HILBERT'S 5TH PROBLEM

LOU VAN DEN DRIES AND ISAAC GOLDBRING

1. Introduction

A Lie group is a topological group G for which inversion $x \mapsto x^{-1} : G \to G$ and multiplication $(x, y) \mapsto xy : G \times G \to G$ are analytic maps with respect to some compatible real analytic manifold structure on its underlying topological space. It is a remarkable fact that then there is only one such real analytic manifold structure. This uniqueness falls under the slogan

$$\text{Algebra} \times \text{Topology} = \text{Analysis}$$

Important Lie groups are the vector groups $\mathbb{R}^n$, their compact quotients $\mathbb{R}^n/\mathbb{Z}^n$, the general linear groups $\mathrm{GL}_n(\mathbb{R})$, and the orthogonal groups $\mathrm{O}_n(\mathbb{R})$. For each of these the group structure and the real analytic manifold structure is the obvious one; for example, $\mathrm{GL}_n(\mathbb{R})$ is open as a subset of $\mathbb{R}^{n^2}$, and thus an open submanifold of the analytic manifold $\mathbb{R}^{n^2}$.

Hilbert's 5th problem asks for a characterization of Lie groups that is free of smoothness or analyticity requirements. A topological group is said to be *locally euclidean* if some neighborhood of its identity is homeomorphic to some $\mathbb{R}^n$. A Lie group is obviously locally euclidean, and the most common version of Hilbert's 5th problem (H5) can be stated as follows:

Is every locally euclidean topological group a Lie group?

A positive solution to this problem was achieved in the early fifties by the combined efforts of Gleason [2] and Montgomery & Zippin [12]. Yamabe improved their results in [16] and [17]. Montgomery & Zippin exposed all of this and more in their book [13] on topological transformation groups. Kaplansky has also a nice treatment in Chapter 2 of [10]. Of course, the affirmative solution of H5 gives further substance to our crude slogan.

Locally euclidean topological groups are certainly locally compact. (We include being hausdorff as part of local compactness.) From now on G denotes a locally compact (topological) group, with identity 1, or 1_G if we want to indicate G. Local compactnes yields a powerful analytic tool, namely Haar measure, and we shall need it.

A notion that has turned out to be central in the story is that of having no small subgroups: G is said to have *no small subgroups* (briefly: *G has* NSS) if there is a neighborhood U of 1 in G that contains no subgroup of G other than $\{1\}$. It is also useful to introduce a weaker variant of this property: G is said to have *no small connected subgroups* (briefly: *G has* NSCS) if there is a neighborhood U of 1 in G that contains no connected subgroup of G other than $\{1\}$. Dimension theory also plays a modest role: call a topological space is *bounded in dimension* if for some n no subspace is homeomorphic to the unit cube $[0,1]^n$. We can now formulate the main result as characterizing Lie groups among locally compact groups:

Main Theorem. *Given G, the following are equivalent:*

1. *G is a Lie group;*
2. *G has* NSS*;*
3. *G is locally euclidean;*
4. *G is locally connected and has* NSCS*;*
5. *G is locally connected and bounded in dimension.*

We say that G *can be approximated by Lie groups* if every neighborhood of its identity contains a closed normal subgroup N of G such that G/N (with its quotient topology) is a Lie group. The following result, due to Yamabe, is closely related to the Main Theorem, and is important in the structure theory of locally compact groups.

Theorem. *Every locally compact group has an open subgroup that can be approximated by Lie groups.*

Hirschfeld [6] used nonstandard methods to simplify some tricky parts of the work by Gleason, Montgomery, and Yamabe. We give here an account of [6] with further simplifications, and some corrections. What's more interesting, Goldbring [4] elaborated these methods to solve affirmatively the *local* form of H5. (A 1957 paper in the Annals of Mathematics by Jacoby [9] claimed a solution to local H5, but about 15 years ago it was found that this paper was seriously wrong; see [14].) We shall discuss local H5 in the last two talks of this series. In the rest of this introduction we sketch the solution to (global) H5.

Further relevant history. The clearcut formulation of H5 above became only possible after basic topological notions had crystallized sufficiently in the 1920's to permit the definition of "topological group" by Schreier. The fundamental tool of Haar measure, on any locally compact group, became available soon afterwards. Von Neumann used it to extend the Peter-Weyl theorem for compact Lie groups to all compact groups, and this led to the solution of H5 for *compact* groups. (In our treatment of H5 we use a weak form of this extended Peter-Weyl theorem.) Another important partial solution of H5 is for the case of *commutative* G, due to Pontrjagin, and we shall need this as well. Finally, we are going to use a result of Kuranishi [11]:

if G has a commutative closed normal subgroup N such that N and G/N are Lie groups, then G is a Lie group.

Gleason [3] and Iwasawa [8] proved this result of Kuranishi without assuming commutativity of N, but we don't need this stronger version and instead obtain it as a consequence of the Main Theorem.

More recently, the solution to Hilbert's fifth problem was used by Hrushovski [7] and Breullard-Green-Tao [1] to solve the classification problem for approximate groups. Tao [15] also gives an account of Hilbert's fifth problem and related topics.

One-parameter subgroups. Lie theory provides a precious guide towards solving H5. It tells us that the tangent vectors at the identity of a Lie group are in a natural bijective correspondence with the 1-parameter subgroups of the Lie group. While tangent vectors require a manifold to live on, the notion of 1-parameter subgroup makes sense in any topological group.

A 1-*parameter subgroup* (or 1-*ps*) of G is a continuous group morphism $\mathbb{R} \to G$. The trivial 1-parameter subgroup O of G is defined by $O(t) = 1 \in G$ for all $t \in \mathbb{R}$. We set

$$L(G) := \{X : \mathbb{R} \to G \mid X \text{ is a 1-ps of } G\}.$$

For $r \in \mathbb{R}$ and $X \in L(G)$ we define $rX \in L(G)$ by $(rX)(t) := X(rt)$, and we also denote $(-1)X$ by $-X$. Note that then $0X = O$, $1X = X$, $-X = X^{-1}$, and $r(sX) = (rs)X$ for $r, s \in \mathbb{R}$ and $X \in L(G)$. The operation

$$(r, X) \mapsto rX : \mathbb{R} \times L(G) \to L(G)$$

will be referred to as *scalar multiplication*.

The case of Lie groups. Suppose G is a Lie group. Then each $X \in L(G)$ is analytic as a function from $\mathbb{R}$ to G, and thus determines a velocity vector $X'(0) \in T_1(G)$ at the point $1 \in G$. This gives the bijection

$$X \mapsto X'(0) : L(G) \to T_1(G)$$

mentioned above. It respects scalar multiplication: $(rX)'(0) = rX'(0)$. The addition operation on $L(G)$ that makes this bijection an isomorphism of vector spaces over $\mathbb{R}$ is as follows: for $X, Y \in L(G)$,

$$(X + Y)(t) = \lim_{n\to\infty} \big(X(1/n)Y(1/n)\big)^{[nt]}.$$

We make $L(G)$ a real analytic manifold such that the $\mathbb{R}$-linear isomorphisms $L(G) \cong \mathbb{R}^n$, with $n := \dim G = \dim_{\mathbb{R}} L(G)$, are analytic isomorphisms. Then the so-called *exponential map*

$$X \mapsto X(1) : L(G) \to G$$

yields an analytic isomorphism from an open neighborhood of O in $L(G)$ onto an open neighborhood of 1 in G.

Sketch why NSS implies Lie. These facts about Lie groups suggest that we should try to establish $L(G)$ as a substitute tangent space at 1, towards finding a compatible manifold structure on G. Note in this connection that the exponential map $X \mapsto X(1) : L(G) \to G$ is defined for any G. This is our clue to proving the key implication NSS $\Rightarrow$ Lie in the Main Theorem.

Indeed, we shall take the following steps towards proving this implication. Suppose G has NSS.

(1) Show that for any $X, Y \in L(G)$ there is an $X + Y \in L(G)$ given by
$$(X+Y)(t) = \lim_{n\to\infty} \big(X(1/n)Y(1/n)\big)^{[nt]},$$
and that this addition operation and the scalar multiplication make $L(G)$ a vector space over $\mathbb{R}$.

(2) Equip $L(G)$ with its compact-open topology (defined below) and show that this makes $L(G)$ a *topological* vector space.

(3) Show that the exponential map $X \mapsto X(1) : L(G) \to G$ maps some neighborhood of O in $L(G)$ homeomorphically onto a neighborhood of 1 in G. Then local compactness of G yields local compactness of $L(G)$ and hence the finite-dimensionality of $L(G)$ as a vector space over $\mathbb{R}$. It follows that G is locally euclidean.

(4) Replacing G by the connected component of 1, we can assume that G is connected. Then the adjoint representation (defined below) of G on the finite-dimensional vector space $L(G)$ has as its kernel a commutative closed normal subgroup N of G, and yields an injective continuous group morphism $G/N \to \mathrm{GL}_n(\mathbb{R})$. Since N has NSS, it is locally euclidean by (3). But N is also commutative, and hence a Lie group (Pontrjagin). The injective continous group morphism $G/N \to \mathrm{GL}_n(\mathbb{R})$ makes G/N a Lie group (E. Cartan, von Neumann). Applying the Kuranishi theorem we conclude that G is a Lie group.

Step (1) is tricky, and requires ingenious constructions due to Gleason and Yamabe. Step (2) is easy, and step (3) is of intermediate difficulty. Step (4) is a reduction of the problem to a situation that that was well-understood before 1950.

New in our treatment is that we carry out steps (1) and (2) without requiring NSS: local compactness of G is enough. Some of (3) and (4) can also be done in this generality, and this is the first thing we shall take care of in the next section.

Sketch why every locally euclidean G has NSS. This is the other key implication in the Main Theorem, and it passes through the other equivalent conditions (4) and (5) in the Main Theorem. This goes roughly as follows. When we have done step (1) above for all G, without assuming NSS, we can use this to prove the following implications:

- if G is locally connected and has NSCS, then G has NSS;
- if G does not have NSCS, then G contains a homeomorphic copy of $[0,1]^n$ for all n.

It only remains to observe that if G is locally euclidean, then G is locally connected (trivially), and bounded in dimension (by Brouwer).

2. Preliminaries

Throughout we let m, n range over $\mathbb{N} = \{0, 1, 2, \dots\}$ and let G and H denote locally compact groups. Given a closed normal subgroup N of G we give G/N its quotient topology; it makes G/N a locally compact group. We also give $\mathbb{R}$ its usual topology, and each $\mathbb{R}^n$ the corresponding product topology. Any n-dimensional vector space over $\mathbb{R}$ is given the topology that makes the $\mathbb{R}$-linear isomorphisms with $\mathbb{R}^n$ into homeomorphisms.

In this section we state some basic facts on $L(G)$ and its compact-open topology. We also list some some elementary facts concerning NSS-groups, and introduce the nonstandard setting that will enable an efficient account of the solution of H5. Here is some further terminology. A subset U of G is said to be *symmetric* if $U^{-1} = U$.

Generalities on one-parameter groups.

Lemma 2.1. *Suppose $X \in L(G)$ and $X \neq O$. Then either $\ker X = \{0\}$ or $\ker X = \mathbb{Z}r$ with $r \in \mathbb{R}^{>0}$. In the first case X maps each bounded interval $(-a, a)$ $(a \in \mathbb{R}^{>0})$ homeomorphically onto its image in G. In the second case X maps the interval $(\frac{-r}{2}, \frac{r}{2})$ homeomorphically onto its image in G.*

Proof. This follows from two well-known facts: a closed subgroup of the additive group of $\mathbb{R}$ different from $\{0\}$ and $\mathbb{R}$ is of the form $\mathbb{Z}r$ with $r \in \mathbb{R}^{>0}$, and any continuous bijection from a compact space onto a hausdorff space is a homeomorphism. □

In the rest of these notes s ranges over $\mathbb{R}^{>0}$. For $X, Y \in L(G)$ we say that $X + Y$ *exists* if $\lim_{s\to\infty} \big(X(1/s)Y(1/s)\big)^{[st]}$ exists in G for all $t \in \mathbb{R}$. In that case the map

$$t \mapsto \lim_{s\to\infty} \big(X(1/s)Y(1/s)\big)^{[st]} : \mathbb{R} \to G$$

is a 1-ps of G, and we define $X + Y$ to be this 1-ps.

