$\pi_1(|\Gamma|)$: a hyperfinite approach

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1 The problem

2 Nonstandard analysis

3 The Main Theorem

4 An application to homology

$\pi_1(X)$

- Suppose that X is a space and $p \in X$.
- Recall that $\pi_1(X;p)$ is the set of (continuous) loops based at p modulo the relation of two loops being homotopic.
- The operation of concatenating loops based at p induces a group operation on $\pi_1(X;p)$ (with identity being the homotopy class of the constant loop at p).
- If X is pathconnected, then this group is independent of p and is denoted by $\pi_1(X)$, referred to as the *fundamental group* of X.
- The typical example is $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$, where $\mathbb{S}^1$ is the unit circle in $\mathbb{C}$.
- This construction is functorial: if $f : X \to Y$ is continuous, then there is an induced map $f_* : \pi_1(X) \to \pi_1(Y)$ given by $f_*([\alpha]) := [f \circ \alpha]$.
- X is called *simply connected* if it is pathconnected and $\pi_1(X) = \{1\}$.

$\pi_1(\Gamma)$ when Γ is finite

Theorem

Suppose that Γ is a connected, finite graph. Then $\pi_1(\Gamma)$ is a finitely generated free group.

Proof.

- Let T be a spanning tree of Γ.
- Let $\vec{e}_1, \ldots, \vec{e}_n$ be *oriented chords* of T, that is, edges of Γ not in T, given a fixed orientation.
- Given $[\alpha] \in \pi_1(\Gamma)$, let r_α be the reduced word on $\{e_1^{\pm 1}, \ldots, e_n^{\pm 1}\}$ obtained by recording which chords α traverses fully and in which direction.
- The map $[\alpha] \mapsto r_\alpha : \pi_1(\Gamma) \to F_n$ is an isomorphism.

□

End compactifications of finite graphs

- We now consider infinite, locally finite, connected graphs.
- Many results from finite graph theory are plain false for infinite graphs.
- However, by compactifying an infinite graph by adding its "ends," one can obtain topological analogues of theorems from finite graph theory.

Ends

Definition

Let X be a metric space and $p \in X$.

1. For $x, y \in X$, we write $x \propto_n y$ to indicate that x and y are in the same path component of $X \setminus B(p; n)$.
2. For $r_1, r_2 : [0, \infty) \to X$ *proper* rays with $r_1(0) = r_2(0) = p$, we say $\text{end}(r_1) = \text{end}(r_2)$ if and only if:

$$(\forall n \in \mathbb{N})(\exists m_0 \in \mathbb{N})(\forall m \geq m_0)(r_1(m) \propto_n r_2(m)).$$

3. $\text{Ends}(X) := \{\text{end}(r) \mid r \text{ a proper ray starting at } p\}$.
4. $|X| := X \cup \text{Ends}(X)$ is the *end compactification of* X, topologized in such a way so that proper rays converge to their ends.

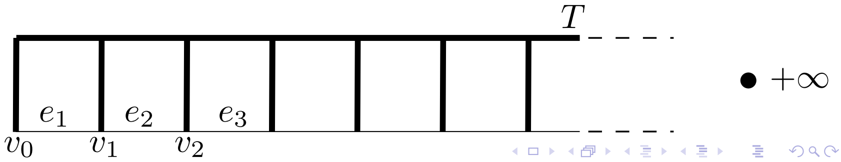
The main problem

Question (Diestel/Sprüssel)

Is there a nice combinatorial characterization of the fundamental group of the end compactification of a locally finite, connected graph in the spirit of the result in the second slide?

An example: the infinite sideways ladder

- Consider the loop α beginning at v_0, going along the bottom rung of the ladder to the end at $+\infty$, and then back again along the bottom rung of the ladder. α is certainly *nullhomotopic* (i.e. homotopic to the constant loop at v_0).
- If we consider the *topological spanning tree* T for Γ pictured below in bold with oriented edges $\vec{e}_1, \vec{e}_2, \ldots$, then the "word" α induces is $(\vec{e}_1 \vec{e}_2 \cdots)^\frown(\cdots \overleftarrow{e_2}\overleftarrow{e_1})$.
- This word is of order type $\omega + \omega^*$ with no consecutive appearances of $\overrightarrow{e_i}$ and $\overleftarrow{e_i}$. So we cannot combinatorially tell that this loop is nullhomotopic.

