

HIGH DENSITY PIECEWISE SYNDETICITY OF PRODUCT SETS IN AMENABLE GROUPS

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ABSTRACT. M. Beiglböck, V. Bergelson, and A. Fish proved that if G is a countable amenable group and A and B are subsets of G with positive Banach density, then the product set AB is piecewise syndetic. This means that there is a finite subset E of G such that EAB is thick, that is, EAB contains translates of any finite subset of G . When $G = \mathbb{Z}$, this was first proven by R. Jin. We prove a quantitative version of the aforementioned result by providing a lower bound on the density (with respect to a Følner sequence) of the set of witnesses to the thickness of EAB . When $G = \mathbb{Z}^d$, this result was first proven by the current set of authors using completely different techniques.

1. INTRODUCTION

In the paper [6], R. Jin proved that if A and B are subsets of \mathbb{Z} with positive Banach density, then $A+B$ is *piecewise syndetic*. This means that there is $m \in \mathbb{N}$ such that $A+B+[-m, m]$ is *thick*, i.e. it contains arbitrarily large intervals. Jin's result has since been extended in two different ways. First, using ergodic theory, M. Beiglböck, V. Bergelson, and A. Fish [1] established Jin's result for arbitrary countable amenable groups (with suitable notions of Banach density and piecewise syndeticity); in [5], M. Di Nasso and M. Lupini gave a simpler proof of the amenable group version of Jin's theorem using nonstandard analysis which works for arbitrary (not necessarily countable) amenable groups and also gives a bound on the size of the finite set needed to establish that the product set is thick. Second, in [4] the current set of authors established a "quantitative version" of Jin's theorem by proving that there is $m \in \mathbb{N}$ such that the set of witnesses to the thickness of $A+B+[-m, m]$ has upper density at least as large as the upper density of A . (We actually prove this result for subsets of \mathbb{Z}^d for any d .) The goal of this article is to prove the quantitative version of the result of Beiglböck, Bergelson, and Fish.

In the following, we assume that G is a countable amenable group¹: for every finite subset E of G and every $\varepsilon > 0$, there exists a finite subset L of G such that, for every $x \in E$, we have $|xL \triangle L| \leq \varepsilon |L|$; such an L is said to be ε -Følner with respect to E . It follows that G is amenable if and only if it has a (left) Følner sequence, which is a sequence $\mathcal{S} = (S_n)$ of finite subsets of G such that, for every $x \in G$, we have $\frac{|xS_n \triangle S_n|}{|S_n|} \rightarrow 0$ as $n \rightarrow +\infty$.

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¹For those not familiar with amenable groups, let us mention in passing that the class of (countable) amenable groups is quite robust, e.g. contains all finite and abelian groups and is closed under subgroups, quotients, extensions, and direct limits. It follows, for example, that every countable virtually solvable group is amenable.

If \mathcal{S} is a Følner sequence and $A \subseteq G$, we define the corresponding upper \mathcal{S} -density of A to be

$$\bar{d}_{\mathcal{S}}(A) := \limsup_{n \rightarrow +\infty} \frac{|A \cap S_n|}{|S_n|}.$$

For example, note that if $G = \mathbb{Z}^d$ and $\mathcal{S} = (S_n)$ where $S_n = [-n, n]^d$, then $\bar{d}_{\mathcal{S}}$ is the usual notion of upper density for subsets of \mathbb{Z}^d .

Following [1], we define the (left) *Banach density* of A , $\text{BD}(A)$, to be the supremum of $\bar{d}_{\mathcal{S}}(A)$ where \mathcal{S} ranges over all Følner sequences of G . One can verify (see [1]) that this notion of Banach density agrees with the usual notion of Banach density when $G = \mathbb{Z}^d$.

Recall that a subset A of G is *thick* if for every finite subset L of G there exists a right translate Lx of L contained in A . A subset A of G is *piecewise syndetic* if FA is thick for some finite subset F of G .

