A NOTE ON ENDS OF GROUPS

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We consider the following question: Suppose that G is a f.g. group with infinitely many ends. Is it possible for G to have an infinite cyclic normal subgroup? Ilya Kapovich explained to me that the answer is no. In fact, he showed me that, using Bass-Serre theory, if G splits over a finite subgroup and has an infinite cyclic normal subgroup, then G is virtually cyclic, whence has two ends; since groups with infinitely many ends admit a nontrivial splitting over a finite subgroup (by Stallings' theorem), this settles our question.

The point of this note is to give a much more elementary proof of this fact under the simplifying assumption that the quotient of G by its infinite cyclic normal subgroup has finitely many ends. We use the nonstandard treatment of ends given in [1].

Theorem 0.1. *Suppose that G has infinitely many ends and H is an infinite cyclic normal subgroup of G. Then G/H has infinitely many ends.*

Proof. For simplicity, we set $C := G/H$ and we let $\pi : G \to C$ be the quotient map. We also suppose that S is a generating set for G with $S = S^{-1}$. We choose $\pi(S)$ to be the generating set for C. Finally, we suppose that a is a generator for H.

We begin with an observation: Suppose that $g \in G_{\text{inf}}$ and $\pi(g) \in C$. Then $\pi(g) = \pi(h)$ for some $h \in G$, so $gh^{-1} \in H^*$. Moreover, $gh^{-1} \in H_{\text{inf}}$, else $g \in G$. Consequently, if $g \in G_{\text{inf}}$ and $\pi(g) \in C$, then $g = a^k b$ for some $k \in \mathbb{Z}^* \setminus \mathbb{Z}$ and some $b \in G$; in this case, $g \propto a^k$. Also note that $a^k \propto a^l$ for all $k, l \in \mathbb{Z}^*$ with $kl > 0$. As a consequence, at most two ends of G have representatives $g \in G_{\text{inf}}$ with $\pi(g) \in C$.

Suppose that $g, g' \in G_{\text{inf}}$ satisfy $\pi(g), \pi(g') \in C_{\text{inf}}$ and $\pi(g) \propto \pi(g')$. Then $\pi(g) = \pi(g')\pi(s_1)\cdots\pi(s_N)$, where $s_1, \ldots, s_N$ is an internal sequence from S and $\pi(g')\pi(s_1)\cdots\pi(s_i) \in C_{\text{inf}}$ for all $i \leq N$. Thus, there is $k \in \mathbb{Z}^*$ such that $ga^k = g's_1 \cdots s_N$. Since $ga^k \propto g$, we see that $g \propto g's_1 \ldots s_N$. Since $g's_1 \cdots s_i \in B_{\text{inf}}$ for all $i \leq N$ (else $\pi(g')\pi(s_1)\cdots\pi(s_i) \in C$), we have that $g's_1 \cdots s_N \propto g'$, whence $g \propto g'$. Consequently, if (g_n) is a sequence from G_{inf} with $\pi(g_n) \in C_{\text{inf}}$ for each n and $g_m \not\propto g_n$ for all distinct m, n, then $\pi(g_m) \not\propto \pi(g_n)$ for all distinct m and n, proving that G/H has infinitely many ends. ∎

References

[1] I. Goldbring, *Ends of groups: a nonstandard perspective*, to appear in the Journal of Logic and Analysis. arXiv 1008.2795.