Lemma 2.2. *Let $X, Y \in L(G)$ and $p, q \in \mathbb{R}$.*

1. *$X + O$ exists and equals X;*
2. *$pX + qX$ exists and equals $(p + q)X$;*
3. *if $X + Y$ exists, then $Y + X$ exists and equals $X + Y$;*
4. *if $X + Y$ exists, then $pX + pY$ exists and equals $p(X + Y)$.*

Proof. We leave (1) and (2) to the reader. Note that (2) yields that $X+(-X)$ exists and equals O. For (3), use that for $a = X(1/s)$ and $b := Y(1/s)$ we have $ba = b(ab)b^{-1}$, so $(ba)^n = b(ab)^n b^{-1}$. Item (4) is easy when $p > 0$. To reduce the case $p < 0$ to this case one first shows that if $X + Y$ exists, then $(-X) + (-Y)$ exists, and equals $-(X + Y)$. □

We define the *adjoint action* of G on $L(G)$ to be the left action

$$(a, X) \mapsto aXa^{-1} : G \times L(G) \to L(G), \qquad (aXa^{-1})(t) := aX(t)a^{-1},$$

of G on the set $L(G)$. Then each $a \in G$ gives a bijection

$$\mathrm{Ad}(a) : L(G) \to L(G), \qquad \mathrm{Ad}(a)(X) := aXa^{-1},$$

and for $r \in \mathbb{R}$ and $X \in L(G)$ we have $\mathrm{Ad}(a)(rX) = r\,\mathrm{Ad}(a)(X)$, and if $X, Y \in L(G)$ and $X + Y$ exists, then $\mathrm{Ad}(a)(X) + \mathrm{Ad}(a)(Y)$ exists and equals $\mathrm{Ad}(a)(X + Y)$.

Corollary 2.3. *Suppose that $X+Y$ exists for all $X, Y \in L(G)$, and that the binary operation $+$ on $L(G)$ is associative. Then $L(G)$ with $+$ as its addition and the usual scalar multiplication is a vector space over $\mathbb{R}$ with O as zero element, and we have a group morphism $a \mapsto \mathrm{Ad}(a) : G \to \mathrm{Aut}\big(L(G)\big)$ of G into the group of automorphisms of the vector space $L(G)$.*

In the situation of this corollary the map $a \mapsto \mathrm{Ad}(a) : G \to \mathrm{Aut}(L(G))$ is called the *adjoint representation* of G.

Next, consider a continuous group morphism $\phi : G \to H$. Then we have a map

$$L(\phi) : L(G) \to L(H), \quad L(\phi)(X) := \phi \circ X,$$

and $L(\phi)(rX) = rL(\phi)(X)$ for all $r \in \mathbb{R}$ and $X \in L(G)$. Also, if $X, Y \in L(G)$ and $X + Y$ exists, so does $L(\phi)(X) + L(\phi)(Y)$ and

$$L(\phi)(X + Y) = L(\phi)(X) + L(\phi)(Y).$$

If ϕ is injective, so is $L(\phi)$. In particular, if G is a subgroup of H with the subspace topology and ϕ is the inclusion map, then we identify $L(G)$ with a subset of $L(H)$ via $L(\phi)$. With $N = \ker(\phi)$ (a closed subgroup of G) and O_H the trivial 1-ps of H we have

$$L(\phi)^{-1}(O_H) \;=\; L(N).$$

Note that assigning to each G the set $L(G)$ and to each ϕ as above the map $L(\phi)$ yields a functor L from the category of locally compact groups and continuous group morphisms into the category of sets.

Generalities on NSS. By "NSS-group" we mean a locally compact group that has NSS. Here are some examples of NSS-groups, and some basic facts about them.

(1) if G is discrete, then G has NSS; the additive group of $\mathbb{R}$ has NSS;
(2) $\mathrm{GL}_n(\mathbb{R})$ has NSS;
(3) if $G_1, \dots, G_n$ are NSS-groups, so is $G_1 \times \cdots \times G_n$;
(4) if $\phi : G \to H$ is a continuous injective group morphism and H has NSS, then G has NSS.
(5) if N is a closed normal subgroup of G such that N and G/N have NSS, then G has NSS.

The nonstandard setting. We assume familiarity with this setting; see [5] for details. Here we just fix notations and terminology. To each relevant "basic" set S corresponds functorially a set $S^* \supseteq S$, the *nonstandard extension* of S. In particular, $\mathbb{N}, \mathbb{R}, G$ extend to $\mathbb{N}^*, \mathbb{R}^*, G^*$, respectively. Also, any (relevant) relation R and function F on these basic sets extends functorially to a relation R^* and function F^* on the corresponding nonstandard extensions of these basic sets. For example, the linear ordering $<$ on $\mathbb{N}$ extends to a linear ordering $<^*$ on $\mathbb{N}^*$, and the group operation $p : G \times G \to G$ of G extends to a group operation $p^* : G^* \times G^* \to G^*$. For the sake of readability we only use a star in denoting the nonstandard extension of a basic set, but drop the star when indicating the nonstandard extension of a relation or function between these basic sets. For example, when $x, y \in \mathbb{R}^*$ we write $x + y$ and $x < y$ rather than $x +^* y$ and $x <^* y$; likewise, the nonstandard extension $X^* : \mathbb{R}^* \to G^*$ of a 1-ps $X : \mathbb{R} \to G$ is usually indicated just by X. Given an ambient hausdorff space S and $s \in S$, the *monad* of s, notation: $\boldsymbol{\mu}(s)$, is by definition the intersection of all $U^* \subseteq S^*$ with U a neighborhood of s in S; the elements of $\boldsymbol{\mu}(s)$ are the points of the nonstandard space S^* that are *infinitely close* to s. The points of S^* that are infinitely close to some $s \in S$ are called *nearstandard*, and S_{ns} is the set of nearstandard points of S^*:

$$S_{\mathrm{ns}} := \bigcup_{s \in S} \boldsymbol{\mu}(s).$$

In particular, $S \subseteq S_{\mathrm{ns}}$. Since S is hausdorff, $\boldsymbol{\mu}(s) \cap \boldsymbol{\mu}(s') = \emptyset$ for distinct $s, s' \in S$. Thus we can define the *standard part* $\mathrm{st}(x)$ of $x \in S_{\mathrm{ns}}$ to be the unique $s \in S$ such that $x \in \boldsymbol{\mu}(s)$. We also introduce the equivalence relation $\sim$ on S_{ns} whose equivalence classes are the monads:

$$x \sim y \ :\Longleftrightarrow\ \mathrm{st}(x) = \mathrm{st}(y) \qquad (\text{“}x \text{ and } y \text{ are infinitely close”}).$$

The following is easy and well-known.

Lemma 2.4. *Suppose S is a regular hausdorff space and X is an internal subset of S^* such that $X \subseteq S_{\mathrm{ns}}$. Then $\mathrm{st}(X) \subseteq S$ is compact.*

Proof. Let for each point $p \in \mathrm{st}(X)$ an open neighborhood $U_p \subseteq S$ of p be given. It suffices to show that then finitely many of the U_p cover $\mathrm{st}(X)$. By regularity we can pick for each $p \in \mathrm{st}(X)$ an open neighborhood $V_p \subseteq S$ of p such that $\mathrm{cl}(V_p) \subseteq U_p$ (and thus $\mathrm{st}(V_p^*) \subseteq U_p$). From $X \subseteq S_{\mathrm{ns}}$ we obtain $X \subseteq \bigcup_{p \in \mathrm{st}(X)} V_p^*$, which by saturation yields $X \subseteq V_{p_1}^* \cup \cdots \cup V_{p_n}^*$ with $p_1, \ldots, p_n \in \mathrm{st}(X)$. Then $\mathrm{st}(X) \subseteq U_{p_1} \cup \cdots \cup U_{p_n}$. □

Note that $G_{\mathrm{ns}} = \bigcup_{g \in G} \boldsymbol{\mu}(g)$ is a subgroup of G^*, and that the standard part map $\mathrm{st} : G_{\mathrm{ns}} \to G$ is a group morphism that is the identity on G. We let $\boldsymbol{\mu} := \boldsymbol{\mu}(1) = \ker(\mathrm{st})$ denote the normal subgroup of *infinitesimals* of G_{ns}. The equivalence relation $\sim$ on G_{ns} is given by:

$$a \sim b \iff ab^{-1} \in \boldsymbol{\mu}, \quad (a, b \in G_{\mathrm{ns}}).$$

We let i, j range over $\mathbb{N}^*$, ν over $\mathbb{N}^* \setminus \mathbb{N}$, k over $\mathbb{Z}^*$. Also, σ will always denote a positive infinite element of $\mathbb{R}^*$, and a, b, sometimes subscripted, range over G^*. We adopt Landau's “big O” and “little o” notation in the following way: for $x, y \in \mathbb{R}^*$ with $y > 0$, $x = \mathrm{o}(y)$ means that $|x| < y/n$ for all $n > 0$, and $x = \mathrm{O}(y)$ means that $|x| < ny$ for some $n > 0$. We also adapt it to G as follows:

$$\begin{aligned}\mathrm{O}[\sigma] = \mathrm{O}_G[\sigma] &:= \{a \in \boldsymbol{\mu} : \ a^i \in \boldsymbol{\mu} \text{ for all } i = \mathrm{o}(\sigma)\},\\ \mathrm{o}[\sigma] = \mathrm{o}_G[\sigma] &:= \{a \in \boldsymbol{\mu} : a^i \in \boldsymbol{\mu} \text{ for all } i = \mathrm{O}(\sigma)\}\\ &= \{a \in \boldsymbol{\mu} : a^i \in \boldsymbol{\mu} \text{ for all } i \le \sigma)\}.\end{aligned}$$

So $\mathrm{o}[\sigma] \subseteq \mathrm{O}[\sigma] \subseteq \boldsymbol{\mu} \subseteq G_{\mathrm{ns}}$, and $\mathrm{o}[\sigma]$ and $\mathrm{O}[\sigma]$ are closed under $a \mapsto a^\ell$, for each $\ell \in \mathbb{Z}$; in particular, these sets are symmetric. It is also clear that if $a \in G_{\mathrm{ns}}$ and $b \in \mathrm{O}[\sigma],\ c \in \mathrm{o}[\sigma]$, then $aba^{-1} \in \mathrm{O}[\sigma]$ and $aca^{-1} \in \mathrm{o}[\sigma]$. A key fact to be proved is that $\mathrm{o}[\sigma]$ and $\mathrm{O}[\sigma]$ are subgroups of G_{ns}.

Lemma 2.5. *If $a \in \mathrm{O}[\sigma]$, then $a^i \in G_{\mathrm{ns}}$ for all $i = \mathrm{O}(\sigma)$.*

Proof. Let $a \in \mathrm{O}[\sigma]$, and take a compact symmetric neighborhood U of 1 in G. If $a^i \in U^*$ for all $i = \mathrm{O}(\sigma)$, then $a^i \in G_{\mathrm{ns}}$ for all $i = \mathrm{O}(\sigma)$, as desired. Suppose $a^j \notin U^*$ for some $j = \mathrm{O}(\sigma)$, and take j minimal with this property. Then $a^i \in G_{\mathrm{ns}}$ with $\mathrm{st}(a^i) \in U$ for $i = 0, \dots, j$. We cannot have $j = \mathrm{o}(\sigma)$, so $\sigma = \mathrm{O}(j)$. Therefore, if $i = \mathrm{O}(\sigma)$, then $i = nj + i'$ with $i' < j$, and thus $a^i = (a^j)^n a^{i'} \in G_{\mathrm{ns}}$. □

The next lemma indicates why $\mathrm{O}[\sigma]$ is of interest: its elements generate the one-parameter subgroups of G in a very intuitive way.

