Scattered sentences have few separable randomizations

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Vaught's conjecture

In this talk:

- L is a countable (classical) first-order language,
- I denotes the set of isomorphism types of countable L-structures,
- for $i \in I$, θ_i denotes a Scott sentence for i,
- φ denotes a sentence of $L_{\omega_1,\omega}$, and
- $I(\varphi)$ denotes the set of isomorphism types of countable models of φ.

Vaught's conjecture

$I(\varphi)$ is either countable or has cardinality $2^{\aleph_0}$.

Theorem (Morley, 1970)

$|I(\varphi)| \leq \aleph_1$ *or* $|I(\varphi)| = 2^{\aleph_0}$.

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The following are equivalent:

- *For each* $\alpha < \omega_1$*, there are only countably many* $\equiv_\alpha$*-classes of models of* φ*;*
- *There is no perfect set of models of* φ*.*

Definition

φ is *scattered* if either of the above equivalent conditions hold.

Theorem (Morley)

- φ *scattered* $\Rightarrow |I(\varphi)| \leq \aleph_1$*.*
- φ *not scattered* $\Rightarrow |I(\varphi)| = 2^{\aleph_0}$*.*

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The pure randomization theory

- The *continuous language* L^R has two sorts: a sort $\mathbb{K}$ for random variables and a sort $\mathbb{E}$ for events.
- For each n-ary L-formula θ, there is a function symbol $[\![\theta(\cdot)]\!] : \mathbb{K}^n \to \mathbb{E}$.
- The *pure randomization theory* P^R has the following axioms:
  - atomless probability algebra axioms;
  - $\sup_{\vec{X}} d([\![(\theta \wedge \psi)(\vec{X})]\!], [\![\theta(\vec{X})]\!] \sqcap [\![\psi(\vec{X})]\!]) = 0$, etc...;
  - $\sup_{\vec{X}} \inf_Y d([\![\exists y(\theta(\vec{X}, y)]\!], [\![\theta(\vec{X}, Y)]\!]) = 0$;
  - $d([\![\sigma]\!], \top) = 0$ for all tautologies σ;
  - $\sup_B \inf_{X,Y} d(B, [\![X = Y]\!]) = 0$;
  - $\sup_{B,C} |d(B, C) - \mu(B \triangle C)| = 0$ and $\sup_{X,Y} |d(X, Y) - \mu[\![X \neq Y]\!]| = 0$.
- Pre-models of P^R are called *randomizations* and models are called *complete randomizations*.

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Borel randomizations

Example

Let $\mathcal{M}$ be a structure with at least two elements. The *Borel randomization of* $\mathcal{M}$ is the structure $(\mathcal{M}^{[0,1)}, \mathcal{L})$, where:

- $\mathcal{M}^{[0,1)}$ is the set of functions $\mathbf{f} : [0,1) \to \mathcal{M}$ with countable range such that $\mathbf{f}^{-1}(t) \in \mathcal{L}$ for all $t \in [0,1)$;
- $\mathcal{L}$ is the family of Borel subsets of $[0,1)$ equipped with Lebesgue measure;
- $[\![\theta(\vec{\mathbf{f}})]\!] := \{t \in [0,1) \ : \ \mathcal{M} \models \theta(\vec{\mathbf{f}}(t))\}$.

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- $[0,1) = \bigcup_n B_n$ is a partition of $[0,1)$ into positive measure Borel sets;
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Then $(\prod_n \mathcal{M}_n^{B_n}, \mathcal{L})$ is called a *basic randomization*.

Basic randomizations are also pre-complete separable randomizations. Their isomorphism type is captured by their *density function*.

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Randomizations of $L_{\omega_1,\omega}$-sentences

Theorem (Keisler)

If $\mathcal{P}$ is a complete separable randomization, then there is a unique mapping $[\![\cdot]\!]^{\mathcal{P}}$ from $L_{\omega_1,\omega}$-sentences to events that agrees with the interpretation of $[\![\cdot]\!]$ on first-order sentences that also respects validity, countable connectives, and quantification. Moreover, the maps are all Lipshitz with bound 1.

Definition

If $\mathcal{N}$ is a separable randomization with completion $\mathcal{P}$, we say that $\mathcal{N}$ is a *randomization of* φ if $\mu^{\mathcal{N}}[\![\varphi]\!] := \mu([\![\varphi]\!]^{\mathcal{P}}) = 1$.

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Suppose that $\mathcal{N}$ is the reduction of the basic randomization $(\prod_n \mathcal{M}_n^{B_n}, \mathcal{L})$. Then $\mathcal{N}$ is a randomization of φ if and only if each $\mathcal{M}_n \models \varphi$, in which case we say that $\mathcal{N}$ is a *basic randomization of* φ.