Definition. Suppose that G is a countable discrete group, \mathcal{S} is a Følner sequence for G , A is a subset of G , and $\alpha > 0$. We say that A is

- *upper \mathcal{S} -thick of level α* if for every finite subset L of G , the set $\{x \in G : Lx \subseteq A\}$ has upper \mathcal{S} -density at least α ;
- *upper \mathcal{S} -syndetic of level α* if there exists a finite subset F of G such that FA is \mathcal{S} -thick of level α .

The following is the first main result of this paper:

Theorem. *Suppose that G is a countable amenable group and \mathcal{S} is a Følner sequence for G . If A and B are subsets of G such that $\bar{d}_{\mathcal{S}}(A) = \alpha > 0$ and $\text{BD}(B) > 0$, then BA is upper \mathcal{S} -syndetic of level α .*

When $G = \mathbb{Z}^d$ and $\mathcal{S} = (S_n)$ where $S_n = [-n, n]^d$, we recover [4, Theorem 14]. We also recover [4, Theorem 18], which states (in the current terminology) that if $A, B \subseteq \mathbb{Z}^d$ are such that $\underline{d}(A) = \alpha > 0$ and $\text{BD}(B) > 0$, then $A + B$ is \mathcal{S}' -syndetic of level α for any subsequence \mathcal{S}' of the aforementioned \mathcal{S} .

If \mathcal{S} is a Følner sequence for G , then the notions of *lower \mathcal{S} -density* $\underline{d}_{\mathcal{S}}$, *lower \mathcal{S} -thick*, and *lower \mathcal{S} -syndetic* can be defined in the obvious ways. In [4, Theorem 19], the current set of authors proved that if $A, B \subseteq \mathbb{Z}^d$ are such that $\underline{d}_{\mathcal{S}}(A) = \alpha > 0$ and $\text{BD}(B) > 0$ (where \mathcal{S} is the usual Følner sequence for \mathbb{Z}^d as above), then for any $\epsilon > 0$, $A + B$ is lower \mathcal{S} -syndetic of level $\alpha - \epsilon$. (An example is also given to show that, under the previous hypotheses, $A + B$ need not be lower \mathcal{S} -syndetic of level α .) In a previous version of this paper, we asked whether or not the amenable group analog of [4, Theorem 19] was true. Using ergodic-theoretic methods, Michael Björklund [2] settled this question in the affirmative. Shortly after, we realized that our techniques readily established the same result and we include our proof here.

In their proof of the amenable group version of Jin's theorem, the authors of [5] give a bound on the size of a finite set needed to witness piecewise syndeticity: if G is a countable amenable group and A and B are subsets of G of Banach densities α and β respectively, then there is a finite subset E of G with $|E| \leq \frac{1}{\alpha\beta}$ such that EAB is thick. In Section 3, we improve upon this theorem in two ways: we slightly improve the bound on $|E|$ from $\frac{1}{\alpha\beta}$ to $\frac{1}{\alpha\beta} - \frac{1}{\alpha} + 1$ and we show that, if \mathcal{S} is any Følner sequence such that $\bar{d}_{\mathcal{S}}(A) > 0$, then EAB is \mathcal{S} -thick of level s for some $s > 0$.

Notions from nonstandard analysis. We use nonstandard analysis to prove our main results. An introduction to nonstandard analysis with an eye towards applications to combinatorics can be found in [7]. Here, we just fix notation.

If r, s are finite hyperreal numbers, we write $r \lesssim s$ to mean $\text{st}(r) \leq \text{st}(s)$ and we write $s \approx r$ to mean $\text{st}(r) = \text{st}(s)$.

If X is a hyperfinite subset of *G , we denote by μ_X the corresponding Loeb measure, which is a probability measure extending the finitely additive probability measure defined on the algebra of internal subsets of X given by $\mu_X(A) := \text{st} \left(\frac{|A|}{|X|} \right)$. If, moreover, $Y \subseteq {}^*G$ is internal, we abuse notation and write $\mu_X(Y)$ for $\mu_X(X \cap Y)$.

