# A NOTE ON PROPERTY (T)

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In this note, we make a small observation about Property (T); this result is known (see Corollary F.2.9 of [1]), but our proof is new and short.

Fix a locally compact group $G$ and a (complex) Hilbert space $H$. Then a *unitary representation* of $G$ in $H$ is a group homomorphism $\pi : G \to U(H)$ from $G$ into the group of unitary operators on $H$ which is strongly continuous, that is, the map $g \mapsto \pi(g)(x) : G \to H$ is continuous for every $x \in H$.

Fix a unitary representation $\pi : G \to U(H)$. If $\epsilon \in \mathbb{R}^{>0}$ and $K$ is a compact subset of $G$, we say that $x \in H$ is an *$(\epsilon, K)$-invariant vector* if $\|\pi(g)(x) - x\| < \epsilon$ for all $g \in K$. We say that $x \in H$ is an *invariant vector* if $\pi(g)(x) = x$ for all $g \in G$.

$G$ is said to have *Kazhdan's property (T)* if whenever $\pi : G \to U(H)$ is a unitary representation which has $(\epsilon, K)$-invariant unit vectors for every $\epsilon \in \mathbb{R}^{>0}$ and every compact $K \subseteq G$, then $\pi$ has a nonzero invariant vector. The point of this note is to show that the only obstruction to property (T) are infinite-dimensional representations:

**Proposition 0.1.** *Suppose that $\pi : G \to U(H)$ is a unitary representation of $G$ in $H$, where $\dim(H) < \infty$. Suppose that $\pi$ has $(\epsilon, K)$-invariant unit vectors for every $\epsilon \in \mathbb{R}^{>0}$ and every compact $K \subseteq G$. Then $\pi$ has an invariant unit vector.*

*Proof.* We use nonstandard analysis to prove this fact. We work in a $\kappa$-saturated nonstandard universe, where $\kappa > |G|$. Fix a positive infinitesimal $\epsilon$. For each $g \in G$, let

$$A_g := \{x \in H^* \mid \|x\| = 1,\ \|\pi(g)(x) - x\| < \epsilon\}.$$

Clearly each $A_g$ is internal. Moreover, by the transfer of our hypothesis, each $A_g$ is nonempty and the family $(A_g \mid g \in G)$ has the finite intersection property. Thus, by saturation, there is $x \in H^*$ such that $\|x\| = 1$ and $\|\pi(g)(x) - x\| < \epsilon$ for each $g \in G$.

Since $\dim(H) < \infty$, there is a (unique) vector in $H$ infinitely close to $x$; we call this vector $\mathrm{st}(x)$. We claim that $\mathrm{st}(x)$ is an invariant vector for $\pi$. Fix $g \in G$. Since $\pi(g) : H \to H$ is continuous, we know that $\pi(g)(\mathrm{st}(x)) \approx \pi(g)(x)$; here $\approx$ means "infinitely close to". We thus have that

$$\pi(g)(\mathrm{st}(x)) \approx \pi(g)(x) \approx x \approx \mathrm{st}(x).$$

Since $\pi(g)(\mathrm{st}(x))$, $\mathrm{st}(x) \in H$, we have that $\pi(g)(\mathrm{st}(x)) = \mathrm{st}(x)$. □

## References

[1] B. Bekka, P. de la Harpe, A. Valette, *Kazhdan's Property (T)*, New Mathematical Monographs (**11**), Cambridge University Press, 2008.