Riemann's Existence Theorem: An elementary approach to moduli

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CHAPTER 1

SCOPE OF THE EXISTENCE THEOREM

This chapter is an overview of this book's three main topics.

- How Riemann's Existence Theorem describes moduli spaces of Riemann surface covers of the Riemann sphere.
- How finite group theory puts practical — for applications — structures into collections of such covers.
- How each finite group generates its own nilpotent theory of fundamental groups, forming systems of moduli spaces with $G_{\mathbb{Q}}$ actions.

1. Context for the book

We note L. Ahlfors' satisfying entwining of the algebra and geometry in 1st year graduate complex variables [**Ahl79**]. This author could do no better than use it as his underpinning. Still, that book leaves the full scope of monodromy a mystery, it prepares little on coordinates describing Riemann surfaces, and none on families of Riemann surfaces.

E. Hille wrote a function theoretic encyclopedia [**Hil62**]. As a graduate student, I enjoyed how relevant were its historical comments to sophisticed mathematics in the 1800s. For example, mathematicians seeking immortality (in private, of course) might ponder its many serious references to H. Schwarz's work [**Sc1890**]. Few present day complex variable enthusiasts know the coherence or context of that work. Two authors, G. Springer [**Spr57**] and R.C. Gunning ([**Gun66**] and [**Gun67**]), did great service bringing Riemann surfaces to graduate students by the 1960s. For the former, that was H. Weyl's uniformization approach (as in his projection lemma). For the latter it was the Cartan-Serre vector bundle view of the algebro-differential geometry that works on Riemann surfaces.

E. Neuenschwanden's perspective answers many questions on what took so long for Riemann surfaces to make their mark [**Ne81**]. He documents contention between Weierstrass's algebraic and Riemann's harmonic function approaches. This is relevant to the relation between Riemann and Abel and Galois. For Weierstrass admits the influence of Abel on his work. Still, one can't see it directly on Riemann. This is despite serious documentation of his intellectual activities, including the direct influence on him of Gauss. Further, [**Ne81**] leaves unanswered other questions about the assimilation of mathematics.

These modern works have little group theory; not even including the original approaches of Abel, Galois and Riemann. Few presented group theory so dramatically as did H. Weyl. Yet, even Weyl (on quantum mechanics) met resilient resistance to group theory. My convictions are here; I advocate using the *power* of group theory. Showing how finite and profinite group theory can handle intricate monodromy and moduli, and apply practically to algebra and complex variables, is my goal. Still, there's a fence to walk. We can't afford to let group theory overwhelm us. Galois was first to note group theory's power. Also, he wrote on its potential to dominate the subject techically.

The introductions of two books, [**MM95**] and [**Vö96**], show they closely connect through group theory with this book. [**Fri94**] and [**Fri95c**] specifically discuss connnections of our topics to [**Se92**]. These three books concentrate on how Riemann's Existence Theorem applies to the Inverse Galois Problem. By contrast, classical topics appear here more often than in the first two. Also, this author uses standard formulations of the Inverse Galois Problemmuch less. Yet, the reader can find here a leisurely track through Riemann surface theory guided by problems requiring little preparation for their statements, a virtue of the Inverse Galois Problem. My choices often have a long literature *before* the connection to Riemann surfaces appeared. By occasional referring to topics from these three books, starting in Chap. 4, I have added efficiency to this liesurely pace.

By being leisurely, we (I and the reader) may also consider the struggle of many generations with whether punctured Riemann surfaces and their moduli variation belong to function theory or to algebra. Since it is leisurely, using a style less sophisticated than my papers in the middle 1970's, it might from its opening chapter be mistaken as curiously old-fashioned. Further, my evident hero worship of Riemann can further confuse those who don't know me. What I have tried is an historical model. I attempt to synthesize in two early chapters what might have been the insights of those famous researchers from the 1800s for whom analytic continuation and its applications to algebraic equations was an open extravaganza of intensely studied equations. The complication of mathematics, that Galois remarked on as often as one can do when one is going to disappear long before maturity, overwhelmed all except the technical giants of the time. Yet, from this came synthesis: Abstract approaches that simplified everything for those who could follow them. The people I admire today tend to admire — by aspiration in their own research — these very same people. If we aim to please and appeal to Abel, Galois and Riemann on this score, we realize — in rational moments — that is an impossibility. Further, since that is a triumvirate of geniuses, such an appeal detracts from showing why even *they* struggled, and despite the time that has passed we too, with the whole topic. There is a serious question for mathematics. When does *mathematics* (versus Riemann) have a firm grasp on a significant subject?

Is it when an elite institution husbands a handful of caretakers of an industry of supporting research? Is it when myriad papers allude to consequent deep theories, even if they don't directly involve the roiling concepts? Is it when some text has nailed the subject completely to a prestigious group's satisfaction? Is it when a blithely confident prestigious group claim the subject's foundations are firm and available to any sincere seeker? Is it when the subject successfully supports several independent and competing schools derived from its basic problems?

We don't know what would convince most research mathematicians of the security of a subject. The author has a point in writing this book; though he cannot easily pick one affirmative viewpoint for the maturity of this book's subject. Its techniques quickly worked to reveal the nature of long standing problems in his hands. On that basis a fair observer might support that the techniques work. Still, there are geniuses beyond Abel, Galois and Riemann who have their viewpoints. Examplars of thinking with great scope and imagination certainly include [**An02**], [**De89**] [**Moc96**]. We end the book at the wealth of analytic questions and applications raised by Modular Towers, a little before the influence of these writers on the author. So, only a shadow of their influence is here.

All, however, support connecting profinite groups to function theory. That leads to final, painful consideration. Will we, and the world outside mathematics, ever be able to tolerate the inundation that often overwhelms us from the connections bridged by mathematical language?

2. A quick summary

A fuller overview follows this sections brief summary.

2.1. A concise description, chapter by chapter. Compact Riemann surfaces as branched covers of a sphere appear in 1st year graduate courses as *elementary* discussions of *multi-valued functions*. We expand the usual brief treatment in Chap. 2. This carefully treats analytic continuation to motivate the geometry behind it. It introduces the Existence Theorem sufficiently to get lessons from the theory of *abelian* algebraic covers of the punctured Riemann sphere (§3.2). It starts with two different definitions of algebraic functions, one from algebraic equations another phrasing from analytic continuation. An imprecise version of Riemann's Existence Theorem is that these describe the same functions. This is an elementary investigation, based on the first half of graduate complex variables.

In this book Riemann's Existence Theorem means the precise statement from Chap. 4. That *really* organizes all algebraic functions (of z). Chap. 4 fully develops Riemann's Existence Theorem. It emphasizes data determining a branched cover of the sphere up to equivalence. Abel and Galois started a tradition. Our version: Translate complex analytic and arithmetic geometry problems into group theory through application of forms of Riemann's Existence Theorem.

Advanced texts often append another statement. It is that *any* compact Riemann surface (Chap. 3) has an analytic (nonconstant) map to $\mathbb{P}^1_z \stackrel{\text{def}}{=} \mathbb{C} \cup \{\infty\}$, the Riemann sphere. Springer's book [**Spr57**] dedicates much space to proving this last statement. We rarely use it; our basic data already includes such a function and (given the Riemann-Roch Theorem) includes Springer's goals (see below).

Suppose, however, $\varphi : X \to \mathbb{P}^1_z$ is such an analytic map. Let z_0 be a particular z value, and consider X_{z_0}, the fiber of φ over z_0. Then, finding algebraic equations for X, necessary for most applications, depends on producing another function $\varphi' : X \to \mathbb{P}^1_w$ that *separates* points of X_{z_0}. The explicit production of such a φ' is a consequence of *uniformization* of X by the appropriate simply-connected domain (disk, plane or sphere). As uniformization plays an important role in advanced applications, say, related to θ functions, we often raise elementary aspects of it.

The globally defined functions, φ and φ' have an algebraic relation $F(\varphi, \varphi') \equiv 0$ between them: $F \in \mathbb{C}[z, w]$. Let $L_{\varphi'} \subset \mathbb{C}$ be the field generated by the ratios of all coefficients of F. Let K be a field containing $L_{\varphi'}$. A frequent application of this relation F is to give meaning to the expression *a K point on X*. From F and $z_0 \in K$, there is an equivalence class of permutation representations of the absolute Galois G_K of K. This comes from its action on points of X over z_0. Refined applications of covers analyze the dependence of this statement on the choice of φ'.

Chap. 4 shows the following. Let $L_{\boldsymbol{z}}$ be the field generated by the symmetric functions in $\boldsymbol{z}$ (with ∞ removed if it appears).

(2.1a) There is a choice of φ' giving K algebraic over $L_{\boldsymbol{z}}$.
(2.1b) The complete set of minimal fields $\mathcal{L}_\varphi$ appearing as $L_{\varphi'}$ in the algebraic closure of $L_{\boldsymbol{z}}$ is an intrinsic (moduli) invariant of $\varphi : X \to \mathbb{P}^1_z$.
(2.1c) Sometimes (the Existence Theorem shows) $\mathcal{L}_\varphi$ consists of a unique field.

When $\boldsymbol{z}$ consists of algebraically independent values, the analysis of $\mathcal{L}_\varphi$ includes the moduli (deformation) theory of a cover. That is Part II of the book. Comparing this case with the case $L_{\boldsymbol{z}} = \mathbb{Q}$ (or some other explicit algebraic number field) is tantamount to approaches to the Inverse Galois Problem.

We assume students with one semester each of a graduate algebra course and a graduate complex variables course. Few students master Galois theory from their algebra courses. Thus, we give an analytic continuation approach to showing the field of convergent Puiseux expansions around a point is algebraically closed. This supports many elementary subtopics that could otherwise be baffling. For example, Riemann's Existence Theorem uses an infinite number of incompatible algebraically closed fields containing the field $\mathbb{C}(z)$. Let $\boldsymbol{z} = (z_1, \dots, z_r)$ be a fixed set of points on the sphere. Denote the complement of $\boldsymbol{z}$ on the sphere by $U_{\boldsymbol{z}}$.

Riemann's Existence Theorem is about algebraic functions *extensible* on $U_{\boldsymbol{z}}$. These are functions with analytic continuations along any path (from an explicit base point) avoiding $\boldsymbol{z}$. At each point z_0, not in $\boldsymbol{z}$, these algebraic extensible functions embed in the algebraically closed field of Puiseux expansions in z_0. Isomorphisms between their different embeddings is coded in the *fundamental groupoid*.

Chap. 2 describes *abelian* functions of z through analytic continuing branches of the log function. It demonstrates many basic definitions and some advanced concepts. Among these is that of a *group* attached to monodromy action. For books motivated by θ functions and their applications, this book is unusually persistent in emphasizing finite group theory.

Chap. 3 has basics on fundamental groups and permutation representations. Though our definitions and first examples of *manifolds* are traditional, our aim is to illustrate practical use of deformations of Riemann surfaces. We concentrate on very explicit manifolds. Chap. 5 produces highly structured *moduli* spaces parametrizing equivalence classes of Riemann surfaces.

Consider the notation around (2.1). For $\boldsymbol{z}$ fixed, and $K = L_{\boldsymbol{z}}$, if $z_0 \in K$, there is an action of G_K on the profinite completion of the fundamental group $\pi_1(U_{\boldsymbol{z}}, z_0)$ (Chap. 4). Moduli parameters appear with the following question.

PROBLEM 2.1. What happens with covers of $U_{\boldsymbol{z}}$ as $\boldsymbol{z}$ varies?

First appearances give the following impression.

(2.2a) The fundamental group of $U_{\boldsymbol{z}}$ doesn't change with $\boldsymbol{z}$.
(2.2b) $G_{\mathbb{Q}}$ action changes drastically if you can even consider it varying with $\boldsymbol{z}$.

Both (2.2a) and (2.2b) are wrong.

Suppose we try to write equations (with coefficients in $\boldsymbol{z}$) for the deformations of an algebraic function $f = f_{\boldsymbol{z}}$ extensible on $U_{\boldsymbol{z}}$ (Chap. 2). Locally in $\boldsymbol{z}$ this is possible. Going, however, around various closed paths in the space for $\boldsymbol{z}$, $f_{\boldsymbol{z}}$ might return to a different extension field of $\mathbb{C}(z)$. Riemann's Existence Theorem tells precisely how to calculate which paths return to the original function field (§5.4.1). *Hurwitz monodromy action* is the phrase for our most important calculations. This produces coordinates for coefficients relating $f_{\boldsymbol{z}}$ algebraically to $(z, \boldsymbol{z})$ (Chap. 5).

Choosing generators and a base point are what allow covering applications of the fundamental group. A response to (2.2a) is that this extra data produces a refined moduli space setup. This motivates a Lie algebra approach to (2.2b) putting the two parts of (2.2) under a common framework. We use ideas from renown papers of Y. Ihara and J.P-Serre and moduli space that give the proper context for the Inverse Galois Problem.

Abelian covers of $U_{\boldsymbol{z}}$ for any $\boldsymbol{z}$ comes from branches of log (Chap. 2). Ihara studied (parts of the) arithmetic of nilpotent covers of $U_{\boldsymbol{z}}$ when $r = 3$ [**Iha86**]. Nilpotent theory appears in applications to the Inverse Galois Problem. Here it starts from nonsplit nilpotent extensions extending data about covers with any given finite (often simple) group G. For p a prime dividing the order of G, a univeral totally nonsplit extension ${}_p\tilde{G}$ of G produces sequences of refined moduli spaces (§8.3).

[**Fri78**] and [**Iha86**] had common elements: use of the theory of *complex multiplication*, and an arithmetic philosophy using the *braid group*. The former used analytic geometry and finite group theory. There is now a natural way to join this to the profinite and function theory approach of the latter. This means joining *Modular Towers* to the *Grothendieck-Teichmüller* technology. The tools include extension of Deligne's tangential base points [**De89**] with insight from Riemann's θ functions.

2.2. Meaning of the word, elementary in the title. The first two chapters are elementary by most perspectives. Still, understanding Chap. 5 on moduli requires mastery of the first two chapters. The approach is elementary because it allows a newcomer into the area through examples and techniques using finite group theory. Traditionally, for example, with modular curves, one must have serious training in complex analysis. The action happens with automorphic functions on the upper half plane.

Here we often use *uniformization from below*, replacing the upper half plane and representations of $\mathrm{SL}_2(\mathbb{R})$ with the Riemann sphere $\mathbb{P}^1_z$ and finite group theory. Then, modular curves and their associated towers are an example of the moduli of dihedral group covers. The same technique works by replacing the dihedral group by any finite group. This opens up applications beyond the traditional modular curve approach.

This modular curve generalization uses a construction attached to each prime p dividing the order of a finite group G: The *universal p-Frattini cover* ${}_p\tilde{G}$ of G. This especially considers those primes p for which G is *p-perfect* (it has no cyclic quotient of order p).

Add to this a collection $\mathbf{C}$ of conjugacy classes from G whose elements have order prime to p. Then, $(G, p, \mathbf{C})$ produces a sequence of *moduli spaces of curves*. Example: G is the dihedral D_p of order $2p$ (p an odd prime) and $\mathbf{C}$ consists of four repetitions of the conjugacy class of involutions. Then, the sequence of moduli spaces is the classical modular curve series $\{Y_1(p^{k+1})\}_{k=0}^{\infty}$: Quotients of the upper half-plane by well-known subgroups denoted $\Gamma_1(p^{k+1})$ of $\mathrm{PSL}_2(\mathbb{Z})$. The kth *level* of the sequence in this case is $Y_1(p^{k+1}$. Introducing the generalizing sequences of spaces, *Modular Towers*, is the book's main advanced topic.

When G is an alternating group A_n ($n \geq 4$), and $p = 2$, Modular Tower properties generalize applications of θ functions. Specifically, in this alternating group case *several* components may appear in a Modular Tower level. This is unlike the dihedral case where all levels are connected. We use modular representations of characteristic quotients of ${}_p\tilde{G}$ (§10.2). This extension of Schur's theory of *universal central extensions* connects these components to the famous mod 2 (*half-canonical class*) invariant from θ functions.

Function theory, as in *cusp forms* and *Eisenstein series* from modular curves also appear here. Since the levels are moduli spaces of curves, we know most about those functions by relating them to θ functions of curves representing points in the moduli spaces. Such varying θ functions produce θ-null automorphic forms. Our main examples illustrate this when the moduli spaces are quotients of the upper half plane, giving covers of the classical j-line. This exactly corresponds (for any (G,p)) to the case $\mathbf{C}$ consists of four conjugacy classes in G.

Modular curves, though a guide, are a small portion of the noncongruence quotients of the upper half plane with a tower structure related to a prime p. New applications reveal the value of a Riemann's Existence Theorem approach. Function wise it generalizes both the *braid group* approach to the Inverse Galois Problem and the *Tate module*.

Early chapters develop detailed motivation for using classical functions. The deeper function theory, however, appears in outline (with exposition on applications related to the literature). Developing this completely is a topic for a later book.

3. Early historical motivation

A renown problem from the early 19th century was to express *in radicals* solutions x of the general nth degree polynomial equation

$$f(x) = x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0, \tag{3.1}$$

with f of degree n in x. The goal specifically asks for solutions x using *known* functions of the coefficients $a_1, \dots, a_n$. The explicitly known functions of the time were what we call radicals.

Traditional books tackle this using Galois theory with pure algebra. They reproduce Galois' Theorem characterizing when a field extension L/K is a subfield of a chain of radical field extensions of K. This happens if the Galois closure of L/K has solvable group.

It is a pretty story. Still, Galois' Theorem is not a common object of mathematical pilgrimage (even if Galois is). This treatment hides ingredients that still seize the imagination of modern mathematicians, as it possessed Abel, Galois and Riemann. Abel and Galois recognized *group theory* for showing, with a_i s and $n \ge 4$, the field of radical sequences in the a_i s do not contain the solutions. Still, these books lack problems motivating present research. Further, the subject's character falls outside the neatly compartmental introduction of rings, groups, modules and elementary classification results of the rest of 1st year graduate algebra. These historically come long after it, leaving the impression Galois theory is both mildly exotic and slightly moribund.

3.1. Consider functions of one variable. To be more explicit turn to complex variables, as did Abel. Instead of $a_1, \dots, a_n$ being general, specialize to functions $a_1(z), \dots, a_n(z)$ of one complex variable z. Assume $a_1(z), \dots, a_n(z)$ are in the field $\mathbb{C}(z)$: rational functions of z with complex coefficients. It is convenient to replace x by a variable w taking complex values. Refer in this specialized form to the equation $f(a_1(z), \dots, a_n(z), w) = m(z, w) = 0$.

The left side of (3.1) does not factor into lower degree polynomials over the field $a_1, \dots, a_n$ generate. The specialized expression $m(z,w) = 0$ may factor over $\mathbb{C}(z)$. To simplify, assume m is an irreducible polynomial in w over $\mathbb{C}(z)$. *Analytic continuation* displays the n solutions in w as n manifestations of one solution. The manifestations cohere through a group. Here is how it arises.

3.2. Motivating integrals. Critical values $\boldsymbol{z} = z_1, \dots, z_r$ of m are places z' where $m(z', w)$ has repeated roots. Fix $z_0 = z$ not equal to a critical value of m. Then the zeros w of $m(z,w)$ have expressions $w_1(z; z_0), \dots, w_n(z; z_0)$, meromorphic functions in z around z_0. This holds for any z_0 outside $\boldsymbol{z}$. So these algebraic functions are *extensible* on $\mathbb{C} \cup \{\infty\} \setminus \{\boldsymbol{z}\} = U_{\boldsymbol{z}}$ (Chap. 2). The *group* of $m(z,w)$ (relative to z) is all permutations of the w_i s from continuation around closed paths in $U_{\boldsymbol{z}}$ based at z_0. Call the w_i s *abelian* if this group is abelian.

This study of zeros jibed nicely with another problem of Abel's day: Analyze elementary antiderivatives, like the watershed example $\int \frac{dx}{\sqrt{x^3+ax+b}}$. Specifically, what is the dependence of these antiderivatives on the parameters a and b?

Here $m(z,w) = w^2 - (z^3 + az + b)$. Write $G(z) = \frac{1}{\sqrt{z^3+az+b}}$ acknowledging (Chap. 2) that plugging in values of z near z_0 requires choosing one of two functions $G(z)$ analytic in a disc about z_0 with $G(z)^{-2} = z^3 + az + b$. Consider $F(z)$, an antiderivative of $G(z)$, locally. An integral gives $F(z)$. So, it has analytic continuations around $U_{\boldsymbol{z}}$. These continuations produce an *abelian group* of periods (Chap. 2). Chap. 4 shows the group is $\mathbb{Z} \times \mathbb{Z}$. Further, its fit with the analytic continuations of $G(z)$ appears in the semidirect product $\mathbb{Z} \times \mathbb{Z} \times^s \{\pm 1\}$ (§8). Let D_n be the dihedral group of order $2n$.

Classical modular curves parametrize four branch point D_n extensions of $U_{\boldsymbol{z}}$. Galois checked with his theorem for which n these modular curve parameters were solvable functions of the classical j parameter [**Rig96**, p. 133]. Properties of $F(z)$ entwine integration and the appearance of abelian extensions:

(3.2) $F(z)$ is a *versal abelian* extensible function on $U_{\boldsymbol{z}}$ with monodromy around $\boldsymbol{z}$ bounded by $G(z)$ (Chap. 4).

Restricting to $U_{\boldsymbol{z}}$ still shows the full scope of Riemann's version of (3.2). The next three sections base a story of his program on analytic continuation.

4. Algebraic functions among extensible functions

Denote Laurent series expansions about z_0 by $\mathcal{L}_{z_0}$. Let $\mathcal{E}(U_{\boldsymbol{z}}, z_0)$ be extensible (meromorphic) elements of $\mathcal{L}_{z_0}$ on $U_{\boldsymbol{z}}$. Call $f \in \mathcal{E}(U_{\boldsymbol{z}}, z_0)$ algebraic if it satisfies $m(z, f(z)) \equiv 0$ with $m \in \mathbb{C}[z,w]$ a nonzero polynomial. Characterizing such f through analytic continuation, the main topic of Chap. 2, is the first step to classifying algebraic functions. Any analytic continuation of f around a closed path in $U_{\boldsymbol{z}}$ also gives a zero of m. So, there are only finitely many analytic continuations of f. Analytic continuations of f along paths whose end points have limits in $\boldsymbol{z}$ take values nowhere dense (a finite set) in the Riemann sphere. This qualitative statement characterizes algebraic f. The full force of Riemann's Existence Theorem is in phrasing this through fundamental group representations (Chap. 4). Denote the algebraic elements of $\mathcal{E}(U_{\boldsymbol{z}}, z_0)$ by $\mathcal{E}(U_{\boldsymbol{z}}, z_0)^{\text{alg}}$.

4.1. One element of $\mathcal{E}(U_{\boldsymbol{z}}, z_0)$ is versal for $\mathcal{E}(U_{\boldsymbol{z}}, z_0)^{\text{alg}}$. There are so many algebraic functions in $\mathcal{E}(U_{\boldsymbol{z}}, z_0) = \mathcal{E}(U_{\boldsymbol{z}})$ (if the cardinality, $r = |\boldsymbol{z}|$ exceeds two).

We can explain little about them by listing their polynomial equations. Yet, there is much structure in this collection.

4.1.1. *Setup for uniformizaton.* Riemann provided such by finding one function $\tilde{f}_{\boldsymbol{z}}$ giving all of $\mathcal{E}(U_{\boldsymbol{z}}, z_0)^{\mathrm{alg}}$ through a type of Galois correspondence. An outline for this appears in Chap. 3.

(4.1a) Recognize each algebraic function $f \in \mathcal{E}(U_{\boldsymbol{z}})$ has an attached topological cover $\varphi_f : X_f \to U_{\boldsymbol{z}}$.

(4.1b) Produce a (uni)versal cover $\varphi_{\boldsymbol{z}} : \tilde{U}_{\boldsymbol{z}} \to U_{\boldsymbol{z}}$ with a discrete group $\pi_1(U_{\boldsymbol{z}}, z_0)$ acting on $\tilde{U}_{\boldsymbol{z}}$.

(4.1c) Show X_f is a topological quotient of $\tilde{U}_{\boldsymbol{z}}$ by a subgroup of $\pi_1(U_{\boldsymbol{z}})$.

(4.1d) Show $\tilde{U}_{\boldsymbol{z}}$ has a complex analytic embedding in $\mathbb{C}$: $h : \tilde{U}_{\boldsymbol{z}} \to \mathbb{C}$.

As in Chap. 3, (4.1d) produces $\tilde{f}_{\boldsymbol{z}}$ as follows. Let U_{z_0} be *any* disk around z_0 (on $U_{\boldsymbol{z}}$). *Cauchy's Theorem* (we return soon to that) shows this:

(4.2) Each $g \in \mathcal{E}(U_{\boldsymbol{z}}, z_0)$ extends to a unique meromorphic function on U_{z_0}.

4.1.2. *$\tilde{U}_{\boldsymbol{z}}$ and Hurwitz equivalence.* Riemann's Existence Theorem shows why $\tilde{U}_{\boldsymbol{z}}$ identifies with the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \Im(z) > 0\}$. Apply the Existence Theorem (see Chap. 4 or §5.1) with a branch cycle description of form

$$g_1 = (1 \ldots s_1) \cdots (s_{t-1}{+}1 \ldots s_t),$$

$s_1 + s_2 + \cdots + s_t = n$; $g_2 = (s_1\, s_2 \ldots s_t)$ and $g_3 = (g_1 g_2)^{-1}$. Count points over branch points: $t + (n - t + 1) + 1 = n + 2$.

Uniformize $U_{\{0,1,\infty\}}$ with the classical λ function (§7.1.1). Choose $n = r - 2$. This produces a genus 0 cover of $\varphi_{\boldsymbol{g}} : X_{\boldsymbol{g}} \to \mathbb{P}^1_z$ unramified over $U_{\{0,1,\infty\}}$ with exactly r points over $\{0, 1, \infty\}$. Further, λ factors through this cover:

$$\mathbb{H} \to X_{\boldsymbol{g}} \setminus \varphi_{\boldsymbol{g}}^{-1}(0, 1, \infty) \to U_{\{0,1,\infty\}}.$$

This uniformizes one copy of $\mathbb{P}^1_z$ minus r points. Deform (differentiably) $X_{\boldsymbol{g}} \setminus \varphi_{\boldsymbol{g}}^{-1}(0, 1, \infty)$ to *any* other copy of $\mathbb{P}^1_z$ minus r points (Chap. 5).

Regard algebraic functions $f = y$ (of z) as giving a relation between two variables x and y. Classical literature often chooses the isomorphism class of the *function field* $\mathbb{C}(z, y)$ as the unique goal of an algebraic relation. If $\mathbb{C}(z, y)$ is isomorphic to $\mathbb{C}(z^*, y^*)$, this views the algebraic relation between (z^*, y^*) (take the minimal polynomial of y^* over $\mathbb{C}(z^*)$) as elementary equivalent to the relation between z and y. The history of considering algebraic relations had its motivation in integrals. There the most telling invariant of a function field $\mathbb{C}(z, y)$ is the *genus* g (maximal number of linearly independent holomorphic differentials §6.2) on the function field.

A connected algebraic space parametrizes all algebraic relations of genus g (Chap. 5). Investigating this and subtler problems about algebraic relations suggest a more delicate equivalence between function fields. In addition to the isomorphism of $\mathbb{C}(z^*, y^*)$ with $\mathbb{C}(z, y)$, this isomorphism includes that $\mathbb{C}(z^*) = \mathbb{C}(z)$. Call this *Hurwitz equivalence*. Even in restricting to genus g function fields there are many components to the parameter spaces of Hurwitz equivalences of algebraic relations. Hurwitz (equivalence) spaces all derive from the elementary notion of deforming points as in the construction above for $\tilde{U}_{\boldsymbol{z}}$.

4.1.3. *The value of $\tilde{f}_{\boldsymbol{z}}$.* Since $\varphi_{\boldsymbol{z}} : \tilde{U}_{\boldsymbol{z}} \to U_{\boldsymbol{z}}$ is a covering space, $\varphi_{\boldsymbol{z}}^{-1}(U_{z_0})$ has countably many connected components $\{U_i\}_{i=1}^{\infty}$, each homeomorphic to U_{z_0} by restriction of $\varphi_{\boldsymbol{z}}$. Let $\varphi_1 : U_1 \to U_0$ be this one-one restriction. Then, (4.1d) produces the function

$$\tilde{f}_{\boldsymbol{z}} = h \circ \varphi_1^{-1} : U_0 \to \mathbb{C}. \tag{4.3}$$

This *one* function distinguishes *homotopy classes* of paths on $U_{\boldsymbol{z}}$ by analytic continuation. It separates homotopy classes of paths (based at z_0) by its values at end points of analytic continuations. Since $\tilde{U}_{\boldsymbol{z}}$ is simply connected and in $\mathbb{C}$, Riemann's mapping theorem says it is analytically isomorphic to a disk (or to $\mathbb{C}$, if $r = 1$ or 2) for each $\boldsymbol{z}$.

4.2. Uniformizing from above versus below. Thus, $\tilde{U}_{\boldsymbol{z}}$ is a domain for parametrizing X_f for all $f \in \mathcal{E}(U_{\boldsymbol{z}}, z_0)$, as $\boldsymbol{z}$ varies. This complements how we use Riemann's Existence Theorem.

4.2.1. *Shortcomings of* $h \circ \varphi^{-1}$. The universal covering space helps organize functions and differential forms. Still, algebraists find it hides phenomena close to their interests. For example, $h \circ \varphi_1^{-1}$ is neither algebraic nor known: Its values at algebraic points of $U_{\boldsymbol{z}}$ are rarely algebraic. Though based on $\lambda(\tau)$ in §4.1.2, it changes with $\boldsymbol{z}$. Yet, it provides no explicit equations for algebraic functions.

Even proving a cover from Riemann's Existence Theorem is algebraic still goes through a hard proof that we now separate from other, more algebraic, observations. Suppose $\varphi : X \to \mathbb{P}^1_z$ is a cover. Let $\varphi_w : X \to \mathbb{P}^1_w$ be any function separating all points on the fiber X_{z_0} over z_0. Then, $X \to \mathbb{P}^1_z \times \mathbb{P}^1_w$ by $x \mapsto (\varphi(x), \varphi_w(x))$ has closed image birational to X in the algebraic variety $\mathbb{P}^1_z \times \mathbb{P}^1_w$. Apply Chow's Lemma (Chap. 4) to get that X is algebraic.

Classical construction of φ_w relies on a uniformization $\mathbb{H} \to U_{\boldsymbol{z}}$ presenting $U_{\boldsymbol{z}}$ as a quotient $\mathbb{H}/H$, H a subgroup of $\mathrm{PSL}_2(\mathbb{R})$. One must find nontrivial H invariant functions on $\mathbb{H}$ [**K72**, Chap. III]. Variants are in [**Spr57**, Chap. 6-10] and [**Vö96**, Chap. 5]. We rely on the treatment from the last of these references — especially well adapted to Riemann's Existence Theorem. How to find φ_w (or some related differential form) algebraically appears in many of our examples.

4.2.2. *Virtues of* $h \circ \varphi^{-1}$. The phrase "abelian theory" means here Riemann's unified generalization of Abel's results. This includes describing functions, abelian covers and the results of integration of differentials on a Riemann surface. It includes Riemann's extension of Cauchy's integral theorem to open Riemann surfaces. We discuss it, and our reason for including a *nilpotent theory* below. There is no denying the value of $h \circ \varphi^{-1}$.

(4.4a) It organizes tool the abelian and nilpotent theory.
(4.4b) It coordinates analyzing real points on moduli spaces of curves.
(4.4c) It is suspiciously close to being algebraic, producing an algebraic object (a flat $\mathbb{P}^1$-bundle) capturing its uniformizing properties.

4.2.3. *The Existence Theorem and classical uniformization meet.* Each item in (4.4) has Existence Theorem and $\tilde{U}_{\boldsymbol{z}}$ aspects: *Uniformization from below versus above.* The literature neglects the former, though it is constructive and practical. The latter has had elegant developments.

Both work best as tools for analyzing properties of families (moduli spaces) of curves. They give enhancements when the moduli spaces themselves fit in natural sequences. The abelian theory gave the first such natural sequences. This shows in modular curve sequences (§8.3, Chap. 5).

4.2.4. *Illustrating with modular curves.* When the parameter r (cardinality of $\boldsymbol{z}$) is 4, the comparison between Modular Towers and modular curves is direct. For example, these properties hold for Modular Towers when $r = 4$.

- Their levels are curves.
- They include modular curve towers and come with an essential prime p: Its powers correspond to Modular Tower levels.
- They lie over the classical j-line and have useful cusps over $j = \infty$.
- All levels are moduli spaces, with variants corresponding to structures going with modular curve notation $X_0(p^{k+1})$, $X_1(p^{k+1})$ and $X(p^{k+1})$.

Any finite group G and prime p dividing $|G|$ produces many Modular Towers; many more than there are modular curve towers. The name Modular Tower comes from this comparison and the group (*modular representation*) theory that appears in their analysis.

An elementary comparison occurs in analyzing *real* points on a Modular Tower. Through Riemann's Existence Theorem this gives the essential data about *cusps*. From that come their geometric properties (Chap. 5), including genuses of their components. This is especially interesting when the finite group G producing the Modular Tower is simple and the prime p is 2. We now discuss the Existence Theorem, then the abelian theory.

5. $\mathcal{E}(U_{\boldsymbol{z}}, z_0)^{\mathrm{alg}}$ and data from groups

Riemann's Existence Theorem (Chap. 4) compactifies $\varphi_f : X_f \to U_{\boldsymbol{z}}$ to a ramified cover of Riemann surfaces $\bar{\varphi}_f : \bar{X}_f \to \mathbb{P}^1_z$. It then turns the process around by using special generators of the fundamental group $\pi_1(U_{\boldsymbol{z}}, z_0)$ of $U_{\boldsymbol{z}}$. From these it produces all elements of $\mathcal{E}(U_{\boldsymbol{z}}, z_0)^{\mathrm{alg}}$.

5.1. Identifying a fundamental group requires generators. Suppose G is a finite transitive subgroup of S_n. A surjective homomorphism $\psi : \pi_1(U_{\boldsymbol{z}}, z_0) \to G$ canonically produces a cover $X_\psi \to U_{\boldsymbol{z}}$ from homotopy classes of paths. We don't need generators of $\pi_1(U_{\boldsymbol{z}})$ to define these covers Chap. 3. They, however, handily *list* all such homomorphisms and therefore all such covers. Convenient listing of covers allows explicitly computing properties of Hurwitz spaces (Chap. 5).

The collections of r paths (based at z_0) we call *classical generators* of $\pi_1(U_{\boldsymbol{z}}, z_0)$ appear in Chap. 3. Points in $\boldsymbol{z}$ produce conjugacy classes $\mathbf{C}_{\boldsymbol{z}}$ in $\pi_1(U_{\boldsymbol{z}}, z_0)$. Classical generators are homotopy classes of paths respectively representing these conjugacy classes. Choose representing paths that pair wise meet only at their beginning and end point z_0. Label one as $\bar{g}_1$. Label the others from their having a clockwise order in leaving the point z_0. These r paths $\bar{g}_1, \dots, \bar{g}_r$ now satisfy

(5.1) $\bar{g}_1\bar{g}_2 \cdots \bar{g}_r = 1$: The *product-one condition*.

Invariants of Hurwitz space components appear from (5.1) (§10.1 illustrates). Classical generators — satisfying these conditions — automatically generate the fundamental group (Chap. 3). Solving for $\bar{g}_r$ presents the fundamental group as a free group on $r - 1$ generators. Yet, that violates the product-one symmetry. So, that free group presentation appears only in stray computations.

This part of Riemann's theory works *very* well. It successfully applies to many problems. These require some finite group theory. It is the center of the first third of the book. Polynomial equations describe algebraic curves. This is what gives structure allowing fields of definitions and interpreting rational points. The

Riemann's Existence Theorem approach, however, emphasizes effective group theory over manipulating explicit equations. Exercises and examples illustrate this (Chap. 3, Chap. 4, Chap. 9).

5.2. Changing classical generators. There is no canonical set of classical generators for $\pi_1(U_{\boldsymbol{z}}, z_0)$. The necessary variation of this choice produces the *braid* and *mapping class* groups (Chap. 5). This complication enriches mathematics. Still, it requires explanation.

The second third of the book organizes collections of elements from $\mathcal{E}(U_{\boldsymbol{z}}, z_0)^{\mathrm{alg}}$. This allows $\boldsymbol{z}$ to vary. Sets of r (unordered) distinct points on $\mathbb{P}^1_z$ have a topology and analytic structure extending that of $\mathbb{P}^1_z$. This set is $\mathbb{P}^r \setminus D_r = U_r$: Projective r-space minus the discriminant locus (Chap. 5). Think of U_r as monic polynomials of degree either r or $r-1$ with distinct roots. Or, consider it the quotient of $(\mathbb{P}^1_z)^r \setminus \Delta_r$ by permutation action of S_r, the symmetric group of degree r, on ordered r-tuples of points. Here Δ_r is r-tuples with distinct coordinates.

The fundamental group of U_r is the degree r *Hurwitz monodromy group* H_r (Chap. 5), an *Artin braid group* quotient. A permutation representation of H_r produces the space of deformations of X_f. These are *unreduced Hurwitz spaces*.

A given function $f \in \mathcal{E}(U_{\boldsymbol{z}}, z_0)^{\mathrm{alg}}$ deforms in many ways as $\boldsymbol{z}$ varies. Local deformation, however, of $\bar{\varphi}_f : \bar{X}_f \to \mathbb{P}^1_z$ is unique along any path. This allows analyzing parameters for these moduli spaces. Yet, it leads further from explicit equations. To paraphrase Joni Mitchell's "Both Sides Now" (from the 60's): *Something's lost and something's gained in putting equations away*. Explicit functions, however, return with the *abelian* and *nilpotent* theory.

5.3. Moving $\boldsymbol{z}$, even with z_0 fixed, forces changing generators. Picture: z_1 and z_2 follow semicircles, producing

$$Q_1 : (\bar{g}_1, \dots, \bar{g}_r) \mapsto (\bar{g}_1\bar{g}_2\bar{g}_1^{-1}, \bar{g}_1, \dots, \bar{g}_r). \tag{5.2}$$

Replacing 1 by $i \le r-1$ gives the full generating collection $Q_1, \dots, Q_{r-1}$ of the *Hurwitz monodromy group* H_r (Chap. 5). The H_r action from (5.2) on classical generators is the technical tool for describing families of covers.