$|\Gamma| =$ 

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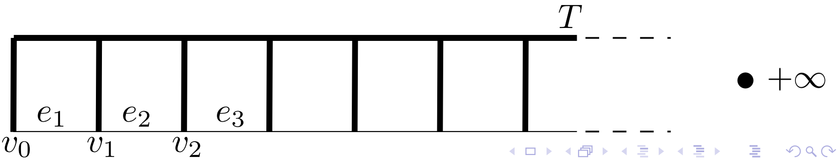
$|\Gamma| =$ 

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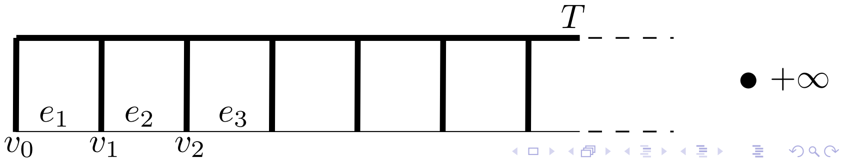
$|\Gamma| =$ 

Diestel and Sprüssel's Result

- Undaunted by the previous example, Diestel and Sprüssel offered the following solution to their question.
- Let Γ be an infinite, locally finite, connected graph with end compactification $|\Gamma|$. Let T be a topological spanning tree for Γ with oriented chords $X = \{\vec{e}_1, \vec{e}_2, \ldots\}$.
- Diestel and Sprüssel consider words on X of arbitrary countable order type (e.g. the order type of $\mathbb{Q}$!) and define a non-wellordered notion of reduction of words.
- If $F(X)$ denotes the group of reduced words (in the above sense), Diestel and Sprüssel show that the map $[\alpha] \mapsto r_\alpha : \pi_1(|\Gamma|) \to F(X)$ is a well-defined injective group homomorphism (although this takes $\geq$ 15 pages!). They also identify the image.
- By considering finite subwords, they construct an injective group morphism $F(X) \to \varprojlim F_n$ into an inverse limit of finitely generated free groups, once again identifying the image. (Algebraic and easy.)

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Can nonstandard analysis help?

After seeing my nonstandard treatment on ends, Diestel asked me the following question:

Question (Diestel)

Can nonstandard analysis make any of this simpler?

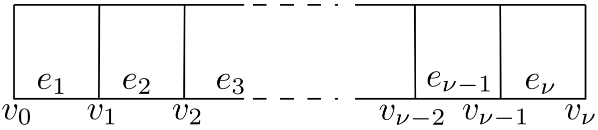
Answer (G., Sisto)

Yes!

The infinite sideways ladder revisited

Let ν be an *infinite natural number*. We can then consider the following *hyperfinite* extension of Γ:

$\Gamma_\nu =$



Our loop α from before "clearly" induces the *hyperfinite word*

$$\overrightarrow{e_1}\overrightarrow{e_2}\cdots\overrightarrow{e_\nu}\overleftarrow{e_\nu}\cdots\overleftarrow{e_2}\overleftarrow{e_1},$$

which "clearly" *internally* reduces to the empty word, exhibiting that α is nullhomotopic.

In this way, we get an injective group morphism $\pi_1(|\Gamma|) \hookrightarrow \pi_1(\Gamma_\nu)$, where $\pi_1(\Gamma_\nu)$ is the *internal fundamental group of* Γ_ν, which is a hyperfinitely generated internally free group on ν generators.

