything together

Question

Is LLP omitting types?

Sinclair and I have shown that LLP is axiomatizable by axioms of the form $\sup_x \inf_v \inf_l \sup_m \chi_{l,m}(v,x) = 0$ with χ existential.

Theorem

Suppose that LLP is omitting types. Then either KEP and LLPEP are equivalent or else the weak QWEP conjecture has a positive solution.

Proof.

Suppose that the LLPEP holds. Then we can run the omitting types theorem to get an e.c. C* algebra A that is LLP. Sinclair and I showed that e.c. C* algebras have WEP. If A is nuclear, then KEP holds. Otherwise, A is a witness to the weak QWEP conjecture. □

References

- C. Eagle, I. Farah, E. Kirchberg, A. Vignati, *Quantifier elimination in C^* algebras*, preprint.
- I. Goldbring and T. Sinclair, *On Kirchberg's Embedding Problem*, to appear in the Journal of Functional Analysis. arXiv 1404.1861