Let G be a fixed finite group. Assume these further ingredients.

(5.3a) $\boldsymbol{z}'$ is a specific point of U_r.
(5.3b) $\psi_{\boldsymbol{z}'} : \pi_1(U_{\boldsymbol{z}'}, z_0) \to G$ is a specific surjective homomorphism to G using classical generators $\bar{g}_1, \dots, \bar{g}_r$ (§5.1).
(5.3c) $T : G \to S_n$, n an integer, is a faithful permutation representation.

Then, $\psi_{\boldsymbol{z}'}$ gives a finite (ramified) cover $\varphi_{G,T,\boldsymbol{z}'} = \varphi_{\boldsymbol{z}'} : X_{\boldsymbol{z}'} \to \mathbb{P}^1_z$ of Riemann surfaces of degree n. The images of $\bar{g}_1, \dots, \bar{g}_r$ give generators $g_1, \dots, g_r$ of $G \le S_n$, with an associated set of r conjugacy classes $\mathbf{C}$ in G. Riemann's Existence Theorem labels covers by $g_1, \dots, g_r$ (branch cycles). It gives $\varphi_{G,T,\boldsymbol{z}'}$ as an equivalence relation on homotopy classes of paths based at z_0. Suppose $\boldsymbol{z}'$ moves to nearby $\boldsymbol{z}''$, with $z_0 \in \mathbb{P}^1_z$ and paths representing $\bar{g}_1, \dots, \bar{g}_r$ fixed. Then, there is a unique isomorphism of $\pi_1(U_{\boldsymbol{z}'}, z_0)$ and $\pi_1(U_{\boldsymbol{z}''}, z_0)$ commuting with their maps to G.

An automorphism α of $\pi_1(U_{\boldsymbol{z}}, z_0)$ sends generators to new generators, changing $\psi_{\boldsymbol{z}}$ to $\psi_{\boldsymbol{z}} \circ \alpha$. Inner automorphisms of $\pi_1(U_{\boldsymbol{z}}, z_0)$, however, produce covers equivalent to the old cover. It is moduli of covers we use; equivalence two homomorphisms if they differ by an inner automorphism. Further, only automorphisms from the Hurwitz monodromy group H_r send classical generators to classical generators (possibly changing the intrinsic order of the paths). Such automorphisms arise from deforming the pair $(\boldsymbol{z}, z_0)$ along closed paths in U_r. They preserve the conjugacy classes of classical generators. So, $\mathbf{C}$, the conjugacy class set in G, is an H_r invariant of any given homomorphism ψ.

5.4. The moduli spaces appear. The *Nielsen class* of $(G, \mathbf{C})$ (Chap. 5) consists of r-tuples $(g_1, \dots, g_r)$ satisfying the product-one condition attached to $(G, \mathbf{C})$. The Existence Theorem uses classical generators of $\pi_1(U_{\boldsymbol{z}}, z_0)$ to produce equivalence class of covers.

5.4.1. *Writing equations in* $\boldsymbol{z}$. The Nielsen class $\mathrm{Ni}(G, \mathbf{C})$ has entries in a set of conjugacy classes $\mathbf{C}$ in G, independent of the braid action. Thus, H_r acts on elements of $\mathrm{Ni}(G, \mathbf{C})$ (similar to (5.2)). An aside: We need to quotient by conjugation from G. Here is how to think of this action.

Let $\varphi_0 : X_0 \to \mathbb{P}^1_z$ be a cover from the Existence Theorem using $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C})$. Take the branch points to be $\boldsymbol{z}_0$. What if someone asks for *explicit* equations for this cover? That could mean either:

(5.4a) equations just for φ_0; or
(5.4b) equations for $\varphi_{\boldsymbol{z}} : X_{\boldsymbol{z}} \to \mathbb{P}^1_z$, with branch points $\boldsymbol{z}$, valid for $\boldsymbol{z}$ near $\boldsymbol{z}_0$ (where it specializes to φ_0).

Don't those seem like asking too little? Why concentrate on one set of branch points $\boldsymbol{z}_0$, or even on a neighborhood of $\boldsymbol{z}_0$? You'd want $\varphi_{\boldsymbol{z}}$ valid for *all* $\boldsymbol{z} \in U_r$. If, however, this were possible, then analytically continuing $\varphi_{\boldsymbol{z}}$ around any closed path $\mathcal{P}$ in U_r would return you to φ_0.

The homotopy class of $\mathcal{P}$ is an element $Q_{\mathcal{P}}$ of H_r. Further, Chap. 5 shows the cover at the end of $\mathcal{P}$ has a branch cycle description $(\boldsymbol{g})Q_{\mathcal{P}}$. (Compute that with the starting classical generators of $\pi_1(U_{\boldsymbol{z}_0})$.) So, finding equations for $\varphi_{\boldsymbol{z}}$ valid for all $\boldsymbol{z}$ requires $(\boldsymbol{g})Q_{\mathcal{P}}$ be $\boldsymbol{g}$ (modulo conjugation by G or closely related). This you can check: Is $(\boldsymbol{g})Q$ essentially $\boldsymbol{g}$ for all $Q \in H_r$. Example: Consider

$$\boldsymbol{g} = ((1\,2\,3), (3\,2\,1), (1\,4\,5), (1\,5\,4)) \in \mathrm{Ni}(A_5, \mathbf{C}_{3^4})$$

(§10.1). Then $(\boldsymbol{g})Q_2 = ((1, 2\,3), (2\,4\,5), (3\,2\,1), (1\,5\,4))$. This is not conjugate to $\boldsymbol{g}$ even under S_5. So, as typical when $r \geq 4$, there are no such equations for $\varphi_{\boldsymbol{z}}$.

5.4.2. *Analytic continuations of* $\varphi_{\boldsymbol{z}_0}$. Nontrivial H_4 action means coefficients of equations for $\varphi_{\boldsymbol{z}}$ act as coordinates for a nontrivial cover of U_r. What cover?

It comes from the action of H_r, the fundamental group of U_r, on $\mathrm{Ni}(G, \mathbf{C})$ produced by covering space theory. Notation for this cover depends on the equivalence used for elements of the Nielsen class (as in (5.5)). Typical notation is $\mathcal{H}(G, \mathbf{C}, T)$. Each point of $\mathcal{H}(G, \mathbf{C}, T)$ corresponds to an equivalence class of covers: A point over $\boldsymbol{z} \in U_r$ is an element from $\mathrm{Ni}(G, \mathbf{C})$ attached to $\boldsymbol{z}$. Then, $\mathcal{H}(G, \mathbf{C}, T)$ itself covers the space U_r of distinct unordered r-tuples of points from $\mathbb{P}^1_z$ (Chap. 5).

Various equivalences among covers produce different versions of this space. Two predominate in early applications. Denote the subgroup of S_n normalizing G and permuting the conjugacy classes in $\mathbf{C}$ by $N_{S_n}(G, \mathbf{C})$.

(5.5a) $\mathcal{H}(G, \mathbf{C})^{\mathrm{in}}$: T is the *regular representation* and the Galois cover comes with a fixed isomorphism between its Galois group and G (inner spaces).
(5.5b) $\mathcal{H}(G, \mathbf{C}, T)^{\mathrm{abs}}$: T any faithful representation, with r-tuples equivalenced by $N_{S_n}(G, \mathbf{C})$ conjugation (absolute spaces).

See the main example of this chapter at §10.1.

The combinatorial groups in Chap. 5 have long histories: the *Artin braid group*, the *Hurwitz monodromy group* and the *mapping class group*. As in Chap. 4, we give formal proofs. Pictures appear only to convey conceptual symbolic data. Absolute spaces are the work horses in applications (Chap. 9). Inner spaces, however, directly connect the Inverse Galois Problem to generalizations of *modular curves* (§7.4).

5.4.3. *Statics and dynamics of a cover.* In the game of mentally *writing equations* for a cover, why would one cover be more significant than another? Many historical applications, such as the Inverse Galois Problem, consider a cover with equations over $\mathbb{Q}$ as most significant. For example, many arithmetic problems gain solutions if one can produce a cover with a particular monodromy group over $\mathbb{Q}(z)$ or over $\mathbb{Q}$. Such a cover provides solutions to related problems over another field by extending its equations to that field.

We picture such a cover $\varphi_0 : X_0 \to \mathbb{P}^1_z$ as being at the crossroads of a network of roads. The real points on $\mathcal{H}(G, \mathbf{C})^{\mathrm{in}}$ would go through the point corresponding to φ_0, as would all p-adic points for every prime p. Concentrate on a real point, $\boldsymbol{p}_0 \in \mathcal{H}(G, \mathbf{C})^{\mathrm{in}}$ corresponding to a cover φ_0 over $\mathbb{R}$. To get a measure of the potential energy of this point we measure its distance from boundary points on $\mathcal{H}(G, \mathbf{C})^{\mathrm{in}}$. Developing such a measure, depends on measuring something that goes to 0 as we deform φ_0 along a real component going to a boundary point, and the measuring coordinates must be canonical functions of the coordinates of the point $\boldsymbol{p}$ as it moves from $\boldsymbol{p}_0$ to the chosen boundary point.

The theory of abelian covers on $\bar{X}_0$ gives classical functions that we can use for making such measurements. As easily this could be on $\bar{X}_0$ minus a finite number of points, as with $U_{\boldsymbol{z}}$. Still, in the compact case, functions in $\mathcal{E}(\bar{X}_0)$ with finitely many analytic continuations are algebraic.

6. Abelian theory on $\bar{X}_f$ and integration

Let $f \in \mathcal{E}(U_{\boldsymbol{z}}, z_0)$. Suppose analytic continuations f_γ of $f(z)$ have this property.

(6.1) $f_\gamma(z) = f(z)$ for each closed path γ based at z_0.

Rather than extensible, Chap. 2 calls f *extendible*. Denote extendible elements of $\mathcal{E}(U_{\boldsymbol{z}})$ by $\mathcal{E}(U_{\boldsymbol{z}})^{\mathrm{ext}}$.

Consider $f \in \mathcal{E}(U_{\boldsymbol{z}})^{\mathrm{ext}}$. Cauchy's Theorem in $U_{\boldsymbol{z}}$ shows precisely the nature of integrals $f(z)\,dz$ around certain closed paths. Since these are integrals, assume without further mention the paths miss any poles of $f\,dz$. Let $\boldsymbol{z}_{f,\infty}$ be the set of these poles. Assume for simplicity it is a finite set (appropriate for algebraic functions) which may include ∞: $z^n\,dz$ has a pole of order $n+2$ at ∞.

The definition of integral makes sense. Let $F(z)$ be an antiderivative of f in a neighborhood of z_0. For any (simplicial) path $\gamma : [0,1] \to U_{\boldsymbol{z}}$, take the indefinite integral to the end point of γ to be F_γ (Chap. 2).

Cauchy's Residue Theorem: Let γ be a closed path *homologous* to 0 in $U_{\boldsymbol{z}}$. Compute $\int_\gamma f(z)\,dz$ from the *winding number* of γ and residue of f at each $z' \in \boldsymbol{z}_{f,\infty}$ (Chap. 2). Winding numbers are values of integrals $\int_\gamma \omega$ where ω is a differential form —*logarithmic*, or of *3rd kind*—taking the shape $\frac{1}{2\pi i}\frac{dz}{z-z'} = \omega_{z'}$ with $z' \in \boldsymbol{z}_{f,\infty}$. Also, winding numbers appear in the definition of being homologous to 0: The path has winding number 0 about each point in $\boldsymbol{z}$.

6.1. Changes of significance for algebraic f. Here is a paraphrase of Cauchy. Suppose (6.1) holds. Then, poles of f and the map $\gamma \to (\int_\gamma \omega_{z_1}, \dots, \int_\gamma \omega_{z_r})$ determine $\int_\gamma f(z)\,dz$ when γ is closed and homologous to 0.

Suppose, however, f is both algebraic and extendible. That means it is a rational function on $\mathbb{P}^1_z$. Then, there is no significant difference between the points in $\boldsymbol{z}$ and those in $\boldsymbol{z}_{f,\infty}$. By combining them both in the set $\boldsymbol{z}_{f,\infty}$ this allows dropping the homologous to 0 condition. We may consider integrals around any closed path.

Riemann made a more abstract change. Antiderivatives of ω_{z_i} are (up to an additive constant) *branches of* $\log(z - z_i)$, $i = 1, \dots, r$. Recognizing Cauchy's Theorem as a statement entirely about integrals of meromorphic differentials (not of functions) immediately allowed generalizations. Here is what the abelian theory does for $U_{\boldsymbol{z}}$ (Chap. 2).

(6.2a) It gives explicit differentials providing details on integrals of any meromorphic differentials around any closed paths.

(6.2b) It describes elements of $\mathcal{E}(U_{\boldsymbol{z}})^{\text{alg}}$ with associated group abelian.

Chap. 2 does (6.2b) by corresponding such functions to an r-tuple in $(\mathbb{Q}/\mathbb{Z})^r$ with entries summing to 0.

6.2. Extending Cauchy's Theorem to $\bar{X}_f$. Riemann extended Cauchy's Theorem to $f \in \mathcal{E}(U_{\boldsymbol{z}}, z_0)^{\text{alg}}$ not satisfying (6.1). Compatible with (6.2), he extended it to meromorphic differentials on $\bar{X}_f$. This emphasis on differentials over functions didn't throw functions out. They were still there through the definition of df, the *differential* and df/f, the *logarithmic* differential of f (Chap. 3).

The serious step was analyzing the space of *holomorphic* (or *first kind*) differentials on $\bar{X}_f$ (Chap. 3, Chap. 4): differentials with no poles anywhere. Standard notation for this $g = g(\bar{X}_f)$ dimensional space over $\mathbb{C}$ is $\Gamma(\bar{X}_f, \Omega^1)$: Global sections of the sheaf of holomorphic differentials on $\bar{X}_f$. The genus g of $\bar{X}_f$ now attaches to $f = f(z)$, toward pinning its place among algebraic functions of z.

Guidance came from the Abel-Jacobi-Legendre differentials like $\frac{dz}{\sqrt{z^3+az+b}}$ from §3.2. Just giving the dimension of $\Gamma(\bar{X}_f, \Omega^1)$ called for a more abstract approach. Riemann needed a full basis to solve the *Jacobi-Inversion problem*. Relying on coordinates from $\mathbb{P}^1_z$ was a confining kludge.

With points removed from $\bar{X}_f$, add further *logarithmic* (or *3rd kind*) differentials. In $U_{\boldsymbol{z}}$, the (vector-)space of logarithmic differentials has a preferred basis by reference to classical generators of $\pi_1(U_{\boldsymbol{z}}, z_0)$ (§5.1).

Extending this to $\bar{X}_f$ still leaves an infinite set of choices for a $\Gamma(\bar{X}_f, \Omega^1)$ basis, with all choices related by the action of a group: The *symplectic group* $\mathrm{Sp}_{2g}(\mathbb{Z})$. Different basis choices correspond to different choices of $2g$ closed paths whose homology classes determine integration of any meromorphic differential around closed paths. This is an imprecise statement of Cauchy's Theorem on $\bar{X}_f$.

As with $U_{\boldsymbol{z}}$, there is a notion of classical generators. With $U_{\boldsymbol{z}}$ the paths were nonintersecting, except at the base point. On $\bar{X}_f$ classical generators signifies normalizing information about the intersection of these $2g$ paths. Given classical generators for $U_{\boldsymbol{z}}$ there is a process for producing classical generators on X_f. This provides explicit actions of appropriate subgroups of H_r on the homology of $\bar{X}_f$. Suppose $\bar{X}_f$ appears in the moduli space of curves of genus g. Then, the whole action may well give $\mathrm{Sp}_{2g}(\mathbb{Z})$. On Hurwitz spaces, however, the data is more refined. The significant group action may be much smaller.

6.3. Jacobians and generalizing Abel's Theorem. Suppose $\omega_1, \dots, \omega_g$ is a specific $\Gamma(\bar{X}_f, \Omega^1)$ basis. As with $U_{\boldsymbol{z}}$, Cauchy's Theorem on $\bar{X}_f$ builds from this data an abelian group. In this case it is a compact complex torus $J(\bar{X}_f)$, the *Jacobian* of $\bar{X}_f$. Follow Mumford's view [**Mum76**, p. 58-67]. Consider the space of locally defined holomorphic tangent vectors for $\bar{X}_f$ as dual to locally defined holomorphic differential forms (Chap. 3). Then, paths are dual to holomorphic differentials (by integration). The problem is to interpret a dual to *global* holomorphic differentials. This generalizes the Abel-Jacobi approach to Cauchy's Theorem and it produces an abelian covering theory.

Let h be any meromorphic function on $\bar{X}_f$ (Chap. 3) of degree u. Then, $h : \bar{X}_f \to \mathbb{P}^1_z$ has zeros $x^0_1, \dots, x^0_u$ and poles $x^\infty_1, \dots, x^\infty_u$. A mysterious identification then occurs: $\bar{X}_f$ appears in $J(\bar{X}_f)$. So, each zero x^0_i and pole of h produces a point in $J(\bar{X}_f)$. List these as $\boldsymbol{p}_{x^0_i}, \boldsymbol{p}_{x^\infty_i}$, $i = 1, \dots, u$.

6.3.1. *Logarithmic differentials.* Yet, finding the $\boldsymbol{p}$s doesn't require giving h. It only needs points $x^0_1, \dots, x^0_u$ and $x^\infty_1, \dots, x^\infty_u$ on $\bar{X}_f$ viewed as inside $J(\bar{X}_f)$. Define $[D_{\boldsymbol{x}}] = [D(\boldsymbol{p}_{x^0_i}, \boldsymbol{p}_{x^\infty_i}, i = 1, \dots, u)]$ as the sum of all the $\boldsymbol{p}_{x^0_i}$s minus all the $\boldsymbol{p}_{x^\infty_i}$s on $J(\bar{X}_f)$. To say $[D]$ is zero means it is the origin of $J(\bar{X}_f)$. Abel's Theorem (generalized) says existence of h with these zeros and poles characterizes exactly when $[D]$ is zero.

If h exists, consider the logarithmic derivative dh/h. This is a meromorphic differential of 3rd kind with pure imaginary periods. Even if h doesn't exist, given the divisor $D_{\boldsymbol{x}}$ above, the following holds.

(6.3) There is a unique differential $\omega_{\boldsymbol{x}}$ with residue divisor $D_{\boldsymbol{x}}$ having pure imaginary periods (Chap. 4).

6.3.2. *Coordinates from holomorphic differentials.* Suppose $[D]$ is not zero, but $m[D]$ is zero on $J(\bar{X}_f)$ for some integer m. Then, repeating all the zeros and poles m times produces a function h on $\bar{X}_f$. The mth root of h defines an abelian unramified cover $Y \to \bar{X}_f$. So, the abelian theory of $\bar{X}_f$ appears from this version of Cauchy's Theorem. Riemann produced $\theta = \theta_{\bar{X}_f}$ functions to provide global coordinates (uniformization) for this construction. They are functions on $\mathbb{C}^g$ (§6.5).

Many mathematical items on $\bar{X}_f$ appear constructively from this. This includes functions and meromorphic differentials (with particular zeros and poles). This was a central goal in generalizing Abel's Theorem: To provide Abel(-Jacobian) constructions for a general Riemann surface. For the function h it has this look:

$$h(x) = \prod_{i=1}^{u} \theta(\int_{x^0_i}^{x} \boldsymbol{\omega}) / \prod_{i=1}^{u} \theta(\int_{x^\infty_i}^{x} \boldsymbol{\omega}). \tag{6.4}$$

In θ you see g coordinates; the ith entry is $\int_{x^0_i}^{x} \omega_i$. Each holds an integral over one basis element from $\boldsymbol{\omega}$. Integration paths join respective points on $\bar{X}_f$'s universal covering space. The integrals make sense up to integration around closed paths. So, they define a point in $J(\bar{X}_f)$.

Even if h doesn't exist, the logarithmic differential of (6.4) does. It gives the third kind differential from (6.3). Here you see the differential equation defining θ functions. In the expression for h, replace $\int_{x^\infty_i}^{x} \boldsymbol{\omega}$ by a vector $\boldsymbol{w}$ in the universal covering space of the Jacobian. Form the logarithmic differential of it: $d\theta(\boldsymbol{w})/\theta(\boldsymbol{w})$. Translations by periods will change it by addition of a constant. With ∇ the gradient in $\boldsymbol{w}$, $\nabla(\nabla\theta(\boldsymbol{w})/\theta(\boldsymbol{w}))$ is invariant under the lattice of periods.

Thus, $J(\bar{X}_f)$ provides transparent coordinates for differentials, and their periods, through a mysterious embedding of $\bar{X}_f$ in it. Then, objects from the abelian structure on $J(\bar{X}_f)$ restrict to $\bar{X}_f$ (§10.6). To use, however, Riemann's theory an algebraist faces two major complications.

6.4. Complication 1: The role of f. Suppose $\bar{X}_f$ varies in the Hurwitz space $\mathcal{H}(G, \mathbf{C})$ attached to $(G, \mathbf{C})$. It moves along a path in U_r with the coordinates for $\boldsymbol{z}$. Is Riemann's theory sufficiently algebraic to express the changes using equations with coefficients in the the point of $\mathcal{H}(G, \mathbf{C})$ corresponding to $\bar{X}_f$. Answer: It is algebraic in many ways, though rarely will coordinates from $\mathcal{H}(G, \mathbf{C})$ support all the identifications. Here is why.

6.4.1. *The Picard components.* There are three geometric ingredients in Riemann's theory: $J(\bar{X}_f)$, $\bar{X}_f$ and the zero (Θ) divisor of the function $\theta = \theta_{\bar{X}_f}$ (§6.5). The first identifies with divisor classes $\mathrm{Pic}^0(\bar{X}_f) = \mathrm{Pic}^0_f$ of degree 0 on $\bar{X}_f$ (Chap. 4). The second embeds naturally (algebraically) in Pic^1_f, divisor classes of degree 1 on $\bar{X}_f$. Then, Θ_f is the dimension $g-1$ variety of positive divisor classes in Pic^{g-1}_f.

Further, Pic^g_f interprets the Riemann-Roch Theorem and the *Jacobi Inversion Problem* geometrically (Chap. 4). It takes its group structure from adding two positive divisors of degree g together modulo linear equivalence. Weil used this for an algebraic construction of Pic^0_f years after his thesis. His principle: The nearly well defined addition on positive divisors produced a unique complete algebraic group on the homogeneous space of divisor classes. Therefore Pic^0_f is almost the symmetric product of $\bar{X}_f$ taken g times. Riemann's theory was an inspiration to Weil's 1928 thesis (§10.6). Still, Weil was not certain until later that Pic^0_f and $\bar{X}_f$ have the same field of definition. This reminds that what now looks obvious is the result of many mathematical stories.

6.4.2. *Half-canonical classes.* All Picard components Pic^k_f are pair wise analytically isomorphic. Yet, finding an isomorphism analytic in the Hurwitz space coordinates may require moving to a cover of the Hurwitz space (§10.6).

Applying Riemann's theory directly requires having $\bar{X}_f$ and the Θ_f divisor on Pic^0_f. For example, suppose there is an analytic assignment of a divisor class of degree $g-1$ on each curve $\bar{X}_f$ in the Hurwitz family. Then, translation of Θ_f by this divisor class puts it in Pic^0_f. Here it would be available to construct the θ function. Convenient for this might be a *half-canonical* class: two times gives divisors for meromorphic differentials (Chap. 4).

Places marked by $\oplus$ in the Constellation Table of §10.1 signify *inner* Hurwitz spaces components that support such an assignment of half-canonical classes. This example shows how the *Schur multiplier* of a finite group appears in describing connected components of Hurwitz spaces (§10.2). It is a taste of the nilpotent theory arising in Modular Towers (§8.3). One last subtlety, however, occurs. Only some half-canonical translates work to give a formula like (6.4). They must be *odd*; the linear system has odd dimension (Chap. 4). This includes that $\theta(\mathbf{0}) = 0$: When you plug in $x = x_i^0$ you expect $h(x_i^0) = 0$. For the correct multiplicity of a zero on the right of (6.4), the gradient of the θ at $\mathbf{0}$ also must be nonzero. Such half-canonical classes always exist (Chap. 4).

Half-canonical classes, however, attached to $\oplus$ components in §10.1 are *even*. Sometimes they provide nontrivial *θ-nulls* along the moduli space.

Riemann was even less algebraic in relating $\bar{X}_f$ and its Jacobian. He used coordinates from $\tilde{X}_f$, its universal covering space, to uniformize $\bar{X}_f$.

6.5. Complication 2: $\tilde{X}_f$ and nilpotent covers. The analytic isomorphism class of $\tilde{X}_f$ depends on the genus g of $\bar{X}_f$. If $g = 0$ it is the sphere, if $g = 1$ it is $\mathbb{C}$ and it is *the upper half plane* $\mathbb{H}$ (or disk) if $g \geq 2$. As with $U_{\boldsymbol{z}}$ (§4.1), suppose we accept that $\tilde{X}_f$ is an analytic subspace of the Riemann sphere. Then, this comes from the Riemann mapping theorem. Still, it is not the uniformizing space we would expect. That would be $\tilde{X}_f^{\mathrm{ab}}$, the quotient of $\tilde{X}_f$ by the subgroup of $\pi_1(\bar{X}_f)$ generated by commutators. This is the maximal quotient of $\tilde{X}_f$ that is an abelian cover of $\bar{X}_f$.

6.5.1. *Abelian Frattini covers.* Mathematics rarely looks directly at $\tilde{X}_f^{\mathrm{ab}}$. It embeds in the universal covering space $\mathbb{C}^g$ of $J(\bar{X}_f)$. It is on $\mathbb{C}_g$ that $\theta_{\bar{X}_f}$ lives with its zeros, the Θ divisor, meeting $\tilde{X}_f^{\mathrm{ab}}$ transversally. Periods of differentials on $\bar{X}_f$ translate $\tilde{X}_f^{\mathrm{ab}}$ into itself. Yet, it is sufficiently complicated there seems to be no device for picturing it.

There are two models for picturing this. A standard picture shows the complex structure on a complex torus (like the Jacobian). It is of a fundamental domain (parallelpiped) in $\mathbb{C}^g$. Then, $2g$ vectors representing generators of the lattice defining the complex torus (Chap. 3) give the sides of the parallelpiped. Inside this sits the pullback of $\bar{X}_f$. The geometry for this picture uses geodesics (straight lines) from the flat (Euclidean) metric defining distances on the complex torus.

Assume the genus of $\bar{X}_f$ is at least 2. Then, the universal covering $\tilde{X}_f$ of $\bar{X}_f$ is the upper half plane $\tilde{X}_f$. A standard picture for $\bar{X}_f$ appears by grace of this having the structure of a negatively curved space. Geodesics here provide a polygonal outline of a set representing points of $\bar{X}_f$ (Chap. 4). Since $\tilde{X}_f \to \tilde{X}_f^{\mathrm{ab}}$ is unramified, $\tilde{X}_f^{\mathrm{ab}}$ inherits a metric tensor with constant negative curvature. Yet, it sits snuggly in a flat space. Every finite abelian (unramified) cover Y of $\bar{X}_f$ is a quotient of $\tilde{X}_f^{\mathrm{ab}}$; it is a minimal cover of $\bar{X}_f$ with that property. Recall: We started with $\varphi_f : \bar{X}_f \to \mathbb{P}^1_z$. Assume it is a Galois cover, with group G.

Let $\mathcal{G}_f$ denote the abelian covers $\psi : Y \to \bar{X}_f$ with $\psi_f = \psi \circ \varphi_f : Y \to \mathbb{P}^1_z$ also Galois. Call ψ_f a (relatively abelian) *Frattini cover* if the following holds. For any sequence $Y \to W \to \mathbb{P}^1_z$, of covers with $W \neq \mathbb{P}^1_z$, there is always a proper cover of $\mathbb{P}^1_z$ that $W \to \mathbb{P}^1_z$ and $\bar{X}_f \to \mathbb{P}^1_z$ factor through. A Frattini cover has no differentials and functions that pull back from covers disjoint from φ_f, so its function theory isn't accessible by knowing smaller degree covers. The most mysterious quotients of $\tilde{X}_f^{\mathrm{ab}}$ are these *relatively abelian* Frattini covers.

This Frattini cover notion does not require an abelian cover ψ. Still, a Frattini cover arises always from ψ being a Galois cover with *nilpotent* (a product of its p-Sylows) group.

6.5.2. *No universal nilpotent cover.* Relatively nilpotent Frattini covers produce natural sequences of moduli spaces generalizing sequences of modular curve covers (§8.3). Further, these moduli space sequences interpret many expectations about the regular version of the Inverse Galois Problem (§8). Relatively nilpotent covers and especially relatively Frattini covers bring up a combination of group theory and function theory. This includes many problems around new aspects of the abelian theory using the Frattini property. This book explores aspects of it through these sequences of moduli spaces.

A complete understanding of all nilpotent (versus abelian) covers of $\bar{X}_f$ requires new, recent, ideas. An immediate difficulty is that there is no $\tilde{X}_f^{\mathrm{nil}}$ similar to $\tilde{X}_f^{\mathrm{ab}}$. Equivalently, no nontrivial subgroup of $\Gamma_0 = \pi_1(\bar{X}_f, x_0)$ is in the intersection of all iterates of commutators in this group.

That is, let $\Gamma_k < \pi_1(\bar{X}_f, x_0)$ be elements of form $(g_1(g_2(\dots g_{k-1}, g_k)\dots)$ with $g_1, \dots, g_k \in \Gamma_0$. Only $\{1\}$ is in all the Γ_k s. So, putting structure on the complete collection of algebraic nilpotent covers of $\bar{X}_f$ requires profinite limits. First consider how profinite limits appear in $G_{\mathbb{Q}}$ acting on points of the moduli spaces.

7. Acting with $G_{\mathbb{Q}}$

What changes in replacing $\boldsymbol{z}$ by $\boldsymbol{z}'$, another r-tuple of elements? You might expect the fundamental group of $U_{\boldsymbol{z}}$ to tell nothing about changes. As a group it remains the same. We don't, however, use it as an abstract group. Its generators appear directly in applications. Changing $\boldsymbol{z}$ forces changing generators. Yet, we understand the braiding changes from H_r (§5.2). From elementary principles they give a profinite guide for action of $G_{\mathbb{Q}}$.

7.1. Acting on Laurent series. Suppose $\sigma \in G_{\mathbb{Q}}$ and $z_0 \in \mathbb{Q}$. Assume $f(z) = \sum_{n=N}^{\infty} a_n(z-z_0)^n$ has coefficients in $\bar{\mathbb{Q}}$. Then, σ acts on the a_n s, producing f_σ. The hypothesis, however, of algebraic coefficients won't hold for $\tilde{f}_{\boldsymbol{z}}$ from (4.3).

7.1.1. *Setup for a test Case:* $r = 3$. Suppose z_1, z_2, z_3 are in $\mathbb{Q}$. Change the variable z by an element of $\mathrm{SL}_2(\mathbb{Z})$ to map $\{z_1, z_2, z_3\}$ in some order to $\{0, 1, \infty\}$. Six different permutations $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ do this, depending on the order we choose. Composing $\tilde{U}_{\boldsymbol{z}} \to U_{\boldsymbol{z}}$ with one of these produces $\lambda : \tilde{U}_{\boldsymbol{z}} \to U_{0,1,\infty} = \mathbb{P}^1_\lambda \setminus \{0, 1, \infty\}$. Riemann's uniformization appears from a classical function, $\lambda : \mathbb{H} \to U_{0,1,\infty}$ (Chap. 4).

7.1.2. *Uses for* $\lambda(\tau)$. Periods of an antiderivative of $F(z)$ form an additive subgroup of $\mathbb{C}$ isomorphic to $\mathbb{Z} \times \mathbb{Z}$ (§3.2). In that notation, consider

$$m(z, w) = w^2 - z(z-1)(z-\lambda)$$

with $\lambda \in \mathbb{P}^1_z \setminus \{0, 1, \infty\}$. Choose $\tau \in \mathbb{H}$ so the function λ takes τ to the value λ (appearing in $m(z, w)$). Identify $\mathbb{Z} \times \mathbb{Z}$ with the subgroup H_τ of $\mathbb{C}$ that 1 and τ generate. Other choices of τ give the same lattice H_τ. It only depends on λ. Let $\Gamma(2)$ be the group of integral matrices congruent to the identity matrix modulo 2. Suppose $\lambda(\tau_0) = \lambda_0$. Then, $\tau \mapsto \lambda(\tau)$ has as preimage of λ_0 the set $\Gamma(2)(\tau_0) = \{\alpha(\tau_0) \mid \alpha \in \Gamma(2)\}$: λ uniformizes $\mathbb{H}/\Gamma(2)$.

Picard used λ to show any nonconstant function $f(z)$ meromorphic on $\mathbb{C}$ excludes at most three values. Assume otherwise, and $f(\mathbb{C})$ excludes $0, 1, \infty$. Then the *monodromy theorem* (Chap. 3) analytically continues $\lambda^{-1} \circ f$ to a function $\mathbb{C} \to \mathbb{H}$. The maximum modulus principle prevents existence of nonconstant holomorphic function maps $\mathbb{C}$ into the upper half plane. This contradiction shows f must be constant [**Ahl79**, p. 307].

7.1.3. *Another valuable function.* Ordering the coordinates of $\boldsymbol{z}$ violates some of our goals. The origins of the subject kept that in mind. Use the notation $U_{\lambda:0,1,\infty}$ when the variable for $U_{0,1,\infty}$ is λ. Six elements of $\mathrm{PSL}_2(\mathbb{Z})$, forming a subgroup S, leave stable the set $\{0, 1, \infty\}$. Then, S acts on $U_{\lambda:0,1,\infty}$. The quotient is $\mathbb{P}^1_j \setminus \{\infty\} = U_{j:\infty}$. The composite from $\mathbb{H} \to U_{j:\infty}$ is a Galois cover with group $\mathrm{PSL}_2(\mathbb{Z})$ (Chap. 4). It is ramified (not a topological cover) over fixed points of elements in $\mathrm{SL}_2(\mathbb{Z})$ with eigenvalues 4th or 6th roots of 1. We use $j(\tau)$ to display how Modular Towers of reduced Hurwitz spaces when $r = 4$ (four elements in $\mathbf{C}$) generalize classical modular curves.

7.1.4. *$G_{\mathbb{Q}}$ won't directly act on λ and j.* A theorem of Schneider-Siegel says $\tau(z_0)$ and z_0 are simultaneously algebraic only if τ is the ration of periods for an elliptic curve with complex multiplication. Therefore, even the constant term in the expansion of $\lambda^{-1}(z)$ around z_0 won't often by algebraic. That illustrates the extent previous generations sought to prove properties of $\lambda(\tau)$. Here, however, it shows using $\tilde{f}_{\boldsymbol{z}}$ directly for the action of $G_{\mathbb{Q}}$ won't work.

7.2. Profinite fundamental groups. Suppose $X \to U_{\boldsymbol{z}}$ is a finite (unramified) cover, and $\boldsymbol{z}$ consists of algebraic points. Then, $X = X_f$ where f has the following properties (Chap. 4).

(7.1a) It is defined by a nontrivial polynomial equation $m(z, f(z)) \equiv 0$.
(7.1b) $m = m(z, w)$ has algebraic coefficients.
(7.1c) $\frac{\partial m}{\partial w}(z_0)$ and $m(z_0, w)$ have no simultaneous zeros.

Apply the implicit function theorem (Chap. 2). It says $m(z, w)$ has $\deg_w(m)$ distinct zeros in $\mathcal{L}_{z_0}$. Conclude: Coefficients of $f(z)$ around z_0 are algebraic.

7.2.1. *Grothendieck's Alternative.* Define $\sigma \in G_{\mathbb{Q}}$ acting on a path γ through what the result does to algebraic functions f:

$$f \mapsto f_{\sigma^{-1} \circ \gamma \circ \sigma} = f_{\gamma^{\sigma}}.$$

In words: Apply σ^{-1} to the coefficients of f, analytically continue f around γ and then apply σ to the coefficients of the result. The effect of γ on algebraic functions determines it. So this determines γ^{σ}.

PROBLEM 7.1. What does γ^{σ} look like?

Only if σ is complex conjugation $\mathfrak{c}$ will there be a *path* γ' (independent of f) so that represent $f_{\gamma^{\sigma}} = f_{\gamma'}$. To see this, apply the theorem of Artin-Schreier: σ, if not complex conjugation $\mathfrak{c}$, either has infinite order or it is $\mu \mathfrak{c} \mu^{-1}$ where all powers of μ give distinct conjugates of $\mathfrak{c}$. Further, σ and μ generate an *uncountable* subgroup of $G_{\mathbb{Q}}$. If all the γ^{σ} s were paths, $\{\gamma^{\sigma'}\}_{\sigma' \in \langle \sigma \rangle}$ would have to be a countable, therefore finite, set. Simple considerations show this is impossible.

7.2.2. *Where can we put γ^{σ}?* Let $\bar{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. The collection $\mathcal{E}(U_{\boldsymbol{z}}, z_0)^{\mathrm{alg}}$ is in the Laurent series about z_0. With no loss we're allowed to assume the coefficients are in $\bar{\mathbb{Q}}$.

This gives an ordering: $f \le g$ if $\bar{\mathbb{Q}}(z, g) \supset \bar{\mathbb{Q}}(z, f)$. Action of a path on $\bar{\mathbb{Q}}(z, g)$ determines its action on $\bar{\mathbb{Q}}(z, f)$. So, paths act on the equivalence classes and respect this ordering. Each equivalence class defines a specific function field inside $\mathcal{L}_{z_0}$. It is the exact data you get from a cover and a point on the cover over z_0. The ordering allows considering $\mathcal{P}_{z_0}$, projective systems of (algebraic) points over z_0. Thus, paths act on $\mathcal{P}_{z_0}$ (Chap. 4 or [**Ihar91**, p. 104]).