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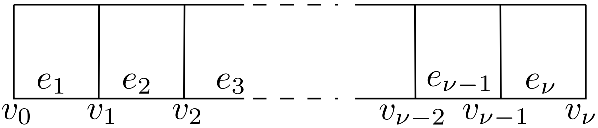

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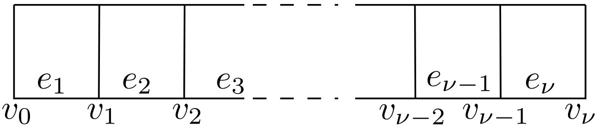

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NSA in a nutshell

- Every set X gets enlarged, in a functorial fashion, to a set X^*, the *nonstandard extension of* X.
- X^* "logically behaves" like X (Transfer Principle), but contains new "ideal" elements, e.g. $\mathbb{R}^*$ contains infinitesimal and infinite numbers.
- In a natural way, $\mathcal{P}(X)^*$ embeds into $\mathcal{P}(X^*)$. The subsets of $\mathcal{P}(X^*)$ that belong to $\mathcal{P}(X)^*$ are called the *internal* subsets of X^*; noninternal subsets of X^* are called *external*.
- The similarity in logical behavior applies only to *internal subsets* of X^*. For example, internal subsets of $\mathbb{R}^*$ that are bounded above have suprema; it follows that the set of infinitesimal numbers is external.

The ultraproduct approach

- Suppose that $\mathcal{U}$ is a *nonprincipal ultrafilter* on $\mathbb{N}$, that is, $\mathcal{U}$ is a $\{0,1\}$-valued measure on $\mathcal{P}(\mathbb{N})$ such that finite sets get measure 0.
- For $f, g : \mathbb{N} \to X$, write $f \sim_{\mathcal{U}} g$ to mean $f = g$ a.e.
- Set $X^{\mathcal{U}} := X^{\mathbb{N}} / \sim_{\mathcal{U}}$, the *ultrapower of X with respect to $\mathcal{U}$*.
- This construction is easily seen to be functorial and the fact that $X^{\mathcal{U}}$ behaves "logically" like X is known to model theorists as *Łos' theorem*.
- In this setting, $A \subseteq X^{\mathcal{U}}$ is internal if there are $A_n \subseteq X$ such that $A = \prod_{\mathcal{U}} A_n := (\prod_n A_n)/ \sim_{\mathcal{U}}$.
- $N := [(1, 2, 3, \ldots)]_{\mathcal{U}} \in \mathbb{N}^*$ is a *positive infinite number* whose reciprocal $\frac{1}{N} = [(1, \frac{1}{2}, \frac{1}{3}, \ldots] \in \mathbb{R}^*$ is a *positive infinitesimal*.
- If $A := \prod_{\mathcal{U}} A_n$ with each A_n finite, then we say that A is *hyperfinite* with *internal cardinality* $[(|A_n|)] \in \mathbb{N}^*$.

Nonstandard metric spaces

- If (X, d) is a metric space, then (X^*, d) is almost a metric space except for the fact that the metric takes values in $\mathbb{R}^*$ rather than in $\mathbb{R}$.
- There are two important subsets of X^* to consider:
  - $X_{\text{ns}} := \{a \in X^* \mid \text{ there is } b \in X \text{ with } d(a, b) \text{ infinitesimal}\}$.
  - $X_{\text{fin}} := \{a \in X^* \mid \text{ there is } b \in X \text{ with } d(a, b) \text{ finite}\}$.
- Clearly $X_{\text{ns}} \subseteq X_{\text{fin}}$ with equality holding if and only if X is a *proper* metric space, that is, closed balls are compact.
- If X is proper, then a ray $r : [0, \infty) \to X$ is proper if and only if $r(\sigma) \in X_{\text{inf}}$ for all infinite elements σ of $\mathbb{R}^*$.

The nonstandard approach to ends

- Suppose that (X, d) is a proper, *geodesic* metric space and $p \in X$.
- For $x, y \in X^*$, write $x \propto y$ to mean there is $\alpha \in C([0,1], X)^*$ (an *internal* path in X^*) such that $\alpha(0) = x$, $\alpha(1) = y$, and $\alpha(t) \in X_{\text{inf}} := X^* \setminus X_{\text{fin}}$ for all $t \in [0,1]^*$.
  "x and y are in the same path component at infinity."

Theorem (G.)