Definition

We say that φ has *few separable randomizations* if every complete randomization of φ is isomorphic to a basic randomization.

Natural Question

Which sentences have few separable randomizations?

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Scattered sentences and few separable randomizations

Theorem (Keisler)

If φ has few separable randomizations, then φ is scattered.

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Assume that Lebesgue measure is $\aleph_1$-additive (e.g. assume MA($\aleph_1$)). If φ is scattered, then φ has few separable randomizations.

Theorem (Andrews, G., Hachtman, Keisler, Marker)

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A representation theorem

Theorem (Keisler)

Every complete separable randomization of φ is isomorphic to the completion of a countable randomization $\mathcal{N} = (\mathcal{K}, \mathcal{B})$ such that for some atomless probability space $(\Omega, \mathcal{E}, \nu)$ and family of countable models $\langle \mathcal{M}_t \rangle_{t \in \Omega}$ of φ we have:

(a) $\mathcal{K} \subseteq \prod_{t \in \Omega} M_t$ *and* $\mathcal{B} \subseteq \mathcal{E}$.

(b) $M_t = \{\boldsymbol{f}(t) \mid \boldsymbol{f} \in \mathcal{K}\}$ *for each* $t \in \Omega$.

(c) $(\Omega, \mathcal{E}, \nu)$ *is the (*σ*-additive) probability space generated by* $(\Omega, \mathcal{B}, \mu)$.

(d) *For each* $L_{\omega_1 \omega}$*-formula* $\psi(\cdot)$ *and tuple* $\vec{\boldsymbol{f}}$ *in* $\mathcal{K}$,

$$\mu^{\mathcal{N}}([\![\psi(\vec{\boldsymbol{f}})]\!]) = \nu(\{t \in \Omega \mid \mathcal{M}_t \models \psi(\vec{\boldsymbol{f}}(t))\}).$$

If, in addition, φ *is scattered, then we may take* $(\Omega, \mathcal{E}, \nu) = ([0,1), \mathcal{L}, \lambda)$.

Few separable randomizations implies scattered

- Suppose that φ is not scattered, so there is a perfect set $\langle \mathcal{M}_t \rangle$ of nonisomorphic models of φ.
- By the Borel isomorphism theorem, we might as well assume $t \in [0, 1)$.
- From this data, we can then build a countable randomization $\mathcal{N}$ as in the representation theorem.
- For any $i \in I$, $|\{t \in [0, 1) \ : \ \mathcal{M}_t \models \theta_i\}| \leq 1$, whence $\mu^{\mathcal{N}}([\![\theta_i]\!]) = 0$.
- But in a basic randomization $\mathcal{P}$, there is $i \in I$ such that $\mu^{\mathcal{P}}([\![\theta_i]\!]) > 0$.
- It follows that the completion of $\mathcal{N}$ is not isomorphic to $\mathcal{P}$ and thus φ does not have few separable randomizations.

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A test for being isomorphic to a basic randomization

Lemma (Keisler)

Suppose we have:

- *a countable subset $J \subseteq I$;*
- *for each $j \in J$, a structure $\mathcal{M}_j$ with isomorphism type j;*
- *a basic randomization $\mathcal{P} = (\prod_{j \in J} \mathcal{M}_j^{A_j}, \mathcal{L})$, and*
- *a separable randomization $\mathcal{N}$.*

Then $\mathcal{N} \cong \mathcal{P}$ if and only if: for each $j \in J$, we have $\mu^{\mathcal{N}}([\![\theta_j]\!]) = \lambda(A_j)$.

Corollary (Keisler)

φ has few separable randomizations if and only if every separable randomization $\mathcal{N}$ of φ satisfies property (S): there is $i \in I$ such that $\mu^{\mathcal{N}}[\![\theta_i]\!] > 0$.

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Scattered implies few separable randomizations (assuming MA($\aleph_1$))

- Suppose that φ is scattered and let $\mathcal{N}$ be a separable randomization of φ with representation as in the theorem.
- For each $i \in I(\varphi)$, let $B_i := \{t : \mathcal{M}_t \models \theta_i\} \in \mathcal{L}$.
- Note that $|I(\varphi)| \leq \aleph_1$ and $[0,1) = \bigsqcup_{i \in I(\varphi)} B_i$.
- Let $J := \{i \in I(\varphi) : \lambda(B_i) > 0\}$. Then $|I(\varphi) \setminus J| \leq \aleph_1$ so $\lambda(\bigcap_{j \notin J} B_j) = 0$ and hence $\lambda(\bigsqcup_{j \in J} B_j) = 1$ by MA($\aleph_1$).
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- Suppose, in V, that φ is scattered and that $\mathcal{N}$ is a countable randomization of φ.
- We show that $\mathcal{N}$ has property (S) in V.
- Go to a forcing extension $V[G]$ with the same ordinals such that MA($\aleph_1$) holds.
- By Shoenfield absoluteness, φ is still scattered in $V[G]$, whence has few separable randomizations in $V[G]$ by Keisler's theorem.
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Preservation of countable randomizations