Suppose that $\mathcal{S} = (S_n)$ is a Følner sequence for G . For ν an infinite hypernatural number, we denote by S_ν the value at ν of the nonstandard extension of \mathcal{S} . It follows readily from the definition that $\bar{d}_\mathcal{S}(A)$ is the maximum of $\mu_{S_\nu}({}^*A)$ as ν ranges over all infinite hypernatural numbers.

The following definition is central.

Definition. Suppose that G is a group and $X \subseteq {}^*G$ is hyperfinite. We say that X is a *Følner approximation* of G if $|gX \triangle X|/|X|$ is infinitesimal for every $g \in G$.

It follows immediately from the nonstandard characterization of limit that if G is a countable amenable group and (S_n) is a Følner sequence for G , then S_ν is a Følner approximation of G for every $\nu > \mathbb{N}$.²

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2. HIGH DENSITY PIECEWISE SYNDETICITY

In this section, G denotes a countable amenable group. We begin with a combinatorial lemma.

Lemma 2.1. *Suppose that $\mathcal{S} = (S_n)$ is a Følner sequence for G , Y a Følner approximation for G , and $A \subseteq G$.*

(1) *If $\bar{d}_\mathcal{S}(A) \geq \alpha$, then there is $\nu > \mathbb{N}$ such that*

$$\frac{|{}^*A \cap S_\nu|}{|S_\nu|} \gtrsim \alpha \quad \text{and} \quad \frac{1}{|S_\nu|} \sum_{x \in S_\nu} \frac{|x({}^*A \cap S_\nu)^{-1} \cap Y|}{|Y|} \gtrsim \alpha. \quad (\dagger)$$

(2) *If $\underline{d}_\mathcal{S}(A) > \alpha$, then there is $\nu_0 > \mathbb{N}$ such that (\dagger) holds for all $\nu \geq \nu_0$.*

Proof. For (1), first apply transfer to the statement “for every finite subset E of G and every natural number k , there exists $n \geq k$ such that

$$\frac{1}{|S_n|} |A \cap S_n| > \alpha - 2^{-k} \quad \text{and} \quad \frac{1}{|S_n|} \sum_{x \in E} |x^{-1}S_n \triangle S_n| < 2^{-k}.”$$

²Conversely, if G admits a Følner approximation X , then G is amenable. Indeed, given finite $E \subseteq G$ and $\epsilon > 0$, observe that X is ϵ -Følner with respect to E , whence an actual (standard) ϵ -Følner set for E exists by downward transfer.

Fix $K > \mathbb{N}$ and let ν be the result of applying the transferred statement to Y and K . Set $C = {}^*A \cap S_\nu$ and let χ_C denote the characteristic function of C . We have

$$\begin{aligned} \frac{1}{|S_\nu|} \sum_{x \in S_\nu} \frac{|xC^{-1} \cap Y|}{|Y|} &= \frac{1}{|S_\nu|} \sum_{x \in S_\nu} \frac{1}{|Y|} \sum_{y \in Y} \chi_C(y^{-1}x) \\ &= \frac{1}{|Y|} \sum_{y \in Y} \frac{|C \cap y^{-1}S_\nu|}{|S_\nu|} \\ &\geq \frac{|C|}{|S_\nu|} - \sum_{y \in Y} \frac{|y^{-1}S_\nu \Delta S_\nu|}{|S_\nu|} \\ &\approx \alpha. \end{aligned}$$

For (2), apply transfer to the statement “for every finite subset E of G and every natural number k , there exists $n_0 \geq k$ such that, for all $n \geq n_0$,

$$\frac{1}{|S_n|} |A \cap S_n| > \alpha - 2^{-n_0} \quad \text{and} \quad \frac{1}{|S_n|} \sum_{x \in E} |x^{-1}S_n \Delta S_n| < 2^{-n_0}.”$$

Once again, fix $K > \mathbb{N}$ and let ν_0 be the result of applying the transferred statement to Y and K . As above, this ν_0 is as desired. \square

Suppose that $B \subseteq G$ is such that $\text{BD}(B) \geq \beta$. It is a routine saturation argument to see that there is then a Følner approximation Y of G such that $\frac{|{}^*B \cap Y|}{|Y|} \gtrsim \beta$. It will become important to demand an extra condition on such a Følner approximation.