Lemma 2.6. *Let $a \in \mathrm{O}[\sigma]$. Then the map $X_a : \mathbb{R} \to G$ defined by $X_a(t) := \mathrm{st}(a^{[\sigma t]})$ is a 1-ps of G. Moreover:*
1. *$X_{a^\ell} = \ell X_a$ for all $\ell \in \mathbb{Z}$;*
2. *$b \in \boldsymbol{\mu} \implies X_{bab^{-1}} = X_a$;*
3. *$X_a = O \iff a \in \mathrm{o}[\sigma]$;*
4. *$L(G) = \{X_b : b \in \mathrm{O}[\sigma]\}$.*

Proof. It is clear that X_a is a group morphism. To show continuity at $0 \in \mathbb{R}$, let U be a neighborhood of 1 in G. Take a neighborhood V of 1 in G such that $\mathrm{cl}(V) \subseteq U$. Since $a^k \in \boldsymbol{\mu} \subseteq V^*$ for all $k = \mathrm{o}(\sigma)$, we have $n > 0$ such that $a^k \in V^*$ whenever $|k| < \sigma/n$. Also $a^k \in G_{\mathrm{ns}}$ for such k, so $\mathrm{st}(a^k) \in \mathrm{cl}(V)$ whenever $|k| < \sigma/n$. Hence $X_a(t) = \mathrm{st}(a^{[\sigma t]}) \in U$ whenever $t \in \mathbb{R}$ and $|t| < 1/n$.

The remaining assertions follow easily. In connection with (4) we note that for $X \in L(G)$ and $b := X(1/\sigma)$ we have $b \in \mathrm{O}[\sigma]$ and $X = X_b$. □

The compact-open topology. Let P be a locally compact space, Q a hausdorff space, and $C(P, Q)$ the set of continuous maps $P \to Q$. For compact $K \subseteq P$ and open $U \subseteq Q$, put

$$O(K, U) := \{f \in C(P, Q) : f(K) \subseteq U\}.$$

We equip $C(P, Q)$ with its *compact-open topology*; this is the topology on $C(P, Q)$ that has the finite intersections of these sets $O(K, U)$ as basic open sets; it makes $C(P,Q)$ into a hausdorff space, and makes the evaluation map

$$\Phi : C(P,Q) \times P \to Q, \qquad \Phi(f,p) := f(p),$$

continuous. Let A be any subset of P and F be a closed subset of Q. Then

$$\{f \in C(P,Q) : \ f(A) \subseteq F\}$$

is closed, since its complement in $C(P,Q)$ is the union over all $a \in A$ of the open sets

$$\{f \in C(P,Q) : f(a) \notin F\}.$$

A nonstandard view of the compact-open topology is as follows: Let $f \in C(P,Q)$ and $g \in C(P,Q)^*$; then

$$g \in \boldsymbol{\mu}(f) \iff g(p') \in \boldsymbol{\mu}(f(p)) \text{ for all } p \in P \text{ and } p' \in \boldsymbol{\mu}(p).$$

We apply this to the case where $P = \mathbb{R}$ is the real line and $Q = G$. Then $L(G)$ is closed in $C(\mathbb{R}, G)$, and below $L(G)$ is given the topology induced on it by the (compact-open) topology of $C(\mathbb{R}, G)$. Let $I := [-1,1] \subseteq \mathbb{R}$. Let $X \in L(G)$. Then every neighborhood U of 1 in G determines a neighborhood

$$N(U) := \{Y \in L(G) : \ Y(t) \in X(t)U \text{ for all } t \in I\}$$

of X in $L(G)$, and the collection

$$\{N(U) : \ U \text{ is a neighborhood of 1 in } G\}$$

is a neighborhood base of X in $L(G)$. (These facts are easy to verify using the above characterization of monads in the compact-open topology.)

Lemma 2.7. *The following maps are continuous:*

(1) *the exponential map* $X \mapsto X(1) : \ L(G) \to G$*;*
(2) *the scalar multiplication map* $(r, X) \mapsto rX : \ \mathbb{R} \times L(G) \to L(G)$*;*
(3) *the adjoint action map* $G \times L(G) \to L(G)$*.*

Proof. Item (1) follows from the continuity of evaluation in the compact-open topology. To prove (2), let $X \in L(G)$ and $r \in \mathbb{R}$, and let $X' \in L(G)^*$ and $r' \in \mathbb{R}^*$ be such that $X' \in \boldsymbol{\mu}(X)$ and $r' \in \boldsymbol{\mu}(r)$; it suffices to show that then $r'X' \in \boldsymbol{\mu}(rX)$. Let $t' \in \mathbb{R}^*$ with $t' \in \boldsymbol{\mu}(t), \ t \in \mathbb{R}$; then $r't' \in \boldsymbol{\mu}(rt)$, so

$$(r'X')(t') = X'(r't') \in \boldsymbol{\mu}(X(rt)) = \boldsymbol{\mu}((rX)(t)).$$

This argument shows that $r'X' \in \boldsymbol{\mu}(rX)$, as desired. □

Lemma 2.8. *Suppose* $\mathcal{U} \subseteq G$ *is a compact neighborhood of* 1 *in* G *and contains no subgroups of* G *other than* $\{1\}$*. Then the set*

$$\mathcal{K} := \{X \in L(G) : \ X(I) \subseteq \mathcal{U}\}$$

is a compact neighborhood of O *in* $L(G)$*.*

Proof. Let $Y \in \mathcal{K}^*$, that is, $Y \in L(G)^*$ and $Y(I^*) \subseteq \mathcal{U}^*$. If $\epsilon \in \mathbb{R}^*$ is infinitesimal, then $\mathrm{st}\big(Y(\mathbb{Z}\epsilon)\big) \subseteq \mathcal{U}$ is a subgroup of G, so $Y(\epsilon) \in \boldsymbol{\mu}$. Hence for each neighborhood V of 1 in G there is $n > 0$ such that $Y(r) \in V^*$ for all $r \in \mathbb{R}^*$ with $|r| < 1/n$. Consequently, $X : \mathbb{R} \to G$ defined by $X(t) = \mathrm{st}(Y(t))$ is a 1-ps with $X(I) \subseteq \mathcal{U}$, and $Y \in \boldsymbol{\mu}(X)$. □

3. Generating compact connected subgroups

We say that a is *degenerate* if $a^i \in \boldsymbol{\mu}$ for all i. Note: G has NSS iff G^* has no degenerate elements other than 1. We do not yet restrict our attention to NSS-groups, so we do allow degenerate elements $\neq 1$ in $\boldsymbol{\mu}$. In some sense, nondegenerate elements in $\boldsymbol{\mu}$ generate nontrivial connected subgroups of G. This depends on the following elementary fact.

Lemma 3.1. *Let $a_1, \ldots, a_\nu$ be an internal sequence such that $a_i \in \boldsymbol{\mu}$ and $a_1 \cdots a_i \in G_{\text{ns}}$ for all $i \in \{1, \ldots, \nu\}$. Then the set*

$$S := \{\text{st}(a_1 \cdots a_i) : \ 1 \leq i \leq \nu\} \subseteq G$$

is compact and connected (and contains 1).

Proof. The compactness of S follows from Lemma 2.4. Assume S is not connected. Then we have disjoint open subsets U and V of G such that $S \subseteq U \cup V$ and S meets both U and V. We can assume that $1 \in U$, so $a_1 \in U^*$. There are $i \leq \nu$ such that $\text{st}(a_1 \cdots a_i) \in V$, and $a_1, \ldots, a_i \in V^*$ for such i. Take $i \leq \nu$ minimal such that $a_1 \cdots a_i \in V^*$. Then $i \geq 2$ and $a_1 \cdots a_{i-1} \in U^*$. Now $a := \text{st}(a_1 \cdots a_{i-1}) = \text{st}(a_1 \cdots a_i) \in S$. If $a \in U$, this gives $a_1 \cdots a_i \in U^*$, and if $a \in V$, it gives $a_1 \cdots a_{i-1} \in V^*$, and we have a contradiction in either case. □

Till further notice U is a compact symmetric neighborhood of 1 in G. If $a^{\mathbb{N}^*} \subseteq U^*$ (in particular, if a is degenerate), then we set $\text{ord}_U(a) = \infty$; if $a^{\mathbb{N}^*} \not\subseteq U^*$, then we let $\text{ord}_U(a)$ be the largest j such that $a^i \in U^*$ for all $i \leq j$. Thus $\text{ord}_U(a) = 0$ iff $a \notin U^*$, and $\text{ord}_U(a) > \mathbb{N}$ if $a \in \boldsymbol{\mu}$.

Lemma 3.2. *Suppose $a \in \boldsymbol{\mu}$ and $a^i \notin \boldsymbol{\mu}$ for some $i = \text{o}\big(\text{ord}_U(a)\big)$. Then U contains a nontrivial connected subgroup of G.*

Proof. By the previous lemma the set

$$G_U(a) := \{\text{st}(a^k) : \ k = \text{o}\big(\text{ord}_U(a)\big)\}$$

is a union of connected subsets of U, each containing 1, and is thus itself a connected subset of U. It is also a subgroup of G. □

An element $a \in \boldsymbol{\mu}$ is said to be *U-pure* if it is nondegenerate and $a \in \text{O}[\nu]$ for $\nu := \text{ord}_U(a)$. If U contains no nontrivial connected subgroup of G, then by the last lemma every nondegenerate $a \in \boldsymbol{\mu}$ is U-pure.

An element $a \in \boldsymbol{\mu}$ is said to be *pure* if it is V-pure for some compact symmetric neighborhood V of 1 in G. Thus:

Corollary 3.3. *If G has* NSCS*, then every nondegenerate $a \in \boldsymbol{\mu}$ is pure.*

Lemma 3.4. *Let $a \in \boldsymbol{\mu}$. Then a is pure iff there is ν such that $a \in \text{O}[\nu]$ and $a^\nu \notin \boldsymbol{\mu}$.*

Proof. If a is U-pure, say, then for $\nu = \mathrm{ord}_U(a)$ we have $a \in \mathrm{O}[\nu]$ and $a^\nu \notin \boldsymbol{\mu}$. Conversely, let ν be such that $a \in \mathrm{O}[\nu]$ and $a^\nu \notin \boldsymbol{\mu}$. If $\mathrm{ord}_U(a) = \mathrm{O}(\nu)$, then a is U-pure. If $\nu = \mathrm{O}\left(\mathrm{ord}_U(a)\right)$, then $a^\nu \in G_{\mathrm{ns}}$, and we can take a compact symmetric neighborhood V of 1 in G such that $a^\nu \notin V^*$, and then a is V-pure. □

Let Q range over internal symmetric subsets of G^* such that $1 \in Q \subseteq \boldsymbol{\mu}$. We define Q^i to be the internal subset of G^* consisting of all $a_1 \cdots a_i$ where $a_1, \ldots, a_i$ is an internal sequence in Q. Thus

$$Q^\infty := \bigcup_i Q^i$$

is the internal subgroup of G^* internally generated by Q.