1. $\text{end}(r_1) = \text{end}(r_2)$ *if and only if for all (equiv. for some)* $\sigma, \tau \in \mathbb{R}^{>0}_{\text{inf}}$, $r_1(\sigma) \propto r_2(\tau)$.
2. *Set* $\text{IPC}(X) := \{[x] \mid x \in X_{\text{inf}}\}$, *where* $[x]$ *denotes the equivalence class of* x *with respect to* $\propto$. *Fix* $\sigma \in \mathbb{R}^{>0}_{\text{inf}}$. *Then the map* $\text{end}(r) \mapsto [r(\sigma)] : \text{Ends}(X) \to \text{IPC}(X)$ *is a bijection.*

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Collapsing graphs

- From now on, Γ is an infinite, locally finite, connected graph.
- Let $\theta_n : \Gamma \to \Gamma_n$ be the map which collapses path components of $\Gamma \setminus B(p; n)$ to points. Note Γ_n is a finite graph.
- It is straightforward to check that θ_n extends continuously to $\theta_n : |\Gamma| \to \Gamma_n$.
- Set $\Gamma_{\text{hyp}} := \prod_{\mathcal{U}} \Gamma_n$, a *hyperfinite* graph.
- Γ_{hyp} arises from the internal map $\theta : \Gamma^* \to \Gamma_{\text{hyp}}$ arising from collapsing internal path components of $\Gamma^* \setminus B(p; N)$ to points, where $N := [(1, 2, 3, \ldots)] \in \mathbb{N}^* \setminus \mathbb{N}$.
- θ extends to an *internally continuous* $\theta : |\Gamma^*| \to \Gamma_{\text{hyp}}$, where $|\Gamma^*|$ denotes the *internal end compactification of* Γ^*.

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Collapsing graphs (cont'd)

- $\theta : |\Gamma^*| \to \Gamma_{\text{hyp}}$.
- Digesting the definitions, one sees that $|\Gamma^*| = |\Gamma|^*$.
- By the Transfer Principle applied to the functoriality of the fundamental group, we get an internal map $\Theta : \pi_1(|\Gamma|^*) \to \pi_1(\Gamma_{\text{hyp}})$, where the π_1's here denote *internal fundamental groups*.
- More digesting of notation reveals $\pi_1(|\Gamma|^*) = (\pi_1(|\Gamma|))^*$, so $\pi_1(|\Gamma|)$ is a subgroup of $\pi_1(|\Gamma|^*)$.

Theorem (G., Sisto)

$\Theta \restriction \pi_1(|\Gamma|) : \pi_1(|\Gamma|) \to \pi_1(\Gamma_{\text{hyp}})$ *is injective.*

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Collapsing graphs (cont'd)

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Remarks

1. Θ is generally not injective.
2. The result does not imply that $\pi_1(|\Gamma|)$ is free. (This happens if and only if every end is contractible.) Indeed, internally free groups (e.g. $\pi_1(\Gamma_{hyp})$) need not be free. For example, $\mathbb{Z}^*$ is internally free on one generator, while, for infinite $M, N \in \mathbb{N}^*$ with $\frac{M}{N}$ infinite, we have the map $(a, b) \mapsto aM + bN : \mathbb{Z}^2 \to \mathbb{Z}^*$ is injective. If $\mathbb{Z}^*$ were free, then $\mathbb{Z}^2$ would be free.
3. $\pi_1(|\Gamma|)$ has the same universal theory as the theory of free groups. (If $\pi_1(|\Gamma|)$ were finitely generated, we would say it is a *limit group*.)

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Universal covers

- Given a topological space X, a *cover of X* is a topological space C and a surjective, continuous map $p : C \to X$ such that, for every $x \in X$, there is an open neighborhood U of x such that $p^{-1}(U)$ is a disjoint union of open sets in C, each of which is mapped homeomorphically onto U by p.
- A *universal cover of X* is a cover of X whose associated topological space is simply connected. "Nice" spaces have universal covers.
- For example, the universal cover of $\mathbb{S}^1$ is $\mathbb{R}$.
- If G is a finite graph, its universal cover is a tree.
- If $p : C \to X$ is a cover of X and γ is a path in X, then for every $c \in p^{-1}(\gamma(0))$, there is a unique path in C lying over γ starting at c. If p is the universal cover of X and γ is a loop, then this unique path in C is also a loop.