Definition 2.2. Suppose that $B \subseteq G$ is such that $\text{BD}(B) \geq \beta$. We say a Følner approximation Y of G is *good for B* if

$$\frac{|{}^*B \cap Y|}{|Y|} \gtrsim \beta$$

and, for any standard $\varepsilon > 0$, there exists a finite subset F of G such that

$$\frac{|{}^*(FB) \cap Y|}{|Y|} > 1 - \varepsilon.$$

The following fact is central to our arguments and its proof is contained in the proof of [3, Lemma 4.6].

Fact 2.3. *If $B \subseteq G$ is such that $\text{BD}(B) > 0$, then there is a Følner approximation Y of G that is good for B .*

We are now ready to prove the main results of this section.

Theorem 2.4. *Suppose that G is a countable amenable group, $\mathcal{S} = (S_n)$ a Følner sequence for G , and $A, B \subseteq G$. If $\bar{d}_{\mathcal{S}}(A) \geq \alpha$ and $\text{BD}(B) > 0$, then BA is upper \mathcal{S} -syndetic of level α .*

Proof. Let Y be a Følner approximation for G that is good for B . Let ν be as in part (1) of Lemma 2.1 applied to Y and A . Once again, set $C := {}^*A \cap S_\nu$. Consider the μ_{S_ν} -measurable function

$$f(x) = \text{st} \left(\frac{|xC^{-1} \cap Y|}{|Y|} \right).$$

Observe that

$$\int_{S_\nu} f d\mu_{S_\nu} = \text{st} \left(\frac{1}{|S_\nu|} \sum_{x \in S_\nu} \frac{|xC^{-1} \cap Y|}{|Y|} \right) \geq \alpha,$$

whence there is some standard $r > 0$ such that $\mu_{S_\nu}(\{x \in S_\nu : f(x) \geq 2r\}) \geq \alpha$. Setting $\Gamma = \left\{x \in S_\nu : \frac{|xC^{-1} \cap Y|}{|Y|} \geq r\right\}$, we have that $\mu_{S_\nu}(\Gamma) \geq \alpha$. Since Y is good for B , we may take a finite subset F of G such that

$$\frac{|*(FB) \cap Y|}{|Y|} > 1 - \frac{r}{2}.$$

Fix $g \in G$. Since Y is a Følner approximation for G , we have that

$$\frac{|*(gFB) \cap Y|}{|Y|} = \frac{|*(FB) \cap g^{-1}Y|}{|Y|} \approx \frac{|*(FB) \cap Y|}{|Y|},$$

whence $\frac{|*(gFB) \cap Y|}{|Y|} > 1 - r$. Thus, for any $x \in \Gamma$, we have that $xC^{-1} \cap *(gFB) \neq \emptyset$. In particular, if L is a finite subset of G , then $\Gamma \subseteq *\left(\bigcap_{g \in L} gFBA\right)$. Therefore

$$\bar{d}_{\mathcal{S}}\left(\bigcap_{g \in L} gFBA\right) \geq \mu_{S_\nu}\left(*\left(\bigcap_{g \in L} gFBA\right)\right) \geq \mu_{S_\nu}(\Gamma) \geq \alpha.$$

It follows that BA is upper \mathcal{S} -syndetic of level α . \square

As mentioned in the introduction, Theorem 14 and Theorem 18 of [4] are immediate consequences of Theorem 2.4, after observing that the sequence of sets $[-n, n]^d$ as well as any of its subsequences is a Følner sequence for \mathbb{Z}^d . Example 15 of [4] shows that the conclusion in Theorem 2.4 is optimal, even when G is the additive group of integers and \mathcal{S} is the Følner sequence of intervals $[1, n]$.