We say that Q is *degenerate* if $Q^\infty \subseteq \boldsymbol{\mu}$. If $Q^\infty \not\subseteq U^*$, then we let $\mathrm{ord}_U(Q)$ be the largest j such that $Q^j \subseteq U^*$, and if $Q^\infty \subseteq U^*$, then we set $\mathrm{ord}_U(Q) := \infty$. Thus $e := \mathrm{ord}_U(Q) > \mathbb{N}$. We set

$$G_U(Q) := \{\mathrm{st}(a) : \ a \in Q^i \text{ for some } i = \mathrm{o}(e)\}.$$

Lemma 3.5. *If $e \neq \infty$, then $\mathrm{st}(Q^e) \not\subseteq \mathrm{int}(U)$.*

Proof. Assume $e \neq \infty$, and take $b \in Q^e$ such that $bq \notin U^*$ for some $q \in Q$. One checks easily that then $\mathrm{st}(b) \notin \mathrm{int}(U)$. (check) □

Lemma 3.6. *$G_U(Q)$ is a compact connected subgroup of G contained in U. In particular, if $Q^i \not\subseteq \boldsymbol{\mu}$ for some $i = \mathrm{o}(e)$, then U contains a nontrivial compact connected subgroup of G.*

Proof. The set $G_U(Q)$ is the union of the increasing family of subsets $\mathrm{st}(Q^i)$ of G with $i = \mathrm{o}(e)$. But there is only a "small" number of subsets of G, so saturation gives $i_0 = \mathrm{o}(e)$ such that $\mathrm{st}(Q^i) = \mathrm{st}(Q^{i_0})$ for all $i = \mathrm{o}(e)$ with $i \geq i_0$. Then $G_U(Q) = \mathrm{st}(Q^{i_0})$, so $G_U(Q)$ is compact. As in the proof of Lemma 3.2, $G_U(Q)$ is a union of connected subsets of U, each containing 1, and is thus itself a connected subset of U. It is also a subgroup of G. □

4. COMPACT GROUPS

Theorem 4.1. *Let G be compact and U an open neighborhood of 1 in G. Then there is a continuous injective group morphism $G/N \to \mathrm{GL}_n(\mathbb{R})$ for some closed normal subgroup N of G contained in U.*

Proof. The Peter-Weyl theorem yields for any $a \neq 1$ a continuous group morphism $\phi_a : G \to \mathrm{GL}_{n_a}(\mathbb{R})$ that does not have a in its kernel N_a. As a varies over $G \setminus H$, the open sets $G \setminus N_a$ cover $G \setminus U$, so there are $a_1, \ldots, a_m \in G \setminus U$ such that $N := N_{a_1} \cap \cdots \cap N_{a_m}$ is contained in U. Then the desired result holds for this N and $n := n_{a_1} + \cdots + n_{a_m}$. □

Corollary 4.2. *Let G be compact and U a neighborhood of 1 in G. Then there is a closed normal subgroup N of G contained in U and an open set V in G such that $N \subseteq V \subseteq U$ and every subgroup of G contained in V is contained in N.*

Proof. We can assume that U is open, and then we take N as in the previous theorem, so that G/N has NSS. Take an open neighborhood W of the identity in G/N that contains no nontrivial subgroup of G/N. Let $V := \pi^{-1}(W) \cap U$, where $\pi : G \to G/N$ is the natural map. Then V has the desired property. □

5. Gleason-Yamabe Lemmas and their Consequences

This is the most technical part of the story. The leading idea is to make G act by isometries on its space of real-valued continuous functions with compact support, and to use the Haar integral on this space.

Gleason-Yamabe Lemmas. Throughout this subsection we fix a compact symmetric neighborhood $\mathcal{U}$ of 1 in G and a continuous function $\tau : G \to [0,1]$ such that

$$\tau(1) = 1, \qquad \tau(x) = 0 \text{ for all } x \in G \setminus \mathcal{U}.$$

Let $Q \subseteq \mathcal{U}$ be symmetric with $1 \in Q$ and let e be a positive integer with $Q^e \subseteq \mathcal{U}$. Define the function $\Delta = \Delta_{Q,e} : \ G \to [0,1]$ by

(i) $\Delta(1) = 0$;
(ii) $\Delta(x) = i/(e+1)$ if $x \in Q^i \setminus Q^{i-1},\ 1 \le i \le e$;
(iii) $\Delta(x) = 1$ if $x \notin Q^e$.

Then for all $x \in G$,

(iv) $\Delta(x) = 1$ if $x \notin \mathcal{U}$;
(v) $|\Delta(ax) - \Delta(x)| \le 1/e$ for $a \in Q$.

Now use τ to smooth $1 - \Delta$: define $\theta = \theta_{Q,e} : \ G \to [0,1]$ by

$$\theta(x) = \sup_{y \in G} \big(1 - \Delta(y)\big)\tau(y^{-1}x) = \sup_{y \in \mathcal{U}} \big(1 - \Delta(y)\big)\tau(y^{-1}x).$$

The following properties are easy consequences:

(1) θ is continuous, and $\theta(x) = 0$ outside $\mathcal{U}^2$;
(2) $0 \le \tau \le \theta \le 1$;
(3) $|\theta(ax) - \theta(x)| \le 1/e$ for $a \in Q$;

For continuity of θ, note that if $a \in \boldsymbol{\mu}$ and $x \in G$, then $\theta(xa) - \theta(x)$ is infinitesimal in $\mathbb{R}^*$. To prove (3), let $a \in Q$, and note that for all $x, y \in G$,

$$|\big(1 - \Delta(a^{-1}y)\big) - \big(1 - \Delta(y)\big)| \le 1/e,$$

and $y^{-1}ax = (a^{-1}y)^{-1}x$, so

$$|\big(1 - \Delta(y)\big)\tau(y^{-1}ax) - \big(1 - \Delta(a^{-1}y)\big)\tau((a^{-1}y)^{-1}x)| \le 1/e,$$

which gives (3).

Let C be the real vector space of continuous functions $G \to \mathbb{R}$ with compact support, with norm given by

$$\|f\| = \sup\{|f(x)| : \ x \in G\}.$$

We have a left action $G \times C \to C$ of G on C given by

$$(a, f) \mapsto af, \qquad (af)(x) = f(a^{-1}x).$$

More suggestively, $(af)(ax) = f(x)$ for $a, x \in G$, $f \in C$. It is clear that for $a \in G$ the map $f \mapsto af$ is an $\mathbb{R}$-linear isometry of C onto itself, and thus,

$$\|abf - f\| \le \|af - f\| + \|bf - f\| \qquad (a, b \in G,\ f \in C).$$

We have the following useful equicontinuity result:

(4) for each $\varepsilon \in \mathbb{R}^{>0}$ there is a neighborhood V_ε of 1 in G, independent of (Q, e), such that $\|a\theta - \theta\| \le \varepsilon$ for all $a \in V_\varepsilon$.

To see why, let $\varepsilon \in \mathbb{R}^{>0}$. Uniform continuity of τ gives a neighborhood U of 1 in G such that $|\tau(g) - \tau(h)| < \varepsilon$ for all $g, h \in G$ with $gh^{-1} \in U$. Now, take a neighborhood V_ε of 1 in G such that $y^{-1}ay \in U$ for all $(y, a) \in \mathcal{U} \times V_\varepsilon$. Then $|\tau(y^{-1}ax) - \tau(y^{-1}x)| < \varepsilon$ for all $x \in G$, $y \in \mathcal{U}$ and $a \in V_\varepsilon$. This gives (4).

A second smoothing will be done by integration. Take the unique left-invariant Haar measure μ on G such that $\mu(\mathcal{U}^2) = 1$. Here left-invariance means that for all $a \in G$ and $f \in C$ we have $\int f(ax)d\mu(x) = \int f(x)d\mu(x)$. Then

(5) $0 \le \int \theta(x)d\mu(x) \le 1$, by (1) and (2).

We now introduce the continuous function $\phi = \phi_{Q,e} :\ G \to \mathbb{R}$ by $\phi(x) := \int \theta(xu)\theta(u)\ d\mu(u)$. Then

(6) $\phi(x) = 0$ outside $\mathcal{U}^4$;
(7) $\phi(1) \ge \int \tau(u)^2\ d\mu(u) > 0$, by (2);
(8) $\|a\phi - \phi\| \le \|a\theta - \theta\|$ for all $a \in G$;
(9) if $a \in Q$, then $\|a\phi - \phi\| \le 1/e$, by (3) and (8).

The significance of (7) is that the positive lower bound $\int \tau(u)^2\ d\mu(u)$ on $\phi(1)$ is independent of (Q, e).

Lemma 5.1. *Let $\varepsilon \in \mathbb{R}^{>0}$. Then there is a neighborhood $U = U_\varepsilon \subseteq \mathcal{U}$ of 1 in G, independent of (Q, e), such that for all $a \in Q$ and $b \in U$,*

$$\|b \cdot (a\phi - \phi) - (a\phi - \phi)\| \ \le\ \frac{\varepsilon}{e}.$$

Proof. Let $a \in Q$, $b \in \mathcal{U}$. Then, with $x \in G$ and $y := b^{-1}x$,

$$(a\phi - \phi)(x) \ =\ \int [\theta(a^{-1}xu) - \theta(xu))]\theta(u)\ d\mu(u)$$

$$b(a\phi - \phi)(x) = (a\phi - \phi)(y) \ =\ \int [\theta(a^{-1}yu) - \theta(yu))]\theta(u)\ d\mu(u).$$

By the left-invariance of our Haar measure we can replace u by $x^{-1}yu$ in the function of u integrated in the first identity, so

$$(a\phi - \phi)(x) \ =\ \int [\theta(a^{-1}yu) - \theta(yu))]\theta(x^{-1}yu)\ d\mu(u).$$

Taking differences gives

$$[b \cdot (a\phi - \phi) - (a\phi - \phi)](x) = \int [(a\theta - \theta)(yu)][(\theta - y^{-1}x\theta)(u)]\ d\mu(u).$$

If the left hand side here is nonzero, then $x \in \mathcal{U}^4$ or $a^{-1}x \in \mathcal{U}^4$ or $b^{-1}x \in \mathcal{U}^4$ or $a^{-1}b^{-1}x \in \mathcal{U}^4$, and thus $x \in \mathcal{U}^6$ in all cases. Also $y^{-1}x = x^{-1}bx$, so by (4) we can take the neighborhood $U_{c,\varepsilon} \subseteq \mathcal{U}$ of 1 in G so small that for all $b \in U_\varepsilon$ and $x \in \mathcal{U}^6$ we have $y^{-1}x \in \mathcal{U}$ and $\|\theta - y^{-1}x\theta\| < \varepsilon/\mu(\mathcal{U}^3)$. Then U_ε has the desired property. □

Lemma 5.2. *With $\varepsilon \in \mathbb{R}^{>0}$, let $U = U_\varepsilon$ be as in the previous lemma and let $a \in Q$ and $n > 0$ be such that $a^i \in U$ for $i = 0, \ldots, n$. Then*

$$\|(a^n\phi - \phi) - n(a\phi - \phi)\| \ \leq\ \frac{n\varepsilon}{e}.$$

Proof. We have $a^n\phi - \phi = \sum_{i=0}^{n-1} a^i(a\phi - \phi)$, so

$$(a^n\phi - \phi) - n(a\phi - \phi) = \sum_{i=0}^{n-1} a^i(a\phi - \phi) - (a\phi - \phi).$$

By the previous lemma we have for $i = 0, \ldots, n-1$,

$$\|a^i(a\phi - \phi) - (a\phi - \phi)\| \ \leq\ \frac{\varepsilon}{e},$$

which gives the desired result by summation. □

Suppose now that Q is a symmetric internal subset of G^* with $1 \in Q$ and $Q \subseteq \boldsymbol{\mu}$. Let $e \in \mathbb{N}^*$ be such that $e \geq 1$ and $Q^e \subseteq \mathcal{U}^*$. Then the constructions and results above transfer automatically to the nonstandard setting and yield internally continuous functions

$$\theta = \theta_{Q,e} :\ G^* \to [0,1]^*, \qquad \phi = \phi_{Q,e} :\ G^* \to \mathbb{R}^*$$

satisfying the internal versions of (1)-(9) and Lemmas 5.1 and 5.2. With these assumptions we have

Corollary 5.3. *Suppose $a \in Q$, $\nu = \mathrm{O}(e)$, and $a \in \mathrm{o}[\nu]$. Then*

$$\nu\|a\phi - \phi\| \ \sim\ 0.$$

Proof. By Lemma 5.2 we have for each $\varepsilon \in \mathbb{R}^{>0}$,

$$\|(a^\nu\phi - \phi) - \nu(a\phi - \phi)\| \ \leq\ \frac{\nu\varepsilon}{e},$$

so the lefthand side in this inequality is infinitesimal. Also, by (8) and (4) we have $\|a^\nu\phi - \phi\| \leq \|a^\nu\theta - \theta\|$, so $\|a^\nu\phi - \phi\|$ is infinitesimal. □

Consequences of the Gleason-Yamabe Lemmas.