PROPOSITION 7.2. *This action on $\mathcal{P}_{z_0}$ determines paths in $\pi_1(U_{\boldsymbol{z}}, z_0)$. The collection $\{\gamma^{\sigma}\}_{\gamma \in \pi_1(U_{\boldsymbol{z}}, z_0), \sigma \in G_{\mathbb{Q}}}$ also acts on $\mathcal{P}_{z_0}$. Define π_1^{alg} to be the projective completion of this action. Then, π_1^{alg} is the completion of π_1 by all normal subgroups of finite index. Further, $G_{\mathbb{Q}}$ acts on this.*

7.3. Extending $G_{\mathbb{Q}}$ action. Extend the homomorphism $\pi_1(U_{\boldsymbol{z}'}, z_0) \to G$ to $\psi_{\boldsymbol{z}',z_0} : \pi_1(U_{\boldsymbol{z}'}, z_0)^{\text{alg}} \to G$. As a profinite group, $\pi_1(U_{\boldsymbol{z}'}, z_0)^{\text{alg}}$ is also a free group on r (topological) generators modulo one relation. Here, however, there are many more sets of *classical generators*.

For $G_{\mathbb{Q}}$ to act requires $\boldsymbol{z}$ is stable under $G_{\mathbb{Q}}$. Then, $G_{\mathbb{Q}}$ acts on $\pi_1(U_{\boldsymbol{z}'}, z_0)^{\text{alg}}$ through Ihara's *pro-braid group* if $z_0 \in \mathbb{Q}$. Again, recognize this action through its effect on classical generators of $\pi_1(U'_{\boldsymbol{z}}, z_0)^{\text{alg}}$. Dependence on z_0 is so subtle, that any two distinct choices of z_0 give different actions. One remedy is to consider only the induced action of $G_{\mathbb{Q}}$ modulo inner automorphisms by $\pi_1(U_{\boldsymbol{z}',z_0})^{\text{alg}}$. Two further points guide investigations.

(7.2a) Unless the cover $X_{\boldsymbol{z}',z_0} \to \mathbb{P}^1_z$ coming from $\psi_{\boldsymbol{z}',z_0}$ is Galois and defined (with its automorphisms) over $\mathbb{Q}$, the action of $G_{\mathbb{Q}}$ won't respect $\psi_{\boldsymbol{z}',z_0}$.

(7.2b) The action is so big, interesting properties of $G_{\mathbb{Q}}$ are hard to detect at the level of finite covers.

7.4. Motivation from the Inverse Galois Problem**.** Consider a finite group G and the *regular* version of the Inverse Problem. It says for some $\boldsymbol{z}$, G should be the group of a cover of $U_{\boldsymbol{z}}$ with it and its automorphisms over $\mathbb{Q}$. That is, G should be an r-branch point realization over $\mathbb{Q}$. To find r and this cover needs structure.

You won't want to do one group at a time. So, we look at various quotients of $\pi_1(U_{\boldsymbol{z}',z_0})^{\text{alg}}$ with classical generators up to an action by H_r. Then, use $G_{\mathbb{Q}}$ action to investigate when there might be a value of r and a corresponding $\boldsymbol{z}'$ to realize such a quotient over $\mathbb{Q}$. Rather, however, than taking finite group quotients of $\pi_1(U_{\boldsymbol{z}',z_0})^{\text{alg}}$, take them maximally *Frattini*. Then dependence of $G_{\mathbb{Q}}$ action on $\boldsymbol{z}'$ has some uniformity. This gives the application generalizing modular curves Chap. 5 calls *Modular Towers*.

Start with a finite group G. Call a surjective homomorphism $\mu : H \to G$ *Frattini* if for any subgroup $H^* \le H$, $\mu(H^*) = G$ implies $H^* = H$. This is the exact group translation of the cover property from §6.5.1. Suppose μ corresponds to a sequence of covers $\mu^* : X \to X/\ker(\mu) \to X/H$. Then, any proper cover W appearing in the factorization $X \to X/H$ must factor properly through the cover $X/\ker(\mu) \to X/H$. A profinite group $\tilde{G}$ gives the maximal Frattini cover of G. All other group covers of $\mu : H \to G$ are targets for the map $\tilde{G} \to G$. Given $\psi : \pi_1(U_{\boldsymbol{z}})^{\text{alg}} \twoheadrightarrow G$, a significant geometric invariant of ψ is the set of maximal Frattini quotients of $\pi_1(U_{\boldsymbol{z}})^{\text{alg}}$ (quotients of $\tilde{G}$) appearing as factors of ψ. These *Frattini invariants* interpret properties of the levels of Modular Towers. Their simplest instances refine Riemann's theory of θ characteristics (§10.1). They give many implications for the Inverse Galois Problem.

Conjugacy classes $\mathbf{C}$ hit by classical generators separate these homomorphisms discretely. This data gives structure to the problem. A preliminary investigation with $(G, \mathbf{C})$ from the Branch Cycle Lemma (Chap. 9, see §8.2) produces a necessary condition for a $(G, \mathbf{C})$ realization (over $\mathbb{Q}$). It is that $\mathbf{C}$ be a *rational union* of conjugacy classes.

8. Extensible nilpotent functions and the group $\tilde{G}$

We explain the *universal Frattini cover* $\tilde{G}$ of G following the guide of Abel. He solved an inverse problem to part of the expression by radicals problem. This produced dihedral group extensions, labeled by parameters still appearing in treatments of modular curves. For a prime p, $\mathbb{Z}_p$ denotes the p-adic numbers. Suppose A and B are two abelian groups. Assume elements of A act as automorphisms of B: $a \in A$ acts on $b \in B$ giving $a(b)$. Then, form a group on $A \times B$ (called $A \times^s B$) using multiplication of 2×2 matrices:

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a' & b' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a+a' & ab'+b \\ 0 & 1 \end{pmatrix}.$$

8.1. A guide from dihedral groups. Case: $G = D_p = \mathbb{Z}/p \times^s \{\pm 1\}$ has $\mathbb{Z}_p \times^s \{\pm 1\}$ and $\mathbb{Z}_p \times^s \mathbb{Z}_2$ as the pieces of its universal Frattini cover. Patch these together as a fiber product over D_p. This generalizes: For each prime p dividing $|G|$, there is a universal p-Frattini cover ${}_p\tilde{G}$ (Chap. 5). You can deal with one prime at a time. So, for investigating the arithmetic properties of quotients of $\pi_1(U_{\boldsymbol{z}})^{\text{alg}}$, consider the biggest quotients compatible with r and $\mathbf{C}$ satisfying the Branch Cycle Lemma. Let p be a prime. Recall: A conjugacy class in a finite group is called p' if its elements have order prime to p.

Certain properties of ${}_p\tilde{G}$ suggest levels of a tower of moduli spaces.

(8.1a) ${}_p\tilde{G} \to G$ has a pro-free pro-p group $\ker_0$ as kernel.
(8.1b) It has a characteristic sequence of quotients G_k, $k = 0, 1, \dots$.
(8.1c) Each p'-conjugacy class of G lifts uniquely to a p'-conjugacy class of ${}_p\tilde{G}$.
(8.1d) Elements of G_k whose images in G generate, already generate G_k.

Form $\ker_1$ as the closed subgroup of $\ker_0$ generated by $\ker_0^p$ and the commutators $(\ker_0, \ker_0)$. This gives G_1 in (8.1b) as the quotient ${}_p\tilde{G}/\ker_1$. Continue inductively to form the other G_k s.

8.2. Applying the Branch Cycle Lemma. When there is profinite data, or over $\mathbb{R}$ or $\mathbb{Q}_p$, the explicit formula from the Branch Cycle Lemma is valuable.

Suppose $\sigma \in G_{\mathbb{Q}}$ maps to $n_\sigma \in \hat{\mathbb{Z}}^* = G(\mathbb{Q}^{\text{cyc}}/\mathbb{Q})$. Find $\pi \in S_r$ to satisfy $z_i^\sigma = z_{(i)\pi}$. Then, a $(G, \mathbf{C})$ realization (over $\mathbb{Q}$ at $\boldsymbol{z}$) implies

$$C_{(i)\pi}^{n_\sigma} = C_i, \ i = 1, \dots, r. \tag{8.2}$$

Suppose the following:

(8.3) $\mathbf{C}$ consists of r conjugacy classes whose elements have orders prime to p.

Note: Classes $\mathbf{C}$ from G uniquely extend to p' classes in all G_k s. Also, suppose $(G, \mathbf{C})$ passes Branch Cycle test (8.1). Then, so does $(G_k, \mathbf{C})$ for *all* values of k. This illustrates a phenomenon: The groups G_k are similar. So, they produce a guiding question.

QUESTION 8.1. Are the G_k s so similar their realizations fall to the Inverse Galois Problem with a k-free bound on the number of branch points?

The answer is conjecturally "No!" If you bound the number of branch points, there should be a bound on the values of k for which G_k has a K regular realization where K is a number field. Making this bound explicit, however, is another matter. The Mazur-Merel Theorem is well-known. It says, for any number field K, there is an explicit bound C_K on p^{k+1} so that for $p^{k+1} > C_K$, there are no non-cusp rational points on the modular curve $X_1(p^{k+1})$. Below we see this interprets as the easiest special case of this conjecture: There are but finitely many four branch point, dihedral group involution realizations. The first step in the process forces us into investigating the structure of some Modular Tower. An H-M (*Harbater-Mumford*) representative of $(G, \mathbf{C})$ is an r-tuple $\boldsymbol{g} \in \mathbf{C}$ with this property:

(8.4) $\langle \boldsymbol{g} \rangle = G$ and $g_{2i-1} = g_{2i}^{-1}$, $i = 1, \dots, s$ with $r = 2s$.

Approach the following statement by considering r' to be very large (say, two trillion). Then, consider if you can see a difference between the following cases.

(8.5a) G is the *monster* (or use your favorite simple group) and $p = 2$.
(8.5b) G is D_5 and $p = 5$.

THEOREM 8.2. *Fix r'. Suppose there are $(G_k, \mathbf{C}_k)$ realizations over $\mathbb{Q}$ with $r_k \le r'$ conjugacy classes in $\mathbf{C}_k$, for each $k \ge 0$. Then, there exists $r \le r'$ and p'-conjugacy classes $\mathbf{C}$ with $(G_k, \mathbf{C})$ realizations over $\mathbb{Q}$ for all k.*

If $p = 2$, each $(G_k, \mathbf{C})$ realization falls on a Hurwitz space component corresponding to an H_r orbit containing H-M representatives.

8.3. Thm. 8.2 and Modular Towers. Thm. 8.2 (Chap. 5) produces p' conjugacy classes $\mathbf{C}$ in ${}_p\tilde{G}$ and a sequence $\{\boldsymbol{z}_k\}_{k=0}^{\infty}$ of $\mathbb{Q}$-stable unordered r-tuples of distinct points from $\mathbb{P}^1_z$. This sequence has the property that $\boldsymbol{z}_k$ lies under a $(G_k, \mathbf{C})$ realization. Further, suppose $p = 2$. Then, the attached homomorphisms $\pi_1(U_{\boldsymbol{z}_k})^{\mathrm{alg}} \to {}_p\tilde{G}$ send classical generators of $\pi_1(U_{\boldsymbol{z}_k})^{\mathrm{alg}}$ to H-M representatives in ${}_p\tilde{G}$ so the induced quotient to G_k has $G_{\mathbb{Q}}$-stable kernel.

Chap. 5 shows how this system of realizations fits into a system of moduli spaces generalizing classical modular curves. Consider all maps $\pi_1(U_{\boldsymbol{z}}, z_0) \to {}_p\tilde{G}$ with generators $\boldsymbol{g}$ mapping to $\mathbf{C}$ as $\boldsymbol{z}$ runs over U_r ($z_0 \notin \boldsymbol{z}$). For each k this produces an affine algebraic variety $\mathcal{H}_k$. Its $\mathbb{C}$ points correspond to equivalence classes of maps $\pi_1(U_{\boldsymbol{z}}, z_0) \to G_k$ (with $\boldsymbol{z}$ variable). The group $\mathrm{GL}_2(\mathbb{C})$ acts on these spaces. The quotient is another affine variety $\mathcal{H}_k^{\mathrm{rd}}$, level k of the Modular Tower for $(G, \mathbf{C}, p)$.

A significant case: $G = D_p$ (p odd), p the prime and $\mathbf{C}$ is $r = 4$ repetitions of the conjugacy class of involutions (elements of order 2) in D_p. Then, $\mathcal{H}_k^{\mathrm{rd}}$ is the modular curve $X_1(p^{k+1})$ minus its cusps. Each case with $r = 4$ produces a tower of curves, respective quotients of the upper half plane by finite index subgroups of $\mathrm{PSL}_2(\mathbb{Z})$. Usually the Modular Tower levels are noncongruence covers. They always have a useful moduli space structure.

8.4. A diophantine view of a nilpotent theory. Generalizations of theorems of Mazur and Serre now have formulations through the action of $G_{\mathbb{Q}}$ on projective systems of points on the spaces

$$\cdots \to \mathcal{H}_{k+1}^{\mathrm{rd}} \to \mathcal{H}_k^{\mathrm{rd}} \to \cdots \to \mathcal{H}_0^{\mathrm{rd}} \to U_r^{\mathrm{rd}} = J_r. \tag{8.6}$$

CONJECTURE 8.3 (Main Conjecture). Suppose $(G, \mathbf{C}, p)$ is data for a Modular Tower. Assume G is centerless and does not have $\mathbb{Z}/p$ as a quotient. For k large, $\mathcal{H}_k^{\mathrm{rd}}$ has no $\mathbb{Q}$ points.

8.4.1. *Interpreting the Main Conjecture.* Thus, $\mathbb{Q}$ realizations of G_k require increasing large sets of conjugacy classes for k large. This is more refined information than from any known versions of the Branch Cycle Lemma. If $p = 2$, Thm. 8.2 says rational points will appear only on H-M components of the sequence, and this refines the problem immensely. Changing $\mathbb{Q}$ to another number field K requires significant generalization (Chap. 5).

Here is a response to the setup of cases from (8.5). Both require information on the geometry of Modular Tower levels we don't know yet. The dihedral group case (with r equal two trillion) looks easier because it translates to statements about classical moduli spaces: The moduli of cyclic 5^{k+1} degree covers of hyperelliptic curves (of genus 1,000,000,000,000-1). No one knows if this space is without $\mathbb{Q}$ points for large k. Suppose the curves in the family have genus 1. Then we know much since the Modular Tower levels are modular curves.

Yet, with the monster, there could be surprises. For example, for $(A_6, \mathbf{C}_{3^5})$ with $p = 2$, there are no $\mathbb{Q}$ points at level 1 of the Modular Tower. Reason: There are no points at level 1 at all, the result of the $\otimes$ symbol at (6,5) in the Constellation Table of §10.1. The case $r = 4$ gets much attention for problems that immediately generalize those for modular curves (§10.5).

8.4.2. *Nilpotency from projective systems of points.* Let X be a compact Riemann surface. Denote the pro-p quotient of the fundamental group of X by $\pi_1(X)^{(p)}$. When this group appears only up to inner automorphism, we drop the notation for the base point. Thm. 8.2 includes a nilpotent theory. Consider one of the homomorphisms $\psi_{\boldsymbol{z}} : \pi_1(U_{\boldsymbol{z}}, z_0)^{\text{alg}} \to {}_p\tilde{G}$ mapping a fixed set of classical generators of into the p'-conjugacy classes $\mathbf{C}$.

Let $X_0 \to \mathbb{P}^1_z$ be the G quotient cover from this homomorphism. For investigating all possible such maps $\psi_{\boldsymbol{z}}$, note it factors through a smaller quotient group of $\pi_1(U_{\boldsymbol{z}}, z_0)^{\text{alg}}$. This is an extension $M_{\boldsymbol{z}}$ (independent of $\psi_{\boldsymbol{z}}$ as a group extension) of $G = G_0$ by $\pi_1(X_0)^{(p)}$.

Call two such homomorphisms $M_{\boldsymbol{z}} \to {}_p\tilde{G} \to G_0$ *inner equivalent* if they differ by inner automorphisms from $\ker_0$ in (8.1a). Suppose $X_0 = X_{\boldsymbol{p}} \to \mathbb{P}^1_z$ corresponds to $\boldsymbol{p} \in \mathcal{H}_0^{\text{rd}}$. Projective systems of points on the Modular Tower over $\boldsymbol{p}$ correspond to inner homomorphism classes of $M_{\boldsymbol{z}} \to {}_p\tilde{G} \to G_0$. Shorten this phrase to a *point* on the Modular Tower. In this case refer to $M_{\boldsymbol{z}}$ as $M_{\boldsymbol{p}}$. Let the set of inner homomorphism classes be $\mathcal{T}_{\boldsymbol{p}}$.

Homomorphisms factoring through ${}_p\tilde{G}$, surjective to G_0, map surjectively to ${}_p\tilde{G}$ (from (8.1d)). Let $g = g(X_{\boldsymbol{p}})$ be the genus of $X_{\boldsymbol{p}}$ — transparent from $\mathbf{C}$ by the Riemann-Hurwitz formula. So, $\pi_1(X_{\boldsymbol{p}})^{(p)}$ is a free pro-p group on $2g$ generators modulo one *commutator* relation.

8.4.3. $G_{\mathbb{Q}}$ *action on* $\text{Ni}({}_p\tilde{G}, \mathbf{C})^{\text{in}}$. The notion of Nielsen class (§5.4) applies uniformly to $({}_p\tilde{G}, \mathbf{C})$. Its absolute and inner versions inherit an H_r action. Orbits for this action correspond to projective systems of components at the levels of the Modular Tower. Reducing this action modulo $\ker_0$ maps each orbit to an H_r orbit at level 0. Components of $\mathcal{H}_k^{\text{rd}}$ (over $\bar{\mathbb{Q}}$) map among each other by $G_{\mathbb{Q}}$ acting on the coefficients of their equations.

We don't often see equations for these moduli spaces. So, figuring this action from the data is one of our main problems. From this, regard $G_{\mathbb{Q}}$ as acting on the H_r orbits in $\text{Ni}({}_p\tilde{G}, \mathbf{C})^{\text{in}}$. In the A_n examples of §10.1, there are components at a finite level k that have no projective system of components above them. This could happen with any $(G, \mathbf{C})$. The invariant in §9.1 catches these *obstructed components* precisely, when you can compute it (Chap. 5).

PROBLEM 8.4. Compute the $G_{\mathbb{Q}}$ action on H_r orbits of $\text{Ni}({}_p\tilde{G}, \mathbf{C})^{\text{in}}$. Also, compute the pattern of chains of obstructed components.

8.4.4. *A nilpotent Tate Grassmanian.* For G *any* finite group this theory has a large pro-nilpotent part. Thus, it generalizes the abelian theory setup.

Suppose $\boldsymbol{z} \in U_r$ lies below a $\mathbb{Q}$ point $\boldsymbol{p} \in \mathcal{H}_0^{\mathrm{rd}}$. Then, $G_{\mathbb{Q}}$ acts on $\pi_1(X_{\boldsymbol{p}})^{(p)}$ (modulo inner automorphisms) as a quotient of the action on $\pi_1(X_{\boldsymbol{p}})^{\mathrm{alg}}$. Act by $G_{\mathbb{Q}}$ on the quotient of $\pi_1(X_{\boldsymbol{p}})^{(p)}$ by the closed subgroup of commutators. Denote this quotient by $T_{\boldsymbol{p}}$, the Tate module for p. This gives the theory of abelian covers of $X_{\boldsymbol{p}}$ with group order a power of p. Its relation to the Jacobian of $X_{\boldsymbol{p}}$ is clear. It is the projective system of points of p-power order on the Jacobian.

Continue the actions of $G_{\mathbb{Q}}$. Suppose α is in $\mathcal{T}_{\boldsymbol{p}}$. Then, $\sigma \in G_{\mathbb{Q}}$ acts on α (on the right) through the composition $\alpha \circ \sigma$ (Chap. 9). There is a *Lie algebra structure* on $\pi_1(X_{\boldsymbol{p}})^{(p)}$. Using it and the *Weil pairing* allows dualizing these maps. The result is $\mathcal{T}_{\boldsymbol{p}}^*$, a nilpotent version of $G_{\mathbb{Q}}$ acting on a *Grassmannian* of a *Tate module* of the Jacobian for $X_{\boldsymbol{p}}$ (Chap. 5).

One goal of Modular Towers is to provide *small* actions for $G_{\mathbb{Q}}$. Modular Towers retains the feel of finite groups. Though a generalization of modular curves, the group theory reminds of situations yielding groups as Galois groups. Chap. 9 reviews achievements of that program, appearing in detail in [**Se92**], [**MM95**] and [**Vö96**] (see [**Fri94**]). In particular, the Dettweiler-Völklein generalization of Katz's *rigid tuples* [**DVo98**] pushes realization of Chevalley simple groups to a new place. It produces many cases with G_0 simple where $\mathbb{Q}$ points are dense in $\mathcal{H}_0^{\mathrm{rd}}$.

These give a setting for $\widehat{\mathcal{G}T}$ relations close to the Inverse Galois Problem territory. Yet, the pro-finite elements of Modular Towers are like those of modular curve towers, suitable for checking the effect of these constraints. One goal is to see if $\widehat{\mathcal{G}T}$ relations force significant quotients of ${}_p\tilde{G}$ to have $\mathbb{Q}$ realizations.

9. The Grothendieck-Teichmüller group

When $G_{\mathbb{Q}}$ acts on fundamental groups related to moduli spaces, that action preserves underlying geometry. Often that geometry is not obvious to us. So, asking what to expect from a $G_{\mathbb{Q}}$ action has us delving more deeply to where the geometry appears. The principle everyone uses occurs in divining components of a moduli space. The expectation is $G_{\mathbb{Q}}$ should map these components among each other, unless a geometric reason prevents it.

9.1. Moduli spaces with several components. The Constellation Table of §10.1 illustrates this. Superficially the two components appearing at the locus (n, r) $(r \geq n)$ have much in common. Action of $G_{\mathbb{Q}}$, however, on their equations leaves them fixed. Setup: The only alternative is it maps one of them to the other, because their union is a moduli space. Finish: The Schur multiplier invariant gives a geometric condition separating the components (§10.2.2).

Does $G_{\mathbb{Q}}$ have relations appearing everywhere in moduli space actions? These would induce relations for $G_{\mathbb{Q}}$ acting on all related moduli spaces (Chap. 5). The Grothendieck-Teichmüller group offers such relations. We discuss now the implication of these for the Inverse Galois Problem. Recall the space $J_r = U_r/\mathrm{PGL}_2(\mathbb{C})$ and its relative $\Lambda_r = (\mathbb{P}_z^1)^r \setminus \Delta_r$ when $r = 4$: $\Lambda_4 = U_{\lambda:0,1,\infty}$ (§7.1.1).

§4.1 has a description of the extensible algebraic functions $\mathcal{E}(\Lambda_4, \lambda_0)^{\mathrm{alg}}$. Each starts from a Laurent series in λ_0 that analytically continues along any path in Λ_4.

9.2. Deligne's tangential base points. Deligne suggested an extra structure to $\mathcal{E}(\Lambda_4, \lambda_0)^{\mathrm{alg}}$ by expanding the choices of base point [**De89**]. The elements of $\mathcal{L}_{\lambda_0}$ sit inside an algebraically closed field $\mathcal{P}_{z_0}$, convergent Puiseux expansions around λ_0 (Chap. 2). They look like Laurent series in $(\lambda - \lambda_0)^{1/e}$ for some integer e. They don't, however, work as functions in a neighborhood of λ_0 (Chap. 2).

Give the special case $\lambda^{1/e}$ meaning by making it take positive values along the real axis pointing from 0 to 1. This produces an analytic expression convergent in a neighborhood of any point on the *positive* real axis between 0 and 1. An alternative would ask $\lambda^{1/e}$ to take positive values along the real axis in the negative direction from 0 to $-\infty$.

Distinguish between those two choices. Extend the meaning of the first to all Puiseux expansions about 0 using the notation $\mathcal{P}_{\overline{0\,1}}$. Each produces a meromorphic function defined near 0 to the right of 0. Similarly, for the second choice use the notation $\mathcal{P}_{\overline{0\,\infty}}$. Each element in this defines a meromorphic function near 0 to the left of 0. To be explicit, choose an open disk (on $\mathbb{P}^1_\lambda$). It should be symmetric about the real axis, tangent to the imaginary axis and contain part of the real axis from 0 to 1 (Chap. 2). Denote this disk $D_{\overline{0\,1}}$.

For any i and j, distinct elements from $\{0, 1, \infty\}$ form the similar set of *functions* $\mathcal{P}_{\overline{i\,j}}$. The ordering from §7.2.2 on algebraic functions in $\mathcal{L}_0$ extends to algebraic elements of $\mathcal{P}_{\overline{0\,1}}$. So does the action of $G_{\mathbb{Q}}$ extend (Chap. 4).

Denote the set of ordered arrows by $\mathbb{B}$. Label the linear fractional transformations that permute $\{0, 1, \infty\}$: $t_{\overline{i\,j}}$ takes i to 0, j to 1 and k to ∞. Apply $t_{\overline{i\,j}}{}^{-1}$ to $D_{\overline{0\,1}}$ to get similar disks $D_{\overline{i\,j}}$ attached to $\mathcal{P}_{\overline{i\,j}}$.

PRINCIPLE 9.1 (Branch Extensibility). *Consider $f \in \mathcal{E}(\Lambda_4, \lambda_0)^{\mathrm{alg}}$ and i, j distinct elements from $\{0, 1, \infty\}$. Suppose $\gamma : [0, 1] \to \Lambda_4$ is a path with $\gamma(0) = \lambda_0$ and $\gamma(1)$ in $D_{\overline{i\,j}}$. Then, there exists a unique $F_{f_\gamma} \in \mathcal{P}_{\overline{i\,j}}$ restricting to f_γ. The collection of order preserving maps on the equivalence classes of fields $\mathbb{C}(\lambda, F_{f_\gamma})$ is $\pi_{\overline{0\,1}} = \pi_1(\Lambda_4, \overline{0\,1})^{\mathrm{alg}}$. It has a natural $G_{\mathbb{Q}}$ action (*Chap. 4*).*

Let x be a clockwise circle ([**Ihar91**] takes counterclockwise; see comments of §11) around 0 meeting $D_{\overline{0\,1}}$. It represents an element of $\pi_{\overline{0\,1}}$ from Princ. 9.1. For example, suppose in the definition of F_{f_γ} that $\gamma(1)$ is on x. Take $F = F_{f_\gamma}$ equal to $h(\lambda^{1/e})$ with h meromorphic around 0. Let $\zeta_e = e^{\frac{2\pi i}{e}}$.

The effect of x on F is the substitution $\lambda^{1/e} \mapsto \zeta_e^{-1}\lambda^{1/e}$. So, $\sigma^{-1} \circ x \circ \sigma$ (following §7.2.1) gives this sequence of operations on a power series. Act on coefficients with σ^{-1}, then substitute $\zeta_e^{-1}\lambda^{1/e}$, then act by σ on the resulting coefficients. Use the notation of §8.2: n_σ is restriction of σ to cyclotomic numbers. The total effect is the substitution $\lambda^{1/e} \mapsto \zeta_e^{-n_\sigma} z^{1/e}$. So, $x^\sigma = x^{n_\sigma}$.

9.3. The first two relations. Following [**AnIh88**], the $t_{\overline{i\,j}}$s act on Puiseux expansions. So, they give maps among the fundamental groups $\pi_{\overline{i\,j}}$.

9.3.1. *Continuations from $\overline{0\,1}$ to $\overline{1\,0}$.* Extend this to the *fundamental groupoid* (Chap. 3), to give $\pi_{\overline{0\,1}\,\overline{1\,0}} = \pi_1(\Lambda_4; \overline{0\,1}, \overline{1\,0})$. Let $\gamma_p : [0, 1] \to \Lambda_4$ be a path running along $\mathbb{R} \cup \{\infty\}$ from 0 toward 1, with $\gamma_p(0) \in D_{\overline{0\,1}}$ and $\gamma(1) \in D_{\overline{1\,0}}$. As with x it defines an element of $\pi_{\overline{0\,1}\,\overline{1\,0}}$.

Let x' be the transform of x by $t_{\overline{1\,0}}$ $(\lambda \mapsto 1 - \lambda)$. Take $y = \gamma_p \circ x' \circ \gamma_p^{-1}$. Then, y represents an element of $\pi_{\overline{0\,1}}$. Even easier than x, $\sigma^{-1} \circ y \circ \sigma$ has the effect of $\gamma_p^\sigma (x')^{n_\sigma} (\gamma^{-1})^\sigma$. Let $m_\sigma = \gamma_p^\sigma \gamma_p^{-1}$. Then y^σ equals $m_\sigma y^{n_\sigma} m_\sigma^{-1}$.

Since x and y are topological generators of $\pi_{\overline{0\,1}}$, the effect of σ on them determines the action of σ. It makes sense to write $m_\sigma(x, y)$. If P_1 and P_2 are two both homotopy classes of paths with the same end points, then they are conjugate.

Even though this is a profinite group, apply this to γ_p and γ_p^{-1}. Therefore, m_σ is a commutator in the pro-free group x and y generate.

9.3.2. *The product-one relation.* Most significant is what σ does to xy. Equivalently: What is z^σ, with $z=(xy)^{-1}$ the 3rd element in a product-one relation (as in (5.1)). The formula for this comes from the first two Drinfeld-Ihara relations:

(9.1a) $m_\sigma(x,y)m_\sigma(y,x)=1$; and with $u_\sigma=\frac{n_\sigma-1}{2}$,
(9.1b) $m_\sigma(z,x)z^{u_\sigma}m_\sigma(y,z)y^{u_\sigma}m_\sigma(x,y)x^{u_\sigma}=1$.

Apply $t_{\overline{10}}$ to $m_\sigma(x,y)$ to see (9.1a). Let r be the half-circle from the center of $D_{\overline{10}}$ to the center of $D_{\overline{1\infty}}$ going clockwise. Then, r defines an element of $\pi_{\overline{10},\overline{1\infty}}$. Expression (9.1b) comes from applying σ to the geometric relation

$$t_{\overline{1\infty}}{}^2(r\circ\gamma_p)\circ t_{\overline{1\infty}}(r\circ\gamma_p)\circ(r\circ\gamma_p)=1.$$

We left out the famous *5-cycle relation* [**Ihar91**, p. 107]. It forcefully appears soon.

9.3.3. *Return of the j-line.* There is a conspicuous quotient of the fundamental group of $\pi_1(\mathbb{P}^1_{j:0,1,\infty})$ (§7.1.3). It has generators $\boldsymbol{\gamma}$:

$$\gamma_0=q_1q_2,\ \gamma_1=q_1q_2q_1 \text{ and } \gamma_\infty=q_2$$

from a quotient of H_4 (Chap. 5; see §5.3). These satisfy

(9.2) $\gamma_0^3=1$, $\gamma_1^2=1$, $\gamma_0\gamma_1\gamma_\infty=1$; the group $\langle\gamma_0,\gamma_1\rangle$ is $\mathrm{PSL}_2(\mathbb{Z})$.

When $r=4$ a reduced Hurwitz space has a Riemann's Existence Theorem description coming from these generators acting on a *reduced Nielsen class* (Chap. 5). The geometry of the reduced Hurwitz spaces $\{\mathcal{H}_k^{\mathrm{rd}}\}_{k=0}^\infty$ shows from analyzing $\boldsymbol{\gamma}$. Most crucial are disjoint cycles of $\gamma_{\infty,k}$, the result of γ_∞ in its action on $\mathrm{Ni}_k^{\mathrm{rd}}$.

PRINCIPLE 9.2 (Cusp Principle). *Each disjoint cycle of $\gamma_{\infty,k}$ corresponds to a cusp point for $\bar{\mathcal{H}}_k^{\mathrm{rd}}$ over $j=\infty$. Further, each cusp has its own geometry.*

9.4. Detecting $\widehat{\mathcal{G}T}$ at the level of a Modular Tower. Relations (9.1) have versions for action of $G_{\mathbb{Q}}$ on $\boldsymbol{\gamma}$. Yet, we must generalize them beyond their present shape to have them suit the geometry of a Modular Tower. Here is why.

9.4.1. *Viewing tangential base points from $\mathbb{P}^4$.* Deligne's tangential base points come from components of real points on $(\mathbb{P}^1_z)^4\setminus\Delta_4=U^4$. An example is R_{z_1,z_2,z_3,z_4}: 4-tuples of distinct points on $\mathbb{R}\cup\{\infty\}=\mathbb{R}_\infty$ where the four points are in the same order as $(0,1,\infty,-1)$ around the circle. Rearrangements from permuting the elements $\{z_1,z_2,z_3\}$ produce new connected components. To get to $\mathbb{B}$, mod out by the subgroup of $\mathrm{PSL}_2(\mathbb{R})$ stabilizing each component.

Formulas similar to (9.1) allow working directly with H_4. Replace elements of $\mathbb{B}$ with the image of R_{z_1,z_2,z_3,z_4} in $\mathbb{P}^4\setminus D_4=U_4$. This is in [**IM95**] which also treats higher values of r.

9.4.2. *Other real component configurations.* The sets R_{z_1,z_2,z_3,z_4} often fail to capture the cusp geometry on a Modular Tower. Here is an example. Real points on level 1 of the $(A_5,\mathbf{C}_{3^4})$ Modular Tower (§10.1, §10.3) lie on the genus 12 component of $\bar{\mathcal{H}}_1^{\mathrm{rd}}$. Denote that $\bar{\mathcal{H}}_1^+$.

Real points on $\bar{\mathcal{H}}_1^+$ collect in eight disjoint components, each associated to a cusp (of width 20). Four attach to H-M representatives in this Nielsen class (§8.2). Let $CP_{z_1,z_2,z_3,z_4}=\{z_1,z_2\in\mathbb{H}\mid z_3=\bar z_1,\ z_4=\bar z_2\}$: Two sets of complex conjugate pairs of points, with the first two in the upper half plane. The preimage in the inner Hurwitz space of these eight components is 32 real components. Each lies over a locus of real points in U_4 with preimage in U^4 of type CP_{z_1,z_2,z_3,z_4}.

There are $\binom{4}{2} = 6$ choices for which two coordinates to put in the upper half plane. Then, counting possible lower half plane pairings with z_1 gives a total of 12 such real components of U_4. Action of $\mathrm{PSL}_2(\mathbb{R})$ on CP_{z_1,z_2,z_3,z_4} has a hyperbolic description (Chap. 2). Put z_1 at i under this action, so orbits of z_2 are points at a fixed (hyperbolic) distance from i.

To choose an explicit representative from each orbit, take a (hyperbolic) circle from i to $i+1$: A half circle perpendicular to the real axis through i and $i+1$. Then, the interval description for Deligne's tangential base points has as analog the portion of the circle from i to the right of i going through $i+1$.

9.5. Variants of the Drinfeld-Ihara relations Chap. 9**.** There was a first definition of $\widehat{\mathcal{G}T}$. It was a subgroup of the automorphisms $\mathrm{Aut}(\hat{F}_2)$ of the profree group on generators x and y.

9.5.1. *$\widehat{\mathcal{G}T}$: A moving target.* $\widehat{\mathcal{G}T}$'s elements are automorphisms of the form $(x, y) \mapsto (x^n, my^n m^{-1})$: $n \in \hat{\mathbb{Z}}^*$ and $m \in (\hat{F}_2, \hat{F}_2)$ satisfy relations 9.1 (with the 5-cycle relation). The composition of two is another automorphism. That this composition also satisfies the relations is more serious. This gives a group structure to such pairs (n, m).

After the first definition, there was speculation $\widehat{\mathcal{G}T}$ might contain $G_{\mathbb{Q}}$ as an open subgroup. These days, however, $\widehat{\mathcal{G}T}$ presents a moving target. Recent joint work of Nakamura and Schneps reveals new relations satisfied by the image of $G_{\mathbb{Q}}$ in $\widehat{\mathcal{G}T}$. Its unclear whether to relabel $\widehat{\mathcal{G}T}$ appropriately for these relations or to start indexing a sequence of $\widehat{\mathcal{G}T}$-like groups. Yet, there are still only few of them and each is precious.

9.5.2. *Cusps producing other base points.* Consider $G_{\mathbb{Q}}$ acting as permutations on H_r orbits of a reduced Nielsen class (§8.4.2).

Several steps are necessary to include $\widehat{\mathcal{G}T}$ type relations (Chap. 9). First: Develop corresponding relations from tangential base points using components like CP_{z_1,z_2,z_3,z_4} (as suggested in [**Fri95a**, App. C-D]).

Second: Complete comparing with $\widehat{\mathcal{G}T}$ by extending this action to $\mathrm{Ni}({}_p\tilde{G}, \mathbf{C})^{\mathrm{rd}}$. This works because H_4 acting on generators of the 4-punctured sphere identifies with a subgroup H_5' of H_5. We explain.

As in §5.2, consider $(\mathbb{P}^1_z)^5 \setminus \Delta_5 = U^5$. There is a *fibration*, $U^5 \to U^4$ by projection on the first four coordinates. Embed S_4 in S_5 as the permutations leaving 5 fixed. Then, S_4 acts to give a new fibration, $U_4 \times \mathbb{P}^1_{z_5} \setminus D_5' \to U_4$ with D_5' the image of Δ_5 in $\mathbb{P}^4 \times \mathbb{P}^1$ (Chap. 5, [**BF82**], [**DFr99**] for the $\mathbb{R}$ analysis). Even without this quotient, analogs of all $\widehat{\mathcal{G}T}$ relations would appear. The fiber is a copy of $\mathbb{P}^1$ minus four points, with classical generators identified with words in $Q_1, \dots, Q_5$. So, even analogs of the 5-cycle relation (Chap. 9) show in identifying the $G_{\mathbb{Q}}$ action on $\mathrm{Ni}({}_p\tilde{G}, \mathbf{C})^{\mathrm{rd}}$ when $r = 4$.

Comparison between $\widehat{\mathcal{G}T}$ and Modular Towers then has these practical goals. Use all cusps on a Modular Tower to define the $\widehat{\mathcal{G}T}$ attached to that Modular Tower.

PROBLEM 9.3. What do $\widehat{\mathcal{G}T}$ relations applied to Modular Towers detect about $\mathbb{Q}$ orbits on $\mathrm{Ni}({}_p\tilde{G}, \mathbf{C})^{\mathrm{rd}}$. Compare with the Branch Cycle Lemma and ω (§10.2.2) invariant combination?

We conclude by tying together four advanced goals of the research motivating this book. It is convenient to do this by joining classical θ-functions to Modular

Towers. Each diophantine element of this section gives specific detailed results on the Modular Towers of this example (Chap. 5).

10. Combining the Existence Theorem and θ functions

The first Hurwitz spaces were moduli spaces of simple branched covers. In this case the Hurwitz spaces are connected. An easy application of the Riemann-Roch theorem then shows connectedness of the moduli space of curves of genus g.