- Suppose that $\mathcal{N} = (\mathcal{K}, \mathcal{E})$ is a countable randomization of φ in V. We want this to remain true in $V[G]$.
- Let $\mathcal{P}$ and $\mathcal{Q}$ be the completions of $\mathcal{N}$ in V and $V[G]$ respectively.
- The desired result follows from the more general fact that $[\![\psi(\mathbf{f})]\!]^{\mathcal{P}} = [\![\psi(\mathbf{f})]\!]^{\mathcal{Q}}$ for all $L_{\omega_1,\omega}$-formulae ψ and all tuples $\mathbf{f} \in \mathcal{K}$.
- One proves this fact by induction on complexity of formulae.
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Absoluteness of property (S)

Lemma

Let $\mathcal{N} = (\mathcal{K}, \mathcal{B})$ be a countable randomization. Then the following two statements are equivalent:

(S) *there is $i \in I$ such that $\mu^{\mathcal{N}}[\![\theta_i]\!] > 0$*

(S') *there is a countable $\mathcal{M}$ with $|M| \geq 2$ and a positive measure set C in the completion of $\mathcal{B}$ such that $\mathcal{M}^{\mathcal{L}} \cong \mathcal{N}|C$.*

Here, $\mathcal{N}|C$ is the completion of the randomization $\mathcal{N}$ with μ replaced by the conditional measure $\mu(\cdot|C)$.

Proof.

($\Rightarrow$) Take $\mathcal{M} \models \theta_i$ and $C := [\![\theta_i]\!]$.
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$$\mu^{\mathcal{N}}[\![\theta_i]\!] = \mu^{\mathcal{N}|C}[\![\theta_i]\!] \cdot \mu(C) = \mu(C) > 0. \quad \square$$

Absoluteness of property (S)

Lemma

Let $\mathcal{N} = (\mathcal{K}, \mathcal{B})$ be a countable randomization. Then the following two statements are equivalent:

(S) *there is $i \in I$ such that $\mu^{\mathcal{N}}[\![\theta_i]\!] > 0$*

(S') *there is a countable $\mathcal{M}$ with $|M| \geq 2$ and a positive measure set C in the completion of $\mathcal{B}$ such that $\mathcal{M}^{\mathcal{L}} \cong \mathcal{N}|C$.*

Here, $\mathcal{N}|C$ is the completion of the randomization $\mathcal{N}$ with μ replaced by the conditional measure $\mu(\cdot|C)$.

Proof.

($\Rightarrow$) Take $\mathcal{M} \models \theta_i$ and $C := [\![\theta_i]\!]$.
($\Leftarrow$) Take i such that $\mathcal{M} \models \theta_i$. Then

$$\mu^{\mathcal{N}}[\![\theta_i]\!] = \mu^{\mathcal{N}|C}[\![\theta_i]\!] \cdot \mu(C) = \mu(C) > 0. \quad \square$$

Absoluteness of property (S)

Lemma

Let $\mathcal{N} = (\mathcal{K}, \mathcal{B})$ be a countable randomization. Then the following two statements are equivalent:

- (S) *there is $i \in I$ such that $\mu^{\mathcal{N}}[\![\theta_i]\!] > 0$*
- (S') *there is a countable $\mathcal{M}$ with $|M| \geq 2$, a sequence $B : \mathbb{N} \to \mathcal{B}$, and double sequences $\alpha : \mathbb{N} \times \mathbb{N} \to \mathcal{M}^{\mathcal{A}}$ and $\beta : \mathbb{N} \times \mathbb{N} \to \mathcal{K}$ such that:*
  1. *(B_n) is Cauchy and $\lim_n \mu(B_n) > 0$;*
  2. *for each m, $(\alpha_{m,n})$ and $(\beta_{m,n})$ are Cauchy;*
  3. *for each $x \in \mathcal{M}^{\mathcal{A}}$, there is $m_x \in \mathbb{N}$ such that $\alpha_{m_x,n} = x$ for all n and likewise for $\mathcal{K}$ and β;*
  4. *for each L-formula $\psi(v_1, \ldots, v_k)$, we have*

$$\lim_n \mu^{\mathcal{M}^{\mathcal{A}}}([\![\psi(\vec{\alpha_n})]\!]) = \lim_n \mu^{\mathcal{N}}([\![\psi(\vec{\beta_n})]\!] \sqcap B_n)/\mu^{\mathcal{N}}(B_n).$$

(S') is easily seen to be Σ^1_1 with parameter $\mathcal{N}$.

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