Lemma 5.4. *Let $a_1, \ldots, a_\nu$ be an internal sequence in G^* such that all $a_i \in \mathrm{o}[\nu]$. Then $a_1 \cdots a_\nu \in \boldsymbol{\mu}$.*

Proof. Put $Q := \{1, a_1, \dots, a_\nu, a_1^{-1}, \dots, a_\nu^{-1}\}$, and towards a contradiction, suppose that $Q^\nu \not\subseteq \boldsymbol{\mu}$. Take a compact symmetric neighborhood U of 1 in G such that $Q^{\nu+1} \not\subseteq U^*$, so $\mathrm{ord}_U(Q) \le \nu$. By decreasing ν if necessary, and Q accordingly, we arrange that $\mathrm{ord}_U(Q) = \nu$.

Consider first the special case that $Q^i \subseteq \boldsymbol{\mu}$ for all $i = \mathrm{o}(\nu)$. (This occurs if G has NSCS). Take $b \in Q^\nu$ such that $\mathrm{st}(b) \ne 1$, and then take a compact symmetric neighborhood $\mathcal{U} \subseteq U$ of 1 in G such that $\mathrm{st}(b) \notin \mathcal{U}^4$, and put $e := \mathrm{ord}_{\mathcal{U}}(Q)$, so $\nu = \mathrm{O}(e)$. The previous subsection yields an internally continuous function $\phi = \phi_{Q,e} : G^* \to \mathbb{R}^*$ satisfying the internal versions of (6)-(9) and Lemma 5.3. In particular, $\phi(x) = 0$ outside $(\mathcal{U}^*)^4$ (hence $\phi(b^{-1}) = 0$), and $\phi(1)$ is not infinitesimal. Then $\|b\phi - \phi\|$ is not infinitesimal. Take an internal sequence $b_1, \dots, b_\nu$ in Q such that $b = b_1 \cdots b_\nu$. Then Lemma 5.3 yields

$$\|b\phi - \phi\| \le \sum_{i=1}^{\nu} \|b_i\phi - \phi\| \sim 0,$$

and we have a contradiction.

Next, assume that $Q^i \not\subseteq \boldsymbol{\mu}$ for some $i = \mathrm{o}(\nu)$. Then we set

$$H := G_U(Q) = \{\mathrm{st}(b) : \ b \in Q^i \text{ for some } i = \mathrm{o}(\nu)\},$$

so H is a nontrivial compact subgroup of G contained in U. By Corollary 4.2 we can take a proper closed normal subgroup N of H and a compact symmetric neighborhood $V \subseteq U$ of 1 in G such that $N \subseteq \mathrm{int}(V)$ and every subgroup of H contained in V is contained in N. Put $\mu := \mathrm{ord}_V(Q)$, so $\mathbb{N} < \mu \le \nu$, and we have the compact subgroup

$$G_V(Q) = \{\mathrm{st}(b) : \ b \in Q^i \text{ for some } i = \mathrm{o}(\mu)\}$$

of H with $G_V(Q) \subseteq V$, so $G_V(Q) \subseteq N$. By Lemma 3.5 we can take $b \in Q^\mu$ with $\mathrm{st}(b) \notin \mathrm{int}(V)$. Then $\mathrm{st}(b) \notin N$, so we can take a compact symmetric neighborhood $\mathcal{U}$ of 1 in G such that $N \subseteq \mathrm{int}(\mathcal{U})$, $\mathcal{U}^4 \subseteq V$ and $\mathrm{st}(b) \notin \mathcal{U}^4$.

Put $e := \mathrm{ord}_{\mathcal{U}}(Q)$. If $e = \mathrm{o}(\mu)$, then $\mathrm{st}(Q^e) \subseteq G_V(Q) \subseteq N$, contradicting $\mathrm{st}(b) \notin N$. This shows $\mu = \mathrm{O}(e)$. The rest of the proof now proceeds as in the special case considered earlier, with ν replaced by μ, and $b_1, \dots, b_\nu$ by an internal sequence $b_1, \dots, b_\mu$ in Q such that $b = b_1 \cdots b_\mu$. □

Corollary 5.5. *Let $a_1, \dots, a_\nu$ be an internal sequence in G^* such that all $a_i \in \mathrm{O}[\nu]$. Then $a_1 \cdots a_\nu \in G_{\mathrm{ns}}$.*

Proof. If $a_1 \cdots a_\nu \in \mathcal{U}^*$, we are done. Assume otherwise. Take the least j with $a_i \cdots a_{i+j} \notin \mathcal{U}^*$ for some i with $1 \le i < i+j \le \nu$. Then by the previous lemma we cannot have $j = \mathrm{o}(\nu)$, and this gives $n > 0$ with $nj \le \nu < (n+1)j$. Hence

$$a_1 \cdots a_\nu = (a_1 \cdots a_j)(a_{j+1} \cdots a_{2j}) \cdots (a_{nj+1} \cdots a_\nu) \in (\mathcal{U}^*)^n \subseteq G_{\mathrm{ns}}.$$

□

Lemma 5.6. *If $a \in \mathrm{O}[\nu]$ and $b \in \mathrm{o}[\nu]$, then $(ab)^i \sim a^i$ for all $i \le \nu$.*

Proof. Set $b_i := a^i b a^{-i}$. Then $(ab)^i = b_1 \cdots b_i \cdot a^i$. Assuming $a \in \mathrm{O}[\nu]$ and $b \in \mathrm{o}[\nu]$, we have $b_i \in \mathrm{o}[\nu]$ for $i \le \nu$, so $b_1 \cdots b_i \in \boldsymbol{\mu}$ for all $i \le \nu$, by Lemma 5.4. □

Lemma 5.7. *Suppose that* $a, b \in \mathrm{O}[\nu]$ *and* $a^i \sim b^i$ *for all* $i \le \nu$. *Then* $a^{-1}b \in \mathrm{o}[\nu]$.

Proof. If $a \in \mathrm{o}[\nu]$, then $b \in \mathrm{o}[\nu]$, so $(a^{-1}b)^i \sim a^{-i} \in \boldsymbol{\mu}$ for all $i \le \nu$, and we are done. So we can assume that $a \notin \mathrm{o}[\nu]$, and then, replacing ν by an element of $\mathbb{N}^*$ of the same archimedean class, we have $a^\nu \notin \boldsymbol{\mu}$. Let $Q := \{1, a, a^{-1}, b, b^{-1}\}$. Then $Q^i \subseteq \boldsymbol{\mu}$ for all $i = \mathrm{o}(\nu)$ by Lemma 5.4, and $Q^\nu \subseteq G_{\mathrm{ns}}$ by Corollary 5.5. Suppose towards a contradiction that $(a^{-1}b)^j \notin \boldsymbol{\mu}$, where $j \le \nu$. Then $\nu = \mathrm{O}(j)$. Take a compact symmetric neighborhood $\mathcal{U}$ of 1 in G such that $a^\nu \notin \mathcal{U}$ and $(a^{-1}b)^j \notin \mathcal{U}^4$, and put $e = \mathrm{ord}_{\mathcal{U}}(Q)$, so e and ν have the same archimedean class. As before we have the internally continuous function $\phi = \phi_{Q,e} : G^* \to \mathbb{R}^*$ satisfying the internal versions of (6)-(9) and Lemma 5.3. Then $\phi((a^{-1}b)^j) = 0$ and $\varepsilon := \phi(1) > 0$ is not infinitesimal, and thus

$$\begin{aligned} \varepsilon &\le \|(b^{-1}a)^j\phi - \phi\| \le j\|(b^{-1}a)\phi - \phi\| \\ &= j\|a\phi - b\phi\| = j\|(a\phi - \phi) - (b\phi - \phi\|. \end{aligned}$$

The desired contradiction will be obtained by showing that

$$j\|(a\phi - \phi) - (b\phi - \phi)\| < \varepsilon.$$

Let $\delta \in \mathbb{R}^{>0}$; then Lemma 5.2 gives a compact symmetric neighborhood $U \subseteq \mathcal{U}$ of 1 in G such that if $k > 0$ and $a^i, b^i \in U^*$ for all $i \le k$, then

$$\|(a^k\phi - \phi) - k(a\phi - \phi)\| \le k\delta/e, \quad \|(b^k\phi - \phi) - k(b\phi - \phi)\| \le k\delta/e, \text{ so}$$

$$\|\frac{j}{k}(a^k\phi - \phi) - j(a\phi - \phi)\| \le j\delta/e, \quad \|\frac{j}{k}(b^k\phi - \phi) - j(b\phi - \phi)\| \le j\delta/e.$$

Choose $\delta \in \mathbb{R}^{>0}$ such that $j\delta/e < \varepsilon/3$, and put $k := \min(\mathrm{ord}_U(a), \mathrm{ord}_U(b))$. Then $k < \nu$ and $a^i, b^i \in U^*$ for all $i \le k$, and therefore

$$\|\frac{j}{k}(a^k\phi - \phi) - j(a\phi - \phi)\| < \varepsilon/3, \quad \|\frac{j}{k}(b^k\phi - \phi) - j(b\phi - \phi)\| < \varepsilon/3.$$

Also $\nu = \mathrm{O}(k)$, and hence $j/k < n$ for some n. Since $a^k \sim b^k$, this gives

$$\|(j/k)(a^k\phi - \phi) - (j/k)(b^k\phi - \phi)\| = (j/k)\|a^k\phi - b^k\phi\| \sim 0.$$

In view of the earlier inequalities, this yields

$$\|j(a\phi - \phi) - j(b\phi - \phi)\| < \varepsilon,$$

as promised. □

Theorem 5.8. *The sets* $\mathrm{O}[\sigma]$ *and* $\mathrm{o}[\sigma]$ *have the following properties:*

1. $\mathrm{O}[\sigma]$ *and* $\mathrm{o}[\sigma]$ *are normal subgroups of* G_{ns}*;*
2. *if* $a \in \mathrm{O}[\sigma]$ *and* $b \in \boldsymbol{\mu}$, *then* $[a, b] := aba^{-1}b^{-1} \in \mathrm{o}[\sigma]$*;*
3. $\mathrm{O}[\sigma]/\mathrm{o}[\sigma]$ *is commutative, and* $\mathrm{O}[\sigma]/\mathrm{o}[\sigma] \subseteq \mathrm{center}\big(\boldsymbol{\mu}/\mathrm{o}[\sigma]\big)$.

Proof. As to (1), let $a, b \in \mathrm{O}[\sigma]$. Then $(ab)^i \in \boldsymbol{\mu}$ for all $i = \mathrm{o}(\sigma)$ by Lemma 5.4, so $ab \in \mathrm{O}[\sigma]$. Thus $\mathrm{O}[\sigma]$ is a normal subgroup of G_{ns}. For $i = \mathrm{O}(\sigma)$ this argument shows that $\mathrm{o}[\sigma]$ is a normal subgroup of G_{ns}. As to (2), this follows from the previous lemma. Item (3) is immediate from (2). □

$L(G)$ **as a topological vector space.** It follows from Lemma 5.6 that for $a \in \mathrm{O}[\sigma]$ and $b \in \mathrm{o}[\sigma]$ we have $X_a = X_{ab}$, so we have a surjective map

$$a\,\mathrm{o}[\sigma] \mapsto X_a : \ \mathrm{O}[\sigma]/\,\mathrm{o}[\sigma] \to L(G).$$

By Lemma 5.6 we also have for $a, b \in \mathrm{O}[\sigma]$ that if $X_a = X_b$, then $a^{-1}b \in \mathrm{o}[\sigma]$, so the above map is a bijection. We make $L(G)$ into an abelian group with group operation $+_\sigma$ so that this bijection is a group isomorphism $\mathrm{O}[\sigma]/\,\mathrm{o}[\sigma] \to L(G)$, in other words, $X_a +_\sigma X_b = X_{ab}$ for $a, b \in \mathrm{O}[\sigma]$. Note that $X_a +_\sigma X_a = 2X_a$ for $a \in \mathrm{O}[\sigma]$. To show that this operation $+_\sigma$ is independent of σ, we need the next lemma. In its proof we use that for $g, h \in G$ and $[g, h] := ghg^{-1}h^{-1}$ we have $gh = [g, h]hg$.