10.1. Theta functions and Hurwitz spaces. An example with many applications comes from covers with alternating groups A_n as monodromy groups. Take A_n, $n \geq 4$, with the prime $p = 2$ and 3-cycles (r of them) as data for a Modular Tower. The usual representation T_n gives an absolute space of degree n covers with group A_n. There is a corresponding inner space of Galois covers (as in (5.5)). The following diagram displays the complete set of inner Hurwitz space components at level 0 of their Modular Tower.

Locations in this diagram have an attached integer pair (n, r). The location shows components of the inner Hurwitz space for $(A_n, \mathbf{C}_{3^r})$. Abbreviate this to $\mathcal{H}^{\rm in}_{n,r}$. The corresponding absolute spaces would be for the data $(A_n, \mathbf{C}_{3^r}, T_n)$, or $\mathcal{H}^{\rm abs}_{n,r}$. The group is the alternating group A_n. Conjugacy classes are r repetitions of 3-cycles. There is a famous *spin* group cover of A_n, $\tilde{A}_n$ where $\tilde{A}_n \to A_n$ is a central nonsplit extension with kernel $\mathbb{Z}/2$. The universal 2-Frattini cover of A_n (as in (8.1) automatically factors through $\tilde{A}_n$. This is a special case of a general phenomenon. The universal p-Frattini cover ${}_p\tilde{G}$ of any perfect group G factors through the universal central extension of G.

Labels for rows are by the genuses of the degree n covers. The relation between the spaces $\mathcal{H}^{\rm abs}_{n,r}$ and $\mathcal{H}^{\rm in}_{n,r}$ comes from a corollary in [**Fri96**].

PROPOSITION 10.1 (Absolute-Inner). *The natural map $\mathcal{H}^{\rm in}_{n,r} \to \mathcal{H}^{\rm abs}_{n,r}$ has degree 2. Each component of the former maps to a corresponding component of the latter.*

10.1.1. *Explanation of the symbols.* Two primitive icons appear. The symbol $\otimes$ corresponds to a(n irreducible) component whose points represent covers $\hat{X} \to \mathbb{P}^1$ with this property. A special degree two unramified cover $\hat{Y} \to \hat{X}$ satisfies

(10.1) $\hat{Y} \to \mathbb{P}^1$ composed from $\hat{Y} \to \hat{X}$ and $\hat{X} \to \mathbb{P}^1$ is Galois with group $\tilde{A}_n$.

Then, $\oplus$ denotes a component of covers in $\mathcal{H}^{\rm in}_{n,r}$ having no such $\hat{Y}$ cover. Excluding the genus 0 row, all rows have exactly two components. One is of $\otimes$ type,

TABLE 1. **Constellation of spaces $\mathcal{H}(A_n, \mathbf{C}_{3^r})^{\rm in}$**
Components correspond to lifting invariant values.

Genus at (n, r) of degree n cover: $g = r - n + 1$
of the Galois cover: $g^* = \frac{(r-3)n!}{6}$

$\xrightarrow{g \geq 1}$	$\otimes\oplus$	$\otimes\oplus$	...	$\otimes\oplus$	$\otimes\oplus$	$\xleftarrow{1 \leq g}$
$\xrightarrow{g=0}$	$\otimes$	$\oplus$	...	$\otimes$	$\oplus$	$\xleftarrow{0=g}$
$n \geq 4$	$n = 4$	$n = 5$	...	n even	n odd	$4 \leq n$

the other of $\oplus$ type. The spin group cover of alternating groups reveals its presence in components of Hurwitz spaces.

10.1.2. *Subtleties about Schur multipliers.* This phenomenon holds in general. Schur multipliers of finite groups produce distinct components of the Hurwitz space. For each conjugacy class C in $\mathbf{C}$, let b_{C} be its multiplicity of appearance in $\mathbf{C}$. A generalization of a Conway-Parker result has as hypothesis that b_{C} is suitably large for all C in $\mathbf{C}$. Conclusion: Distinct components in level k of a Modular Tower correspond exactly to elements in a subgroup of the Schur multiplier.

Yet, whether b_{C} is suitably large depends on G_k (or on k) with $G = G_0$ fixed. This is the issue of §10.2. The Constellation Table shows level 0 of Modular Towers for all alternating groups with $p = 2$ and 3-cycle conjugacy classes.

Further, covers in one component differ from those in another in a simple striking way. Suppose $\hat{\varphi}_{\boldsymbol{p}} : \hat{X}_{\boldsymbol{p}} \to \mathbb{P}^1_z$ is a cover attached to $\boldsymbol{p} \in \mathcal{H}^{\mathrm{in}}_{n,r}$. Then the differential $d\hat{\varphi}_{\boldsymbol{p}}$ has a divisor of form $2\hat{D}_{\boldsymbol{p}}$. (This happens whenever all elements in the conjugacy classes $\mathbf{C}$ have odd order.) The divisor $\hat{D}_{\boldsymbol{p}}$ is canonically defined over $\boldsymbol{p}$. Let $\dim(\hat{D}_{\boldsymbol{p}})$ be the dimension of the space of meromorphic functions h on $\hat{X}_{\boldsymbol{p}}$ for which $(h) + D_{\boldsymbol{p}} \ge 0$ (Chap. 3, Chap. 4).

So, $\hat{D}_{\boldsymbol{p}}$ defines a half-canonical divisor at each point on $\mathcal{H}^{\mathrm{in}}_{n,r}$, and a half-canonical divisor class on $\mathcal{H}^{\mathrm{in,rd}}_{n,r}$. A formula of Fried-Serre ([**Fri96**], [**Ser90b**]) says the components of $\mathcal{H}^{\mathrm{in,rd}}_{n,r}$ separate according to $\dim(\hat{D}_{\boldsymbol{p}})$ modulo 2. For $r \ge n$, there is an $\oplus$ component of *even* half-canonical classes, the other of *odd*. For the components of $\mathcal{H}^{\mathrm{abs}}_{n,r}$, define a similar divisor $D_{\boldsymbol{p}}$. Then, the formula for even or odd half-canonical classes is $\dim(D_{\boldsymbol{p}}) + r$ [**Ser90b**, Thm. 3]. Note: When $X_{\boldsymbol{p}}$ has genus 0, $\dim(D_{\boldsymbol{p}})$ is 0. Alternating $\otimes$ and $\oplus$ signs in the first row of the Constellation Table correspond exactly to r. §10.2 shows this is a small piece of an invariant applying to every Modular Tower.

10.2. Conjugacy class products. Examples show the Branch Cycle Lemma and ω invariant (§10.2.2) combination work well in this profinite context. Still, computing ω is not yet easy.

10.2.1. *How modular representations appear.* Computing the ω invariant for a Modular Tower relies on modular representation theory. The ω invariant is trivial for the usual modular curve tower. Here the kernel of ${}_p\tilde{G} \to G$ is one dimensional (${}_p\tilde{G} = \mathbb{Z}_p \times^s \{\pm 1\}$ and $\ker_0 = p\mathbb{Z}_p$). It is, however, more interesting for Modular Towers in the Constellation Table of §10.1.

Consider the location $(5,4)$. Four repetitions of the conjugacy class C_3 of 3-cycles appear there. Here consider it a conjugacy class in ${}_2\tilde{A}_5$ (${}_p\tilde{G}$ for A_5 and $p = 2$). As above, let C_3^4 be all products of four elements from C_3. Let M_k be G_k/G_{k+1}, the G_k module associated to level k. For any G_k submodule M'_k of M_k there is a quotient $G_{k+1}/M'_k = G'_k$. A special case is when $M_k/M'_k = W_k$ is maximal for G_k acting trivially on it.

Suppose G_0 is a perfect group (includes any simple group). Then, W_k is the maximal exponent p Schur multiplier of G_k and $G'_k = R_k$, the *representation* cover of G_k (Chap. 5). This A_5 case has $p = 2$ and $R_k \to G_k$ has kernel $\mathbb{Z}/2$ for each k.

Let O be an H_r orbit of $\mathrm{Ni}(G_k, \mathbf{C})$ with $\boldsymbol{g}$ a representative. Since $R_k \to G_k$ is a central extension, $\boldsymbol{g}$ has a unique lift to $\tilde{\boldsymbol{g}} \in R_k^r \cap \mathbf{C}$. If the product-one condition holds for $\tilde{\boldsymbol{g}}$, then it is in $\mathrm{Ni}(R_k, \mathbf{C})$. Otherwise let $s(\boldsymbol{g}) \in W_k$ be the product of the

$\tilde{\boldsymbol{g}}$ entries. Running over all such orbits O creates a subset $\mathbf{Obs}_{1,k} = \mathbf{Obs}_1(G_k, \mathbf{C})$ of W_k not containing the identity.

Suppose O is an H_r orbit with $s(\boldsymbol{g}) = 1$. Consider $M_{k+1} \lneq M'_k \lneq W_k$, with M'_k a G_k submodule. Call O *obstructed* at M'_k if these two properties hold.

(10.2a) $\boldsymbol{g}$ lifts to $\mathrm{Ni}(G'_k, \mathbf{C})$, but not to $\mathrm{Ni}(G_{k+1}, \mathbf{C})$.
(10.2b) M'_k is minimal with this property.

From [**FrK97**, §2] (or Chap. 5), G_{k+1}/M'_k has a nontrivial p part to its Schur multiplier. Also, M'_k contains a proper submodule distinct from $\mathbf{1}_{G_k}$. Under these assumptions (running over allowable orbits O) put M'_k in the set $\mathbf{Obs}_{2,k}$. We state a problem only $\mathbf{C} = \mathbf{C}_{3^r}$ (general formulation in Chap. 5). The answer is not known even if $r = 4$.

PROBLEM 10.2 (Commutator Problem). Fix $r \geq 4$ even. What are the elements of $\mathbf{Obs}_{1,k} = \mathrm{C}_3^r \cap W_k \setminus \{1\}$? Suppose k is large. Is this set just the identity? Then, the same question for $\mathbf{Obs}_{2,k}$ where we ask if it is empty for k large.

10.2.2. *Interpreting Problem 10.2.* Notice the problem is about commutators. Suppose r is even and C is any conjugacy class with $\mathrm{C} = \mathrm{C}^{-1}$. Then, elements of C^r are products of $r/2$ commutators of form (g, g') with $g, g' \in \mathrm{C}$. Now assume G_0 is a perfect group. Then, so are the G_k s for all k. Therefore, for r large, all elements of G_k are in C^r. The crucial elements are in W_k? For example, make a graph on the group G_k. Elements of G_k are the vertices, and edges are pairs $g_1, g_2 \in R_k$ with $g_1 g_2^{-1} \in \mathrm{C}$. As a function of k, what is the minimal distance between 1 and $W_k \setminus \{1\}$?

The sets $\mathbf{Obs}_{1,k}$ and $\mathbf{Obs}_{2,k}$ give a version of the ω invariant (§10.2.2, Chap. 5, [**Fri95a**, Part III], [**Ser90a**]). This *big* invariant $\omega(O)$ is a collection of conjugacy classes in the kernel of ${}_p\tilde{G} \to G_0$. An H_4 orbit that contributes to the sets **Obs** is obstructed; O has nothing above it at level $k+1$. Suppose we know these sets and they determine the $\bar{\mathbb{Q}}$ components of $\mathcal{H}_k^{\mathrm{rd}}$. Then, it is easy to compute definition fields of obstructed components contributing to $\mathbf{Obs}_{i,k}$ $i = 1, 2$.

10.3. The diophantine effect of few components. Take $r = 4$. Chap. 5 shows the genus of components in the sequence (8.6) goes up with k. That suffices to prove Conj. 8.3 when $r = 4$. For example, level 0 of the $(A_5, \mathbf{C}_{3^4})$ Modular Tower (§10.1) has one genus 0 component. Yet, level 1 has two components of respective genuses 12 and 9. The latter is obstructed [**BFr02**].

This one example illustrates the influence of Schur multipliers (equivalent to distinguishing θ characteristics). Why no obstructed component at level 0, and then such appears at level 1? The Schur multiplier presence at level 1 comes from two same length (1152) H_4 orbits on $\mathrm{Ni}_1^{\mathrm{in}}$. So, the inner Hurwitz space has two absolutely irreducible components of the same degree as covers of U_4. Yet, they aren't conjugate under $G_{\mathbb{Q}}$. The H_4 orbits gave distinct permutation representations that show profoundly in the cusps of the reduced spaces cover $\mathbb{P}^1_j$.

Suppose $r = 4$ and all components at some level of a Modular Tower have genus least 2. This assures only finitely many points (no matter what is the number field K) at some level k. That does come from Falting's Theorem (the former Mordell Conjecture [**Fal83**]), though there are other older techniques that are more explicit about computing the exceptional values of k [**Fr02**, §5].

What, however, will help analyze levels of a Modular Tower when $r \geq 5$; they are no longer curves? We don't know. It would be valuable to show level k components are varieties of *general type* for large k. Then, according to a conjecture of Lang, rational points on that level would lie in a lower dimensional subset. That would be progress, though not the quality of Conjecture 8.3.

More to the point would be a *canonical height* on a Modular Tower. Having in print background for developing this is an important goal of this book.

10.4. Height functions. Let K be a number field. Let $\mathcal{H}_k^\dagger$ be the unobstructed components of $\mathcal{H}_k^{\rm rd}$ (§10.2.1). The goal is a function $H_{G,\mathbf{C}} : \mathcal{H}_0^{\rm rd} \to \mathbb{R}$ whose properties prove Main Conjecture 8.3. That's simple enough and too much to expect. So, following [**Fal83**], aim for a finiteness result. Consider finding functions $H_k : \mathcal{H}_0^{\rm rd} \to \mathbb{R}$, $k = 0, \dots,$ with these properties.

(10.3a) $H_k(\boldsymbol{p})$ is nondecreasing in k for each fixed $\boldsymbol{p}$.
(10.3b) For k large it is positive on a nontrivial Zariski open subset V_k of $\mathcal{H}_0^{\rm rd}$.
(10.3c) H_k is a sum of local height functions, one for each prime dividing $|G|$.
(10.3d) There are no K points on $\mathcal{H}_k^\dagger$ over V_k.
(10.3e) When $r = 4$, $\mathcal{H}_k^\dagger$ consists of finitely many curves. For k large, H_k should detect that the genus of all components of $\mathcal{H}_k^\dagger$ has gone beyond one.

Should such a function be effective? Bounding k with H_k not positive on an open set is only one critical problem. As important is to describe the *nonordinary* (see §10.5) locus that is the intersection of $\cap_k(\mathcal{H}_0^\dagger \setminus V_k)$. There also must be an overall measure using the branch points. The primes dividing $|G|$ contribute heavily to a measure of how branch points behave.

10.5. Introducing nonordinary points. We prefer to think of Conj. 8.3 as the Main *Operating* Conjecture. It's value is to find failures in nonobvious places. These would provide astounding realizations for Inverse Galois Problem. [**FKVo98**] has an example of a Chevalley group $G_0 = \mathrm{PGL}_n(p)$ (with certain special p and n and conjugacy classes $\mathbf{C}$ (with $r = 5$). The p-adic version $G^\dagger$ is a p-Frattini cover of G_0 (a common phenomenon, attested to in [**Ser86**]). It has characteristic quotients $G_k^\dagger$ formed as in (8.1). Then, there is a projective system of $(G_k^\dagger, \mathbf{C})$ realizations (over some number field K).

Since $G^\dagger$ is a p-Frattini cover of G_0, it is the image of a map ${}_p\tilde{G} \to G^\dagger$. Let $\ker^*$ be the kernel of this map. So, this gives a K point on a significant Modular Tower *quotient*. There is exactly one point in $\mathcal{H}^{\rm in,rd}(G_0, \mathbf{C})$ under a K point in the tower. It would be proper to call such a point *extraordinary*. The literature, however, uses the name *nonordinary point*. Justifying that name, and locating nonordinary points and there corresponding Modular Tower quotients is a topic motivated by classical problems.

10.5.1. $\mathbb{R}$ *contribution to height.* Cusps of $\mathcal{H}_k^{\rm rd}$ guide us to the behavior at the real prime. Cusps attached to H-M representatives give a degeneracy that goes with $\mathbb{R}$ contribution to the height function. This is what happens at level 1 of the $(4, 5)$ location. Elementary techniques of [**BFr02**] and [**DFr90**] use uniformization of $\mathbb{R}$ points on Hurwitz spaces.

The less elementary part comes from interpreting them with group theory. Combining this with tangential base points as in §9.4.2 allows analyzing new functions on a Modular Tower. This includes the *even θ-nulls* from §6.4.2, which relate to other functions:

(10.4a) half-canonical differentials on the space $\mathcal{H}^{\mathrm{in,rd}}$; and
(10.4b) Scholl's Eisenstein series associated with cusps [**Scho86**].

The cusp tangential base point geometry allows quantifying the amount of degeneracy as points of the moduli space approach the cusp. Cusps attached to H-M representatives (as in (8.4)) support a total degeneracy. Including contributions for *all* cusps is still an open problem.

10.5.2. *Combining geometry and function theory.* Here is one development with modular curve precedents. Consider a Modular Tower (with $r = 4$) and a degree 0 divisor D supported in cusps of a component at some level of the tower. Sometimes such a D generates a torsion group on the Jacobian of the tower component. Cases include when the support of D consists of cusps associated to H-M representatives (as in $\oplus$ components of §10.1). We give a brief outline.

Under the hypotheses, consider the automorphic function θ_0 on the reduced Hurwitz space coming from the θ-nulls along the fibers of the family. Scholl associates to D a sum E_D of Eisenstein series. Since D is a divisor on the curve giving the Modular Tower component, it corresponds to a logarithmic differential on this curve (§6.3.1). This is E_D.

So, following (6.4), our goal is to evaluate E_D using θ functions. An example place would be the level 0 component $\mathcal{H}^{\mathrm{rd}}_{0:(5,4)} = \mathcal{H}^{\mathrm{rd}}_0$ of the Modular Tower at locus $(5, 4)$ of the §10.1 Constellation Table. This component has genus 0. Its Jacobian is trivial. So we don't mean a θ function on $\mathcal{H}^{\mathrm{rd}}_0$ (or on $\mathcal{H}^{\mathrm{odd}}_0$ where this computation really happens, see §10.6). Yet, it is much more than a genus 0 curve. It is a moduli space from whose points we gather data.

Evaluate a significant 3rd kind differential such as E_D from θ_0 at each cusp tangential base point (as in §9.4.2) in the support of D. As θ_0 is canonically defined for a family over $\mathbb{Q}$, its expansion at the cusps has algebraic coefficients. A Theorem of Waldschmidt [**Wa79**] interprets this algebraic coefficient statement. It is equivalent to D generating a torsion group in the Jacobian.

Since these components are moduli spaces, this has interpretations for the Inverse Galois Problem. Here is a low-brow corollary of the geometry in this story. There are exactly three regular $\mathbf{C}_{3^4}$ realizations (up to $\mathrm{SL}_2(\mathbb{Q})$ action) of the spin group cover of A_5. These realizations correspond to three points on the genus 1 pullback of $\mathcal{H}^{\mathrm{rd}}_0$ to the λ-line. The cusps there generate a group of order 12 over $\mathbb{Q}$. Nine of those points are cusps, but three are not.

A bigger story, however, requires considering a curve $\hat{X}_{\boldsymbol{p}}$ (of genus 21) corresponding to a point $\boldsymbol{p}$ in the real locus of a tangential base point of §9.4.2 type. Calculation of E_D gives a measure of how $\hat{X}_{\boldsymbol{p}}$ degenerates (into unions of copies of $\mathbb{P}^1_z$) as it pushs along toward evaluation at the tangential base point. It is a bigger story because function theory informs about cusps on all projective systems of components in the Modular Tower. Height data involves all levels of a Modular Tower. Chap. 9 tells that story, related to [**R77**], [**CTT98**] and [**GR78**].

This focused example brings together function theory, geometry and arithmetic on a Modular Tower. It illustrates many potential applications of Modular Towers.

10.5.3. *p contribution to the height.* This investigation comes from restricting the action of $G_{\mathbb{Q}}$ to $G_{\mathbb{Q}_p}$, p the prime of the Modular Tower. After Hasse's invariant, the idea of *nonordinary* points for p started with Serre-Tate theory ([**Se72**], [**Se68**]). Ihara used Hasse's invariant in examples that still inform us [**Ihar00**]. Mochizuki's use of canonical Frobenius elements defines the meaning of ordinary (and nonordinary?) directly [**Moc96**]. His theory, however, must descend from the moduli space of curves of genus g to the moduli spaces in a Modular Tower. Defining and identifying *nonordinary* points on a Modular Tower is at the top of the problems this text aims at (Chap. 9).

In Ihara's approach the theory is entirely nilpotent. He has p-adic versions of classical functions. Especially, such have appeared from the action of $G_{\mathbb{Q}}$ on the second commutator quotient of $\pi_{\overline{0\,1}} = \pi_1(\Lambda_4, \overline{0\,1})^{(p)}$ (§9.1). Coordinates arise from going to the induced Lie algebra actions. The rubric comes from *Gassner-Magnus matrices*. These give coordinates for the Lie algebra of an automorphism group acting on the second graded term of the Lie algebra of $\pi^{(p)}_{\overline{0\,1}}$. Abelian covers of Λ_4 are Fermat curves. Similar to the discussion of §8.4.4, this is a p-adic Lie algebra acting on the p-Tate module of Fermat curves. [**Ihar91**] describes the appearance of Jacobi sum grössencharacters.

These use partials (in the Lie algebra) of $m_\sigma(x, y)$ from (9.1). The Ihara-Drinfeld relations are vital here. Nakamura connects Ihara's example and another case: Replace Λ_4 by an elliptic curve minus one point. When it is an elliptic curve with bad degeneration at p, [**Na98**] produces a *Tate Eisenstein series*. This is a logarithmic partial of Ihara's series. For some examples from the Constellation Table, the real Eisenstein series of §10.5.1 have p-adic parallels to Nakamura's examples. This is what we mean by function theory on the nilpotent part of Modular Towers.

The nonlinear part, from G_0 still has a classical function relation as with θ invariants in §10.1. The nilpotent part, in examples, produces global functions on the moduli space. Specifically we expect these functions, at least those from H-M representative cusps, to tell us about nonordinary points.

10.6. Weil's distributions. Look at (6.4) again. Weil's thesis constructed an analog of it: $(h(x)) \equiv \prod_{i=1}^u \theta^w_{x_i^0}(x)/\prod_{i=1}^u \theta^w_{x_i^\infty}(x)$. Here is its meaning. Both sides are fractional ideals in the ring of integers $\mathcal{O}_K$ of a number field K. The $\equiv$ sign means the left and right are equal up to a bounded fractional ideal. The left side is the principal fractional ideal that $h(x)$ generates. Most important, of course, are the functions $\theta^w_{x'}\colon x \mapsto \theta_{x'}(x)$ maps K points x into integral ideals. This function is defined only up to $\equiv$. Weil's distribution theorem allowed he (and Siegel [**Si29**]) to perform diophantine magic.

This works to define part of the height data for the commutative quotient of a Modular Tower. We explain. Denote the commutator subgroup of a profinite group H by (H, H). Replace inner homomorphism classes of $M_{\boldsymbol{p}} \to {}_p\tilde{G}$ in §8.4.2 by the sequence $M_{\boldsymbol{p}}/(\pi_1(X_{\boldsymbol{p}}), \pi_1(X_{\boldsymbol{p}})) \to {}_p\tilde{G}/(\ker_0, \ker_0)$. The question is now a refined question about subspaces of the Tate module of $J(X_{\boldsymbol{p}})$.

[**Si29**] starts with a crude set of reductions by going to a finite extension of K. Doing this point-by-point along a Hurwitz space would be a disaster. Canonical heights avoid this. Here is a related allusion to the odd half-canonical classes.

Following a comment from §6.4.2, we should replace $\mathcal{H}_0^{\mathrm{rd}}$ by its pullback to a space $\mathcal{H}_0^{\mathrm{odd}}$. Points of $\mathcal{H}_0^{\mathrm{odd}}$ are pairs, $\boldsymbol{p} \in \mathcal{H}_0^{\mathrm{rd}}$ with an *odd* half-canonical class on

$X_{\boldsymbol{p}}$. When the general point of $\mathcal{H}_0^{\rm odd}$ carries a non-degenerate half-canonical class (§6.4.2) this starts an effective analysis. We still don't know what to do in the general case.

Here is a final word on even half-canonical classes. The locations in the Constellation Table with $\oplus$ support even half-canonical classes varying analytically with the coordinates of the Hurwitz space. Suppose the attached $\theta_{\boldsymbol{p}}$ is not zero at the origin of $J(\hat{X}_{\boldsymbol{p}})$. Then, taking its value at the origin provides an *automorphic form* (the meaning is precise and conventional when $r = 4$) on $\mathcal{H}_0^{\rm in,rd}$ whose value appears in inspecting properties of the cusps.

10.7. Prelude to the general case? Level 1 of Constellation Table Modular Towers has further surprises related to the Schur multipliers of the level 1 groups. These illustrate practical applications of the nilpotent extension theory of covers (Chap. 9). There are lessons for group theory and geometry.

One is that nilpotent extensions (of any given group, simple or solvable) occur in many constructions with underlying geometric meaning. Such events don't naturally extend to solvable extensions much less to general (pro-)finite group theory. Consider lessons from the dihedral group D_p and its association with the modular curve case of Modular Towers. It has a natural series of groups by changing the prime p to any other prime: vary p among primes. That isn't, however, so special.

10.7.1. *Hecke operators.* Consider the notation arising from §8.1 for the dihedral group $D_p = \mathbb{Z}/p \times^s \{\pm 1\}$. Let p' be a prime distinct from 2 or p. The famous *Hecke Operators* of modular curve theory come from there being several values of $j(\tau_1), \dots, j(\tau_{p'+1})$ for $\tau \in \mathbb{H}$ where $j(p'\tau_j)$ is a particular value. This produces an algebraic correspondence represented by a curve $T_{p'}$ on $X_0(N) \times X_0(N)$. A natural correspondence automatically induces an action on holomorphic differentials and cohomology, etc. Significantly, this correspondence produces a lift of the Frobenius correspondence from characteristic p': *The Eichler-Shimura congruence formula.*

Here is how to interpret this from a Modular Tower viewpoint. Consider the Nielsen class $\mathrm{Ni}(D_p, \mathbf{C}_{2^4})^{\rm abs} = \mathrm{Ni}_0$. Suppose $\boldsymbol{g} \in \mathrm{Ni}_0$ is the branch cycle description of a cover $f_{\boldsymbol{g}} : \mathbb{P}^1_w \to \mathbb{P}^1_z$ with D_p as monodromy group and involutions as branch cycles. This description comes from a choice of classical generators of $\pi_1(U_{\boldsymbol{z}}, z_0)$. Then, the Galois closure of $f_{\boldsymbol{g}}$ is an elliptic curve E which has a canonical degree p isogeny to another elliptic curve E'. Let $A_{p'}$ be any cyclic subgroup of p' order on E and let $A'_{p'}$ be its image in E'. The morphism $E/A_{p'} \to E'/A'_{p'}$ modulo multiplication by -1 produces a new rational function $f_{\boldsymbol{g},p'}$. This is the genesis of the Hecke theory. It won't extend easily to a general Modular Tower. Yet, there are other candidates for constructions like the above.

Let H be any finite group acting irreducibly on a $\mathbb{Z}$ module V of rank m. Consider conjugacy classes $\mathbf{C}$ of H. (Take $H = \{\pm 1\}$ and $V = \mathbb{Z}$ to get the dihedral group situation.) Consider the semi-direct product $V \times^s H$ and then for each prime p, take $V/pV \times^s H = H_p$. Suppose $(p, |H|) = 1$. Extend the conjugacy classes to H_p. Then, apply the Modular Tower construction to $(H_p, \mathbf{C}, p)$.

Let p' be a prime distinct from those dividing $|H|$ and p. Add in $V/p'V$ with an H action to get $V/pGV \times V/p'V \times^s H$ with an extension of the branch cycles $\mathbf{C}$ to this. This produces situations analogous to that for Hecke operators. This remains unexplored territory. A few examples will encourage further exploration. Examples of this type should give Modular Towers uniformizing natural collections of varieties defined over $\mathbb{Q}$, when the Branch Cycle Lemma conditions imply $\mathbb{Q}$

structures (§8.2). Given H what varieties have such a natural uniformization? We haven't developed the expertise to consider this in detail. The value of making such a formulation is that all the arithmetic (including rational point statements) will fall under a uniform rubric. This would include using the Main Conjecture 8.3 on Modular Towers.

10.7.2. *Separating the nilpotent tail and the nonnilpotent quotient.* Group extensions of a given G_0 by a solvable group behave no better than general extensions of G_0. Roughly, the only distinction between solvable (excluding nilpotent) and general groups is that only cyclic groups appear as simple composition factors in solvable groups. That is the author's belief. With it goes the feeling that each finite nonnilpotent group G_0 generates its own intrinsic geometry. The discrete invariants of §10.2 capture much of this.

Then, there is a rich function theory appearing in the geometry from the nilpotent tail of a Modular Tower (as in §10.5). Together they separate the nilpotent tail from the nonnilpotent quotient. We believe this separation is natural and inevitable, and will never be breached. Further, our diophantine experience with problems involving solvable groups is that they belong more with the nonnilpotent quotient than with the nilpotent tail. We intend these comments to raise questions about modern understanding of Galois' famous theorem.

11. Aids to the reader and choice of actions

Expression numbers go from the left margin and most running lists use latin letters. For example, item 3 of expression 2 of section 5 of chapter 4 is (5.2c). Reference to it in another chapter would use the variant Chap. 4 (5.2c). Lemmas, corollaries, theorems, remarks, definitions, and examples fall under one collection of numbers: Definition 3 of section 9, written Def. 9.3, might follow Ex. 9.2. Figures have their own numbering system. Exercises appear as the last section in each chapter. References to these follow a special notation: Exercise 3, part c) appearing in section 9 appears as [9.3c]. Again, the chapter is given if it is in another chapter than that being read. Bibliographical items have notational shorthand for the author's(s') name(s), followed by a pinpoint reference, the usual LaTeX scheme. Like [**Ahl79**, p. 31].

There is sufficient material for a year course around two themes: fundamental groups in complex analytic geometry and families of Riemann surfaces. A third semester of complex analysis might cover just Chap. 2, Chap. 3 and Chap. 4. One year of complex analysis and one semester of graduate algebra are sufficient background. We assume undergraduate topology, as in a junior-senior analysis course, for proper background for the treatment of fundamental groups (Chap. 3).

The author spent much time considering on which side permutation groups would act. He chose the *right side* as the primary action side. That is, when $g \in S_n$ is an element of the symmetric group acting on integers $\{1, \dots, n\}$, usually we write g applied to i as $(i)g$. It is not possible to be universally consistent. It is so typical to act with matrices on the *left* that with matrix groups we follow the usual convention. In making this decision there were these problems:

- Eventually, no matter the starting side, situations force simultaneous action on the other side.
- Group products in fundamental groups work with permutation representations only if you act on the right.

- Finite group theorists in the United States act on the right.

Many students trained by such books as [**Lan71**] and [**Jac85**] put group actions on the left. Neither book, however, does enough group theory training to dissuade from the need to spend considerable further time. Of course, there are always notational ways around the difficulties in any one situation.

The exposition on Riemann-Roch and the Picard groups in Chap. 4 quotes such sources as [**FaKr90**], [**Mum76**] and [**Se59**]. In addition, later examples quote finite group theory results outside the scope of this book. This goes with the book's aspiration to teach group theory *interpretation*, rather than detail. It simplifies exposition on examples to use references to [**Vö96**], in place of lengthy computations. The differences between the two books are large, ours geometric, while [**Vö96**] is more group theoretic. We, however, spend as much time on group theory. Our intention is to teach its use through examples to a generation of students interested in using Riemann surfaces who have little training in group theory. Still, the reader will recognize the two authors had more than a passing acquaintance.

12. Poetry and Mathematics

In the solipsistic world of mathematics, there are still many who find the subject matter of moduli of covers —that this book tackles —*beautiful*. The author agrees, with reservations.

Mathematics isn't poetry though Keats gave us hope it might be!

> A thing of beauty is a joy for ever: its loveliness increases; it will never Pass into nothingness; but still will keep a bower quiet for us . . . : *From Endymion*

12.1. The grandest virtues. The grandest virtue of mathematics is its modularity; That it builds from pieces. Second: That it lasts so long. An ingredient here is its independence of the framing secular language used. One easily sees the appearance of pythagorean triples in the Rind Papyrus. Yet, few would appreciate that the pyramid architect Imhotep was a *god* to the Egyptian Middle Kingdom.

Still, the converse of Keats' rhyme may not hold. The persistence of mathematics does not imply its beauty. The Durants suggest:

> Poetry makes of language and feeling a music that cannot be heard across the frontiers of speech. [**Du54**, p. 77]

Independent of my abilities with written and spoken German, I can thrill to the simplifying structure Riemann brought to algebraic functions. Though I never think to tack a new verse onto *Endymion*, adding consequentially to Abel, Galois and Riemann is an ever present goal.

12.2. The eye of the beholder. Mathematical colleagues often don't appreciate the goals of other areas. One θ function adherant can't imagine the value of preoccupation with diophantine properties of large fields, and vice-versa. I'm speaking of co-writers I've known for over 30 years. It is one example of many.

If mathematicians sincerely fail to see the beauty of each others' grand enterprises, how could the world at large have the language and intellectual base to agree with what we think beautiful? In practice it is extremely difficult to explain the beauty of mathematics, even on occasion to a Nobel Prize winning Chemist; or to nonmathematical graduates of our elite institutions. Our perceptions can fail from not appreciating the depth of what we already know before we address our papers. Failure to recognize the absorbed contribution of previous generations has much to do with the present hubris of today's mathematical community.

In particular, we (collectively) learned much from Abel, Galois and Riemann, though the first two produced very few theorems, and the third influenced mathematics through something strikingly beyond theorems. Abel and Galois used group theoretic interpretation to bring simplicity to an area littered with facts labeled as theorems. Riemann created coordinates for analyzing the details of a world of baffling geometries. All inherited and enhanced the goal of synthesizing algebra and geometry that Lagrange first articulated. In the age of specialization, we still recognize the coherency of mathematics in large part because of these people.

Mathematics is the *only* language supporting rich neologisms that bears its topics unadulterated to other areas and other generations. It overwhelms us locally in our seminars and colloquiums. Our students rail against what they think its incoherence, though its free inundating associations cause far more problems. The world, however, slowly accustoms to it, long forgetting —especially in related sciences—what a miracle of persistence is wrought by the foundation of clear definition. Definition that more than spotlights a resonant example; fluid definition that takes on new shape in each generation. In its fluidity it lasts and lasts and lasts. So we are certain, will the ideas of Abel, Galois and Riemann.

12.3. Two afterthoughts. The following found its motivation from θ functions and diophantine properties of large fields. There is an exact sequence [**FrV92**]:

$$1 \to \tilde{F}_\omega \to G_{\mathbb{Q}} \to \prod_{n=2}^{\infty} S_n \to 1.$$

The group on the left is the profree group on a countable number of generators. The group on the right is the direct product of the symmetric groups, one copy for each integer. The absolute Galois group is caught between two known groups.

Here is a paraphrase from [**Fri99**, Acknowledgements]

> The 20th century of mathematics belongs to group theory applications; I don't mean just Lie groups or classifications.

CHAPTER 2

ANALYTIC CONTINUATION

1. Why Riemann's Existence Theorem?

We start with two different definitions of algebraic functions. An imprecise version of Riemann's Existence Theorem is that these describe the same set of functions. Chap. 2 has two goals. First: To define and show the relevance of *analytic continuation* in defining algebraic functions. Second: To illustrate points about Riemann's Existence Theorem in elementary situations supporting the main ideas. Our examples are *abelian* algebraic functions. They come from analytic continuation of a branch of the log function. This also shows how integration relates algebraic functions to crucial functions that aren't algebraic. These examples depend only on homology classes, rather than homotopy classes, of paths. The slow treatment here quickens in Chap. 4 to show how Riemann's approach organized algebraic functions without intellectual inundation.

1.1. Introduction to algebraic functions. The complex numbers are $\mathbb{C}$, the nonzero complex numbers $\mathbb{C}^*$ and the reals $\mathbb{R}$. We start with analytic (more generally, meromorphic) functions defined on an open connected set D, a *domain* on $\mathbb{P}^1_z = \mathbb{C} \cup \{\infty\}$, the Riemann sphere: §4.6 defines *analytic* and *meromorphic*. The standard complex variable is z. When D is a disk, a function $f(z)$ analytic on D has a presentation as a convergent power series about the center z_0 of D. The first part of the book describes *algebraic functions* (of z). Let D be any domain in $\mathbb{P}^1_z$ and $z_0, z' \in D$. Denote (continuous) paths beginning at z_0 and ending at z' by $\Pi_1(D, z_0, z')$ (§2.2.2). Use $\Pi_1(D, z_0)$ for closed paths in D based at z_0. For any finite set $\boldsymbol{z} = \{z_1, \dots, z_r\} \subset \mathbb{P}^1_z$ denote $\mathbb{P}^1_z \setminus \{\boldsymbol{z}\}$ by $U_{\boldsymbol{z}}$.

1.1.1. *Riemann's definition of algebraic functions.* Suppose $f(z)$ is analytic in a neighborhood of z_0. Call f *algebraic* if some finite set $\boldsymbol{z} \subset \mathbb{P}^1_z$ has these properties.

(1.1a) An *analytic continuation* (Def. 4.1) of $f(z)$ along each $\lambda \in \Pi_1(U_{\boldsymbol{z}}, z_0)$ exists. Call this $f_\lambda(z)$. Let $\mathcal{A}_f(U_{\boldsymbol{z}})$ be the collection $\{f_\lambda\}_{\lambda \in \Pi_1(U_{\boldsymbol{z}}, z_0)}$.
(1.1b) The set $\mathcal{A}_f(U_{\boldsymbol{z}})$ is finite.
(1.1c) For $z' \in \boldsymbol{z}$, limit values of f_λ along $\lambda \in \Pi_1(U_{\boldsymbol{z}}, z_0, z')$ is a finite set.

1.1.2. *Standard definition of algebraic functions.* There is another definition of algebraic function (of z). Suppose $f(z)$ is analytic on a disk D. It is algebraic if some polynomial $m(z, w) \in \mathbb{C}[z, w]$ (nonconstant in w) satisfies

(1.2) $m(z, f(z)) \equiv 0$ for all $z \in D$.