Theorem 2.5. *Suppose that G is a countable amenable group, $\mathcal{S} = (S_n)$ a Følner sequence for G , and $A, B \subseteq G$. If $\underline{d}_{\mathcal{S}}(A) \geq \alpha$ and $\text{BD}(B) > 0$, then BA is lower \mathcal{S} -thick of level $\alpha - \varepsilon$ for every $\varepsilon > 0$.*

Proof. Without loss of generality, we may suppose that $\underline{d}_{\mathcal{S}}(A) > \alpha$. Fix a Følner approximation Y for G that is good for B and $\nu_0 > \mathbb{N}$ as in part (2) of Lemma 2.1 applied to Y and A . Fix $\nu \geq \nu_0$ and standard $\varepsilon > 0$ with $\varepsilon < \alpha$. Set

$$\Gamma_\nu = \left\{x \in S_\nu : \frac{|xC^{-1} \cap Y|}{|Y|} \geq \varepsilon\right\}$$

and observe that $\frac{|\Gamma_\nu|}{|S_\nu|} > \alpha - \varepsilon$. Since Y is good for B , we may fix a finite subset F of G such that

$$\frac{|*(FB) \cap Y|}{|Y|} > 1 - \frac{\varepsilon}{2}.$$

Fix $g \in G$. Since Y is a Følner approximation of G , arguing as in the proof of the previous theorem, we conclude that

$$\frac{|*(gFB) \cap Y|}{|Y|} > 1 - \varepsilon.$$

Fix $L \subseteq G$ finite. Once again, it follows that $\Gamma_\nu \subseteq *\left(\bigcap_{g \in L} gFBA\right)$ whence

$$\frac{|*\left(\bigcap_{g \in L} gFBA\right) \cap S_\nu|}{|S_\nu|} \geq \frac{|\Gamma_\nu|}{|S_\nu|} > \alpha - \varepsilon.$$

Since the previous inequality held for every $\nu \geq \nu_0$, by transfer we can conclude that

$$\underline{d}_{\mathcal{S}}\left(\bigcap_{g \in L} gFBA\right) \geq \alpha - \varepsilon.$$

It follows that BA is lower \mathcal{S} -syndetic of level $\alpha - \epsilon$. \square

As mentioned in the introduction, Theorem 2.5 is a generalization of [4, Theorem 19] and was first proven by M. Björklund using ergodic-theoretic methods.

3. A BOUND ON THE NUMBER OF TRANSLATES

The following theorem is a refinement of [5, Corollary 3.4]. In particular, we improve the bound on the number of translates, and also obtain an estimate on the \mathcal{S} -density of translates that witness the thickness of EBA .

Theorem 3.1. *Suppose that G is a countable amenable group, $\mathcal{S} = (S_n)$ a Følner sequence for G , and $A, B \subseteq G$. If $d_{\mathcal{S}}(A) \geq \alpha$ and $\text{BD}(B) \geq \beta$, then there exists $s > 0$ and a finite subset $E \subseteq G$ such that $|E| \leq \frac{1}{\alpha\beta} - \frac{1}{\alpha} + 1$ and EBA is \mathcal{S} -thick of level s .*

Proof. Reasoning as in the proof of Theorem 2.4, one can prove that there exist $\nu \in {}^*\mathbb{N} \setminus \mathbb{N}$, a Følner approximation Y of G , a standard $r > 0$, and an infinitesimal $\eta \in {}^*\mathbb{R}_+$ such that, if $C = {}^*A \cap S_\nu$, $D = {}^*B \cap Y$, and

$$\Gamma := \left\{ x \in S_\nu : \frac{1}{|Y|} |x C^{-1} \cap D| \geq \alpha\beta - \eta \right\},$$

then $|\Gamma| \geq r |S_\nu|$. Fix a family $(p_g)_{g \in G}$ of strictly positive standard real numbers such that $\sum_{g \in G} p_g \leq \frac{1}{2}$.