Lemma 5.9. *Let $a, b \in \mathrm{o}[\nu]$ and $a^\nu \in \mathrm{O}[\sigma]$. Then $(ab)^\nu = ca^\nu b^\nu$ with $c \in \mathrm{o}[\sigma]$. Likewise, $(ba)^\nu = b^\nu a^\nu d$ with $d \in \mathrm{o}[\sigma]$.*

Proof. We define $c_i := [a^{i-1}, [b^{i-1}, a]][b^{i-1}, a] \in \boldsymbol{\mu}$ for $i = 1, \ldots, \nu$, so $c_1 = 1$. We claim that then $(ab)^i = c_1 \cdots c_i a^i b^i$. This is clear for $i = 1$. Assume the claim holds for a certain $i < \nu$. Then

$$\begin{aligned}(ab)^{i+1} &= c_1 \cdots c_i a^i b^i ab = c_1 \cdots c_i a^i [b^i, a] a b^{i+1} \\ &= c_1 \cdots c_i [a^i, [b^i, a]][b^i, a] a^{i+1} b^{i+1} = c_1 \cdots c_{i+1} a^{i+1} b^{i+1}\end{aligned}$$

This proves our claim. Now $a^\nu \in \mathrm{O}[\sigma]$ gives $a \in \mathrm{O}[\nu\sigma]$, so $[b^i, a] \in \mathrm{o}[\nu\sigma]$ for $0 \le i < \nu$, hence $c_i \in \mathrm{o}[\nu\sigma]$ for $1 \le i \le \nu$. Put $c := c_1 \cdots c_\nu$. Then for $1 \le j \le \sigma$, the element $c^j = (c_1 \cdots c_\nu)^j$ is a product of $j\nu \le \nu\sigma$ elements, each in $\mathrm{o}[\nu\sigma]$, so $c^j \in \boldsymbol{\mu}$ by Lemma 5.4, and thus $c \in \mathrm{o}[\sigma]$, as desired.

With a^{-1}, b^{-1} in place of a, b, this yields the second part. □

Lemma 5.10. *Let $X, Y \in L(G)$. Then $X + Y$ exists and equals $X +_\sigma Y$.*

Proof. It suffices to show that $X +_\sigma Y = X +_\tau Y$ for all positive infinite $\tau \in \mathbb{R}^*$. Consider first the case $\tau = \nu\sigma$ and set

$$a := X(1/\tau), \quad b := Y(1/\tau), \quad a_\sigma := X(1/\sigma), \quad b_\sigma := Y(1/\sigma),$$

so $a_\sigma = a^\nu$, $b_\sigma = b^\nu$. We have $a, b \in \mathrm{o}[\nu]$, so $a^\nu b^\nu = c(ab)^\nu$ with $c \in \mathrm{o}[\sigma]$ by Lemma 5.9. Put $d := (ab)^\nu$, so $a_\sigma, b_\sigma, d \in \mathrm{O}[\sigma]$, with $a_\sigma b_\sigma = cd$, hence

$$X +_\sigma Y = X_{a_\sigma b_\sigma} = X_{cd} = X_d,$$

and thus for all $t \in \mathbb{R}$,

$$(X +_\sigma Y)(t) = \mathrm{st}(d^{[\sigma t]}) = \mathrm{st}\left((ab)^{[\tau t]}\right) = (X +_\tau Y)(t).$$

Next we consider the case $\tau = (1+\varepsilon)\sigma$ with infinitesimal $\varepsilon \in \mathbb{R}^*$. With a, b, a_σ, b_σ defined as before, we have $a, b, a_\sigma, b_\sigma \in \mathrm{O}[\sigma] = \mathrm{O}[\tau]$ and

$$a_\sigma = a \cdot X(\varepsilon/\tau), \qquad b_\sigma = b \cdot Y(\varepsilon/\tau)$$

with $X(\varepsilon/\tau), Y(\varepsilon/\tau) \in \mathrm{o}[\sigma] = \mathrm{o}[\tau]$, so $a_\sigma b_\sigma = abc$ with $c \in \mathrm{o}[\sigma]$. For $t \in \mathbb{R}^{>0}$ we have $[\sigma t] = [\tau t] + k$ with $k = \mathrm{o}(\sigma)$, so

$$(a_\sigma b_\sigma)^{[\sigma t]} \sim (a_\sigma b_\sigma)^{[\tau t]} = (abc)^{[\tau t]} \sim (ab)^{[\tau t]},$$

and thus $X +_\sigma Y = X +_\tau Y$. For arbitrary positive infinite $\tau \in \mathbb{R}^*$ we reduce to the previous two cases by taking $\nu, \nu' \in \mathbb{N}^* \setminus \mathbb{N}$ such that $\nu'\tau = (1+\varepsilon)\nu\sigma$ with infinitesimal $\varepsilon \in \mathbb{R}^*$. □

By Lemma 5.10 we now have the real vector space $L(G)$ as indicated in Lemma 2.3. In Section 2 we gave it the topology induced by the compact-open topology of $C(\mathbb{R}, G)$. Note also that for $X, Y \in L(G)$ and $r \in \mathbb{R}$ we have $(X+Y)(r) = (rX + rY)(1)$, that is,

$$(X+Y)(r) = \lim_{s\to\infty} \big(X(\frac{1}{s})Y(\frac{1}{s})\big)^{[rs]} = \lim_{s\to\infty} \big(X(\frac{r}{s})Y(\frac{r}{s})\big)^{[s]}.$$

Corollary 5.11. *$L(G)$ is a topological vector space over $\mathbb{R}$.*

Proof. Lemma 2.7 gives the continuity of scalar multiplication, so it remains to establish the continuity of $+$. Let $X, Y \in L(G)$, and let W be a neighborhood of $X+Y$ in $L(G)$. It suffices to obtain neighborhoods P and Q of X and Y in $L(G)$ such that for all $X' \in P$ and $Y' \in Q$ we have $X' + Y' \in W$. To get such P, Q, take a compact neighborhood U of 1 in G so small that for all $Z \in L(G)$, if $Z(t) \in (X+Y)(t)U$ for all $t \in I$, then $Z \in W$. Next, let $X', Y' \in L(G)^*$ and $X \sim X'$ and $Y \sim Y'$. Fix some $\nu > \mathbb{N}$, and put

$$a := X(1/\nu), \quad a' := X'(1/\nu), \quad b := Y(1/\nu), \quad b' := Y'(1/\nu),$$

so $a, a', b, b' \in G(\nu)$ and $a^i \sim a'^i$ and $b^i \sim b'^i$ for all $i \le \nu$, so $a\,\mathrm{o}[\nu] = a'\,\mathrm{o}[\nu]$ and $b\,\mathrm{o}[\nu] = b'\,\mathrm{o}[\nu]$. Hence $(ab)^k \sim (a'b')^k$ whenever $|k| \le \nu$, so for all such k,

$$(X'(\frac{1}{\nu})Y'(\frac{1}{\nu}))^k \in \ \big((X(\frac{1}{\nu})Y(\frac{1}{\nu}))^k\big)U^*.$$

By overspill this gives neighborhoods P and Q of X and Y in $L(G)$ such that for all $X' \in P$ and $Y' \in Q$ we have

$$(X'(\frac{1}{\nu})Y'(\frac{1}{\nu}))^k \in \big((X(\frac{1}{\nu})Y(\frac{1}{\nu}))^k\big)U^*$$

whenever $|k| \le \nu$. It follows that for all $X' \in P$ and $Y' \in Q$ we have

$$(X'+Y')(t) \in (X+Y)(t)\cdot U \qquad \text{for all } t \in I.$$

This gives $X' + Y' \in W$ for all $X' \in P$ and $Y' \in Q$. □

Corollary 5.12. *Suppose the exponential map of G maps some open neighborhood of O in $L(G)$ homeomorphically onto an open neighborhood of* 1 *in G. Then G is locally euclidean and has* NSS.

Proof. Since G is locally compact, so is $L(G)$. It follows that $L(G)$ has finite dimension as vector space over $\mathbb{R}$, and so we can put a norm on $L(G)$. With respect to this norm we take an open ball B centered at O that is homeomorphic to an open neighborhood U of 1 in G via the exponential map of G. Take $n > 1$ such that $V := \{X(1) : X \in \frac{1}{n}B\}$ satisfies $V^2 \subseteq U$. We claim that then V contains no subgroup of G other than $\{1\}$. To see why, let $a \in V$, $a \neq 1$. Take $X \in \frac{1}{n}B$ with $a = X(1)$, and take $m > 1$ such that $mX \in B \setminus \frac{1}{n}B$. Then $(mX)(1) = a^m \in U \setminus V$, so $a^{\mathbb{Z}} \not\subseteq V$. □

6. Consequences of NSS

In this section we assume that our G has NSS. We shall now carry out step (3) from the sketch in the Introduction.

Lemma 6.1. *There is a neighborhood U of 1 such that for all $x, y \in U$, $x^2 = y^2 \implies x = y$.*

Proof. Towards a contradiction, let $x, y \in \boldsymbol{\mu}$, $x \neq y$ and $x^2 = y^2$. Then

$$y^{-1}(xy^{-1})y = y^{-1}x = (xy^{-1})^{-1},$$

so with $a := xy^{-1}$ we get $y^{-1}ay = a^{-1}$. Then $y^{-1}a^k y = a^{-k}$ for all k. Take a compact symmetric neighborhood U of 1 in G that contains no non-trivial subgroup of G. Take positive k such that $a^i \in U^*$ for $0 \leq i \leq k$ and $a^{k+1} \notin U^*$. Set $b := \mathrm{st}(a^k)$, so $b \neq 1$, $b \in U$, and $b = b^{-1}$, so $\{1, b\}$ is a non-trivial subgroup of G contained in U, a contradiction. □

By a *special neighborhood of G* we mean a compact symmetric neighborhood U of 1 in G such that U contains no non-trivial subgroup of G and for all $x, y \in U$, $x^2 = y^2 \implies x = y$.

In the rest of this section we fix a special neighborhood $\mathcal{U}$ of G (which exists by the lemma above), and we set $\mathrm{ord}(a) := \mathrm{ord}_{\mathcal{U}}(a)$.

Corollary 6.2. *Suppose G is not discrete. Then $L(G) \neq \{O\}$.*

Proof. Take $a \in \mu$ with $a \neq 1$, and set $\sigma := \mathrm{ord}(a)$. Then $a \in \mathrm{O}[\sigma]$ and $a \notin \mathrm{o}[\sigma]$, so $X_a \in L(G)$, $X_a \neq O$ where X_a is defined as in Lemma 2.5. □

We set $\mathcal{K} := \{X \in L(G) : X(I) \subseteq \mathcal{U}\}$, so $\mathcal{K}$ is a compact neighborhood of O in $L(G)$, by Lemma 2.8. Note that for any $X \in L(G)$ there is $\lambda \in \mathbb{R}^{>0}$ such that $\lambda X \in \mathcal{K}$. Put $K := \{X(1) : X \in \mathcal{K}\}$, so K is compact by Lemma 2.7. Note also that $K = \bigcup_{X \in \mathcal{K}} X(I)$, so K is pathconnected.

Corollary 6.3. *The vector space $L(G)$ has finite dimension, and the exponential map $X \mapsto X(1) : L(G) \to \mathbb{R}$ maps $\mathcal{K}$ homeomorphically onto K.*

Proof. The first assertion follows from Riesz's theorem that a locally compact topological vector space over $\mathbb{R}$ has finite dimension. For the second assertion it suffices that the exponential map is injective on $\mathcal{K}$. Let $X, Y \in \mathcal{K}$ and $X(1) = Y(1)$. Then $(X(1/2))^2 = (Y(1/2))^2$, so $X(1/2) = Y(1/2)$, and by induction, $X(1/2^n) = Y(1/2^n)$ for all n, and thus $X(i/2^n) = Y(i/2^n)$ for all $i \in \mathbb{Z}$ and n. By density this gives $X = Y$. □

Lemma 6.4. *Let $a \in G^*$. Then* $\operatorname{ord} a$ *is infinite iff* $a \in \boldsymbol{\mu}$.