This chapter explains (1.1) and its equivalence with (1.2) (Prop. 7.3).

Simple examples illustrate (1.1) and (1.2). These often appear briefly in a first course in complex variables. Though they give only algebraic functions with abelian monodromy group, they hint how Chap. 4 *lists* all algebraic functions.

We review elementary field theory as it applies to $f(z)$ satisfying (1.2). With no loss assume $m(z, w)$ in (1.2) is irreducible in the ring $\mathbb{C}[z, w]$ ([9.8]) and $f(z)$ satisfies (1.2). Any graduate algebra book is proper for this review, including [**Lan71**], [**Jac85**] and [**Isa94**]. The latter, with the best treatment of permutation representations and group theory, will be our basic reference. [**Isa94**, Chap. 17] contains material supporting the comments of §1.2.

1.2. Equivalence of algebraic functions of z. Let $\mathbb{C}(z)$ be the field of the rational functions in z. Its elements $u(z)$ consist of ratios $P_1(z)/P_2(z)$ with $P_1, P_2 \in \mathbb{C}[z]$. Standard notation denotes the greatest common divisor of P_1 and P_2 as (P_1, P_2). Suppose P_1 and P_2 have no common nonconstant factor: Write this as $(P_1, P_2) = 1$. Then the integer *degree* of $u(z)$, $\deg(u)$, is $\max(\deg(P_1), \deg(P_2))$. The Euclidean algorithm finds the greatest common divisor of P_1 and P_2. Factor this out to compute $\deg(u)$. This degree is also the *degree* of the field extension $\mathbb{C}(z)$ over $\mathbb{C}(u(z))$: $[\mathbb{C}(z) : \mathbb{C}(u(z))]$ [9.3].

Suppose L and K are fields with $K \subset L$. The degree $[L : K]$ of L/K is the dimension of L as a vector space over K. Assume $L = K(\alpha)$ for some $\alpha \in L$. Then, $[L : K]$ is the maximal number of linearly independent powers of α over K: the *degree* of α over K. This degree is also the minimal positive degree of an irreducible polynomial $f_\alpha(w) \in K[w]$ having α as a zero. Up to multiplication by a nonzero element of K, $f_\alpha(w)$ is unique. If L/K is a field extension, $\alpha \in L$ is *algebraic* over K if $[K(\alpha) : K] < \infty$.

1.2.1. *The degree of* $\mathbb{C}(z)/\mathbb{C}(u(z))$. Introduce variables z' and w'. Write $u(w')$ as $P_1(w')/P_2(w')$ with $(P_1, P_2) = 1$, and

$$m(z', w') = P_1(w') - z'P_2(w') \in \mathbb{C}[z', w'].$$

Then, $m(z', w')$ is irreducible of degree $n = \max(\deg(P_1), \deg(P_2))$ [9.3]. Consider $m(z', w')$ as a polynomial in w' with coefficients in the field $\mathbb{C}(z')$. Let w'' be a zero of this polynomial in *some* algebraic closure of $\mathbb{C}(z') = K$. Then, $L = \mathbb{C}(z')(w'') = \mathbb{C}(w'')$ is the quotient field of the integral domain $R = K[w']/(m(z', w'))$. It is a degree n extension of $\mathbb{C}(z')$. Now $\mathbb{C}(z')$ is isomorphic to $\mathbb{C}(u(z))$: map z' to $u(z)$. Map w'' to z to extend this to an isomorphism of L with $\mathbb{C}(z)$.

1.2.2. *Degree of function fields over* $\mathbb{C}(z)$. §1.2.1 uses Cauchy's abstract production of $\mathbb{C}(z')(w'')$ with w'' a zero of $m(z', w')$ [**Isa94**, Lem. 17.18]. It, however, explicitly identifies w'' with z and z' with $u(z)$. Putting L in $\mathbb{C}(z)$, a *genus* 0 or *pure transcendental* field over $\mathbb{C}$, is convenient for seeing the algebraic relation between functions —like z' and w''.

Now assume $f(z)$ is any algebraic function according to (1.2). Similarly construct $L = \mathbb{C}(z, f(z))$, a degree $\deg_w(m(z, w))$ field extension of the rational functions $\mathbb{C}(z)$. This is the *algebraic function field* of m (or of f). Call any $f^* \in L$ with $L = \mathbb{C}(z, f^*)$ a *primitive generator* of $L/\mathbb{C}(z)$. (Or, f is just a primitive generator when reference to z is clear.)

1.2.3. *Equivalence of presentations of* $L/\mathbb{C}(z)$. Infinitely many algebraic functions f gives the same field L up to isomorphism as an extension of $\mathbb{C}(z)$. Within a fixed algebraic closure of $\mathbb{C}(z)$ it is abstractly easy to list all primitive generators of L. They have the form $f^* = g(z, f_k)$ with f_k any other zero of $m(z, w)$ and $g(z, u) \in \mathbb{C}(z)[u]$. To assure $\mathbb{C}(z, f^*) = L$ add that $[\mathbb{C}(z, f^*) : \mathbb{C}(z)] = [L : \mathbb{C}(z)]$. Riemann's Existence Theorem lists algebraic extensions of $\mathbb{C}(z)$ efficiently by listing the isomorphism class of extensions $L/\mathbb{C}(z)$ and not specific algebraic functions.

Suppose $\mathbb{C}(f(z))$ contains z. Then, $L = \mathbb{C}(f(z))$ is *pure transcendental*. So, it is easy to list (without repetition) generating algebraic functions. Even, however, when the total degree of m is as small as 3, L usually is not pure transcendental field [9.10g]. While listing generating functions of L is then harder, it isn't our main problem. To identify when two function field extensions $L_1/\mathbb{C}(z)$ and $L_2/\mathbb{C}(z)$ are (or are not) isomorphic is more important. Two questions arise: Is L_1 isomorphic to L_2? If so, does the isomorphism leave $\mathbb{C}(z)$ fixed?

Abel handled these questions for cubic equations. His results would have been easy if L was pure transcendental. This book includes applying Riemann's extension of Abel's Theorems. Riemann's Existence Theorem is the start of this extension.

Riemann's Existence Theorem foregoes having all algebraic functions within one convenient algebraic closure. There may be no unique algebraic closure of $\mathbb{C}(z)$ so useful as $\mathbb{C}$. §1.3 introduces the infinite collection of incompatible algebraically closed fields appearing in Riemann's Existence Theorem. Every algebraic function $f(z)$ appears in each of them.

1.3. Puiseux expansions. Consider the *Laurent field* $\mathcal{L}_{z'}$ consisting of series $f(z) = \sum_{n=N}^{\infty} a_n(z - z')^n$, with N any integer, possibly negative, where $f(z)(z-z')^{-N}$ is convergent in some disk about z'. Elements of $\mathcal{L}_{z'}$ define functions meromorphic at z'. Then, $\mathcal{L}_{z'}$ is a field, containing $\mathbb{C}(z)$ and we are familiar with it. It isn't, however, algebraically closed. To remedy that, for any positive integer e form $\mathcal{P}_{z',e}$, convergent series in a variable u_e. Think of u_e as $(z-z')^{1/e}$: $u_e^e = z-z'$.

For $e \mid e^*$ let $t = e^*/e$. Map $\mathcal{P}_{z',e}$ into $\mathcal{P}_{z',e^*}$ by substituting $u_{e^*}^t$ for u_e. Regard the union $\cup_{e=1}^{\infty}\mathcal{P}_{z',e} = \mathcal{P}_{z'}$ as a field, the *direct limit* of the fields $\cup_{e=1}^{\infty}\mathcal{P}_{z',e}$ with its set of compatible generators $\{u_e\}_{e=1}^{\infty}$. Details on the following are in [9.9].

LEMMA 1.1. *Suppose $\mathcal{P}^*/\mathcal{L}_{z'}$ is any field extension generated by a sequence of elements $\{u_e^*\}_{e=1}^{\infty}$ with these properties.*

(1.3a) *u_e^* is a solution of the equation $u^e = z - z'$.*

(1.3b) *$(u_{ee'}^*)^{e'} = u_e^*$ for all positive integers e, e': compatibility condition.*

Then, $u_e \mapsto u_e^*$ gives a canonical isomorphism between $\mathcal{P}^*$ and $\mathcal{P}_{z'}$ that is the identity on $\mathcal{L}_{z'}$. In particular, automorphisms of the Galois extension $\mathcal{P}_{z'}/\mathcal{L}_{z'}$ correspond one-one with compatible systems of roots of 1.

The field $\mathcal{P}_{z'}$ of *Puiseux expansions* around z' provides an explicit algebraically closed field extension of $\mathbb{C}(z)$. It is clear fractional exponents are necessary for an algebraic closure. It is harder to see they give an algebraically closed field (Cor. 7.5). The fields $\mathcal{P}_{z'}$ and $\mathcal{P}_{z''}$ are isomorphic. Such an isomorphism, however, restricts to mapping $\mathbb{C}(z) \to \mathbb{C}(z)$ by $z \mapsto z - (z'' - z')$. For comparing all algebraic functions of z we usually must regard these algebraically closed fields as distinct. Each, in its own way contains the field of algebraic functions (using either (1.1) or (1.2)).

Comparing expressions for a given algebraic function embedded in different Puiseux fields leads to our precise version of Riemann's Existence Theorem.

1.4. Monodromy groups and the genus. Both definitions (1.1) and (1.2) readily attach a group G_f to any algebraic function $f(z)$. Using an irreducible $m(z,w)$ from (1.2) (with $m(z, f(z)) \equiv 0$) it is the group of the splitting field of $m(z,w)$ over $\mathbb{C}(z)$ ([**Isa94**, p. 267] and [9.5]). The order of this group is the degree of the splitting field extension over $\mathbb{C}(z)$. Efficient use of group theory gives more structured information than describing field extensions. Knowing something about the Galois group is usually better information than comes from looking at polynomial coefficients.

§4.4.1 gives a geometric construction for G_f. Chap. 4 has this group as its main theme. This group reveals $\mathcal{A}_f(D)$ from (1.1) as the complete set of zeros w of $m(z,w)$ (Prop. 6.4). Then, G_f acts through analytic continuation. This representation of G_f on $\mathcal{A}_f(D)$ (of degree $\deg_w(m(z,w))$ is discrete data from f. *Discrete* here means the group G_f does not change with continuous changes in $\boldsymbol{z}$.

Every algebraic function f has another integer attached to it, the *genus* of its function field (Chap. 4). If $L = \mathbb{C}(z, f(z))$ is isomorphic to $\mathbb{C}(t)$ for some $t \in L$, it has *genus 0* as above. This means all genus 0 function fields are abstractly isomorphic. Note: The integer $[L : \mathbb{C}(z)]$ is rarely a good clue for computing the genus [9.3]. Abel's results allow viewing genus 1 function fields as similar to genus 0 function fields, though that similarity has limits. Crucial: Unlike genus 0 fields, there are many isomorphism classes of genus 1 function fields (over $\mathbb{C}$).

Abel's results allow listing isomorphism classes of genus 1 function fields, exactly as we list points of $\mathbb{P}^1_z$. That is, with a classical parameter j replacing z, finite values of j correspond one-one to isomorphism classes of genus 1 function fields. As with $\mathbb{P}^1_z$ the value $j = \infty$ requires special consideration. Even if L has genus 1, we don't easily find where its corresponding j value is in this list. Still, for many problems this is a satisfactory theory.

Riemann generalized much of Abel's Theorem to function fields of all genuses. Most difficult was his analog, for genus greater than 1, of a parameter space for isomorphism classes of fields. Variants on its study continue today, and this book is an example.

1.5. Advantages of Riemann's definition. Defining branches of $z^{\frac{1}{e}}$ (§8.3) on any disk D in $\mathbb{C}\setminus\{0\}$ gives a practical introduction to analytic continuation. This gives the simplest algebraic functions. Still, how would we have located $w = f(z)$ satisfying $f(z)^5 - 2zf(z) + 1 = 0$ by a similar definition? The field $\mathbb{C}(z, f(z))$, like $\mathbb{C}(z^{\frac{1}{e}})$, is pure transcendental [9.3]. Yet, this is not obvious from a Puiseux expansion of $f(z)$ around some point.

Suppose $f(z)$ is a convergent power series satisfying (1.1). Can we expect to find data appropriate to its description?: The set $\boldsymbol{z}$ of exceptional values, and the finite group expressing there are but finitely many analytic continuations around closed paths. Excluding elementary examples, the Riemann's Existence Theorem approach suggests it doesn't pay to give functions by their power series. *Elliptic functions* (Chap. 4 §7.1) are a good example where the functions are explicit, though power series don't give their definition. Riemann's Existence Theorem uses **group data** to replace power series information about $f(z)$.

This is practical, computable information about algebraic equations making Riemann's approach useful to the rest of mathematics. Especially it gives a way to track complete collections of related algebraic functions. This is the story of *moduli* of families of covers. Abel used the modular function classical texts call $j(\tau)$ where τ is a complex number in the upper half plane. We refine and generalize this theme.

2. Paths

We assume elementary properties of the *complete fields*, the *real numbers* $\mathbb{R}$ and the *complex numbers* $\mathbb{C}$ as in [**Rud76**, Chap. 1], [**Ahl79**, §1.1-1.3].

2.1. Notation from calculus. For each positive integer n, let $\mathbb{R}^n$ (resp. $\mathbb{C}^n$) be the set of ordered n-tuples $\boldsymbol{x} = (x_1, \dots, x_n)$ (resp. $\boldsymbol{z} = (z_1, \dots, z_n)$) of real (resp. complex) numbers. The set $\mathbb{R}^n$ is a *vector space* over $\mathbb{R}$: addition of $\boldsymbol{x} = (x_1, \dots, x_n)$ and $\boldsymbol{y} = (y_1, \dots, y_n)$ gives $(x_1 + y_1, \dots, x_n + y_n)$; and scalar multiplication of $\boldsymbol{x}$ by $r \in \mathbb{R}$ gives $r\boldsymbol{x} = (rx_1, \dots, rx_n)$. The zero element (*origin*) of $\mathbb{R}^n$ is $\mathbf{0} = (0, \dots, 0)$. The *inner product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is $\boldsymbol{x} \cdot \boldsymbol{y} = \sum_{i=1}^n x_i y_i$. The *law of cosines* (from high school trigonometry) interprets the dot product $\cdot$ to give the expression $|\boldsymbol{x}||\boldsymbol{y}|\cos(\theta)$ where θ is the (counter clockwise) angle from the side from $\mathbf{0}$ to $\boldsymbol{x}$ to the side from $\mathbf{0}$ to $\boldsymbol{y}$ in (a/the) plane containing $\mathbf{0}$, $\boldsymbol{x}$, $\boldsymbol{y}$. Define the *distance between* points $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ to be

$$|\boldsymbol{x} - \boldsymbol{y}| = \sqrt{(\boldsymbol{x} - \boldsymbol{y}) \cdot (\boldsymbol{x} - \boldsymbol{y})}.$$

Here are simple properties of the distance function.

(2.1a) $|\boldsymbol{x}| \ge 0$ and $|\boldsymbol{x}| = 0$ if and only if $\boldsymbol{x} = \mathbf{0}$.
(2.1b) $|\boldsymbol{x} - \boldsymbol{z}| \le |\boldsymbol{x} - \boldsymbol{y}| + |\boldsymbol{y} - \boldsymbol{z}|$ for $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^n$: the *triangle inequality*.

Thus, the distance function gives a *metric* on $\mathbb{R}^n$.

2.2. Elementary properties and paths. Multiplication of complex numbers is crucial, especially that each nonzero complex number has a multiplicative inverse. Still, vector calculus often appears in the study of analytic functions using the topological identification of $\mathbb{R}^2$ with $\mathbb{C}$. In standard coordinates: $(x, y) \in \mathbb{R}^2 \mapsto x + i\,y = z \in \mathbb{C}$. Rephrase multiplication of complex numbers on elements of $\mathbb{R}^2$: $z_1 \leftrightarrow (x_1, y_1)$ and $z_2 \leftrightarrow (x_2, y_2)$ gives the association $z_1 z_2 \leftrightarrow (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$. Beyond these properties we gradually introduce statements from a one semester graduate course in complex variables. Paths and integration, however, are so important, we pause for notation around integration of 1-forms and Riemannian metrics.

For $a, b \in \mathbb{R}, a < b$, $[a, b]$ denotes the closed interval of $\mathbb{R}$ with a and b as end points. A path in $\mathbb{R}^n$ consists of a *continuous* map $\gamma : [a, b] \to \mathbb{R}^n$ for some choice of a and b with $a < b$. That is, for each $t \in [a, b]$, there is a *range* value $\gamma(t)$, the point on the path at time t.

Integration around paths turns computations into first year calculus integrals or derivatives. Such integration extends to manifolds (Chap. 3) because they are pieces of $\mathbb{R}^n$ tied together. Since $\gamma(t)$ is a point of $\mathbb{R}^n$, it has coordinates. One standard notation for these coordinates is $(f_1(t), \dots, f_n(t))$ (f is for function). Another possible notation is $(x_1(t), \dots, x_n(t))$. We prefer $(\gamma_1(t), \dots, \gamma_n(t))$. The points $\gamma(a)$ and $\gamma(b)$ are, respectively, the *initial* and *end* points of the path. The path γ is *closed* if $\gamma(a) = \gamma(b)$.

2.2.1. *Derivatives of a path.* Call γ differentiable if

$$\frac{d\gamma(t)}{dt} = \Big(\frac{d\gamma_1(t)}{dt}, \dots, \frac{d\gamma_n(t)}{dt}\Big),$$

the *tangent vector* to γ at t, exists and is continuous for each $t \in [a, b]$. (Use one-sided limits at the end points.) Reminder: $\frac{d\gamma(t)}{dt}$ is a point in $\mathbb{R}^n$. Interpret it as giving a direction and speed (length of the vector $\frac{d\gamma(t)}{dt}$) of travel along the path γ at time t. We always insist γ is continuous (to be a path).

DEFINITION 2.1. Let $\gamma : [a, b] \to \mathbb{R}^n$ be a path. For $a \le a' < b' \le b$ denote the restriction of γ to $[a', b']$ by $\gamma_{[a',b']}$. Call γ *simplicial* if for some integer m there exist

$t_0 = a < t_1 < \cdots < t_{m-1} < t_m = b$ with $\gamma_{|_{[t_i,t_{i+1}]}}$ differentiable, $i = 0, \ldots, m-1$. This includes $\gamma_{|_{[t_i,t_{i+1}]}}$ having a one-sided derivative at the end points.

2.2.2. *Paths and connectedness.* The notation $\Pi_1(X, x_0, x_1)$ denotes the collection of (continuous) paths in a topological space X, starting at x_0 and ending x_1. Write $\Pi_1(X, x_0)$ when $x_0 = x_1$. We often need paths in integrals to be simplicial. When necessary, the text assumes this implicitly for γ, though we may merely write $\gamma \in \Pi_1(X, x_0, x_1)$. For analytic continuation, or integrating meromorphic differentials, simplicialness is necessary only for paths satisfying explicit conditions as in (Rem. 4.4). One subtle use of simplicial paths is to give *classical generators* of the fundamental group of $U_{\boldsymbol{z}}$ (Chap. 4).

If $\Pi_1(X, x_0, x_1)$ is nonempty, then x_1 is *path-connected* to x_0. This is an equivalence relation, and the equivalence classes are the path-connected components of X. For subspaces of manifolds (Chap. 3; in particular, subspaces of $\mathbb{R}^n$), the path-connected components are the same as the connected components. Further, for our examples, using simplicial paths would define the same components. [**Ahl79**, p. 54-58] discusses connectedness at greater length.

2.3. Integrals along a simplicial path. Using simplicial paths guarantees existence of various integrals, including *arc length* and *line integrals* along γ. We explain this. Let γ be a simplicial path in $\mathbb{R}^n$. Consider $T_\gamma : [a, b] \to \mathbb{R}^{2n}$ defined by $t \mapsto (\gamma(t), \frac{d\gamma(t)}{dt})$. Suppose $F = F(\boldsymbol{x}, \boldsymbol{y}) = F(x_1, \ldots, x_n, y_1, \ldots, y_n)$ is defined and continuous on an open set containing the range of T. The integral

$$\int_\gamma F \overset{\text{def}}{=} \int_a^b F \circ T_\gamma \, dt \tag{2.2}$$

exists, though $\frac{d}{dt}(\gamma_i(t))$ may be undefined for finitely many t [**Rud76**, p. 126]. Here are two traditional cases.

(2.3a) $F = \sqrt{\boldsymbol{y} \cdot Q(\boldsymbol{x})(\boldsymbol{y})}$ with $Q(\boldsymbol{x}) : \mathbb{R}^n \to \mathbb{R}^n$ by $\boldsymbol{y} \mapsto Q(\boldsymbol{x})(\boldsymbol{y})$ linear in $\boldsymbol{y}$, where $Q(\boldsymbol{x})$ is a symmetric and positive definite matrix for each $\boldsymbol{x}$.

(2.3b) $F = G(\boldsymbol{x}) \cdot \boldsymbol{y}$ with $G = (G_1(\boldsymbol{x}), \ldots, G_n(\boldsymbol{x})) : \mathbb{R}^n \to \mathbb{R}^n$ a continuous function (*vector field*) defined on the range of γ.

Definition 2.2. Suppose γ is a one-one function onto its range. Case (2.3a) of (2.2) is the *arc length* of γ relative to the *infinitesimal metric* $Q(\boldsymbol{x})$ at $\boldsymbol{x}$. [9.19] explains the value of *tensor* form for metrics. In case (2.3b), (2.2) is the line integral of the *differential one form* $G \cdot d\boldsymbol{x} = \sum_{i=1}^n G_i(\boldsymbol{x})\, dx_i$ along γ.

Here is the crucial point of these examples. Suppose we change γ to another parameterization γ^* of the same set. Then, (2.2) doesn't change modulo these conditions: γ^* is one-one in case (2.3a); and γ^* has the same beginning and end points as γ in case (2.3b). Proving this uses Lemma 2.3 [9.19b].

Recall from vector calculus, the physical meaning of (2.3b). It is the *work* done in moving a particle along the path parametrized by γ against the force field G. Here is the formula for computing integrals of such differential expressions along γ:

$$\int_\gamma \sum_{i=1}^n G_i(\boldsymbol{x})\, dx_i \overset{\text{def}}{=} \sum_{i=1}^n \int_a^b G_i(\gamma_1, \ldots, \gamma_n) \frac{d\gamma_i}{dt}\, dt. \tag{2.4}$$

Tensor form of a metric defines distance along γ from an integral of positive functions [9.19]. The triangle inequality is automatic: $\int_a^b f(t)\, dt + \int_b^c f(t)\, dt \geq \int_a^c f(t)\, dt$ if $f(t) \geq 0$ for $t \in [a, c]$.

LEMMA 2.3 (Change of Variable Formula). *Let $\gamma : [c, d] \to \mathbb{R}$ be a simplicial path. Assume $f : \mathbb{R} \to \mathbb{R}$ is continuous, defined on the range of γ and $a = \gamma(c)$, $b = \gamma(d)$. Then,*

$$\int_a^b f(x)\, dx = \int_c^d f(\gamma(t))\, \frac{d}{dt}(\gamma(t))dt.$$

PROOF. This is a variant on [**Apo57**, p. 216]. Let $F(x) = \int_a^x f(t)\, dt$ for x in the range of γ, and $H(x) = \int_c^x f(\gamma(t)) \frac{d}{dt}(\gamma(t))\, dt$. The functions $F(\gamma(x))$ and $H(x)$ are both continuous. Excluding finitely many x, the chain rule shows they have the same derivatives. Thus $H(x) - F(\gamma(x))$ is a constant evaluated by taking $x = c$:

$$H(c) - F(\gamma(c)) = H(c) - F(a) = 0 - 0 = 0.$$

The formula follows by taking $x = d$. □

Apostol notes: "Many texts prove the preceding theorem under the added hypothesis that $\frac{d\gamma(t)}{dt}$ is never zero on $[c, d]$. The interval joining a to b need not be the image of $[c, d]$ under γ."

2.4. Relation between integrals and analytic functions. Integration theory is the heart of complex variables. Equations, algebraic or differential, with coefficients analytic on a *domain* D, define the classical functions of complex variables. By a domain we mean an open connected topological subspace of a given topological space. The first examples of the subject are domains in $\mathbb{C}$, the complex plane. As we use them, we will remind of most basics from a first semester graduate complex variables course.

This chapter refers to basic material of [**Ahl79**]. The notation $\mathcal{H}(D)$ denotes the ring (integral domain [9.8a]) of functions analytic (equivalently, *holomorphic*) on D. With R any ring, let $R[w]$ be polynomials in w with coefficients in R.

2.4.1. *Analytic Functions.* The definition of analytic function reflects how the chain rule works for a composition of an analytic function and a path. Assume $\lambda : [a, b] \to D$ is any differentiable path: $t \mapsto \lambda_1(t) + i\lambda_2(t)$ has $\lambda_1 = \Re(\lambda)$ and $\lambda_2 = \Im(\lambda)$, differentiable on the interval $[a, b]$.

DEFINITION 2.4. Suppose $z_0 \in D$, $t_0 \in [a, b]$ and $\lambda : [a, b] \to D$ is any path, differentiable at t_0, for which $\lambda(t_0) = z_0$. Then, $f(z)$ defined on D is analytic at z_0 if there exists a complex number $M + iN$ dependent only on f and z_0 with

$$\frac{d}{dt}(f \circ \lambda)(t_0) = (M + iN)\frac{d\lambda}{dt}(t_0). \tag{2.5}$$

To compute the derivative on the left, assume $f(z) = u(x, y) + iv(x, y)$ has partial derivatives (not necessarily continuous) and use the chain rule.

Apply (2.5) to $t \mapsto z_0 + (t - t_0)\boldsymbol{v}$ in two cases: $\boldsymbol{v} = 1$ and $\boldsymbol{v} = i$. This produces *two* expressions for each of M and N. That M and N could satisfy both expressions is equivalent to the Cauchy-Riemann equations:

$$M = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } N = -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \tag{2.6}$$

with each expression evaluated at $\lambda(t_0)$.

2.4.2. *The notation $f'(z)$.* To accentuate that the expression $M + iN$ comes from f alone, denote it by $f'(z)$ or $\frac{df}{dz}$. It only, however, *exists* for functions satisfying the Cauchy-Riemann equations. Here are ways it is like a derivative.

(2.7a) It fits in the chain rule for $\frac{d}{dt}$ of $f(\lambda(t))$ like a derivative.
(2.7b) Directional derivative $D_{\boldsymbol{v}}$ of $f(z)$ in the direction $\boldsymbol{v}$ works as does the gradient for a general function $F : \mathbb{R}^2 \to \mathbb{R}^2$: $D_{\boldsymbol{v}}(f)(z_0) = f'(z_0)v$ is $\frac{d}{dt}(f(z_0 + tv))(0)$. Check equivalence of this with being analytic!
(2.7c) Analytic composites $\mathbb{C}\xrightarrow{h}\mathbb{C}\xrightarrow{g}\mathbb{C}$ have a simple chain rule [**Con78**, p. 35]:

$$\frac{d}{dz}(g \circ h)(z) = \frac{dg}{dw}(w)_{|w=h(z)}\frac{dh}{dz}(z).$$

(2.7d) $f'(z)\,dz$ acts like the differential 1-form $h'(x)\,dx$ in first year calculus.

2.5. More explanation of differential forms. First, consider (2.7d) in more detail. The fundamental theorem of calculus says $\int_a^b h'(x)\,dx = h(b) - h(a)$. A partial analog for integration on $\mathbb{C}$ considers $f'(z)\,dz$, with f analytic. We say f is a *primitive* (or antiderivative) of f'. The outcome is the same. Let z_a and z_b be two points in D. Then, let $\lambda : [a, b] \to D$ be a piecewise differentiable path from z_a to z_b. [**Con78**, Ch. IV, Th. 1.18]:

$$\int_\lambda f'(z)\,dz = \int_a^b f'(\lambda(t))\frac{d}{dt}(\lambda(t))\,dt = f(z_b) - f(z_a). \tag{2.8}$$

Definition 2.5 (Differential forms). Suppose $m, n : \mathbb{C} \to \mathbb{C}$ are continuous on D, though maybe not analytic. The symbol $m(z)\,dx + n(z)\,dy$ is a differential (complex 1-form) on D. *Closed*, *locally exact* and *exact* differentials appear later.

A differential 1-form is analytic (or *holomorphic*) if on each disc in D it has the form $f(z)\,dz$ with $f(z)$ analytic. We also use *meromorphic* differentials: f is meromorphic on D. [**Con78**, p. 63] introduces only the differential 1-forms $m(z)\,dz$, ($m(z)$ may not be analytic). It often uses $\int_\lambda f$ to substitute for $\int_\lambda f\,dz$. These have the form above: Write dz as $dx + idy$. They don't, however, include all differential 1-forms $m(z)\,dx + n(z)\,dy$.

It is convenient to change variables from (x, y) to $(z, \bar{z})$ to write differentials in the form $u(z)\,dz + n(z)\,d\bar{z}$ with $\bar{z} = x - iy$ (and $d\bar{z} = dx - idy$). Chap. 3 Lem. 5.6 formulates the several complex variable version of the next lemma. Call a function *anti-holomorphic* if about each point it has a power series expression in $\bar{z}$.

Lemma 2.6. *The operator $\frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$ maps z to 1 and $\bar{z}$ to 0. So, it extends the action of $\frac{\partial}{\partial z}$ on holomorphic functions, and it kills anti-holomorphic functions. Similarly, $\frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$ extends the action of $\frac{\partial}{\partial \bar{z}}$ from anti-holomorphic functions to all differentiable functions.*

If f is a differentiable function, the expression for the total differential $df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy$ is the same as $\frac{\partial f}{\partial z}\,dz + \frac{\partial f}{\partial \bar{z}}\,dz$.

Proof. Everything is from the definitions. The sums defining $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ act on differentiable functions. For the last equality in differentials, check that $\frac{\partial f}{\partial z}\,dz + \frac{\partial f}{\partial \bar{z}}\,dz$, written in x and y, gives $\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy$. □

3. Branch of $\log(z)$ along a path

Let D be a domain in $\mathbb{C}^*$. Denote a path $\gamma : [a, b] \to D$ by just γ. A power series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ defines the *exponential function* e^z.

3.1. How e^z defines branches of $\log(z)$. The exponential has properties so valuable for explicit computation that many parts of mathematics find functions generalizing it. This chapter practices with the exponential function how that works. Here are basic properties of e^z.

(3.1a) $e^0 = 1$ and $e^{z_1+z_2} = e^{z_1}e^{z_2}$: e^z gives a homomorphism $\mathbb{C} \to \mathbb{C}^*$.
(3.1b) $e^{x+i\,y} = e^x(\cos(y) + i\,\sin(y))$.

In particular, the exact values $w \in \mathbb{C}$ with $e^w = 1$ are in the set $\{n2\pi i \mid n \in \mathbb{Z}\}$. Variants of the following definition appear throughout this chapter.

Definition 3.1. Suppose $h(t)$ is a continuous function defined on $[a, b]$ satisfying $e^{h(t)} = \gamma(t)$. Call h a *branch of* $\log(z)$ (or, of log) along γ.

For $z_0 \in D$, let $\gamma : [a, b] \to z_0$ be the constant path. Suppose $w = w_0$ is one solution of $e^w = z_0$. Then, all solutions are $\{w_0 + n2\pi i\}$: possible values of a branch of $\log h(z)$ at z_0. An easier definition is of a branch of log on the domain D. This is a continuous function $H : D \to \mathbb{C}$ satisfying $e^{H(z)} = z$ for all $z \in D$: a right inverse to the exponential function. It is necessary to assure $0 \notin D$; $e^{H(0)} = 0$ has no solution $H(0)$ because e^z never equals 0.

3.2. Questions about branches of log. The two definitions raise the following questions. Variants apply to the general topic of analytic continuation.

(3.2a) What is the relation between Def. 3.1 and the definition of H?
(3.2b) When does a branch of log exist along γ, and if it exists how many such branches are there?
(3.2c) How does Def. 3.1 give a simple criterion for the existence of H (on D)?
(3.2d) What integrals naturally associate with interpreting existence of $H(z)$?
(3.2e) What natural geometric relation between $\mathbb{C}^*$ and $\mathbb{C}$ codifies the answers to the previous questions?

Prop. 3.2 answers questions (3.2a), (3.2b) and (3.2c). Then, Prop. 3.5 answers those remaining. These arguments motivate the theory of Riemann surface covers and their moduli. We never use classical language referring to *branch cuts* (except in a simple example for its historical utility). In the proposition, unless otherwise said, assume $[a, b]$ is the domain of any path.

Proposition 3.2. *Suppose $H(z)$ is a branch of* log *on D. Fix $z_0 \in D$. Then, $h^\dagger(t) = H(\gamma(t))$ is a branch of* log *along γ. Further, suppose $h(t)$ is a branch of* log *along γ. Then, for $t_0 \in [a, b]$ there is a branch H of* log *on a neighborhood of $\gamma(t_0)$ with $H(\gamma(t)) = h(t)$ for t close to t_0.*

Even if there is no branch of log on D, the following hold.

(3.3a) *There is always a branch $h(t)$ of* log *along γ.*
(3.3b) *For $h^*(t)$ any branch of* log *along γ, $h(t) - h^*(t)$ is constant on $[a, b]$.*
(3.3c) *$h(t) + 2\pi i m$, $m \in \mathbb{Z}$, gives the complete set of branches of log along γ.*
(3.3d) *There is a branch $H(z)$ of* log *on D precisely if for each $\gamma \in \Pi_1(D, z_0)$, $h(b) = h(a)$ for h some branch of* log *along γ.*

3.3. Proof of Prop. 3.2. If $e^{H(z)} \equiv z$ for $z \in D$, then $e^{H(\gamma(t))} \equiv \gamma(t)$ for $t \in [a,b]$ as in the proposition statement. Thus, $h^{\dagger}(\gamma(t))$ is a branch of log along γ.

Now suppose $h^*(t)$ is any branch of log along γ. Then,

$$e^{h(t)}/e^{h^*(t)} = e^{h(t)-h^*(t)} = \gamma(t)/\gamma(t) \equiv 1$$

for $t \in [a,b]$. So, the continuous function $F(t) = h(t) - h^*(t)$ maps the connected set $[a,b]$ into the topological subspace $2\pi i\,\mathbb{Z}$ of $i\,\mathbb{R}$. The range of a connected set under a continuous function is connected. This shows the range of $F(t)$ is a single point; $F(t)$ is constant on $[a,b]$.

Suppose $z_0 \in \mathbb{C}$ satisfies $e^{z_0} = \gamma(a)$. The rest of the proof has three parts, corresponding to patching pieces of branches of log along γ.

3.3.1. *Extending a branch of log on a subpath.* Suppose $[a',b'] \subset [a,b]$. Then, restriction of γ to $[a',b']$ produces a new path, $\gamma_{[a',b']}$. Let $h_{t_0}(t)$ be a branch of log along $\gamma_{[a,t_0]}$ for $t_0 \in [a,b]$ with $t_0 < b$.

A classical construction produces a branch $H(z)$ of log in any sector

$$S_{\theta_1,\theta_2} = \{re^{i\,\theta} \mid \theta_1 < \theta < \theta_2\} \text{ with } \theta_2 - \theta_1 \le 2\pi \text{ [9.7}a\text{]}.$$

Any disk in $\mathbb{C}^*$ is in some sector. Restrict H to a disk around $\gamma(t_0) = z_0$ and translate it by an integer multiple of $2\pi i$ to assume $H(z_0) = h(t_0)$. From above, $H(\gamma(t))$ is a branch of log along γ restricted to $[t_0-\epsilon, t_0+\epsilon]$ for $\epsilon > 0$ small. Since $H(z_0) = h(t_0)$, these two branches of log are equal on $\gamma_{[t_0-\epsilon,t_0]}$. If $t_0+\epsilon \le b$, this defines a branch of log along $\gamma_{[a,t_0+\epsilon]}$:

$$h_{t_0+\epsilon}(t) = \begin{cases} h_{t_0}(t) & \text{for } t \in [a,t_0] \\ H(\gamma(t)) & \text{for } t \in [t_0, t_0+\epsilon]. \end{cases} \tag{3.4}$$

We say $h_{t_0+\epsilon}$ *extends* h_{t_0}.

3.3.2. *Sequences of extensions of branch of log.* Suppose $t_0 < t_1 < \cdots < b$ and $h_i(t)$ is a branch of log along $\gamma_{[a,t_i]}$, with $h_i(a) = z_0$ for each i. Then, from the first part of the proof, h_{i+1} extends h_i. As the t_i s are increasing and bounded, they have a limit point, t^*. Define h_{t^*} by this formula: for $t < t^*$, $h_{t^*}(t) = h_i(t)$ where $t < t_i$; and $h_{t^*}(t^*) = \lim_i h_i(t_i)$. Note: The left side is independent of i. The right side has a limit because it is a Cauchy sequence.

3.3.3. *Completing existence of branch of log.* §3.3.2 shows there is a maximal t' having a branch of log $h_{t'}$ along $\gamma_{[a,t']}$. Then, if $t' < b$, §3.3.1 gives an extension to $\gamma_{[a,t'+\epsilon]}$ for some $\epsilon > 0$. Thus, $t' = b$. That completes proving existence of the extension. Criterion (3.3d) for a branch of log on a domain is a special case of Lemma 4.12. This depends only on the notion of multiplying paths.

Suppose, as in Prop. 3.2, h is a branch of log along γ. For $t \in [a,b]$ there is a neighborhood D_t of $\gamma(t)$ and a branch $H_t(z)$ of log on D_t satisfying this property.

(3.5) $H(\gamma(t')) = h(t')$ for t' close to t.

This matches Def. 4.1: There is an analytic continuation of $H_a(z)$ along γ.

Example 3.3 (Branch of log along a circle). The function $t \mapsto e^{2\pi i t} = \gamma(t)$, $t \in [0,1]$, parametrizes the counterclockwise unit circle. Let $\epsilon > 0$ be small. As in [9.7a], $H_\epsilon(re^{2\pi i t}) = \ln(|r|) + 2\pi t$ is a branch of log for all z of form $re^{2\pi i t}$, $0 \le t \le 1-\epsilon$. So, $h_\epsilon(t) = 2\pi i t$ is a branch of log along $\gamma_{[0,1-\epsilon]}$. Like the proof of Prop. 3.2, $h(t) = 2\pi i t$ extends h_ϵ to be a branch of log along γ.