We now define a sequence of subsets (H_n) of G and a sequence (s_n) from G . Define

$$H_0 := \left\{ g \in G : \frac{1}{|\Gamma|} |\{x \in \Gamma : gx \notin {}^*(BA)\}| > p_g \right\}.$$

If H_n has been defined and is nonempty, let s_n be any element of H_n and set

$$H_{n+1} := \left\{ g \in G : \frac{1}{|\Gamma|} |\{x \in \Gamma : gx \notin {}^*({s_0, \dots, s_n} BA)\}| > p_g \right\}.$$

If $H_n = \emptyset$ then we set $H_{n+1} = \emptyset$.

We claim $H_n = \emptyset$ for $n > \left\lfloor \frac{1}{\alpha\beta} - \frac{1}{\alpha} \right\rfloor$. Towards this end, suppose $H_n \neq \emptyset$. For $0 \leq k \leq n$, take $\gamma_k \in \Gamma$ such that

$$s_k \gamma_k \notin {}^*({s_0, \dots, s_{k-1}} BA).$$

Observe that the sets $s_0 D$, $s_1(\gamma_1 C^{-1} \cap D)$, ..., $s_n(\gamma_n C^{-1} \cap D)$ are pairwise disjoint. In fact, if

$$s_i D \cap s_j \gamma_j C^{-1} \neq \emptyset$$

for $0 \leq i < j \leq n$, then

$$s_j \gamma_j \in s_i D C \subseteq {}^*({s_0, \dots, s_{j-1}} BA),$$

contradicting the choice of γ_j . Therefore we have that

$$\begin{aligned} 1 &\gtrsim \frac{1}{|Y|} |s_0 D \cup s_1((\gamma_1 C^{-1} \cap D) \cup \dots \cup s_n((\gamma_n C^{-1} \cap D))| \\ &\geq \frac{1}{|Y|} \left(|D| + \sum_{i=1}^n |\gamma_i C^{-1} \cap D| \right) \\ &\geq \beta + \alpha\beta n. \end{aligned}$$

It follows that $n \leq \left\lfloor \frac{1}{\alpha\beta} - \frac{1}{\alpha} \right\rfloor$.

Take the least n such that $H_n = \emptyset$. Note that $n \leq \left\lfloor \frac{1}{\alpha\beta} - \frac{1}{\alpha} \right\rfloor + 1$. If $n = 0$ then BA is already \mathcal{S} -thick of level r , and there is nothing to prove. Let us assume that $n \geq 1$, and set $E = \{s_0, \dots, s_{n-1}\}$. It follows that, for every $g \in G$, we have that

$$\frac{1}{|\Gamma|} |\{x \in \Gamma : gx \in {}^*(EBA)\}| \geq 1 - p_g.$$

Suppose that L is a finite subset of G . Then

$$\frac{1}{|\Gamma|} |\{x \in \Gamma : Lx \subseteq {}^*(EBA)\}| \geq 1 - \sum_{g \in L} p_g \geq 1 - \sum_{g \in G} p_g \geq \frac{1}{2}.$$

Therefore

$$\frac{1}{|S_\nu|} |\{x \in S_\nu : Lx \subseteq {}^*(EBA)\}| \geq \frac{r}{2}.$$

This shows that EBA is \mathcal{S} -thick of level $\frac{r}{2}$. \square

With a similar argument using Markov's inequality [8, Lemma 1.3.15], one can also prove the following result. We omit the details.

Theorem 3.2. *Suppose that G is a countable amenable group, $\mathcal{S} = (S_n)$ a Følner sequence for G , and $A, B \subseteq G$. If $d_{\mathcal{S}}(A) \geq \alpha$ and $\text{BD}(B) \geq \beta$, then for every $\gamma \in (0, \alpha\beta]$ there exists $E \subseteq G$ such that $|E| \leq \frac{1-\beta}{\gamma} + 1$ and EBA is \mathcal{S} -syndetic of level $\frac{\alpha\beta-\gamma}{1-\gamma}$.*

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