Proof. Suppose $\operatorname{ord} a$ is infinite. Then $a \in \mathcal{U}^*$ and $a^{\mathbb{Z}} \subseteq \mathcal{U}^*$, so $(\operatorname{st} a)^{\mathbb{Z}} \subseteq \mathcal{U}$, and thus $\operatorname{st} a = 1$. □

We put $\operatorname{ord}(Q) := \operatorname{ord}_{\mathcal{U}}(Q)$. We extend this to the standard setting: for any symmetric $P \subseteq G$ with $1 \in P$ we let $\operatorname{ord}(P)$ be the largest n such that $P^n \subseteq \mathcal{U}$ if there is such an n, and set $\operatorname{ord} P := \infty$ if $P^n \subseteq \mathcal{U}$ for all n.

We set $\mathcal{U}_n := \{x \in G : \operatorname{ord} x \geq n\}$ for $n \geq 1$, so $\mathcal{U}_n \supseteq \mathcal{U}_{n+1}$.

Lemma 6.5. *The sets $\mathcal{U}_n$ have the following properties:*

(1) *each $\mathcal{U}_n$ is a compact symmetric neighborhood of 1 in G;*
(2) *$\{\mathcal{U}_n : n \geq 1\}$ is a (countable) neighborhood basis of 1 in G;*
(3) $\operatorname{ord} \mathcal{U}_n \geq cn$ *for all $n \geq 1$ and some $c > 0$ independent of n.*

Proof. Given $n \geq 1$, it is clear that $\mathcal{U}_n \subseteq \mathcal{U}$, that the complement of $\mathcal{U}_n$ in G is open, and that $\mathcal{U}_n$ is a neighborhood of 1 in G. This gives (1). For each ν we consider the internal set

$$\mathcal{U}_\nu := \{g \in G^* : \operatorname{ord} g \geq \nu\}.$$

Since $\nu > \mathbb{N}$ by convention, we have $\mathcal{U}_\nu \subseteq \boldsymbol{\mu}$ by Lemma 6.4. It follows that for any neighborhood U of 1 in G we have $\mathcal{U}_n \subseteq U$ for all sufficiently large n; this gives (2). From $\mathcal{U}_\nu \subseteq \boldsymbol{\mu}$ we also obtain $\mathcal{U}_\nu \subseteq \mathrm{O}[\nu]$, hence $(\mathcal{U}_\nu)^i \subseteq \boldsymbol{\mu}$ for all $i = \mathrm{o}(\nu)$ by Lemma 5.4, so $\operatorname{ord} \mathcal{U}_\nu \geq c\nu$ for some $c \in \mathbb{R}^{>0}$. This gives (3): nonexistence of c as in (3) gives ν with $\operatorname{ord} \mathcal{U}_\nu < c\nu$ for all $c \in \mathbb{R}^{>0}$. □

Because 1 has a countable neighborhood basis in G, the topology of G is induced by some metric on G. Given such a metric d on G we obtain also a metric d on $L(G)$ by

$$d(X, Y) := \max\{d(X(t), Y(t)) : |t| \leq 1\},$$

and one verifies easily that this metric induces the same topology on $L(G)$ as the compact-open topology of $C(\mathbb{R}, G)$. We do not need this metric, but it may help in visualizing some arguments.

Proof that G is locally euclidean. Let $X \in L(G)^*$. We say that X is *infinitesimal* if $X \in \boldsymbol{\mu}(O)$, the monad of O in $L(G)^*$. Therefore,

$$X \text{ is infinitesimal} \iff X(I^*) \subseteq \boldsymbol{\mu} \iff X(1) \in \boldsymbol{\mu},$$

by the definitions and Corollary 6.3.

Lemma 6.6. *Let $X, Y \in L(G)^*$ be infinitesimal, with $Y(1) \in \mathrm{O}[\sigma]$. Then*

$$X(1)Y(1) = (X + Y)(1) \cdot z \qquad \text{with } z \in \mathrm{o}[\sigma].$$

Proof. Put $a := X(1)$, $b := Y(1)$, $c := (X + Y)(1)$. Take an open neighborhood U of 1 in G with $U \subseteq \mathcal{U}$ and take ν with $\sigma = o(\nu)$ and put

$$W := \{w \in G^* : w^i \in U^* \text{ for } i = 1, \ldots, \nu\}.$$

Then W is internally open in G^* and $1 \in W \subseteq \mathrm{O}[\nu] \subseteq \mathrm{o}[\sigma]$. By the definition of c and using transfer we have $\left(X(\frac{1}{e})Y(\frac{1}{e})\right)^e \in cW$ for all sufficiently large $e \in \mathbb{N}^* \setminus \mathbb{N}$, so

$$\left(X(\frac{1}{e})Y(\frac{1}{e})\right)^e = cw_e, \qquad w_e \in \mathrm{o}[\sigma],$$

for all sufficiently large $e \in \mathbb{N}^* \setminus \mathbb{N}$. But also, by Lemma 5.9,

$$\left(X(\frac{1}{e})Y(\frac{1}{e})\right)^e = abd_e, \qquad d_e \in \mathrm{o}[\sigma],$$

for all $e \in \mathbb{N}^* \setminus \mathbb{N}$. Hence $ab = c(wd^{-1})$ with $w, d \in \mathrm{o}[\sigma]$. □

Lemma 6.7. *K is a neighborhood of* 1 *in G.*

Proof. It is enough to show that $\boldsymbol{\mu} \subseteq K^*$. Let $a \in \boldsymbol{\mu}$ and suppose towards a contradiction that $a \notin K^*$. Since K^* is internally compact, we have $K^* = \bigcap_\nu K^*\mathcal{U}_\nu$, so we can take ν maximal with $a \in K^*\mathcal{U}_\nu$. Then $a = bc$ with $b \in K^*$ and $c \in \mathcal{U}_\nu \subseteq \boldsymbol{\mu}$, and $\operatorname{ord} c = \nu$. With $X := X_c \in L(G)$ defined by $X(t) = \mathrm{st}(c^{[\nu t]})$ we have $X \in \mathcal{K}$, and thus for $d := X(1/\nu) \in K^*$ we have $c^i \sim d^i$ for all $i \leq \nu$, and thus $c = du$ with $u \in \mathrm{o}[\nu]$ by Lemma 5.7. Hence $a = bdu$. By Lemma 6.6 we have $bd = gh$ with $g \in K^*$ and $h \in \mathrm{o}[\nu]$. Hence $a = g(hu)$ with $\nu = \mathrm{o}(\operatorname{ord}(hu))$, contradicting the maximality of ν. □

Corollary 6.8. *G is locally euclidean of dimension* $\dim_{\mathbb{R}} L(G)$.

Proof. Take an open neigborhood U of 1 in G such that $U \subseteq K$, and let V be the subset of $\mathcal{K}$ such that $U = \{X(1) : X \in V\}$. Then V is a neighborhood of O in $L(G)$, so contains an open neighborhood V' of O in $L(G)$. Then V' is also open in V, so $U' := \{X(1) : X \in V'\}$ is open in U and thus is an open neighborhood of 1 in G homeomorphic to the open subset V' of $L(G)$. □

The adjoint representation. Take an $\mathbb{R}$-linear isomorphism $L(G) \cong \mathbb{R}^n$ of vector spaces. It induces a group isomorphism

$$\mathrm{Aut}(L(G)) \cong \mathrm{GL}_n(\mathbb{R}) \subseteq \mathbb{R}^{n^2},$$

and we give the set $\mathrm{Aut}(L(G))$ the topology that makes this bijection into a homeomorphism, and thus into an isomorphism of topological groups. It is clear that this topology on $\mathrm{Aut}(L(G))$ does not depend on the initial choice of $\mathbb{R}$-linear isomorphism $L(G) \cong \mathbb{R}^n$.

Let G^0 be the connected component of 1 in G. It is the subgroup of G generated by the elements $X(t)$ with $X \in L(G)$ and $t \in \mathbb{R}$.

Lemma 6.9. *The group morphism* $\mathrm{Ad} : G \to \mathrm{Aut}(L(G))$ *is continuous, and* $\ker(\mathrm{Ad}) = \{a \in G :$ *a commutes with all elements of* $G^0\}$. *In particular, if G is connected, then* $\ker(\mathrm{Ad}) = \mathrm{center}(G)$.

Proof. One checks easily that if $a \in \boldsymbol{\mu}$ and $X \in L(G)$, then $aXa^{-1} \in \boldsymbol{\mu}(X)$ in $L(G)^*$. Applying this to the X from a basis of the vector space $L(G)$, we see that Ad is continuous at 1. Since Ad is a group morphism, it follows that Ad is continuous.

It is clear that $\ker(\mathrm{Ad})$ consists of those $a \in G$ that commute with all elements of the form $X(t)$ with $X \in L(G)$ and $t \in \mathbb{R}$. □

Corollary 6.10. *If G has* NSS*, then G is a Lie group.*

Proof. Replacing G by the connected component of 1, we may assume that G is connected. Let $N := \ker(\mathrm{Ad})$, which, by the previous lemma, coincides with the center of G. In particular, N is commutative, whence N has NSS. Moreover, there is a continuous injection $G/N \to \mathrm{Aut}(L(G))$; since $\mathrm{Aut}(L(G)) \cong \mathrm{GL}_n(\mathbb{R})$, it follows by ??? that G/N is a Lie group. It follows from the result of Kuranishi mentioned in the introduction that G is a Lie group. □

7. Locally Euclidean implies NSS

Lemma 7.1. *Let U be a neighborhood of 1 in G. Then U contains a compact subgroup H of G and a neighborhood V of 1 in G such that H contains every subgroup of G contained in V.*

Proof. Take an internal neighborhood V of 1 in G^* such that $V \subseteq \boldsymbol{\mu}$. Let S be the internal subgroup of G^* that is internally generated by the internal subgroups of G^* that are contained in V. Then $S \subseteq \boldsymbol{\mu}$ by Lemma 5.4, and so the internal closure H of S in G^* is an internally compact internal subgroup of G^* contained in U^*. Now use transfer. □

Corollary 7.2. *If $L(G) = \{O\}$, then there is a neighborhood basis of 1 in G consisting of compact open subgroups of G.*

Proof. Let U be a neighborhood of 1 in G, and take H and V as in the previous lemma. If $L(G) = \{O\}$, then every $a \in \boldsymbol{\mu}$ is degenerate, hence $a^{\mathbb{Z}^*} \subseteq H^*$ for each $a \in \boldsymbol{\mu}$, so H is open. □

Corollary 7.3. *If G is connected and $G \neq \{1\}$, then $L(G) \neq \{O\}$.*

Lemma 7.4. *Let N be a totally disconnected closed normal subgroup of G and let $\pi : G \to G/N$ be the canonical map. Then the induced map $L(\pi) : L(G) \to L(G/N)$ is surjective.*