3.4. Branch of log as a primitive. Let $g : D \to D'$ by $w \mapsto g(w)$ be continuous. Assume $g(w_0) = z_0$ with $w_0 \in D$ and $\gamma : [a,b] \to D'$ has $\gamma(a) = z_0$.

DEFINITION 3.4. Consider $\gamma^* : [a,b] \to D$ with $\gamma^*(a) = w_0$. Call it a *lift* (relative to g) of γ (based at w_0) if $g(\gamma^*(t)) = \gamma(t)$ for all $t \in [a,b]$.

§4.4 has explicit notation for multiplying paths, as in $\gamma \cdot \gamma'$. Let D be a domain in $\mathbb{C}^*$; $f(z) = 1/z$ is analytic in D. Suppose $\gamma \in \Pi_1(D, z_0, z')$ and $\Delta_{z'}$ is a disc in D about z'. For $z \in \Delta_{z'}$ define $F_1(z)$ as $\int_{\gamma\cdot\gamma'} \frac{dz}{z}$ where γ' is any path from z' to z in $\Delta_{z'}$. The discussion before Def. 5.1 has the precise definition of winding number.

PROPOSITION 3.5. *Given γ, $F_1(z) = F_{1,\gamma}(z)$ depends only on the end point of γ'. Also, $\frac{dF_1}{dz} = \frac{1}{z}$ for all $z \in \Delta_{z'}$. In particular, $F_1(z)$ differs by a constant from a branch of* log *along $\gamma \cdot \gamma'$. Suppose γ_1 and γ_2 have the same end points. Then, $F_{1,\gamma_1} - F_{1,\gamma_2} = 2\pi i m$ with m the winding number of $\gamma_1 \cdot \gamma_2^{-1}$ about the origin.*

Consider $\psi : \mathbb{C} \to \mathbb{C}^*$ by $w \mapsto e^w$. Suppose $\gamma : [a,b] \to \mathbb{C}^*$ has beginning point z_0 with $e^{w_0} = z_0$. Then, a branch of log *along γ (with initial value w_0) is a lift of γ (starting at w_0; relative to ψ). Let D^* be the connected component of $\psi^{-1}(D)$ through w_0. Then, there is a branch of* log *on D with value w_0 at z_0 exactly when ψ is one-one to D on D^*.*

The first part requires Cauchy's Theorem ([**Ahl79**, p. 141, Cor. 1], [**Con78**, p. 84]). This typifies how integration of analytic functions arises. Abel and Riemann based information on differentials; in Riemann's Existence Theorem they are a substantial subplot.

PROPOSITION 3.6 (Cauchy's Theorem on a disk). *Suppose D is a domain in $\mathbb{P}^1_z$ and $f(z)$ is analytic on D. Further, assume D is either analytically isomorphic to $\mathbb{C}$ or to a disk. Then, $\int_\gamma f(z)\,dz = 0$ for each closed path in D.*

PROOF OF PROP. 3.5. Integration of $f(z) = 1/z$ along paths in $\mathbb{C}^*$ analytically continues a primitive for f at the initial point. Thus, to prove $F_1(z)$ is independent of γ' only requires showing the integral is 0 for any closed path γ' in $\Delta_{z'}$. This, follows from Prop. 3.6. The remainder follows by plugging in a lift γ^* of γ: $e^{\gamma^*(t)} = \gamma(t)$ for $t \in [a,b]$. By definition γ^* gives a branch of log along γ. □

4. Analytic continuation along a path

Suppose $f(z)$ is a branch of log on a domain $D \subset \mathbb{C}^*$. Since e^z is analytic on $\mathbb{C}$, Def. 3.1 provides *analytic continuation* of $f(z)$ along any path in $\mathbb{C}^*$. It does so using an equation $e^w = z$ to force the desired extension. The following generalizes Def. 3.1 (see §6.1). It requires *no* equation for extending an analytic function.

4.1. Definition of analytic continuation.

4.1. Definition of analytic continuation. Suppose f is analytic in a neighborhood $U_{z_0} \subset D$ of z_0 and $\gamma : [a,b] \to D$ is a path in D based at z_0.

DEFINITION 4.1 (Analytic continuation of f along γ). Let $f^* : [a,b] \to \mathbb{C}$ be a continuous function with the following properties.

(4.1a) $f^*(t) = f(\gamma(t))$ for t close to a (in $[a,b]$).
(4.1b) For each $t' \in [a,b]$, there is a function $h_{t'}(z)$ analytic on a disk $D_{t'}$ about $\gamma(t')$ with $h_{t'}(\gamma(t)) = f^*(t)$ for t near t' (in $[a,b]$).

If such an f^* exists, this definition produces $h_{t'}(z)$. This is the analytic continuation of f *to* t'. It is an analytic function in some neighborhood of $\gamma(t')$. Usually, however, the important reference is to the *end* function $h_b(z)$, analytic in a neighborhood of $\gamma(b)$. This we call $f_\gamma(z) = f_\gamma$, analytic continuation of f (along γ).

Note: $f^*(t)$ determines all data for an analytic continuation. It is unique: its difference from another function suiting (4.1) must be constant (hint of [9.8a]). Again, there is a related definition.

Suppose $\hat{f} : D \to \mathbb{C}$ satisfies $\hat{f}(z) = f(z)$ for all $z \in U_{z_0}$. We call $\hat{f}$ an analytic continuation or *extension* of f to D.

REMARK 4.2. Let $\gamma : [a, b] \to \mathbb{P}^1_z$ be a nonconstant path. Here is an example of a function analytic at $\gamma(a)$ with no analytic continuation along γ. Assume $\gamma(t') \neq \gamma(a)$ for t' close to a and let f be a branch of $\log(z - \gamma(t'))$ about $\gamma(a)$. Algebraic functions, and others, like branches of log, analytically continue along any path missing some finite set $\boldsymbol{z}$ of points on $\mathbb{P}^1_z$. Def. 4.5 introduces $\mathcal{E}(U_{\boldsymbol{z}}, z_0)$, analytic functions around z_0 that are *extensible* if we avoid $\boldsymbol{z}$.

4.2. Practical analytic continuation. Analytic functions have a power series expression around each point of their domain. This converges in any disc not containing a singularity of the analytic function [**Ahl79**, p. 179, Thm. 3].

4.2.1. *Using disks of convergence.* In Def. 4.1, for example, consider γ with range a segment of the real axis. Assume also f^* is real-valued along γ with continuous derivatives of all order. Then, an analytic function restricts to f^* along γ if and only if f^* has a Taylor series around each point. This gives a practical alternative definition of analytic continuation using *polygonal paths* like γ^* in the next lemma. Notation is from Def. 4.1.

LEMMA 4.3. *The following is equivalent to f having an analytic continuation along γ. There exists a partition $a = t_0 < t_0^* < t_1 < t_1^* < \cdots < t_{n-1}^* < t_n = b$ of $[a, b]$, disks D_i centered about $\gamma(t_i)$ and $f_i \in \mathcal{H}(D_i)$ with these properties.*

(4.2a) $D_i \cap D_{i+1} \neq \emptyset$ *and* $f_i(z) = f_{i+1}(z)$ *for* $z \in D_i \cap D_{i+1}$.

(4.2b) $\gamma(t) \in D_i$ *for* $t \in [t_i, t_i^*]$, $\gamma(t) \in D_{i+1}$ *for* $t \in [t_i^*, t_{i+1}]$, $i = 0, \ldots, n-1$.

(4.2c) $f_0(z) = f(z)$ *for* $z \in D_0$.

Further, let γ^* be the path following consecutive line segments $\gamma(t_i)$ to $\gamma(t_i^*)$, then $\gamma(t_i^*)$ to $\gamma(t_{i+1})$, $i = 0, \ldots, n-1$. Then, $f_{\gamma^*} = f_\gamma$.

PROOF. Suppose we have the pairs (D_i, f_i), $i = 1, \ldots, n$, and the partition of $[a, b]$. This gives an analytic continuation of f along γ by the following formula:

$$f^*(t) = \begin{cases} f_i(\gamma(t)) & \text{for } t \in [t_i, t_i^*] \\ f_{i+1}(\gamma(t)) & \text{for } t \in [t_i^*, t_{i+1}]. \end{cases}$$

Then, $f^*(t)$ provides an analytic continuation from Def. 4.1.

Follow notation of §3.3.1. Inductively consider analytic continuation of f to the end point of $\gamma_{[a,t_i]}$ (and $\gamma_{[a,t_i^*]}$). Set up the induction by showing this is analytic continuation of f to the end point of $\gamma^*_{[a,t_i]}$ (and $\gamma^*_{[a,t_i^*]}$). The essential point is f_i exists on a disk containing the range of γ on $[t_i, t_i^*]$. So, f_i in a neighborhood of $\gamma(t_i^*)$ analytically continues f_i (from a neighborhood of $\gamma(t_i)$) along *any* path entirely within D_i. Then, at the end points of γ and γ^*, $f_{\gamma^*} = f_\gamma$.

Now assume we have an analytic continuation of f along γ. Completing the lemma requires creating (D_i, f_i) for a corresponding partition of $[a, b]$. Since the range of γ is compact, the distance between $\gamma(t)$ and $\gamma(t')$ is a uniformly continuous function of (t, t'). So, for $d' > 0$ there exists $d > 0$ with $|\gamma(t) - \gamma(t')| < d'$ if $|t - t'| < d$. Choose d' with the following property.

(4.3) For each $t' \in [a, b]$, there is a disk of radius no more than d' around $\gamma(t')$ supporting analytic $h_{t'}(z)$ as in Def. 4.1.

Compactness of the range of γ produces such a d'. Use d from the above comment. Partition $[a, b]$ so $|t_i - t_i^*|$ and $|t_i^* - t_{i+1}|$ are at most d. Then, inductively show this partition has the desired properties. □

REMARK 4.4 (Nonsimplicial paths). §4.6 extends Lemma 4.3 to $D \subset \mathbb{P}^1_z$. There *geodesic paths* on $\mathbb{P}^1_z$ might replace polygonal paths: its pieces are arcs on longitudinal circles. The proof extends with no change.

Lem. 4.3 makes no assumption paths are simplicial. Chap. 3 applies the lemma to general continuous paths. A simplicial assumption allows integrating general differential 1-forms or for computing arc length. Still, suppose $\omega = f(z)\,dz$ is an analytic 1-form in a neighborhood of z_0 and $\gamma : [a, b] \to \mathbb{C}$ is a (continuous, not necessarily simplicial) path with beginning point z_0.

Let D be any domain containing the range of γ in which f extends analytically along each path. Lemma 4.3 produces a simplicial (or polygonal) path γ^* in D (notice D contains no potential poles of f) along which integration of f is defined. Let $F(z)$ be an antiderivative of $f(z)$. Analytic continuation of $F(z)$ along γ^* allows defining $\int_\gamma \omega$ to be $F(\gamma(b)) - F(\gamma(a))$.

4.2.2. *The word monodromy*. *Monodromy* isn't in Webster's dictionary. It is in [**Ahl79**, p. 295] and [**Con78**, p. 219] in the statement of the *Monodromy Theorem* (§8.2 and Chap. 3 Prop. 6.11). The Oxford English Dictionary references exactly the same theorem. It gives it the following meaning:

> The characteristic property: If the argument returns by any path
> to its original value, the function also returns to its original value.

We extend that to include regions where a function may not return to its original value. For this we add group data that accounts for the nonreturn. The loose name for that structure is *monodromy action*, though we often drop the last word.

The simplest setup for discussing monodromy starts with these elements:

(4.4a) a domain D and $z_0 \in D$
(4.4b) a closed path λ based at z_0
(4.4c) $f(z)$ analytic in a neighborhood of z_0
(4.4d) f has an analytic continuation around λ

Then, analytic continuation around λ produces a (possibly) new function, f_λ analytic in a neighborhood of z_0.

DEFINITION 4.5 (Extensibility). Assume the setup of (4.4) for *every* closed path in D. Call such an f *extensible* in D: $(f, D) = (f, D, z_0)$ is extensible. This is a neologism, differing from the notion f has an extension (is extendible) to D. Denote the complete set of extensible functions in D (based at z_0) by $\mathcal{E}(D, z_0)$.

By assumption $\mathcal{E}(D, z_0) \subset \mathcal{L}_{z_0}$. So, field operations like multiplication and taking ratios make sense. Suppose $f, g \in \mathcal{E}(D, z_0)$. Recall the notation $\mathbb{C}[z, u, v]$ for polynomials in z, u, v. Define $\mathbb{C}[z, f, g] = R$ to be $\{\alpha(z, f, g) \text{ with } \alpha \in \mathbb{C}[z, u, v]\}$.

LEMMA 4.6. *With the above assumptions, the ring R consists of extensible functions. For any $\lambda \in \Pi_1(D, z_0)$, $\alpha_\lambda = \alpha(z, f_\lambda, g_\lambda)$.*

Assume $f \in \mathcal{E}(D, z_0)$ and D is analytically isomorphic to a disk (or to $\mathbb{C}$). Then, f is extendible (restriction of an analytic function) on D.

PROOF. For the first part, show the last result for $f + g$ and fg. Every element in R is built from such algebraic operations. Now consider the case D is a disk. Cauchy's Integral formula for an analytic function says a power series for an analytic function converges up to a singularity on its boundary of convergence. Consider $f \in \mathcal{E}(D, z_0)$ with z_0 the center of the disk D.

Suppose the power series for f converges only on a disk of radius smaller than D. Then, analytic continuation of f to some singular boundary point fails. This is contrary to $f \in \mathcal{E}(D, z_0)$.

More generally, let $\beta : D \to \Delta$ be an analytic isomorphism of D with a disk. Then, $(f \circ \beta^{-1}, \Delta, \beta(z_0))$ extends to $F(z)$, and $F(\beta(z))$ extends f. □

REMARK 4.7. Webster's dictionary defines extensible to mean capable of being extended, whether in length or breadth; susceptible of enlargement. That agrees with our definition. Still, it has extendible as a synonym of extensible, whereas we distinguish between the two words.

4.2.3. *Meromorphic extensibility.* It simplifies many discussions to allow meromorphic functions in $\mathcal{E}(D, z_0)$. Even on $U_{\boldsymbol{z}}$, in considering $f \in \mathcal{E}(D, z_0)$, we eventually remove z' from $\boldsymbol{z}$ if it is only a pole of f. The simplest way is to allow in $\mathcal{E}(D, z_0)$ functions f having for each path γ some $g \in \mathbb{C}(z)$ with $g(z)f(z)$ extensible along γ as in Def. 4.5. Technical proofs would use extensibility of $g(z)f(z)$ and analytic continuation to the end point of γ would be $(g(z)f(z))_\gamma/g(z)$. The result, of course, could have a pole at the end of the path.

In Def. 4.1 there is an auxiliary function $f^* : [a, b] \to \mathbb{C}$: $f^*(t) = f(\gamma(t))$, the values of f along γ. Extending f^* to allow poles requires allowing maps into $\mathbb{P}^1_z$.

For example: If $g(z)$ is a branch of log at $z_0 = 1$, we allow $g(z)/(z-1)$ in $\mathcal{E}(\mathbb{C}^*, 1)$. Unless there is a reason to be careful about poles, most discussions will proceed as with extensibility of analytic functions. Integrals and primitives of a function require such care (§4.3). Occasions may need extending this definition to include infinitely many poles.

4.2.4. *Conjugates of f.* Assume $f \in \mathcal{E}(D, z_0)$. Even if λ isn't closed, f_λ has meaning for any path λ in D based at z_0. This produces *conjugates* of f (in D) or the *monodromy range* of (f, D, z_0):

$$\mathcal{A}_f(D, z_0) = \mathcal{A}_f(D) = \{f_\lambda(z)\}_{\lambda \in \Pi_1(D, z_0)}. \tag{4.5}$$

Regard $f_{\lambda_1}, f_{\lambda_2} \in \mathcal{A}_f(D)$ as equal if are the same function near z_0. As in [9.8a], f_{λ_1} and f_{λ_2} are then equal in any neighborhood of z_0 where they are meromorphic. Prop. 7.3 implies conjugate here is exactly as in basic Galois Theory. Suppose $h \in K[x]$ an irreducible polynomial over a field K and $h(\alpha) = 0$. Then, the full collection of zeros of h are the *conjugates of* α.

Recall the *Laurent series* field $\mathcal{L}_{z_0}$ (about z_0). This consists of ratios of power series convergent around z_0. The ring $\mathcal{A}_f(D, z_0)$ is in $\mathcal{L}_{z_0}$. So we may form the composite field $\mathbb{C}(\mathcal{A}_f(D, z_0))$ these functions generate. Still, not all elements of $\mathbb{C}(\mathcal{A}_f(D, z_0))$ are in $\mathcal{E}(D, z_0)$ unless f is algebraic.

LEMMA 4.8. *If $f \in \mathcal{E}(U_{\boldsymbol{z}}, z_0)$ is algebraic (as in* (1.2)*), then $1/f \in \mathcal{E}(U_{\boldsymbol{z}}, z_0)$. So, the field $\mathbb{C}(z, f)$ that z and f generate in $\mathcal{L}_{z_0}$ is in $\mathcal{E}(U_{\boldsymbol{z}}, z_0)$.*

PROOF. This requires showing extensions of f have only finitely many zeros. Suppose f satisfies an equation $m(z, f(z))$ with $m \in \mathbb{C}[z, w]$. Then, $\deg_z(m)$ bounds the number of solutions of $m(z, 0) = 0$. That shows $f(z)$ has only finitely many zeros among its analytic continuations, so $1/f \in \mathcal{E}(U_{\boldsymbol{z}}, z_0)$. □

Prop. 7.3, showing equivalence of (1.1) and (1.2), lets Lem. 4.8 apply without reservation to algebraic functions.

4.3. A branch of a primitive. Continue notation from §4.1. Suppose $F(z)$ is a primitive of $f(z)$ in U_{z_0}: $\frac{dF}{dz} = f(z)$. This discussion does require care on extensibility of meromorphic functions as in §4.6. If f is meromorphic in D, and $z' \in D$, write f as $h_1(z) + f_1(z)$ with these properties.

(4.6a) f_1 is analytic in a neighborhood of z'.
(4.6b) $h_1(z) = \frac{1}{z-z'} m_{z'}(\frac{1}{z-z'})$ with $m_{z'}(z) \in \mathbb{C}[z]$ ($\equiv 0$ for f analytic at z').

Then, the *residue* of f at $z' \in D$ is $m_{z'}(0)$.

DEFINITION 4.9. Consider $f \in \mathcal{E}(D, z_0)$, $z' \in D$ and a path $\gamma : [a, b] \to D$ based at z_0. Denote the restriction $\gamma_{[a,t]}$ to $[a, t]$ by γ_t. We say f has no residue along γ if f_{γ_t} has no residue for each $t \in [a, b]$.

A (branch of) primitive of $f(z)$ along $\lambda : [a, b] \to D$ is an analytic continuation $\hat{F}_\lambda$ of $F(z)$ along λ. We also label it by $\hat{F} : [a, b] \to D$.

LEMMA 4.10. *Assume $f \in \mathcal{E}(D, z_0)$. Then, f has a primitive in a neighborhood of z_0 when it has no residue at z_0. Let $\gamma : [a, b] \to D$ be a path in D along which f has no residue. Then there exists a primitive $\hat{F} : [a, b] \to \mathbb{C}$ of f along γ. Further, for $c \in \mathbb{C}$, there is a unique such $\hat{F}$ with $\hat{F}(a) = c$.*

PROOF. Get a primitive for f in a neighborhood of z_0 from a primitive for each term in the Laurent series for f around z_0. The function z^k has a primitive $\frac{1}{k+1} z^{k+1}$ if $k \neq -1$. The discussion from §3.4 has done overkill on showing z^{-1} has no primitive. That is, f must have 0 as residue at z_0 to have a primitive. Further, by assumption every analytic continuation of f (in D) has this property.

Let D_0 be a disk centered at z_0 and contained in D. By assumption $f(z)$ has no residue along any path in D. So, it has a primitive $F(z) = F_0(z)$ in this disk; integrate the power series for $f(z)$ term by term. The primitive is unique up to addition of a constant.

Now apply the notation of Lemma 4.3. Similarly, there exists $F_i(z)$, a primitive of $f_i(z)$ in D_i, $i = 1, \ldots, n$. Since $f_i = f_{i+1}$ in $D_i \cap D_{i+1}$, $F_i(z)$ and F_{i+1} have equal derivatives on this intersection. Thus, $F_i - F_{i+1}$ is a constant on $D_i \cap D_{i+1}$. This sets up for an induction. Assume k is an integer for which $F_0(z), \ldots, F_k(z)$ give an analytic continuation of $F(z)$ along $\gamma_{[a,t_k]}$. Let F_{k+1} be the function we just produced, where $F_k - F_{k+1} = b$ for $z \in D_k \cap D_{k+1}$. Now replace F_{k+1} by $F_{k+1} + b$. Continue inductively on k to conclude the result. □

4.4. Continuation along products of paths. Let $\lambda_1 : [a, b] \to D$ be a path where $\lambda_1(a) = z_0$ and $\lambda_1(b) = z_1$. Assume $\lambda_2 : [a^*, b^*] \to D$ is another path and $\lambda_1(b) = \lambda_2(a^*)$. Create a new path $\lambda_1 \cdot \lambda_2 \stackrel{\text{def}}{=} \lambda^\dagger : [a, b + b^* - a^*] \to D$:

$$\lambda^\dagger = \begin{cases} \lambda_1(t) & \text{for } t \in [a, b] \\ \lambda_2(t + a^* - b) & \text{for } t \in [b, b + b^* - a^*]. \end{cases} \tag{4.7}$$

The proof of Lemma 4.12 includes detailed notation for a sequence of analytic continuations. Use that notation for details of the following lemma. Given a path λ, denote the path $t \mapsto \lambda(b - t + a)$, $t \in [a, b]$, by λ^{-1}, the *inverse of* λ. If λ is simplicial so is λ^{-1}. Continue notation for the function f and let $f_1 = f_{\lambda_1}$ be analytic continuation of f along a path λ_1.

LEMMA 4.11. *For paths λ_1, λ_2 and λ_3, assume the end point of λ_i equals the beginning point of λ_{i+1}, $i = 1, 2$. Analytic continuation of f_1 along λ_2, $f_2 = (f_1)_{\lambda_2}$,* *is the analytic continuation* $f_{\lambda_1\cdot\lambda_2}$ *of* f *along* $\lambda_1\cdot\lambda_2$. *Then,* $f_{(\lambda_1\cdot\lambda_2)\cdot\lambda_3} = f_{\lambda_1\cdot(\lambda_2\cdot\lambda_3)}$ *giving unambiguous meaning to* $f_{\lambda_1\cdot\lambda_2\cdot\lambda_3}$. *Also,* $f_{\lambda\cdot\lambda^{-1}} = f$.

As in §2.3, $\int_{\lambda_1\cdot\lambda_2} F\,dz = \int_{\lambda_1} F\,dz + \int_{\lambda_2} F\,dz$, *Further,* $\int_{\lambda\cdot\lambda^{-1}} F\,dz = 0$.

While $\lambda\cdot\lambda^{-1}$ isn't the constant path (at $\lambda(a)$), Lemma 4.11 lists situations where it acts as if it is.

LEMMA 4.12. *Suppose* $f \in \mathcal{E}(D, z_0)$. *Let* λ^* *be any path with beginning point* z_0 *and end point* z_1. *Let* $f_1 = f_{\lambda^*}$. *There is a one-one map between* $\mathcal{A}_f(D, z_0)$ *and* $\mathcal{A}_{f_1}(D, z_1)$. *Also,* f *is extendible to* D *if and only if* $f_\lambda = f$ *for each* $\lambda \in \Pi_1(D, z_0)$.

§4.5 has the proof of Lemma 4.12. It says there is an analytic function $\hat{f}$ on D restricting to f around z_0 exactly when $\mathcal{A}_f(D, z_0)$ has a single element. Then, monodromy action on (f, D), or (if D is clear, on f) is trivial.

4.4.1. *A permutation representation.* For $f \in \mathcal{E}(D, z_0)$ and $\lambda \in \Pi_1(D, z_0)$, Lemma 4.11 gives a permutation of $\mathcal{A}_f(D, z_0)$ by $h \mapsto h_\lambda$ for $h \in \mathcal{A}_f(D, z_0)$. Denote h_λ by $(h)T(\lambda)$ to distinguish $T(\lambda)$ as a permutation of the set $\mathcal{A}_f(D, z_0)$. According to Lemma 4.11,

$$((h)T(\lambda_1))T(\lambda_2) = (h)T(\lambda_1)\circ T(\lambda_2) = (h)T(\lambda_1\cdot\lambda_2), \tag{4.8}$$

for $\lambda_1, \lambda_2 \in \Pi_1(D, z_0)$.

That is, analytic continuation gives a homomorphism from the semi-group (set with multiplication) $\Pi_1(D, z_0)$ to permutations on $\mathcal{A}_f(D, z_0)$. From Lem. 4.11, the permutation $T(\lambda)$ has $T(\lambda^{-1})$ as its inverse permutation. So, the image set of permutations is a group. Call it the *monodromy group* $G_{f,D}$ of (f, D).

Chap. 3 puts an equivalence relation, *homotopy*, on $\Pi_1(D, z_0)$ to produce the *fundamental group* $\pi_1(D, z_0)$. In particular, from those results T produces a permutation representation of $\pi_1(D, z_0)$. This chapter's elementary examples depend only on *homology classes* of $\Pi_1(D, z_0)$ (§5 and [9.12]; Chap. 3 §6.2 has the comparison).

4.5. Proof of Lemma 4.12. We show unique analytic continuation to the end points of each closed path implies f extends analytically to D. First, we construct the map between $\mathcal{A}_f(D, z_0)$ and $\mathcal{A}_{f_1}(D, z_1)$ based on λ^* as in the lemma. Then, $\mathcal{A}_f(D, z_0)$ consists of a single element if and only if $\mathcal{A}_{f_1}(D, z_1)$ does. Then, we construct F, the extension of f.

4.5.1. *Identifying* $\mathcal{A}_f(D, z_0)$ *and* $\mathcal{A}_{f_1}(D, z_1)$. Given $h = f_\lambda \in \mathcal{A}_f(D, z_0)$, apply Lemma 4.11 several times to produce this chain:

$$\begin{aligned} h_{\lambda^*} = {} & f_{\lambda\cdot\lambda^*} = \\ f_{\lambda^*\cdot(\lambda^*)^{-1}\cdot\lambda\cdot\lambda^*} = {} & (f_1)_{(\lambda^*)^{-1}\cdot\lambda\cdot\lambda^*}, \end{aligned} \tag{4.9}$$

since $(\lambda^*)^{-1}\cdot\lambda\cdot\lambda^* \in \Pi_1(D, z_1)$. This gives a map from $\mathcal{A}_f(D, z_0)$ to $\mathcal{A}_{f_1}(D, z_1)$: *Conjugating* paths based at z_0 by λ^*.

Map in the other direction by conjugating by $(\lambda^*)^{-1}$. These maps between $\mathcal{A}_f(D, z_0)$ and $\mathcal{A}_{f_1}(D, z_1)$ are inverse to each other. That is, conjugating $\mathcal{A}_f(D, z_0)$ by $\lambda^*\cdot(\lambda^*)^{-1}$ acts trivially on $\mathcal{A}_f(D, z_0)$ (from in Lemma 4.11). Conclude: Monodromy action on f (in D) is trivial if and only the same holds for f_{λ^*}.

4.5.2. *Extending* f *to be analytic on* D. We prove the last statement of the lemma. Suppose f extends to $\hat{f}$ analytic on D. Then uniqueness of analytic continuation shows $f_\lambda(\lambda(t)) = \hat{f}(\lambda(t))$ for each t near b ($\lambda \in \Pi_1(D, z_0)$).

Now suppose $f_\lambda = f$ for each $\lambda \in \Pi_1(D, z_0)$. For $z' \in D$, assume z is in a disk neighborhood about z' entirely contained in D. Set $\hat{f}(z)$ equal to $f_\lambda(z)$

with $\lambda : [a,b] \to D$ a path where $\lambda(a) = z_0$ and $\lambda(b) = z'$. Lem. 4.6 says f_λ extends to be analytic in the whole disk neighborhood. So this defines $f_\lambda(z)$. Let $\lambda^* : [a^*, b^*] \to D$ be another such path with the same end points. We have only to show $f_{\lambda^*}(z) = f_\lambda(z)$.

Then, $\lambda^\dagger = \lambda^{-1} \cdot \lambda^*$ is a closed path based at $\lambda(b)$. From §4.5.1, analytic continuation of f_λ around $\lambda^\dagger$ equals $f_\lambda(z)$. It also equals analytic continuation of f_λ along λ^{-1} followed by analytic continuation of f along λ^*. The result of these analytic continuations is f_{λ^*}. This proves the desired equalities.

4.6. Extending analytic continuation to $\mathbb{P}^1_z$. Similar definitions work for meromorphic functions in a domain, including analytically continuing meromorphic functions. It simplifies results of Chap. 3 to systematically extend paths into $\mathbb{P}^1_z$. Recall: A neighborhood basis of open sets around each point gives the topology on a space. Around ∞ the neighborhood basis consists of sets of form $N \cup \{\infty\}$ where N is the complement of any closed set in $\mathbb{C}$.

EXAMPLE 4.13 (Meromorphic functions). Suppose for some disc Δ_{z_0} about z_0, $D \cap \Delta_{z_0} = \Delta_{z_0} \setminus \{z_0\}$. That is, z_0 is an *isolated boundary point* of a domain D. Further, assume f is analytic on D and it extends to a meromorphic function at z_0. That means $\lim_{z \mapsto z_0} (z - z_0)^{n+1} f(z) = 0$ for some $n \in \mathbb{Z}$ [**Con78**, p. 109]. The minimal such n allows expressing $f(z)$ as $(z - z_0)^n h(z)$ with h holomorphic and nonzero in a neighborhood of z_0. If the minimal n is negative, then f has a *pole* of order n. Define $F : D \cup \{z_0\} \to \mathbb{P}^1_z$ by this formula:

$$F(z) = \begin{cases} f(z) & \text{for } z \in D \\ \infty & \text{for } z = z_0. \end{cases} \tag{4.10}$$

Continuity of F is equivalent to continuity of $z \mapsto 1/F(z)$ around z_0. This function is continuous at z_0 (taking the value 0). So it is continuous around z_0.

DEFINITION 4.14 (Analytic maps to $\mathbb{P}^1_z$). Suppose $f : D \to \mathbb{P}^1_z$ is analytic. Assume z_0 is an isolated boundary point of D and f extends to be meromorphic in a neighborhood of z_0. Then, we say the extension $F : D \to \mathbb{P}^1_z$ is analytic. If $f(z_0) = \infty$, this means $z \mapsto 1/f(z)$ (with $z_0 \mapsto 0$) is analytic in a neighborhood of z_0. Also, suppose ∞ is an isolated boundary point of D on $\mathbb{P}^1_z$. Let D' be the image of D under $z \mapsto 1/z$. Then, f extends analytically to $F : D \cup \{\infty\} \to \mathbb{P}^1_z$ if $g(z) = f(1/z)$ extends analytically to $D' \cup \{0\}$ in a neighborhood of 0.

Those functions $f : \mathbb{P}^1_z \to \mathbb{P}^1_z$ analytic everywhere are the rational functions $\mathbb{C}(z)$ in z [9.3f]. Extending Lem. 4.10 to allow any D in $\mathbb{P}^1_z$ only requires clarifying what will be the residue at ∞. This allows integrations of analytic functions $f : D \to \mathbb{P}^1_z$ along paths for any domain D in $\mathbb{P}^1_z$.

DEFINITION 4.15. By definition a function $f(z)$ meromorphic in a neighborhood of ∞ is in $\mathcal{L}_\infty$, Laurent series in $1/z$: $f(z) = g(1/z)$ with $g \in \mathcal{L}_0$. The residue at ∞ is the coefficient of z in $\frac{-g(z)}{z^2}$.

For example, $f(z) = 1/z$ has residue -1 at ∞. So, it has no primitive at ∞.

This chapter's examples explicitly compute conjugates of special functions f. Riemann's Existence Theorem turns this around when D is $U_{\boldsymbol{z}} = \mathbb{P}^1_z \setminus \{\boldsymbol{z}\}$. Running over all algebraic $f \in \mathcal{E}(U_{\boldsymbol{z}}, z_0)$, Chap. 4 describes all possible permutations of the sets $\mathcal{A}_f(U_{\boldsymbol{z}}, z_0)$. The goal will be to recognize f by the permutations that come from applying $\Pi_1(U_{\boldsymbol{z}}, z_0)$. Then Riemann's Existence Theorem produces (algebraic) f realizing a given labeling. It doesn't, however, give f explicitly; it only exists.

Given such an f, suppose $g \in \mathbb{C}(z,f)$ and $\mathbb{C}(z,g) = \mathbb{C}(z,f)$: f and g are *primitive generators* of this field (over z; §1.2.2). §1.2 gives $u(w), v(w) \in \mathbb{C}(z)[w]$ with $g = u(f)$ and $f = v(g)$. Here is a particular case of Lem. 4.6.

LEMMA 4.16. *For $\lambda \in \Pi_1(U_{\boldsymbol{z}}, z_0)$, $g_\lambda = u(f_\lambda)$ and $f_\lambda = v(g_\lambda)$.*

5. Winding numbers and homology

Winding numbers appear in §3.4. Here is the formal definition for the winding number of the closed path γ (in $\mathbb{C}$, not passing through z') about z':

$$n_{z'}(\gamma) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-z'}.$$

This definition alone would justify complex variables; it defines this winding for any path avoiding z'.

DEFINITION 5.1. Suppose D is a domain in $\mathbb{C}$, $z_0 \in D$ and $\gamma_1, \gamma_2 \in \Pi_1(D, z_0)$ have the same winding numbers about each point in $\mathbb{C} \setminus D$. We say they are *homologous* (in D). A path is homologous to 0 if all winding numbers for points in $\mathbb{C} \setminus D$ are 0. It is obvious this forms an equivalence relation on $\Pi_1(D, z_0)$. Denote the equivalence classes by $H_1(D)$: the (first) *homology* group of D.

5.1. Extending Def. 5.1.

5.1. Extending Def. 5.1. Suppose $\gamma_1, \gamma_2 \in \Pi_1(D, z_0, z_1)$. Extend the definition of homologous paths: γ_1 and γ_2 are homologous if the closed path $\gamma = \gamma_1 \cdot \gamma_2^{-1}$ is homologous to 0. Suppose γ is a closed path in $\mathbb{C}$. Use the notation $\mathbb{P}^1_z \setminus \gamma$ for the complement of the range of γ in $\mathbb{P}^1_z$. If $z' \in \mathbb{C} \setminus \gamma$, we have a winding number $n_{z'}(\gamma)$ of γ about z'. If $\gamma_1, \gamma_2 \in \Pi_1(D, z_0)$, then $\gamma_1 \cdot \gamma_2$ is homologous to $\gamma_2 \cdot \gamma_1$. This is because all winding numbers are from computations of integrals in Lem. 4.11. For γ a closed path in $\mathbb{P}^1_z$ denote the complement of the range of γ by $\mathbb{P}^1_z \setminus \gamma$.

LEMMA 5.2. *In the previous notation, let $U_1, \dots, U_{r'}$ be the connected components of $\mathbb{P}^1_z \setminus \gamma$. One of these, say $U_{r'}$ includes ∞. Then $n_{z'}(\gamma)$ is a constant function of z' (with γ fixed) as z' runs over a connected component of $\mathbb{P}^1_z \setminus \gamma$. So, if $z' \in U_{r'} \setminus \{\infty\}$, then $n_{z'}(\gamma) = 0$.*

Let $n_i(\gamma)$ be the winding number of γ around any point in U_i, $i = 1, \dots, r'$. Suppose $D \subset \mathbb{C}$ is any domain containing the range of γ. Any connected component of $\mathbb{C} \setminus D$ is in one of the U_i s. Denote the set of integers i with U_i containing a component of $\mathbb{C} \setminus D$ by I_D. Then, the function $i \in I_D \mapsto n_i(\gamma)$ determines the homology class of γ in D.

PROOF. This follows immediately by noticing $g(z') = \int_\gamma \frac{dz}{z-z'}$ is an analytic (and therefore continuous) function on $\mathbb{P}^1_z \setminus \gamma$. Its values are in $2\pi i\mathbb{Z}$, a discrete set. So, it is constant on each connected component of $\mathbb{P}^1_z \setminus \gamma$ (proof of Prop. 3.2).

Now suppose $z' \in U_r \setminus \{\infty\}$. Then, some big disc Δ' contains all of (the range of) γ. Let z'' be any other point in $U_r \setminus \{\infty\}$ outside Δ'. A previous observation shows $n_{z'}(\gamma) = n_{z''}(\gamma)$. Further, $g(z) = 1/(z-z'')$ is analytic in Δ'. Apply Cauchy's Theorem 3.6 to conclude $n_{z''}(\gamma) = 0$.

Finally, consider the function $i \in I_D \mapsto n_i(\gamma)$. This determines the winding numbers of γ on each connected component of $\mathbb{C} \setminus D$. This, in turn determines the homology class of γ. □

Denote the image of γ in $H_1(D)$ by $[\gamma]_h$. We understand that a tuple of integers from Lemma 5.2 may be our best interpretation. Further, additivity of winding numbers gives $[\gamma_1 \cdot \gamma_2]_h = [\gamma_1]_h + [\gamma_2]_h$.

5.2. Homology for domains including ∞. Def. 5.1 doesn't include defining homologous paths if a domain in $\mathbb{P}^1_z$ includes ∞. (This includes allowing the paths to go through ∞.) Several adjustments allow extending the definition. Chap. 3 has a general approach, one that will not put ∞ in a special place. Here we follow implications from a standard complex variables course.

5.2.1. *Use linear transformations.* If $z' \in \mathbb{P}^1_z \setminus D$ and $\infty \in D$, choose a linear (fractional) transformation $\alpha \in \mathrm{PGL}_2(\mathbb{C})$ mapping z' to ∞ [9.14]. Since γ_1, γ_2 are paths in D, $\alpha \circ \gamma_1$ and $\alpha \circ \gamma_2$ don't go through ∞. Now, apply Def. 5.1 to $\alpha \circ \gamma_1$ and $\alpha \circ \gamma_2$ relative to $\alpha(D)$. To justify this, check that $\alpha \circ \gamma_1 \cdot (\alpha \circ \gamma_2)^{-1}$ being homologous to 0 doesn't depend on α [9.14e]. If $D = \mathbb{P}^1_z$, declare all closed paths to be homologous to 0.