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About the proof

Theorem (G., Sisto)

$\Theta \restriction \pi_1(|\Gamma|) : \pi_1(|\Gamma|) \to \pi_1(\Gamma_{\text{hyp}})$ *is injective.*

Idea of Proof

- Suppose $\theta(\alpha)$ is internally nullhomotopic.
- Since Γ_{hyp} is hyperfinite, its internal universal cover $\widetilde{\Gamma_{\text{hyp}}}$ is an internal tree. This passage to universal covering tree is not possible in the standard approach!
- $\theta(\alpha)$ lifts to an internal loop in $\widetilde{\Gamma_{\text{hyp}}}$.
- Using nice geodesic paths in $\widetilde{\Gamma_{\text{hyp}}}$, we can project back onto Γ_{hyp} to construct a homotopy witnessing that α is nullhomotopic.
- We do not need to consider topological spanning trees in Γ, an added bonus since their existence is nontrivial.

Connection with the standard result

- With more effort, we can completely recover the Diestel-Sprüssel result.
- However, we can easily recover the embedding of $\pi_1(|\Gamma|)$ into an inverse limit of f.g. free groups.
- The maps $\theta_n : |\Gamma| \to \Gamma_n$ yield a homomorphism

$$\Psi : \pi_1(|\Gamma|) \to \varprojlim \pi_1(\Gamma_n).$$

- Define $\Phi : \varprojlim \pi_1(\Gamma_n) \to \prod_{\mathcal{U}} \pi_1(\Gamma_n) = \pi_1(\Gamma_{\text{hyp}})$ by $\Phi((x_n)) := [(x_n)]$.
- Check that $\Theta \restriction \pi_1(|\Gamma|) = \Phi \circ \Psi$. Since $\Theta \restriction \pi_1(|\Gamma|)$ is injective, so is Ψ.
- As a result, we see that $\pi_1(|\Gamma|)$ is ω-*residually free*, a property known to be equivalent to being a limit group for finitely generated groups.

1 The problem

2 Nonstandard analysis

3 The Main Theorem

4 An application to homology

The first homology group

Definition

1. If G is a group, its *first homology group* is the group $H_1(G) := G/[G, G]$.
2. If X is a pathconnected space, its first singular homology group is $H_1(X) := H_1(\pi_1(X))$.

- For finite graphs, the first singular homology group coincides with a familiar combinatorial object, the so-called cycle space $\mathcal{C}(\Gamma)$.
- For infinite graphs, Diestel and Sprüssel devised an ad hoc homology theory for $|\Gamma|$, the *topological cycle space* $\mathcal{C}^{\mathrm{top}}(\Gamma)$.
- They wondered if the topological cycle space coincides with the first singular homology.

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- They wondered if the topological cycle space coincides with the first singular homology.

The topological cycle space

- Set $\vec{\mathcal{E}}(\Gamma) := \{\varphi : E^{\text{or}}(\Gamma) \to \mathbb{Z} \mid \varphi(\vec{e}) = -\varphi(\overleftarrow{e})\}$.
- If $(\varphi_n)_{n\in\mathbb{N}} \subseteq \vec{\mathcal{E}}(\Gamma)$ is such that, for all edges e, $\varphi_n(e) \neq 0$ for finitely many n, then we may form the *thin sum* of (φ_n), $\sum_n \varphi_n \in \vec{\mathcal{E}}(\Gamma)$.
- Given a *circle* α in $|\Gamma|$, get $\varphi_\alpha \in \vec{\mathcal{E}}(\Gamma)$ by setting $\varphi_\alpha(\vec{e}) = 1$ if α traverses $\vec{e}$, -1 if it traverses $\overleftarrow{e}$, and 0 otherwise. Call φ_α an oriented circuit.
- $\mathcal{C}^{\text{top}}(\Gamma)$ is the subgroup of $\vec{\mathcal{E}}(\Gamma)$ obtained by taking thin sums of oriented circuits.