Proof. Let $H := G/N$, and $Y \in L(H)$ with $Y(1) \neq 1_H$. Fix ν and put $h := Y(1/\nu) \in \boldsymbol{\mu}(1_H)$. Take a compact symmetric neighborhood V of 1_H in H such that $Y(1) \notin V$. Take a compact symmetric neighborhood U of 1 in G such that $\pi(U) \subseteq V$. Since π is an open map we have $\pi(\boldsymbol{\mu}) \supseteq \boldsymbol{\mu}(1_H)$. Take $a \in \boldsymbol{\mu}$ with $\pi(a) = h$. Then $\pi(a^\nu) = h^\nu = Y(1)$, so $a^\nu \notin U$, so $\sigma := \mathrm{ord}_U(a) \leq \nu$. We have $\pi(\mathrm{st}(a^k)) = \mathrm{st}(h^k) = 1_H$ for all $k = \mathrm{o}(\sigma)$, so the connected subgroup $G_U(a) = \{\mathrm{st}(a^k) : k = \mathrm{o}(\sigma)\}$ of G is contained in N. But N is totally disconnected, so $G_U(a) = \{1\}$, that is, $a \in \mathrm{O}[\sigma]$. Also $a \notin \mathrm{o}[\sigma]$, so $X \neq O$ where $X = X_a \in L(G)$ is defined by $X(t) = \mathrm{st}(a^{[\sigma t]})$. If $\sigma = \mathrm{o}(\nu)$, then $\pi(X(t)) = \mathrm{st}(h^{[\sigma t]}) = 1$ for all t, so $X \in L(N) \subseteq L(G)$, that is $X = O$, a contradiction. Thus $\sigma = (r + \epsilon)\nu$ with $r \in \mathbb{R}^{>0}$ and infinitesimal $\epsilon \in \mathbb{R}^*$. Hence $\pi(X(t)) = \mathrm{st}(h^{[\nu rt]}) = (rY)(t)$ for all $t \in \mathbb{R}$, that is, $L(\pi)(X) = rY$, and thus $L(\pi)(\frac{1}{r}X) = Y$. □

Lemma 7.5. *Suppose G is locally connected and has* NSCS*. Then G has* NSS*.*

Proof. Take a compact symmetric neighborhood U of 1 in G that contains no connected subgroup of G other than $\{1\}$. By Lemma 7.1 we can take an open neighborhood $V \subseteq U$ of 1 in G and a compact subgroup N_1 of G such that $N_1 \subseteq U$ and all subgroups of G contained in V are contained in N_1. Since $N_1 \subseteq U$ we have $L(N_1) = \{O\}$, so N_1 is totally disconnected by Corollary 7.2, so we can take a compact subgroup N of N_1 such that N is open in N_1 and $N \subseteq V$. Take an open subset W of V such that $N = N_1 \cap W$. Note that the set

$$\{a \in W : \ aNa^{-1} \subseteq W\}$$

is open. But if $a \in W$ and $aNa^{-1} \subseteq W$, then $aNa^{-1} \subseteq N_1$, so $aNa^{-1} \subseteq N$. Thus the normalizer G_1 of N in G is open in G. Let $H := G_1/N$, and let $\pi : G_1 \to H$ be the canonical map. It remains to show that H has NSS and N is finite.

Let $a \in \boldsymbol{\mu} \cap G_1^*$. If a is degenerate, then $a \in N_1^*$, so $a \in N^*$ and $\pi(a) = 1_H$. Suppose a is pure, and put $\nu := \mathrm{ord}_U(a)$. Then $a^{\nu+1} \notin U^*$. Take an open neighborhood V' of 1 in G such that $V'N \subseteq V \subseteq U$, so $\pi(a)^{\nu+1} \notin \pi(V')$, while $\pi(a)^i \in \boldsymbol{\mu}(1_H)$ for all $i = \mathrm{o}(\nu)$. Thus $\pi(a)$ is pure in H. Thus all infinitesimals of H other than 1_H are pure, that is, H has NSS. Thus $L(H)$ is finite-dimensional, and by Lemma 7.4 the $\mathbb{R}$-linear map

$$L(\pi) : L(G_1) = L(G) \to L(H)$$

is continuous and surjective with kernel $L(N) = \{O\}$, and thus a homeomorphism. (Have not yet used the local connectedness of G. Of U we only used that $X(\mathbb{R}) \not\subseteq U$ for all $X \neq O$ in $L(G)$.)

Take a special neighborhood $\mathcal{V}$ of H, as defined in Section 6. Take a connected neighborhood $\mathcal{U}$ of 1 in G_1 such that

$$\pi(\mathcal{U}) \subseteq \{Y(1) : Y \in L(H), Y(I) \subseteq \mathcal{V}\}.$$

(Here we used that G is locally connected.) Let $x \in \mathcal{U}$. Then $\pi(x) = Y(1)$ for a unique $Y \in L(H)$ with $Y(I) \subseteq \mathcal{V}$. and there is a unique $X \in L(G_1)$ such that $\pi \circ X = Y$, so $x = X(1)x(N)$ with $x(N) \in N$. The map that assigns to each $x \in \mathcal{U}$ the above $Y \in L(H)$ above is continuous, by ..., Since $L(\pi) : L(G_1) \to L(H)$ is a homeomorphism, it follows that the map $x \mapsto x(N) : \mathcal{U} \to N$ is continuous. But N is totally disconnected and $1(N) = 1$, so $x(N) = 1$ for all $x \in \mathcal{U}$. Hence $\mathcal{U} \subseteq \{X(1) : \ X \in L(G_1)\}$, and thus G has NSS. □

Recall that a topological space is *bounded in dimension* if for some n it does not contain a homeomorphic copy of $[0, 1]^n$.

Lemma 7.6. *If G is bounded in dimension, then G has* NSCS*.*

Proof. Suppose G does not have NSCS. Let U, V range over compact symmetric neighborhoods of 1 in G. We claim that for every n and U some compact subgroup of G contained in U contains a homeomorphic copy of the n-cube $[0,1]^n$. Assume this holds for a certain n and let U be given. By Lemma 7.1 we can take $V \subseteq U$ and a compact subgroup $H \subseteq U$ of G that contains every subgroup of G contained in V. Since V contains a nontrivial connected compact subgroup of G, Corollary 7.3 yields a nontrivial $X \in L(H)$. By decreasing V if necessary we can assume that $X(\mathbb{R}) \nsubseteq V$. Take a compact subgroup $G(V) \subseteq V$ of G with a homeomorphism $Y : [0,1]^n \to Y([0,1]^n) \subseteq G(V)$. Then $X(\mathbb{R}) \nsubseteq G(V)$. Replacing X by rX for a suitable real $r > 0$ we can assume that $X(I) \subseteq V$, X is injective on I and $X(I) \cap G(V) = \{1\}$. Since $G(V) \subseteq H$ we can define $Z : [0,1]^{n+1} = [0,1] \times [0,1]^n \to H$ by $Z(s,t) = X(s)Y(t)$ for $s \in [0,1]$, $t \in [0,1]^n$. It is easy to check that then Z is injective and continuous, and thus a homeomorphism. □

Corollary 7.7. *If G is bounded in dimension and locally connected, then G has* NSS*. In particular, if G is locally euclidean, then G has* NSS*.*

Proof. Use Lemmas 7.6 and 7.5. □

8. Yamabe's Theorem

Lemma 8.1. *Let U be a neighborhood of 1 in G. Then there is an open subgroup G' of G and a compact normal subgroup N' of G' such that $N' \subseteq U$ and G'/N' has* NSS*.*

Proof. By Lemma 7.1 we can take a compact subgroup $H \subseteq U$ of G and W an open neighborhood $W \subseteq U$ of 1 in G such that every subgroup of G contained in W is a subgroup of H. Since H is compact, Theorem 4.1 yields a compact normal subgroup $N' \subseteq W$ of H and an injective group morphism $H/N' \to \mathrm{GL}_n(\mathbb{R})$. Since $\mathrm{GL}_n(\mathbb{R})$ has NSS, it follows that H/N' has NSS. The latter gives an open $W' \subseteq W$ such that $N' \subseteq W'$ and every subgroup of G contained in W' is a subgroup of N'. Set

$$G' := \text{the normalizer of } N' \text{ in } G = \{g \in G : \ gN'g^{-1} = N'\},$$

so G' is a subgroup of G and N' is a normal subgroup of G'. We claim that G' and N' have the desired properties.

Since N' is compact and W' is open, there is a symmetric neighborhood V of 1 in G such that $VN'V \subseteq W'$. Then for all $g \in V$, the subgroup $gN'g^{-1}$ of G is contained in W', so $gN'g^{-1} \subseteq N'$, which by symmetry of V gives $gN'g^{-1} = N'$. Consequently, $V \subseteq G'$ and thus G' is open. It remains to show that G'/N' has NSS. This holds because $VN' \subseteq W'$ is a neighborhood of N' in G', so every subgroup of G' contained in VN' is contained in W' and thus in N'. □

Theorem 8.2. *Suppose G/G_o is compact. Then G can be approximated by locally euclidean groups: every neighborhood of 1 contains a compact normal subgroup N of G such that G/N is locally euclidean.*

Proof. Let U be a neighborhood of 1 in G. By Lemma 8.1 and its proof we obtain G' and N' as in that lemma and an open neighborhood W' of N' such that any subgroup of G contained in W' is a subgroup of N'. Note that $G_o \subseteq G'$ since G' is clopen in G. Consequently, G'/G_o is an open subgroup of the compact group G/G_o, and thus of finite index in G/G_o. Hence G' has finite index in G, so $G = g_1 G' \cup \cdots \cup g_n G'$ where $g_1, \ldots, g_n \in G$. Given $g \in G$ we have $g = g_i a$ with $1 \leq i \leq n$ and $a \in G'$, so $gN'g^{-1} = g_i(aN'a^{-1})g_i^{-1} = g_i N' g_i^{-1}$, since N' is normal in G'. Thus

$$N := \bigcap_{i=1}^{n} g_i N' g_i^{-1} = \bigcap_{g \in G} gN'g^{-1}$$

is a compact normal subgroup of G and $N \subseteq N' \subseteq U$. It remains to show that G/N has NSS. Let

$$W := \bigcap_{i=1}^{n} g_i W' g_i^{-1},$$

an open subset of G containing N. If $H \subseteq W$ is any subgroup of G, then for each i we have $g_i^{-1} H g_i \subseteq W'$, so $g_i^{-1} H g_i \subseteq N'$, and thus $H \subseteq N$. □

References

[1] Breuillard, Green, and Tao, The structure of approximate groups, Publ. Math. IHES **116** (2012), 115-221.

[2] Gleason, Groups without small subgroups, Ann. of Math. **56** (1952), 193–212.

[3] Gleason, On the structure of locally compact groups, Duke Math. J. **18** (1951), 85-104.

[4] Goldbring, Hilbert's fifth problem for local groups, Ann. of Math. **172** (2010), 1269-1314.

[5] Henson, Foundations of Nonstandard Analysis: A Gentle Introduction to Nonstandard Extensions; Nonstandard Analysis: Theory and Applications, L. O. Arkeryd, N. J. Cutland, and C. W. Henson, eds., NATO Science Series C:, Springer, 2001.

[6] Hirschfeld, The nonstandard treatment of Hilbert's fifth problem, Trans. AMS **321** (1990), 379–400.

[7] Hrushovski, Stable group theory and approximate subgroups, Journal of the American Mathematical Society **25** (2012), 189-243.

[8] Iwasawa, On some types of topological groups, Ann. of Math. **2** (1949), 507-557.

[9] Jacoby, Some theorems on the structure of locally compact local groups, Ann. of Math. **66** (1957), 36–69.

[10] Kaplansky, *Lie Algebras and Locally Compact Groups*, University of Chicago Press, 1971.

[11] Kuranishi, On local euclidean groups satisfying certain conditions, Proc. Amer. Math. Soc. **1** (1950), 372-380.

[12] Montgomery & Zippin, Small subgroups of finite-dimensional groups, Ann. of Math. **56** (1952), 213–241.

[13] Montgomery & Zippin, *Topological Transformation Groups*, Interscience, New York, 1955.

[14] Olver, Non-associative local Lie groups, Journal of Lie Theory **6** (1996), 23–51.
[15] Tao, Hilbert's fifth problem and related topics, in preparation.
[16] Yamabe, On the conjecture of Iwasawa and Gleason, Ann. of Math. **58** (1953), 48–54.
[17] Yamabe, A generalization of a theorem of Gleason, Ann. of Math. **58** (1953), 351–365.

University of Illinois, Department of Mathematics, 1409 W. Green Street, Urbana, IL 61801
E-mail address: `vddries@math.uiuc.edu`

University of Illinois at Chicago, Department of Mathematics, Statistics, and Computer Science, Science and Engineering Offices (M/C 249), 851 S. Morgan St., Chicago, IL 60607
E-mail address: `isaac@math.uic.edu`
URL: `http://www.math.uic.edu/~isaac`