There is one obvious problem. Suppose $\psi_{D_1,D_2} : D_1 \subset D_2$ is the inclusion map. Yet, you have already chosen points $z'_i \in \mathbb{C} \setminus D_i$ for reverting homology to a winding number computation, with $z'_1 \neq z'_2$. Then, we lose having an explicit map $\bar{\psi}_{D_1,D_2} : H_1(D_1) \to H_1(D_2)$ induced from paths in D_1 also being paths in D_2.

5.2.2. *Excising* ∞. Assume $\infty \in D$, $z_0 \in D \setminus \{\infty\}$ and Δ_∞ is some closed disk about ∞ lying entirely in D. Regard $\mathbb{P}^1_z$ as an actual sphere (in $\mathbb{R}^3$). Assume the radius of Δ_∞ is one unit (see §5.4.1). Let $\Delta_{\infty,s}$ be the closed disk about ∞ of radius s, $0 < s \le 1$. Let $D_\infty = D \setminus \{\infty\}$. Now, $H_1(D_\infty)$ has meaning from Def. 5.1.

Let $U_1, \dots, U_r$ be the connected components of $\mathbb{C} \setminus D$. Each defines a winding number for $\gamma \in \Pi_1(D_\infty, z_0)$. Use notation from Lemma 5.2:

$$\gamma \in \Pi_1(D_\infty, z_0) \mapsto [\gamma]_h = (n_1(\gamma), \dots, n_r(\gamma)) \in \mathbb{Z}^r.$$

Define $H_1(D)$ by extending $[\gamma]_h$ to paths in $\Pi_1(D_\infty, z_0)$ going through ∞. For this, consider the submodule M_r of $\mathbb{Z}^r$ that $\boldsymbol{v}_r = (1, 1, \dots, 1) \in \mathbb{Z}^r$ generates.

Suppose $\gamma \in \Pi_1(D, z_0)$ goes through ∞. Apply Lemma 4.3 to replace γ by a geodesic path γ^* in D (Rem. 4.4) with these properties.

(5.1a) γ and γ^* have the same end points.

(5.1b) If $f \in \mathcal{E}(D, z_0)$, then $f_\gamma = f_{\gamma^*}$.

If γ^* doesn't go through ∞, precede as below. Otherwise, If γ^* goes through ∞ then it does so only finitely many times. It is the product of a finite number of paths γ' with the property there is a neighborhood of ∞, $\Delta_{s_0} \subset D$, which γ' returns to and leaves just once. With no loss assume there exists $a < t_1 < t_2 < b$ with $\gamma(t) \in \Delta_{s_0}$ for $t \in [t_1, t_2]$ and $\gamma(t) \notin \Delta_{s_0}$ for t outside this interval. Therefore, $\gamma(t_1)$ and $\gamma(t_2)$ are on the boundary $\partial\Delta_{s_0}$ of Δ_{s_0}. There are two paths on $\partial\Delta_{s_0}$ going at constant speed from $\gamma(t_1)$ to $\gamma(t_2)$. Let τ be one of these. Form a new path, γ^* from γ using this formula:

$$\gamma^*(t) = \begin{cases} \gamma(t) & \text{for } t \in [a, t_1] \\ \tau(t) & \text{for } t \in [t_1, t_2] \\ \gamma(t) & \text{for } t \in [t_2, b]. \end{cases} \tag{5.2}$$

Then, $[\gamma^*]_h \in H_1(D_\infty)$.

Definition 5.3. In the above, when $\infty \in D$, define $H(D)$ to be $H_1(D_\infty)/M_r$. Denote the canonical map $H_1(D_\infty) \to H(D)$ by ψ. Extend to $[\gamma]_h$: Take $\psi([\gamma^*]_h)$ to be its image in $H(D)$. Prop. 5.4 completes why this is well defined.

5.3. Computing $H_1(D)$ for explicit domains. The word *explicit* has only subjective meaning. It depends on personally interpreting what it means to *know data*. Still, consider $U_{\boldsymbol{z}} = \mathbb{P}^1_z \setminus \{\boldsymbol{z}\}$ for some set of r points $\boldsymbol{z}$. Then, giving $\boldsymbol{z}$ explicitly has comfortable interpretation from experience.

This generalizes to when $\mathbb{P}^1_z \setminus D$ has r connected components, $C_1, \dots, C_r$. Our treatment tacitly assumes r is finite. Then, interpret *giving D explicitly* as knowing simple closed paths bounding each of the C_i s. Such paths might be circles or polygons with explicit beginning and end points. Given these conditions, computing the homology class of an *explicit* path in D uses calculations within our experience.

The next proposition specializes a statement in Chap. 4 with *homotopy classes* replacing homology classes. It gives ∞ a special status, that Chap. 4 will not. Simple examples, like [9.10], illustrate having ∞ play a special role.

Suppose D is a domain in $\mathbb{P}^1_z$ whose complement $C(D)$ in $\mathbb{P}^1_z$ has $r > 0$ connected components $C_1, \dots, C_r = C(D)_1, \dots, C(D)_r$. Denote this ordering of the components as J_D with the proviso $C_r = C_\infty$ is the component containing ∞ if $D \subset \mathbb{C}$. If $\infty \in D$, add C_∞ by including the empty set $\emptyset$ as the last position. Write D_∞ for $D \setminus \infty$. As in §5.2.1, consider an inclusion map $\psi_{D_1,D_2} : D_1 \subset D_2$.

Each connected component of $\mathbb{P}^1_z \setminus D_2$ is in some connected component of $\mathbb{P}^1_z \setminus D_1$. (If $\infty \in D_2$ regard $\emptyset$ as $C(D_2)_\infty$.) This induces a map $\psi^\dagger_{D_1,D_2} : J_{D_2} \to J_{D_1}$. The module M_r is from §5.2.2. Recall the definition of a residue of a meromorphic function f at a point $z' \in D$ from (4.6).

PROPOSITION 5.4. *Suppose D is a domain in $\mathbb{P}^1_z$ where $C(D)$ has r connected components. Then, $H_1(D)$ is isomorphic to $\mathbb{Z}^{r-1}$. If $\infty \in C(D)$, then $\gamma \in \Pi_1(D, z_0) \mapsto [\gamma]_h$ of Prop. 5.2 and Def. 5.3 gives this isomorphism explicitly. If $\infty \in D$, this identifies $H_1(D)$ with $\mathbb{Z}^r/M_r$ (isomorphic to $H_1(D_\infty)/M_r$), also isomorphic to $\mathbb{Z}^{r-1}$.*

Suppose $C(D')$ has r' components and $D \subset D'$, with $\infty \in C(D')$. Then, these isomorphisms induce $\mathbb{Z}^{r-1} \to \mathbb{Z}^{r'-1}$ where $n_1, \dots, n_{r-1} \mapsto m_1, \dots, m_{r'-1}$ by

$$m_j = \sum_{i \in J_{D'}, \psi^\dagger_{D,D'}(i) = j} n_i.$$

Assume f is meromorphic in D and $\gamma \in \Pi_1(D, z_0)$ passes through no residue of f. Then, $\int_\gamma f(z)\,dz$ depends only on $[\gamma]_h$ and the residues of f at points in D.

5.4. Proof of Prop. 5.4. Let $z_0 \in D$. As above, denote the r connected components of $\mathbb{P}^1_z \setminus D$ by $C_1, \dots, C_r$. First assume $\infty \in C_r$. For each i, $1 \le i \le r-1$, there is a closed path $\gamma_i = \delta_i \cdot \bar\gamma_i \cdot \delta_i^{-1} \in \Pi_1(D, z_0)$ with the following description.

(5.3a) $\delta_i : [0,1] \to D$ and $\bar\gamma_i : [0,1] \to D$ are paths with $\bar\gamma_i$ closed.
(5.3b) $\delta_i(0) = z_0$ and $\delta_i(1) = \bar\gamma_i(0)$.
(5.3c) $\bar\gamma_i$ has winding number 1 around each point in C_i.
(5.3d) $\bar\gamma_i$ has winding number 0 around each point in C_j, $j \ne i$.

5.4.1. *Construction of $\bar\gamma_i$.* Our construction of γ_i is similar to that of [**Ahl79**, p. 140]. Again use the metric topology on $\mathbb{P}^1_z$ identifying it with a sphere in $\mathbb{R}^3$ with coordinates (r, u, v). So, $z_0 \in \mathbb{P}^1_z$ corresponds to $(r_0, u_0, v_0) \in \mathbb{R}^3$. Each point of the sphere has a vector pointing *outward*, perpendicular to the tangent plane to the sphere at (r_0, u_0, v_0). Further, in any disk on the sphere around (r_0, u_0, v_0), the boundary of this disk has a well-defined orientation around (r_0, u_0, v_0). We take it counterclockwise around the outward normal to the disk at its center. This orientation applies to any simple closed path in the disk [9.17].

Components of $C(D)$ are closed, disjoint (and bounded). Let $d(z_i, z_j)$ be the distance (along the minor arc) between $z_i \in C_i$ and $z_j \in C_j$. The function $1/d(z_i, z_j)$ has a minimum on $C_i \times C_j$. Running over all i and j let δ be at most $1/\sqrt{2}$ times the smallest of these minimums. Form a grid on $\mathbb{P}^1_z$ of equally spaced longitudes and latitudes, with spacing at most δ. The closed (spherical) squares (and triangles) of this grid each meet at most one component of $C(D)$.

Let Q be one of the closed grid squares. Its boundary orientation is counter clockwise around any outward normal to an interior point of Q [9.17e]. Define $\bar{Q}_i$ to be the union of all Q s meeting C_i. Such a Q meets none of the C_j s with $j \neq i$. Let $\bar{\gamma}_i$ be the topological boundary of $\bar{Q}_i$. This is the union of bounding sides — oriented counter clockwise from the paths bounding the Q s — to squares of $\bar{Q}_i$. Also, $\bar{Q}_i$ includes only sides appearing in exactly one Q. Such a side has three (or two, if the grid element is by chance a triangle) other sides of grid squares meeting each vertex. Exactly one side is in D and on another square in $\bar{Q}_i$. So, each vertex has an adjoining segment of $\bar{\gamma}_i$; $\bar{\gamma}_i$ is a simple closed (oriented) path.

5.4.2. *Winding numbers of* $\bar{\gamma}_i$. Choose any square Q^* in $\bar{Q}_i$ and any point $z' \in Q^* \cap C_i$. The winding number of $\bar{\gamma}_i$ about z' is

$$n_i(\bar{\gamma}_i) = n_{z'}(\bar{\gamma}_i) = \sum_{Q \in \bar{Q}_i} n_{z'}(\partial Q) = n_{z'}(\partial Q^*) = 1.$$

Similarly, $n_j(\bar{\gamma}_i) = 0$ for $j \neq i$. Winding numbers of the path γ_i with respect to the C_j s are the same as for $\bar{\gamma}_i$. This is from their definition as an integral (5.3); the integral along δ_i cancels with the integral along δ_i^{-1}.

Suppose $\infty \in C(D)$. Let γ be any closed path in D. To γ associate the r-tuple $(n_1(\gamma), \dots, n_r(\gamma)) \in \mathbb{Z}^r$. Then, the path $\prod_{i=1}^r \gamma_i^{n_i}$ is homologous to γ. Thus, the winding number map is onto $\mathbb{Z}^{r-1}$. This completes Prop. 5.4 for $\infty \in C(D)$.

5.4.3. *The case* $\infty \in D$. Consider the map $H_1(D_\infty) \to H_1(D_\infty)/M_r = H_1(D)$. The latter is the definition of $H_1(D)$. So we comment only on why the image of $\gamma \in \Pi_1(D, z_0)$ depends only on the path γ^* from (5.2). There were two stages to forming γ^*. The first replaced γ by a geodesic path where (5.1) gives its relation to γ. Suppose γ_1 and γ_2 are two such choices. Then, $f_{\gamma_1} = f_{\gamma_2}$ for any f extensible to all of D. In particular, this applies to f a branch of $\log(\frac{z-z_i}{z-z_j})$ with $z_i \in C_i$. Its analytic continuations around γ_1 and γ_2 are the same. Therefore, if neither γ_1 nor γ_2 go through ∞, the winding numbers of $\gamma_1\gamma_2^{-1}$ with respect to all components of the complement of D are the same.

Then, we adjusted the geodesic path to a new path γ^* which for certain did not go through ∞. There were, however, two such choices for γ^*. Label these γ_1^* and γ_2^*. Let δ be the parametrized boundary $\partial\Delta_{s_0}$ of Δ_{s_0}. Then $\delta = \tau_1 \cdot \tau_2$ with τ_1 going from $\gamma(t_1)$ to $\gamma(t_2)$ and τ_2 going (in the same direction) from $\gamma(t_2)$ to $\gamma(t_1)$. For simplicity assume δ goes clockwise around ∞ (as in §5.4.1).

Then, $\gamma_1^* = \gamma_{[a,t_1]} \cdot \tau_1 \cdot \gamma_{[t_2,b]}$ and $\gamma_2^* = \gamma_{[a,t_1]} \cdot \tau_2^{-1} \cdot \gamma_{[t_2,b]}$. Integrals determine homology classes in $H_1(D_\infty)$. From Lemma 4.11, γ_2^* and

$$\gamma_2' = \gamma_{[a,t_1]} \cdot \tau_2^{-1} \cdot \tau_1^{-1} \cdot \tau_1 \cdot \gamma_{[t_2,b]}$$

have the same homology class. So, $[\gamma_2^*]_h - [\gamma_1^*]_h$ is $[\tau_1 \cdot \tau_2]_h$. From Cauchy's Theorem 3.6, $[\tau_1 \cdot \tau_2]_h$ is independent of s_0. On the other hand, δ bounds the disk complement of Δ_{s_0} in the counter clockwise direction. By assumption that disk contains all components of $C(D)$. So, $n_{z'}(\delta) = 1$ as z' runs over points in all components of $C(D)$: $[\gamma_2^*]_h - [\gamma_1^*]_h = (1, \dots, 1)$. This shows the images of $[\gamma_1^*]_h$ and $[\gamma_2^*]_h$ in $H_1(D)$ are the same. That is, M_r measures exactly the discrepancy in substituting γ^* for the original path.

5.4.4. *Integrals along homologically trivial paths.* Now assume f is meromorphic in D. It suffices to show the following. If $\gamma_1, \gamma_2 \in \Pi_1(D, z_0)$, and $\gamma = \gamma_1 \cdot \gamma_2^{-1}$ is homologous to 0, then $\int_{\gamma_1} f\,dz - \int_{\gamma_2} f\,dz = \int_\gamma f\,dz$ depends only on the residues of f in D. Let R_f be the poles of f for which f has nonzero residues. If $\infty \in C(D)$, and $\gamma \in \Pi_1(D, z_0)$ is homologically trivial, then Cauchy's Residue Theorem ([**Ahl79**, p. 149] or [**Con78**, p. 112]) says $\int_\gamma f\,dz$ is $\sum_{z' \in R_f} n_{z'}(\gamma)\mathrm{Res}'_z(f)$. This is the result we want, at least if $\infty \in C(D)$. We won't need to consider the possibility of f having infinitely many nonzero residues.

A reduction of the Residue Theorem to the case f is analytic in D is algebraic. Cauchy's Theorem in this case may be the most important result from first year complex variables. We state it and a generalization for use later.

DEFINITION 5.5. Suppose $u, v : D \to \mathbb{C}$ are continuous (though maybe not analytic). The differential 1-form $\omega = u(z)\,dx + v(z)\,dy$ is *locally exact* if for each $z_0 \in D$, there exists $F_{z_0}(z) = F(z)$ in a neighborhood of z_0 with these properties.

(5.4a) $F(z)$ has continuous partial derivatives.
(5.4b) $\frac{\partial F}{\partial x} = u(z)$ and $\frac{\partial F}{\partial y} = v(z)$.

THEOREM 5.6. *Suppose f is analytic in D, and $\gamma \in \Pi_1(D, z_0)$ is homologous to 0 in D. Then, $\int_\gamma f\,dz = 0$. More generally, this holds with any locally exact differential ω on D replacing $f\,dz$* [**Ahl79**, p. 144, Thm. 16].

Thm. 5.6 holds even if $\infty \in D$ [9.13a]. If we only assume $f \in \mathcal{E}(D, z_0)$, then $\int_\gamma f\,dz$, $\gamma \in \Pi_1(D, z_0)$, usually depends on more than the residues of f and $[\gamma]_h \in H_1(D)$ [9.13d].

6. Branch of solutions of $m(z, w) = 0$

This section discusses the *implicit function* theorem. It is the key ingredient for showing a function satisfying (1.2) satisfies (1.1),

6.1. Branch of inverse of $f(z)$.

Suppose $f(z)$ is meromorphic on D and has range D'. A branch of (right) *inverse* of $f(z)$ on D' is a continuous function $g : D' \to D$ with $f \circ g(z) = z$ for $z \in D'$.

DEFINITION 6.1 (Branch of inverse of f along a path). Let $\gamma : [a, b] \to D$ be a path and $f \in \mathcal{E}(D, z_0)$. Let $g(z)$ be a branch of inverse of $f(z)$ in a neighborhood of z_0. Then a branch of (right) inverse of f along γ is an analytic continuation of $g(z)$ along γ.

We now change the variable z to w, and discuss functions analytic in w. This sets notation for the full implicit function theorem. Suppose $f(w)$ is analytic in a neighborhood Δ_{w_0} of w_0, and $f(w_0) = z_0$. For a given fixed z, assume $\partial\Delta_{w_0}$ passes through no zero or pole of $f(w) - z$ (as a function of w). Then,

$$(6.1) \qquad n_z = \frac{1}{2\pi i}\int_{\partial\Delta_{w_0}} \frac{f'(w)\,dw}{f(w) - z} \quad \text{and} \quad g(z) = \frac{1}{2\pi i}\int_{\partial\Delta_{w_0}} \frac{w f'(w)\,dw}{f(w) - z}$$

count the number n_z (resp. the sum $g(z)$) of zeros of $f(w) - z$ in Δ_{w_0}. By Leibniz's theorem, compute the derivative of $g(z)$ by applying $\frac{\partial}{\partial z}$ under the integral sign (see §7.1). So, $g(z)$ is analytic in z for z close to z_0.

LEMMA 6.2. *Suppose $f(w) - z_0$ has exactly one zero (and no poles) in a neighborhood Δ_{w_0} of w_0. For z sufficiently close to z_0, $f(w) - z$ also has only one zero (and no poles). Thus, the second expression of* (6.1) *defines a branch $g(z)$ of the inverse of $f(z)$ locally.*

The proof of the implicit function theorem in §6.2 includes the proof of Lemma 6.2.

6.1.1. *Branch of $f(z)^{\frac{1}{e}}$ along a path.* For e a positive integer, we use the inverse of the eth power map in a general form. This returns to branch of log.

Suppose f is meromorphic in a domain D. Let $\gamma : [a, b] \to D$ be any path whose range misses all zeros and poles of $f(z)$. Then, define a branch of $\log(f(z))$ along γ to be a continuous function $h(t)$, for which $e^{h(t)} = f(\gamma(t))$, $t \in [a, b]$. Existence of a branch of $\log(f(z))$ along such γ follows from Prop. 3.2. It is the same as a branch of log along the path $f \circ \gamma : [a, b] \to f(D)$.

Define a branch of $f(z)^{\frac{1}{e}}$ along γ using $h(t)$ a branch of $\log(f(z))$ along γ:

$$e^{h(t)/e} \overset{\text{def}}{=} \mathrm{Br}((f(z))^{\frac{1}{e}})(\gamma(t)). \tag{6.2}$$

The left side has a clear meaning. Define the right side to be the value of the branch at $\gamma(t)$. Check: The left of (6.2) to the eth power is $f(\gamma(t))$, as expected. As before, there are e such branches.

Applying Prop. 3.2 gives a unique branch $h(t)$ having a specific value $h(a)$ equal to one of the e th roots of $f(\gamma(a))$.

6.1.2. *Local inverses of rational functions.* Suppose $f = f_1/f_2 \in \mathbb{C}(w)$ with $(f_1, f_2) = 1$. Consider the set $X_f = \{(z, w) \in \mathbb{P}^1_z \times \mathbb{P}^1_w \mid f(w) - z = 0\}$. Each point (z_0, w_0) on $\mathbb{P}^1_z \times \mathbb{P}^1_w$ has a basis of open sets; each set in the basis is the product of an open set around z_0 and an open set around w_0. Intersect those open sets with X_f to get neighborhoods of points of X_f. We discuss for which (z_0, w_0) there exists $g(z)$ analytic in a neighborhood of z_0 satisfying

(6.3) $g(z_0) = w_0$ and $f(g(z)) = z$.

That is, g produces a local parametrization of a neighborhood of $(f(w_0), w_0)$ by $z \mapsto (z, g(z))$: $(z, g(z))$ is on X_f because $f(g(z)) - z \equiv 0$.

There is a global parametrization of X_f by $w \mapsto (f(w), w)$: $f(w) - f(w) \equiv 0$. This parametrization, however, isn't as a function of z. It is insistent reference to z as the parameter that gives coherent information about the algebraic function $g(z)$.

Lemma 6.2 says points (z_0, w_0) with a multiplicity one zero w_0 of $f(w) - z_0$ have neighborhoods projecting one-one to the z-line: $(z, g(z)) \mapsto z$. Assume $z_0 \neq \infty$. Then, w_0 is a multiplicity one zero of $f_1(w) - z_0 f_2(w)$. If this doesn't hold, then w_0 is a zero of $f_1(w) - z_0 f_2(w)$ and its derivative $f_1'(w) - z_0 f_2'(w)$ in w. Call it a *critical* value. Eliminate z_0.

(6.4) Critical values of w_0 are zeros of $f_1(w) f_2'(w) - f_2(w) f_1'(w)$.

In particular, there are at most $\deg(f_1) + \deg(f_2) - 1$ critical values of w_0 (or of z_0). [9.4] precisely defines critical values when w_0 is a pole of f.

6.1.3. *Abel's application.* Apply the chain rule to $f(g(z)) \equiv z$:

$$\frac{df}{dw}_{|w=g(z)} \frac{dg}{dz} = 1. \tag{6.5}$$

Therefore, $\frac{dg}{dz} = 1/\frac{df}{dw}_{|_{w=g(z)}}$. This is the complex variable variant of how first year calculus computes an antiderivative of inverse trigonometric functions. Abel applied this to a (right) inverse of a branch of primitive from the following integral

$$\int_\gamma \frac{dz}{(z^3+cz+d)^{\frac{1}{2}}} \tag{6.6}$$

with $c, d \in \mathbb{C}$ (Chap. 4 §7.1). Use (6.2) to interpret $h(z)\,dz = \frac{dz}{(z^3+cz+d)^{\frac{1}{2}}}$ around some base point z_0: $h(z)$ is a branch of $(z^3+cz+d)^{-\frac{1}{2}}$. Let $f(z)$ be a primitive for $h(z)\,dz$. Apply (6.5) to $f(g(z)) = z$ (special case of (7.3)):

$$\frac{dg(z)}{dz} = (g(z)^3 + cg(z) + d)^{\frac{1}{2}}. \tag{6.7}$$

Let $\boldsymbol{z}\{z_1, z_2, z_3, \infty\}$, the three zeros of z^3+cz+d and ∞. Analytic continuation of $(z^3+cz+d)^{-\frac{1}{2}}$ and its primitive $f(z) = f(z; c, d)$ produce the collection $\mathcal{A}_f(U_{\boldsymbol{z}})$. First year calculus computes the inverse of a primitive of $h_1(z) = (z^2+cz+d)^{-\frac{1}{2}}$, recognizing it from the trigonometric function $\sin(z)$. This has a unique analytic continuation everywhere in $\mathbb{C}$. Abel discovered the same was true for the inverse $g(z) = g(z; c, d)$ of $f(z; c, d)$; it extends everywhere in $\mathbb{C}$. Many conclusions follow.

This example will inspire later topics. For example, dependence of $g(z) = g(z; c, d)$ on (c, d) usefully distinguishes between algebraic curves defined by $w^2 - z^3 + cz + d$ as a function of (c, d) (Chap. 4 §7.1). For each (c, d), $g(z; c, d)$ is to the exponential function as (6.6) is to a branch of $\log(z)$.

6.2. Implicit function theorem. Consider $m(z, w) \in \mathcal{H}(D)[w]$ (a polynomial in w with coefficients in $\mathcal{H}(D)$). Suppose $g(z)$ is analytic on D and $m(z, g(z)) \equiv 0$. We discuss paths $\gamma \to D$ along which there is an analytic continuation of $g(z)$. Such paths should exclude z' having a w' with

(6.8) $m(z', w') = 0$ and $\frac{\partial m}{\partial w}(z', w') = 0$.

Riemann's Existence Theorem produces the *Riemann surface* attached to $g(z)$ (Chap. 4). Data for the Riemann surface include information about all embeddings of $\mathbb{C}(z, g(z))$ in Puiseux fields. This important, though lesser data, is available from the proof that Puiseux fields are algebraically closed (§7.3). Given a polynomial $m(z, w)$ it is theoretically possible, though not always practical, to compute exactly the Puiseux embeddings of $\mathbb{C}(z, g(z))$ from m.

6.2.1. *Branch and critical points.* A branch of *solutions to* $m(z, w)$ along γ is an analytic continuation of $g(z)$ along γ. Such analytic continuations avoid points z' having w' satisfying (6.8). Prop. 6.4 references $h_0 \in \mathbb{C}[z]$ in the expression

$$m(z, w) = h_0(z)w^n + h_1(z)w^{n-1} + \cdots + h_n(z). \tag{6.9}$$

If z' is a zero of h_0, $m(z', w)$ has degree lower than n in w.

Definition 6.3 (Branch point of (m, w)). A point (z', w') is *critical* for (m, w) if it satisfies (6.8). Call $z' \in \mathbb{C}$ a *branch* point of (m, w) if either there exists w' with (z', w') a critical point or $\deg(m(z', w)) < \deg_w(m(z, w)) = n$.

Suppose z' is not a branch point of (m, w). Then, there are exactly n distinct values w' with $m(z', w') = 0$. The substitutions $z \mapsto 1/z$ and/or $w \mapsto 1/w$ allows extending the definition of critical points of $m(z, w)$ to include z' and/or w' equal to ∞ (see [9.4] and [9.11]). Use the notation of (6.9) and $U_{\boldsymbol{z}} = \mathbb{P}^1_z \setminus \boldsymbol{z}$.

6.2.2. *Algebraic according to* (1.2) *implies* (1.1). Now we see that algebraic by the equation definition implies algebraic by the analytic continuation definition.

PROPOSITION 6.4. *Suppose* $\boldsymbol{z}$ *includes* ∞ *and all branch points of* (m, w). *Assume* (z_0, w_0) *satisfies the first equation of* (6.8), *but* $z_0 \notin \boldsymbol{z}$. *Then, there is a* $g(z)$ *analytic near* z_0 *with* $m(z, g(z)) \equiv 0$ *and* $g(z_0) = w_0$. *For* $\gamma \in \Pi_1(U_{\boldsymbol{z}}, z_0)$, $g(z)$ *analytically continues along* γ *and* $m(z, g_\gamma(z)) \equiv 0$ *(near the end point of* γ*).*

If $m(z, w) \in \mathbb{C}[z, w]$ *is irreducible, then* $\boldsymbol{z}$ *is a finite set. There are exactly* n *branches of solutions of* $m(z, w)$ *along any* $\gamma \in \Pi_1(U_{\boldsymbol{z}}, z_0)$ *(and exactly* n *elements of* $\mathcal{A}_g(U_{\boldsymbol{z}})$*). Conclude:* $X_m = \{(z, w) \in \mathbb{C} \times \mathbb{C} \mid m(z, w) = 0, z \in U_{\boldsymbol{z}}\}$ *is connected and* g *is algebraic according to* (1.1).

The proof takes up §7.1. Then we get complete equivalence between (1.1) and (1.2).

7. Equivalence of the two definitions of algebraic

We show (m, w) has only finitely many branch points if $m \in \mathbb{C}[z, w]$.

LEMMA 7.1. *Assume* $m \in \mathcal{H}(D)[w]$ *and* $\deg_w(m) = n > 0$. *Suppose there is no domain* $D' \subset D$ *in which all* $z' \in D'$ *are branch points. Then, the branch points of* (m, w) *have no accumulation point in* D. *Further, if* $m \in \mathbb{C}[z, w]$, *either* m *and* $\frac{\partial m}{\partial w}$ *have a common factor, or* (m, w) *has only finitely many branch points.*

PROOF. Suppose the lemma is false, and z' is such an accumulation point. Let $\Delta_{z'} \subset D$ be a disk around z'. So, in this disk there is a sequence of pairs (z_j, w_j), $j = 1, 2, \dots$ with these properties:

(7.1) w_j is a multiple zero of $m(z_j, w)$ and $\lim_{j \mapsto \infty} z_j = z'$.

Let $\mathcal{R}_{z'}$ be the ring of power series in z convergent in a neighborhood of z'. Then, $\mathcal{R}_{z'}$ is a principle ideal domain.

Regard m and $\frac{\partial m}{\partial w}$ as polynomials in w with coefficients in $\mathcal{R}_{z'}$. Apply the Euclidean algorithm [9.11]. It produces the greatest common divisor $m_1(w)$ of m and $\frac{\partial m}{\partial w}$ in the form $a(z, w)m + b(z, w)\frac{\partial m}{\partial w} = m_1(z, w)$, a nonzero polynomial. These polynomials in w have coefficients in $\mathcal{H}(D')$ with D' a neighborhood of z'.

If $\deg_w(m_1) \geq 1$ for each $z' \in D'$, a zero w' of $m_1(z', w)$ gives a common zero of $m(z', w)$ and $\frac{\partial m}{\partial w}(z', w)$. This is contrary to our assumption. So $\deg_w(m_1) = 0$ and the z_j s are zeros of m_1, an analytic function of z, accumulating at z'. So, m_1 is identically zero contrary to a previous observation.

Apply the Euclidean algorithm to the case $m \in \mathbb{C}[z, w]$. Conclude: If m and $\frac{\partial m}{\partial w}$ have no common factor, then m_1 is a polynomial in z, and all branch points are zeros of it. Thus, there are only finitely many such zeros. □

7.1. Proof of Prop. 6.4. Assume (z_0, w_0) is not a critical point of (m, w). Let $g(z)$ be $\frac{1}{2\pi i}\int_C w\frac{\partial m}{\partial w}(z, w)\,dw/m(z, w)$ for each z close to z_0 with C a counter clockwise circle suitably close to w_0. We show there are neighborhoods, U_{z_0} of z_0 and U_{w_0} of w_0, with $U_{z_0} \times U_{w_0}$ free of critical points of (m, w).

To do this, extend Lemma 7.1. Simplify notation by taking $z_0 = 0$ and $w_0 = 0$. Then, $m(0, w) \neq 0$ for $0 < |w| < r_1$. As $z \mapsto 0$, $m(z, w) \mapsto m(0, w)$ uniformly with respect to w. So, there exists $r < r_2 < r_1$ with $|m(z, w) - m(0, w)| < |m(0, w)|$ for $|z| < r_2$ and $|w| < r$. By Rouche's Theorem [**Con78**, p. 125], $m(z, w)$ and $m(0, w)$ have the same number of zeros in $|w| < r$. So, $m(z, w)$ has a single zero in this region and $g(z)$ gives it.

With C fixed and z close to (but not equal) z_0, apply $\frac{\partial}{\partial z}$ under the integral giving $g(z)$ to compute its derivative. The partial derivative of $w\frac{\partial m}{\partial w}(z,w)/m(z,w)$ exists and is continuous. Thus, Leibniz's rule [**Con78**, p. 68] says this gives $\frac{dg}{dz}$, showing it is analytic.

Now consider analytic continuation of $g(z)$ along any path in $U_{\boldsymbol{z}}$. This is the same as the proof of Prop. 3.2 starting at §3.3.1. The key ingredient was analytically continuing $g(z)$ beyond the end point of any given path. We have the tools now for that. If $\gamma : [a,b] \to U_{\boldsymbol{z}}$ is any path, there is a neighborhood of $\gamma(b)$ and $g_1(z)$ analytic in this neighborhood with $g_1(\gamma(b))$ the value of the extension of $g(z)$ to the end point. As in that proof, since $m(\gamma(t), g_1(\gamma(t))) \equiv 0$ for t close to b, $m(z, g_1(z)) \equiv 0$ for all z with $g_1(z)$ defined.

This leaves showing that as γ runs over $\Pi_1(U_{\boldsymbol{z}}, z_0)$, g_γ runs over all n branches $g_1, \dots, g_n$ of solutions of $m(z,w)$ around z_0. Suppose, however, it runs over only the subset $g_1, \dots, g_t$ with $t < n$. Consider

$$M(z,w) \stackrel{\text{def}}{=} \begin{array}{l}\prod_{i=1}^t (w - g_i(z)) = \\ w^t - G_1(z)w^{t-1} + G_2(z)w^{t-2} + \cdots + (-1)^t G_t(z).\end{array} \tag{7.2}$$

Each $G_i(z)$ is a symmetric polynomial $S_i(w_1, \dots, w_t)$ in $w_1, \dots, w_t$ evaluated at $(g_1, \dots, g_t)$. So, $G_i \in \mathcal{E}(U_{\boldsymbol{z}}, z_0)$ (Lem. 4.6).

By assumption, for $\gamma \in \Pi_1(U_{\boldsymbol{z}}, z_0)$, $g_{1,\gamma}, \dots, g_{t,\gamma}$ is a permutation of $g_1, \dots, g_t$. Thus, $G_{i,\gamma} = S_i(g_{1,\gamma}, \dots, g_{t,\gamma}) = S_i(g_1, \dots, g_t)$ (Lem. 4.6). So, $\mathcal{A}_{G_i}(U_{\boldsymbol{z}})$ contains a single element, $i = 1, \dots, t$. Apply Riemann's removable singularity theorem [**Ahl79**, p. 124] exactly as in the proof of Cor. 7.5. Conclude: Singularities of G_i in $\mathbb{P}^1_z$ are at worst poles. So G_i is a rational function in z: $M(z,w) \in \mathbb{C}(z)[w]$.

Plug in $g_1(z) = g(z)$, $M(z, g(z)) \equiv 0$. Therefore, M is an irreducible polynomial for $g(z)$ over $\mathbb{C}(z)$ of degree $t < n$. This is contrary to the function field being of degree n. This contradiction proves the transitivity statement and concludes the proof of Prop. 6.4. The n elements of $\mathcal{A}_g(U_{\boldsymbol{z}})$ give the n values w' satisfying $m(z_0, w) = 0$. So, as λ runs over closed paths for which $g_\lambda(z_0) = w'$, this connects all the points of X_m lying over z_0. Therefore, analytic continuation along the connected set $U_{\boldsymbol{z}}$ connects all the points of X_m. For future use, here is the lemma hidden in this argument.

LEMMA 7.2. *Suppose $f(z)$ is analytic in a neighborhood of $z_0 \notin \boldsymbol{z}$ with $\boldsymbol{z}$ the branch points of $m(z,w) \in \mathbb{C}[z,w]$ and $m(z, f(z)) \equiv 0$. Let $g \in \mathbb{C}(z, f(z))$ and assume $g_\lambda = g$ for each $\lambda \in \Pi_1(U_{\boldsymbol{z}}, z_0)$. Then, $g \in \mathbb{C}(z)$.*

7.2. The converse and integrals along paths. Assume $f \in \mathcal{E}(U_{\boldsymbol{z}}, z_0)$. If f satisfies (1.1) we see it satisfies a nontrivial polynomial equation. Let $f_1, \dots, f_n$ be the conjugates of f. Apply to $f_1, \dots, f_n$ the argument in (7.2) for $g_1, \dots, g_t$.

PROPOSITION 7.3. *The definitions* (1.1) *and* (1.2) *are equivalent.*

Assume $m(z, g(z)) \equiv 0$, as in Prop. 6.4. Analytic continuation of $g(z)$ along $\gamma : [a,b] \to U_{\boldsymbol{z}}$ produces $t \mapsto h(t)$, continuous; $h(t)$ is one of the n distinct values w' of $m(\gamma(t), w') = 0$. For $n_1, n_2 \in \mathbb{C}[z,w]$, let $n_1(z,w)/n_2(z,w) = n(z,w)$. Define the integral of $n(z, g(z))$ along γ:

$$\int_\gamma n(z, g(z))\, dz \stackrel{\text{def}}{=} \int_a^b n(\gamma(t), h(t))\, dt. \tag{7.3}$$

Avoid paths through zeros of n_2 to assure the integral exists.

7.3. $\mathcal{P}_{z'}$ is algebraically closed. Let $\Delta_{z'}$ be a closed disk in $\mathbb{P}^1_z$ centered at z'. Denote $\Delta_{z'} \setminus \{z'\}$ by $\Delta^0_{z'}$. We show analytic continuations of $f(z) \in \mathcal{E}(\Delta^0_{z'}, z_0)$ depend only on analytic continuation of f on a circle about z'. This will show $\mathcal{P}_{z'}$ is algebraically closed. Let δ be the counter clockwise circle about z' through z_0.

PROPOSITION 7.4. *If $\lambda \in \Pi_1(\Delta^0_{z'}, z_0)$ has winding number $n_{z'}(\lambda) = e(\lambda)$, then $f_\lambda = f_{\delta^{e(\lambda)}}$.*

Prop. 7.4 gives the complete theory of Riemann surface covers of a punctured disk (in Chap. 3). The proof of Prop 7.4 is in §7.4.

COROLLARY 7.5. *As in Prop. 7.4, assume $f \in \mathcal{E}(\Delta^0_{z'}, z_0)$ is algebraic over $\mathcal{L}_{z'}$. Let $e = e_f$ be the minimal positive integer with $f_{\delta^e}(z) = f(z)$ (near z_0). Then, $f \in \mathcal{P}_{z',e}$ and $\mathcal{L}_{z'}(f)/\mathcal{L}_{z'}$ is isomorphic to $\mathcal{P}_{z',e}/\mathcal{L}_{z'}$. In particular, the Puiseux expansion field $\mathcal{P}_{z'}$ is algebraically closed. Algebraic functions in $\mathcal{P}_{z',e}$ consist of composites $h(\alpha(z))$ with h algebraic in $\mathcal{L}_{z'}$ and $\alpha(z)$ in the set $\{(z-z')^{1/e}\}_{e=1}^{\infty}$.*

PROOF. If $f(z)$ is algebraic over $\mathcal{P}_{z'}$, then it satisfies an equation of degree n with coefficients in $\mathcal{P}_{z'}$. There are only a finite number of coefficients. With no loss assume these are in $\mathcal{P}_{z',e'}$ for some e'; f is algebraic over $\mathcal{P}_{z',e'}$. We want to show $f \in \mathcal{P}_{z',e'e}$ for some e.