Theorem (Diestel-Sprüssel)

There is a surjective group homomorphism $H_1(|\Gamma|) \to \mathcal{C}^{\text{top}}(\Gamma)$ such that:

- *the homomorphism is an isomorphism when Γ is finite;*
- *if α is a loop in $|\Gamma|$, then the image of $[\alpha]$ is 0 in $\mathcal{C}^{\text{top}}(\Gamma)$ if and only if α traverse each edge the same number of times in each direction.*

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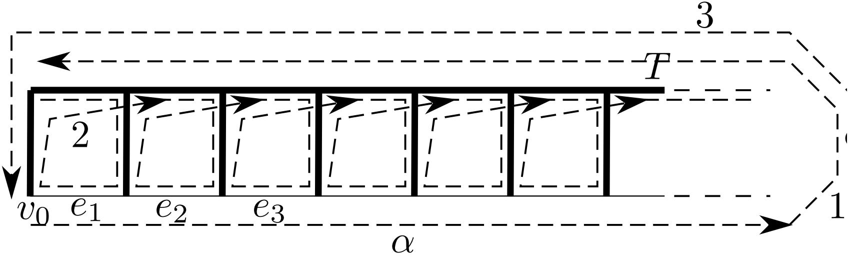
Two different homology theories

Theorem (Diestel/Sprüssel)

The loop α depicted below is trivial in $\mathcal{C}^{\text{top}}(\Gamma)$ but not in $H_1(|\Gamma|)$.

$|\Gamma| =$ 

Why is α not nullhomologous?

- A finite version of α would trace the word

$$\overrightarrow{e_1} \cdots \overrightarrow{e_n}\overleftarrow{e_1} \cdots \overleftarrow{e_n} \in [\pi_1(\Gamma), \pi_1(\Gamma)],$$

whence the finite version of α would be nullhomologous.

- Diestel and Sprüssel give a topological proof that the finite version of the loop α is nullhomologous. They then remark "But we cannot imitate this proof for α and our infinite ladder, because homology classes in $H_1(|\Gamma|)$ are still finite chains: we cannot add infinitely many boundaries to subdivide α infinitely often." (I.e. Wouldn't it be great if nonstandard analysis existed?)
- Instead, Diestel and Sprüssel use their analysis of the fundamental group to attach a complicated invariant to loops which vanish on nullhomologous loops. They then show that the invariant for α is nonzero.

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A simple lemma

Recall our maps $\theta : \Gamma \to \Gamma_{\text{hyp}}$ and $\Theta : \pi_1(|\Gamma|)^* \to \pi_1(\Gamma_{\text{hyp}})$.

Lemma

If the loop α is null-homologous, then $\theta(\alpha)$ has finite commutator length as an element of $\pi_1(\Gamma_{\text{hyp}})$.

Proof.

Let $g \in \pi_1(|\Gamma|)$ be the element represented by α and let $f : G \to H_1(G)$ be the natural map. If $f(g) = 0$, then we can write g as the product of, say, n commutators. As Θ is a group homomorphism, we have that $\Theta(g)$ can be written as a product of n commutators as well. □

A simple proof that α is not nullhomologous

Fact (Goldstein-Turner; G.-Sisto)

If $e_1, \ldots, e_n$ generate a free group, then $e_1 \cdots e_n e_1^{-1} \cdots e_n^{-1}$ has commutator length $\lfloor \frac{n}{2} \rfloor$.

- By transfer, $\theta(\alpha)$ induces the word $e_1 \cdots e_\nu e_1^{-1} \cdots e_\nu^{-1}$ in Γ_{hyp}.
- By transfer of the above fact, $\theta(\alpha)$ has commutator length $\lfloor \frac{\nu}{2} \rfloor > \mathbb{N}$.
- By the previous lemma, α is not nullhomologous.

Remarks about homology

- We have seen that $\theta(\alpha)$ internally nullhomotopic implies α nullhomotopic.
- The preceding example shows that $\theta(\alpha)$ internally nullhomologous does not necessarily imply α nullhomologous.
- In fact, one can check that $\theta(\alpha)$ is internally nullhomologous if and only if α is trivial in $\mathcal{C}^{\text{top}}(\Gamma)$. So, in some sense, their ad hoc homology theory is really the internal version of the ordinary homology theory.

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