Replace $u_{e'} = (z-z')^{1/e'}$ by $z-z'$ everywhere in the equation for $f(z)$ to revert this to where f is algebraic over $\mathcal{P}_{z'}$. Or, use this usual algebra observation: If f is algebraic over $\mathcal{P}_{z',e'}$, since $\mathcal{P}_{z',e'}$ is algebraic over $\mathcal{L}_{z'}$, the degree of f is finite over $\mathcal{L}_{z'}$, equal to $[\mathcal{P}_{z',e'}(f) : \mathcal{P}_{z',e'}][\mathcal{P}_{z',e'} : \mathcal{P}_{z'}]$ (§1.2).

Suppose $f \in \mathcal{E}(\Delta_{z'}, z_0)$. Also, $m(f(z)) \equiv 0$ for $z \in \Delta_{z'}$ with $m(w) \in \mathcal{L}_{z'}[w]$ and $\lambda \in \Pi_1(\Delta_{z'}, z_0)$. Then, f_λ is another zero of $m(w)$ [9.8c]. Let $\deg_w(m(w)) = n$. Then $f_{\lambda^e} = f$ for some integer $e \le n$. Choose e minimal. Then, use δ as in Prop. 7.4. It shows e is the minimal integer with $f_{\delta^e} = f$.

For simplicity, assume $z' = 0$ ($\Delta_{z'} = \Delta_0$) with w_0 a solution of $w_0^e = z_0$. Let Δ_1 be the preimage of Δ_0 by the map $\psi : u \to u^e$: Δ^0_1 the preimage of Δ^0_0. Finally, let δ_1 be the counter clockwise circle through w_0 around 0 in Δ^0_1. Then,

$$f \circ \psi_{\delta_1}(u) = f_{\delta^e}(\psi(u)) = f(\psi(u)).$$

Apply Prop. 7.4 to $(f\circ\psi, \Delta^0_1, w_0)$ to conclude $f\circ\psi_\gamma = f\circ\psi$ for $\gamma \in \Pi_1(\Delta^0_1, w_0)$. Lemma 4.12 implies $f \circ \psi$ is analytic in Δ^0_1. Replace z by u^e in the coefficients of $m(w)$. Let $\mathcal{L}_{0,u}$ be convergent Laurent series in u around $u = 0$. This gives $m_1(w) \in \mathcal{L}_{0,u}[w]$ and $m_1(f \circ \psi(u)) \equiv 0$. So, as $u \mapsto 0$, $f \circ \psi(u)$ goes to one of finitely many values on the Riemann sphere.

Apply Riemann's removable singularity theorem [**Ahl79**, p. 124]: $f\circ\psi$ extends to an analytic function $\Delta_1 \to \mathbb{C} \cup \{\infty\}$. That is, $f \circ \psi$ is analytic in u with $u^e = z$. As in [9.9g], this embeds the function field $\mathbb{C}(z, f(z))$ into $\mathcal{P}_{z',e}$. As $f(z)$ has e conjugates over $\mathcal{L}_{z'}$, $[\mathcal{L}_{z'}(f(z)) : \mathcal{L}_{z'}]$ is at least e. As $\mathcal{L}_{z'}(f(z))$ is a subfield of $\mathcal{P}_{z',e}$, with $[\mathcal{P}_{z',e} : \mathcal{L}_{z'}] = e$, the two fields are equal. This concludes the proof. □

7.4. Proof of Prop. 7.4. Let $\lambda \in \Pi_1(\Delta^0_{z'}, z_0)$ have winding number $n_{z'}(\lambda)$ around z'. The proof is in parts for later use. They consist of preliminary notation and description; explicit contraction of λ to a path having range the points of δ; and an observation on analytic continuation around such a path. Lemma 4.3 assures $f_\lambda = f_{\lambda^*}$ with λ^* a polygonal path. So, with no loss assume λ is polygonal.

7.4.1. *Notational simplifications.* The range of λ is compact, and it does not include z'. So, there is a minimal distance r_0 between z' and the range of λ. Let A be an annulus around z' with inner radius $r' < r_0$ and outer radius R' giving the boundary of $\Delta_{z'}$. For simplicity assume $z' \neq \infty$ and the disk $\Delta_{z'}$ is in the complex plane, rather than on the Riemann sphere. Since circles go to circles by stereographic projection, the only adjustment to use the Riemann sphere would be to compose the description of the sets here with stereographic projection. Also, for simplicity, assume $z_0 - z' = r_0 e^{2\pi\theta_0}$ has $\theta_0 = 0$.

7.4.2. *Description of A.* The point $z_v = z' + r_0 e^{2\pi i v}$ lies on δ. We also use $z_v^- = z' + r' e^{2\pi i v}$ and $z_v^+ = z' + R' e^{2\pi i v}$. The points of the line segment cut by a ray from z' to z_v^+ meet A in the set

$$L_v = \{z_v - s(z_v^- - z_v) \mid s \in [-1,0]\} \cup \{z_v + s(z_v^+ - z_v) \mid s \in [0,1]\}.$$

Thus the annulus is the union of the points on L_v, $v \in [0,1]$. Reference the point on L_v corresponding to $s \in [-1,1]$ by $L_v(s)$.

7.4.3. *Contraction of A to δ.* Define $\Gamma : A \times [0,1] \to A$ by

$$\Gamma(L_v(s), u) = \begin{cases} z_v - (1-u)s(z_v^- - z_v) & \text{for } s \in [-1,0] \\ z_v + (1-u)s(z_v^+ - z_v) & \text{for } s \in [0,1]. \end{cases}$$

Finally, for each $u \in [0,1]$ we have a path $\gamma_u : [a,b] \to A$:

$$t \mapsto \gamma_u(t) = \Gamma(\gamma(t), u).$$

Note: $\gamma_0(t) = \gamma(t)$ and $\gamma_1(t)$ has range in the points of δ. Further, $\gamma_1(t)$, being the contraction of a polygonal path to δ changes direction but finitely many times. Take f as in the statement of Prop. 7.4. Conclude easily: $f_{\gamma_1} = f_{\delta^{e_1}}$ with e_1 the winding number of γ_1 around z'.

7.4.4. *f_{γ_u} constant in $u \in [0,1]$.* For $u \in [0,1]$ consider the continuous function $f_u^*(t)$ giving analytic continuation (according to Def. 4.1) along γ_u. Let $h_{u,t}$ be the analytic function with restriction to $\gamma_u(t')$ giving $f_u^*(t')$ for t' close to t.

Lemma 4.3 says for (u', t') close to (u,t), $h_{u,t}$ restricts to $\gamma_{u'}(t')$ to give $f_{u'}^*(t')$. Since $f_{u'}^*(t')$ is a composition of two continuous functions $\gamma_{u'}(t')$ and $h_{u,t}$, it is continuous. Thus, $f_u^*(b)$ is a continuous function of u. As $f_u^*(b)$ is in the discrete set of end values of the analytic continuations of f in $\Delta_{z'}^0$, it is constant in u.

Since z_0 is not a branch point of the algebraic function f, the end value $f_u^*(b)$ determines f_{γ_u}. So, $f_{\gamma_1} = f_\gamma$, to conclude the proof of the proposition.

7.5. Ramification indices, branch cycles and inertia groups. Consider $L/\mathbb{C}(z)$, a finite extension. Let $z' \in \mathbb{P}_z^1$ and let $\mu : L \to \mathcal{P}_{z'}$ be an embedding of L into Puiseux expansions about z'. As in [9.9], let $\zeta_e = e^{2\pi i/e}$ for $e \geq 1$ an integer.

DEFINITION 7.6. The *ramification index* of (L, z', μ) is the minimal integer $e = e(L, z', \mu)$ for which $\mathcal{P}_{z',e}$ contains $\mu(L)$.

7.5.1. *A crucial automorphism.* Let $\hat{L}$ be the Galois closure of $L/\mathbb{C}(z)$. Cor. 7.5 says there is an integer $\hat{e}$ giving an embedding $\psi : \hat{L} \to \mathcal{P}_{z',e}$ fixed on $\mathbb{C}(z)$. Here is how ψ produces a conjugacy class in $G(\hat{L}/\mathbb{C}(z))$ depending only on z'. Let $g_{z'}$ be the automorphism of $\mathcal{P}_{z',\hat{e}}$ mapping $(z - z')^{1/\hat{e}}$ to $\zeta_{\hat{e}}^{-1}(z - z')^{1/\hat{e}}$. This is restriction of a topological generator of the group of the whole algebraic closure.

Denote invertible integers modulo e by $(\mathbb{Z}/e)^*$. Consider *compatible* sequences of integers $m_e \in \mathbb{Z}/e^*$, $e \geq 1$: $m_{ee'} \bmod e = m_e$ for all integers e, e'. Denote this collection $\hat{\mathbb{Z}}^*$. Similarly, $\hat{\mathbb{Z}}$ is the compatible collection of $m_e \in \mathbb{Z}/e$. Then, $\hat{\mathbb{Z}}$ is a topological ring whose (multiplicative) units are $\mathbb{Z}^*$ [**FJ86**, Chap. 1].

REMARK 7.7 (Use of the p-adics). Here is a reminder of the algebra for writing elements of $\hat{\mathbb{Z}}^*$. First: Consider only e that are powers of a particular prime p. Then, the compatible sequences $\{m'_{p^k}\}_{k=1}^{\infty}$ analogous to $\hat{\mathbb{Z}}$ is $\mathbb{Z}_p$, the *p-adic integers*. These satisfy $m'_k \in \mathbb{Z}/p^k$, with $m'_{k+1} = m'_k \bmod p^k$ with $k = 1, \dots$. The direct product of the $\mathbb{Z}_p$ s over primes p is $\hat{\mathbb{Z}}$. The direct product of the units $\hat{\mathbb{Z}}_p^*$ of $\hat{\mathbb{Z}}$ is $\hat{\mathbb{Z}}^*$. Symbolically write elements of $\hat{\mathbb{Z}}_p^*$ as series $a_0 + a_1 p + a_2 p^2 + \cdots$. Here $1 \le a_0 \le p-1$ and $0 \le a_i \le p-1$ are arbitrary. Without this procedure, excluding 1 and -1, it might be hard to list *any* elements of $\hat{\mathbb{Z}}^*$.

LEMMA 7.8. *The automorphism $g_{z'}$ maps $\mathcal{P}_{z',e}$ into itself for each e. Its effect on $\mathcal{P}_{z',ee'}$ extends its effect on $\mathcal{P}_{z',e}$.*

Let σ be any automorphism of $\mathcal{P}_{z'}$ fixed on $\mathcal{L}_{z'}$. The effect of σ on $\mathcal{P}_{z',e}$ is the same as $g_{z'}^{m_e}$ for some $m_e \in (\mathbb{Z}/e)^*$. So, σ corresponds to an element of $\hat{\mathbb{Z}}^*$.

PROOF. This requires checking the effect of $g_{z'}$ on generators of the field extensions. By definition, $g_{z'}(z-z')^{1/ee'} = \zeta_{ee'}^{-1}(z-z')^{1/ee'}$. Put both sides to the power e' and then apply $g_{z'}$. As $g_{z'}$ is a field automorphism,

$$(g_{z'}((z-z')^{1/ee'}))^{e'} = g_{z'}(((z-z')^{1/ee'})^{e'}) = g_{z'}((z-z')^{1/e}).$$

Yet, $(g_{z'}((z-z')^{1/ee'}))^{e'} = (\zeta_{ee'}^{-1}(z-z')^{1/ee'})^{e'}$. As $\zeta_{ee'}^{e'} = \zeta_e$ (by definition), this concludes the first part.

Powers of $g_{z'}$ give the group of the degree e extension $\mathcal{P}_{z',e}/\mathcal{L}_{z'}$ [9.9d]. So, σ restricted to $\mathcal{P}_{z',e}$ equals $g_{z'}^{m_e}$ for some $m_e \in (\mathbb{Z}/e)^*$. Let σ_e be restriction of σ to $\mathcal{P}_{z',e}$. Compatibility of these m_e s is from σ_e being restriction of $\sigma_{ee'}$ to $\mathcal{P}_{z',e}$. □

7.5.2. *Embeddings and branch cycles.* Continue the discussion starting §7.5.1. Restrict $g_{z'}$ to $\hat{L}$. Since $\hat{L}/\mathbb{C}(z)$ is Galois and $g_{z'}$ fixes $\mathbb{C}(z)$, this gives an automorphism $g_{z',\psi}$ of $\hat{L}$. Denote this element of $G(\hat{L}/\mathbb{C}(z)) = G$ by $g_{z',\psi}$. It depends on ψ, the choice of the embedding. Call it the *branch cycle* attached to the pair (z', ψ).

LEMMA 7.9. *For $z' \in \mathbb{P}_z^1$, $[\hat{L} : \mathbb{C}(z)]$ distinct embeddings $\psi : \hat{L} \to \mathcal{P}_{z',\hat{e}}$ leave $\mathbb{C}(z)$ fixed. As ψ runs over such embeddings, $g_{z',\psi}$ runs over a conjugacy class in G. Suppose $f(z)$, meromorphic about a nonbranch point z_0, satisfies $m(z, f(z)) \equiv 0$, $m \in \mathbb{C}[z,w]$. So, $z' \in \mathbb{P}_z^1$ produces a conjugacy class $\mathrm{C}_{z'}$ of $G = G(\hat{L}/\mathbb{C}(z))$. With $\boldsymbol{z}$ the branch points of (m,w), for each $z' \notin \boldsymbol{z}$, $\mathrm{C}_{z'} = \{1\}$.*

Let δ be a clockwise (closed) circle around $z' \in \boldsymbol{z}$ bounding a closed disk $\Delta_{z'}$. Assume $\Delta_{z'}$ (excluding possibly z') contains no other branch point of (m,z) and $z_0 \in \Delta_{z'}$. Let $f_1, \dots, f_n$ be a complete list of conjugates of f. Denote analytic continuation of f_j around δ by $f_{j,\delta}$. Then, for some choice of ψ, $g_{z',\psi}$ maps this set to $f_{1,\delta}, \dots, f_{n,\delta}$.

PROOF. Cor. 7.5 produces one embedding, $\psi : \hat{L} \to \mathcal{P}_{z',\hat{e}}$. Let α run over the automorphisms of $\hat{L}$ fixed on $\mathbb{C}(z)$. Then, $\psi \circ \alpha : \hat{L} \to \mathcal{P}_{z',\hat{e}}$ runs over $[\hat{L} : \mathbb{C}(z)]$ embeddings of $\hat{L}$ into the algebraic closure of $\mathcal{L}_{z'}$ fixed on $\mathbb{C}(z)$. Galois theory says this is the exact number of embeddings possible. So we have listed them all.

Consider the effect on $g_{z',\psi}$ of composing ψ with α. The new automorphism is

$$g_{z',\psi\circ\alpha} = (\psi \circ \alpha)^{-1} \circ g_{z'} \circ (\psi \circ \alpha) = \alpha^{-1} g_{z',\psi} \alpha.$$

That is, $g_{z',\psi\circ\alpha}$ runs over the conjugacy class of $g_{z',\psi}$ in G as α runs over G.

Regard elements $f_1,\dots,f_n$ as in $\mathcal{L}_{z_0}$. Let $h(z)$ be a branch of $(z-z')^{1/\hat{e}}$ defined in this neighborhood of z_0. Giving an embedding of $\hat{L}$ (fixed on $\mathbb{C}(z)$) into $\mathcal{P}_{z',\hat{e}}$ is equivalent to giving an embedding of $\hat{L}$ mapping $f_1,\dots,f_n$ into power series $g_1(h(z)),\dots,g_n(h(z))$ in $h(z)$, $g_1,\dots,g_n \in \mathcal{L}_0$. Analytic continuation of $g_1(h(z)),\dots,g_n(h(z))$ around δ maps $g_i(h(z))$ to $g_i(\zeta_{\hat{e}}^{-1}h(z))$. This is the effect of restriction of $g_{z'}$ on the embedding of the f_i s in the Puiseux expansions. □

7.5.3. *Branch cycles and inertia groups.* Choosing $\zeta_{\hat{e}}^{-1}$ (rather than $\zeta_{\hat{e}}$) in the definition of $g_{z'}$ is convenient (later). This assures δ in Lem. 7.9 is a clockwise path. The conjugacy class $\mathrm{C}_{z'}$ in Lem. 7.9 is crucial to precise formulations of Riemann's Existence Theorem. This is the branch cycle conjugacy class attached to z'. Using $G \le S_n$, disjoint cycle data (Chap. 3 §7.1) for elements of $\mathrm{C}_{z'}$ is sufficient for some applications, though not for the more serious.

Definition 7.10 (Inertia groups). The branch cycle $g_{z',\psi}$ in Lem. 7.9 generates a group, $I_{z',\psi}$ of $G(\hat{L}/\mathbb{C}(z))$. This is the *inertia group* attached to the embedding ψ. The notation $I_{z'}$ refers to any choice of the groups conjugate to $I_{z',\psi}$. Points $z' \in \mathbb{P}^1_z$ for which $I_{z'}$ is nontrivial are the *branch points* of $L/\mathbb{C}(z)$.

7.5.4. *Two definitions of branch points.* There are now two definitions of branch points. Def. 7.10 gives it for the function field $L/\mathbb{C}(z)$ and §6.2 for the pair (m,z). They are related though they may not be equal [9.11].

Proposition 7.11. *Suppose $m(z,f(z)) \equiv 0$ and $L = \mathbb{C}(z,f(z))$. If $z' \in \mathbb{C}$ is a branch point of $L/\mathbb{C}(z)$, then it is also a branch point of (m,z).*

Proof. Suppose z' is a branch point of $L/\mathbb{C}(z)$. Then, there is an embedding $\psi : \mathbb{C}(z,f(z)) \to \mathcal{P}_{z',e}$ where the image of f is not in $\mathcal{L}_{z'}$. In particular, the power series $\psi(f)$ and $g_{z'}(\psi(f))$ in $(z-z')^{1/e}$ have the same value after substituting 0 for $(z-z')^{1/e}$. Since $(w-\psi(f(z))(w-g_{z'}(\psi(f)))$ divides $m(z,w)$ (in $\mathcal{P}_{z'}[w]$), this shows $m(z',w)$ has multiple zeros. □

8. Abelian functions from branch of log

A branch of log isn't an algebraic function. Still, it allows explicit construction of all the algebraic functions we call *abelian*, the topic of this subsection.

8.1. Further notation around extensible functions. Let $\mathcal{E}(U_{\boldsymbol{z}},z_0)$ be the extensible (meromorphic) functions on $U_{\boldsymbol{z}}$ (as in Def. 4.5; given by elements of $\mathcal{L}_{z_0}$). Denote algebraic elements of $\mathcal{E}(U_{\boldsymbol{z}},z_0)$ (as in Def. 1.1) by $\mathcal{E}(U_{\boldsymbol{z}},z_0)^{\mathrm{alg}}$.

Definition 8.1. Let G be a finite group having a specific property P^*. Say an element $f \in \mathcal{E}(U_{\boldsymbol{z}},z_0)^{\mathrm{alg}}$ has property P^* if its monodromy group G_f (§4.4.1) has this property. This allows referring to abelian, *nilpotent* (G_f is a product of its p-Sylow subgroups), solvable or primitive functions.

Example: Suppose $[\mathbb{C}(z,f):\mathbb{C}(z)] = n$. Then, f is *primitive* if G_f is a primitive subgroup of S_n (Chap. 3 Def. 7.9). Equivalently, by the Galois correspondence, there is no field properly between $\mathbb{C}(z)$ and $\mathbb{C}(z,f)$ [9.5]. Later chapters show this is a very important concept. Unfortunately, the word *primitive* appears in many guises in mathematics (already in this chapter). It has even more meanings in the Webster's dictionary. The closest to our meaning here is this: not derived; as a primitive verb in grammar. So, $\mathbb{C}(z,f)$ is an extension not (even partially) derived from any other proper extension of $\mathbb{C}(z)$. Note that this is different in English than it being generated by a single element over $\mathbb{C}(z)$ (primitive generator). Denote the abelian (resp. nilpotent) functions in $\mathcal{E}(U_{\boldsymbol{z}}, z_0)^{\mathrm{alg}}$ by $\mathcal{E}(U_{\boldsymbol{z}}, z_0)^{\mathrm{ab}}$ (resp. $\mathcal{E}(U_{\boldsymbol{z}}, z_0)^{\mathrm{nil}}$).

8.2. Abelian monodromy. For $e \in \mathbb{Z}$ and $\gamma : [a, b] \to D$ a path whose range misses all zeros and poles of $f(z)$, (6.2) defines branch of $f(z)^{\frac{1}{e}}$ along γ.

Here is data for abelian functions of index e:

- distinct points $\boldsymbol{z} = z_1, \dots, z_r$ in $\mathbb{P}^1_z$: *branch points*
- Δ_{z_0}, a disk neighborhood of z_0: *base point*
- an integer e: *index*
- a branch $g_{i,j}$ of $\log(\frac{z-z_i}{z-z_j})$ in Δ_{z_0}, $1 \le i < j \le r$

Denote the field $\mathbb{C}(z, e^{g_{i,j}/e}, 1 \le i < j \le r)$ by $L_{e,\boldsymbol{z}}$: The field of *abelian functions* (on $\mathbb{P}^1_z$) ramified over $\boldsymbol{z}$ of index dividing e. It is a subfield of $\mathcal{L}_{z_0}$. Any $f \in L_{e,\boldsymbol{z}}$ defines an analytic $f : \Delta_{z_0} \to \mathbb{P}^1_z$ according to notation of §4.6. If some $z_i = \infty$ replace $z - z_i$ by 1 in the definition. In particular, when $z_r = \infty$, $g_{i,r}$ is a branch of $\log(z - z_i)$, $i = 1, \dots, r-1$. This definition includes all algebraic functions having abelian monodromy group. It will give a valuable comparison in Chap. 4. There is a similar definition of algebraic functions on D with any domain D replacing $\mathbb{P}^1_z$.

8.2.1. *Galois group of $L_{e,\boldsymbol{z}}$.* A complete description of $L_{e,\boldsymbol{z}}$ depends only on homology classes of paths in $\Pi_1(U_{\boldsymbol{z}}, z_0)$.

Corollary 8.2. *Assume $\gamma_1, \gamma_2 \in \Pi_1(U_{\boldsymbol{z}}, z_0)$ are homologous and f is an algebraic abelian function on $U_{\boldsymbol{z}}$ corresponding to the data* (8.2)*. Then, the analytic continuations f_{γ_1} and f_{γ_2} (back to z_0) are equal. Monodromy from $\Pi_1(U_{\boldsymbol{z}}, z_0)$ induces a faithful action of $H_1(U_{\boldsymbol{z}})/eH_1(U_{\boldsymbol{z}})$ on $L_{e,\boldsymbol{z}}$ and therefore on $\mathbb{C}(z, \mathcal{A}_f(U_{\boldsymbol{z}}, z_0))$ (§4.2.2). In particular, $L_{e,\boldsymbol{z}}/\mathbb{C}(z)$ is Galois with group $H_1(U_{\boldsymbol{z}})/eH_1(U_{\boldsymbol{z}})$. For $f \in L_{e,\boldsymbol{z}}/\mathbb{C}(z)$, $\mathbb{C}(z, \mathcal{A}_f(U_{\boldsymbol{z}}, z_0))/\mathbb{C}(z)$ is Galois with group a quotient of this group.*

Proof. For simplicity assume $z_r = \infty$. Take $\gamma \in \Pi_1(U_{\boldsymbol{z}}, z_0)$ and

$$f(z) = m_1(e^{g_{1,\gamma}(z)/e}, \dots, e^{g_{r-1,\gamma}(z)/e})/m_2(e^{g_{1,\gamma}(z)/e}, \dots, e^{g_{r-1,\gamma}(z)/e}),$$

where $g_{j,\gamma}$ denotes analytic continuation of g_j around γ. Let m_j be the winding number of γ about z_j. Analytic continuation of g_j around γ adds $2\pi i m_j$ to g_j (Prop. 3.5). Since γ_1 and γ_2 have the same winding numbers around each z_j, this proves the effect of their analytic continuations on f are the same.

Note that $L_{e,\boldsymbol{z}}/\mathbb{C}(z)$ is the composite of the field extensions $\mathbb{C}(z, e^{g_j(z)/e})/\mathbb{C}(z)$. Apply [9.9] using $(e^{g_j(z)/e})^e = z - z_j$. Conclude: $\mathbb{C}(z, e^{g_j(z)/e})\mathbb{C}(z)$ is Galois with group $\mathbb{Z}/(e)$. From [9.5d], the composite of these fields is Galois, with group a subgroup of $\mathbb{Z}/(e) \times \cdots \times \mathbb{Z}/(e)$. The image of $H_1(U_{\boldsymbol{z}})/eH_1(U_{\boldsymbol{z}})$ produces field automorphisms of $L_{e,\boldsymbol{z}}$. We know these explicitly. Let a closed path λ have respective winding numbers $(a_1, \dots, a_{r-1})$ around $(z_1, \dots, z_{r-1})$. If e does not divide a_j, then monodromy action of λ on g_j is nontrivial. So the automorphism group is all of $(\mathbb{Z}/e)^{r-1}$. This shows the result. □

8.3. Deeper into the Monodromy Theorem. Consider $m \in \mathbb{C}[z, w]$ and D a domain in $\mathbb{P}^1_z$. It is a fundamental to decide when some branch of solutions of $m(z, w) = 0$ is a meromorphic function on all of D. Riemann's Existence Theorem gives a satisfactory answer to versions of this question.

8.3.1. *Simple connectedness.* Call a domain in $\mathbb{C}$ *simply connected* if there is at most one connected component in $\mathbb{P}^1_z \setminus D$. Chap. 3 has the usual definition of a simply connected topological space. For open subsets of $\mathbb{P}^1_z$ these definitions describe the same sets. The following is an application of Cauchy's Residue Theorem for later comparison with the general Monodromy Theorem.

THEOREM 8.3 (Monodromy Theorem). *Suppose $D \subset \mathbb{C} \setminus \{z_1, \dots, z_r\}$ is simply connected. Assume f has no residues in D. Then $f(z)$ has a primitive (antiderivative; §2.5) $F(z)$ on D. Suppose $\boldsymbol{z}$ contains the zeros and poles of $f(z)$. Apply this to $\frac{df}{dz}/f$ to conclude there is a branch of $\log(f(z)))$ on D.*

8.3.2. *Homological triviality versus simple connectedness.* Being simply connected has another characterization: the winding number of any closed path in D relative to any point z' outside of D is 0. That is, D is simply connected if all paths in D are homologous to 0. Beware! If D is not simply connected, some paths may be homologous to 0, though not trivial for our applications. For example, any function that isn't abelian has a nontrivial analytic continuation around some path homologous to 0. For, however, abelian functions, most questions use just the Monodromy Theorem in Prop. 7.4. For example, suppose $m(z, g(z)) \equiv 0$, and $\mathbb{C}(z, g(z))/\mathbb{C}$ is an abelian extension (g is *abelian*). Then, we can characterize those D that aren't simply connected on which g is extendible. It is tougher to be so precise about antiderivatives for even abelian functions g along paths in D.

8.4. Primitive tangential base points. Let $f \in \mathcal{E}(U_{\boldsymbol{z}}, z_0)^{\text{alg}}$ and $z' \in \boldsymbol{z}$. Suppose λ in $U_{\boldsymbol{z}}$ goes from z_0 to z_1. Analytic continuation of f produces $f_\lambda \in \mathcal{E}(U_{\boldsymbol{z}}, z_1)$. Consider λ a *restriction* map. Applying λ restricts f to $f_\lambda \in \mathcal{L}_{z_1}$.

How about using a path to *restrict* f to a function around z'? That is, let λ be a path with end point close to z'. Can we consider f_λ restriction of $g \in \mathcal{P}_{z'}$? The simple answer is No!, unless f_λ extends to an analytic function around z'. It is valuable, however, to add data to $\mathcal{P}_{z'}$ so the answer will be Yes!

Choose an open disk D' in $U_{\boldsymbol{z}}$, with z' on its boundary. Let $g_e(z)$ be a branch of $(z - z')^{1/e}$ on D', one for each positive integer e. This always exists from (6.2). Further, we ask the system of these be *compatible*:

(8.1) For all integers (e, e', e'') satisfying $ee' = e''$, $g_{e''}(z)^{e'} = g_e(z)$.

Call this collection $\{g_e\}_{e=1}^\infty = \mathcal{G}(D', z')$ a *system of branches* on (D', z'). The following is a slight enhancement of Lem. 7.8.

PROPOSITION 8.4. *Given $\mathcal{G}(D', z')$, any system of branches on (D', z') corresponds one-one with elements of $\hat{\mathbb{Z}}$ (§7.5.1). Precisely: $\{m_e\} \in \hat{\mathbb{Z}} \mapsto \{\zeta_e^{m_e} g_e(z)\}_{e=1}^\infty$.*

Let $D'' \subset D'$ be any (open) disk tangent to z'. Restriction of $\mathcal{G}(D', z')$ to D'' defines a system of branches $\mathcal{G}(D'', z')$. Let $\boldsymbol{v}$ be the direction from z' along a geodesic on $U_{\boldsymbol{z}}$ toward the center of D'. (Consider $U_{\boldsymbol{z}}$ a subset of the sphere with its metric; geodesics being great circles.) Containment orders disks tangent to z' with $\boldsymbol{v}$ pointed into the disk. There is a maximal element

$$\mathcal{G}(\boldsymbol{v}, z', U_{\boldsymbol{z}}) = \mathcal{G}(\boldsymbol{v}, z') = \mathcal{G}(D_{\boldsymbol{v}}, z') :$$

Take $D_{\boldsymbol{v}}$ the largest disk in $U_{\boldsymbol{z}}$ having radius along $\boldsymbol{v}$ and tangent to z'.

So, the set of branch systems satisfying (8.1) is a *homogeneous space* for $\hat{\mathbb{Z}}$. That is, an action of the group $\hat{\mathbb{Z}}$ on one of them gives all. You still, however, need one choice $\mathcal{G}(D', z')$ to get the process going.

DEFINITION 8.5. Call $\mathcal{G}(\boldsymbol{v}, z') = \hat{\boldsymbol{v}}$ a *primitive* (or *naive*) tangential base point: $\hat{\boldsymbol{v}}$ has an underlying point z', direction $\boldsymbol{v}$ and system of branches on $D_{\boldsymbol{v}}$.

From Cor. 7.5, elements in $\mathcal{P}_{z',e}$ have the form $f^* = h((z-z')^{1/e})$ with $h \in \mathcal{L}_{z'}$. Define $\text{rest}_{\hat{\boldsymbol{v}}}(f^*)$ to be $h(g_e(z))$. For any simply connected subspace Y of $U_{\boldsymbol{z}}$, denote paths in $U_{\boldsymbol{z}}$ from z_0 with endpoint in Y by $\Pi_1(z_0, Y)$.

PROPOSITION 8.6 (Tangential Base Point Restriction). *Assume* $f \in \mathcal{E}(U_{\boldsymbol{z}}, z_0)$ *and* $\gamma \in \Pi_1(z_0, D_{\boldsymbol{v}})$. *There is a unique* $f^* \in \mathcal{P}_{z'}$ *with* $\text{rest}_{\hat{\boldsymbol{v}}}(f^*) = f_\lambda$.

PROOF. Uniqueness of f^* is clear. Existence is from Cor. 7.5. Here are details. Let δ be a clockwise circle bounding a disk $\Delta_{z'}$ with center z' with $\Delta_{z'} \setminus \{z'\} \subset U_{\boldsymbol{z}}$. Assume δ meets $D_{\boldsymbol{v}}$. Connect the end point of λ to some point on δ by a path lying entirely in $D_{\boldsymbol{v}}$. From Cauchy's Theorem (Prop. 3.6), there is a unique function g defined by a power series on $D_{\boldsymbol{v}}$ that restricts to f_λ. So, any analytic continuation of f_λ along a path in $D_{\boldsymbol{v}}$ equals g. Thus it depends only on the end point of this path. Assume with no loss λ ends on δ.

Let $e = e_f$ be the order of the monodromy action of δ on f_λ. Then, Cor. 7.5 says f_λ is $f^* = h(g_{e_f}(z))$ with h holomorphic in the disk δ bounds. $\square$

EXAMPLE 8.7 (Deligne tangential base points). Take $z' = 0$ and $\boldsymbol{v}$ any direction $0 \le \theta < 2\pi$ on $\mathbb{C}_z$ represented by $e^{i\theta}$. Define $g_e(z)$ to be $e^{i\theta/e}$ times the unique branch of $(e^{-i\theta}z)^{1/e}$ taking positive real values along the direction $\boldsymbol{v}$ from 0: [**De89**, §15] or [**Ihar91**, p. 103].

8.5. Describing all algebraic abelian functions. Suppose $f(z)$ is algebraic and $\mathbb{C}(z,f)/\mathbb{C}(z)$ is a Galois extension with abelian Galois group G. Assume $\boldsymbol{z}$ contains the branch points of f and the ramification indices at all points of $\boldsymbol{z}$ divide some integer e. Each $z' \in \boldsymbol{z}$ produces an inertia group $I_{z'}$ (Def. 7.10). More explicitly it produces a well defined conjugacy class $\mathrm{C}_{z'}$ in G (Lem. 7.9). Since, however, G is abelian, this conjugacy class is an element $g_{f,z'} \in G$.

THEOREM 8.8. *Under the above hypotheses,* $g_{f,z'}$*, as* z' *runs over* $\boldsymbol{z}$*, determines the field extension* $\mathbb{C}(z,f)$*. Further, two other properties hold.*

- $\langle g_{f,z'}, z' \in \boldsymbol{z}\rangle = G$*: generation*
- $\prod_{z'\in \boldsymbol{z}} g_{f,z'} = 1$ *: product-one condition*

Conversely, suppose given G *and elements* $g_{z'} \in G$ *for each* $z' \in \boldsymbol{z}$ *satisfying* (8.8). *Then, there exists algebraic* f *(given as above by branches of log) satisfying* $g_{f,z'} = g_{z'}$ *for* $z' \in \boldsymbol{z}$*. Another algebraic function* f^* *produces the same data if and only if* $\mathbb{C}(z,f^*) = \mathbb{C}(z,f)$.

PROOF. There is a standard reduction for showing the field is determined by the data $g_{f,z'}, z' \in \boldsymbol{z}$. Write G as $\prod_{i=1}^u G_i$ where G_i is cyclic of some prime power order. Every finite abelian group has this form ([**Isa94**, p. 90], see [9.15]). Then, $\mathbb{C}(z,f)$ is the composite of field extensions $L_i/\mathbb{C}(z)$ with group G_i, $i = 1,\dots,u$. Further, any subextension $\mathbb{C}(z) < M < L_i$ is Galois with group a quotient of G_i. So, it is cyclic of prime power order. So, with no loss assume $\mathbb{C}(z,f)/\mathbb{C}(z)$ is Galois with group isomorphic to $\mathbb{Z}/p^t$ for some integer t and prime p. List $\boldsymbol{z}$ as $z_1,\dots,z_r$, then list the group data as $(g_1,\dots,g_r)$ with $g_i = g_{f,i}$ attached to z_i. Since $G = \mathbb{Z}/p^t$, identify g_i with an integer $n_i \in \mathbb{Z}/p^t$.

It is easy to produce a cyclic extension that has exactly this attached data. For simplicity, assume $z_r = \infty$. Then, for any z_0 not in $\boldsymbol{z}$, let $h(z) = \prod_{i=1}^{r-1} h_i(z)^{n_i}$

with h_i a branch of $(z-z_i)^{\frac{1}{p^t}}$ in a neighborhood of z_0. The lemma is done if $\mathbb{C}(z,h(z)) = \mathbb{C}(z,f(z))$. Both fields embed in $\mathcal{P}_{z_i}$ and the action of g_{z_i} restricts to both fields the same way. Any function in the fixed field of all the g_i s is extensible over the whole Riemann sphere, as in §7.1. So such a function is a rational function in z. Therefore, the fixed field of $\langle g_1,\dots,g_r\rangle$ in $\mathbb{C}(z,f(z))$ is trivial. Apply [9.5d] to the composite of the two fields and conclude they are equal.

Consider the generation condition. Assume $\langle g_{f,z'}, z' \in \boldsymbol{z}\rangle = H$ is a proper subgroup of G. If $f_1 \in \mathbb{C}(z,f)$ is in the fixed field of H, then $f_{1,\lambda} = f_1$ for all $\lambda \in \Pi_1(U_{\boldsymbol{z}}, z_0)$. Lem. 7.2 implies $f_1 \in \mathbb{C}(z)$. So $\mathbb{C}(z)$ is the exact fixed field of H and $H = G$. The product-one condition appears by recognizing $g_{f,z'}$ as restriction of the $g_{e,z'}$ for the field $L_{e,\boldsymbol{z}}$. Apply the product of the $g_{e,z'}$ to generating functions in $L_{e,\boldsymbol{z}} = \mathbb{C}(z, e^{g_{i,j}/e}, 1 \le i < j \le r)$ (from (8.2)). It comes to showing $g_{e,z_j} g_{e,z_i}(e^{g_{i,j}/e}) = e^{g_{i,j}/e}$. With no loss take $z_i = 0$ and $z_j = \infty$ [9.10a]. □

The full version of Riemann's Existence Theorem generalizes the generation and product-one conditions (8.8) to $\mathbb{C}(z,f(z))$ where f is any algebraic function. When G is abelian, the product-one condition is independent of the order of the elements $g_{f,z'}$. Keep your eye on the analysis that goes into tracking the order of elements appearing in the product-one condition when G is not abelian. This is what produces the significant action of the *Hurwitz monodromy group* in Chap. 5. Further, the converse holds in generality. Without, however, the abelian condition producing the algebraic function f is more mysterious.

Suppose G and G^* are abelian groups and $\boldsymbol{g}_{\boldsymbol{z}}$ and $\boldsymbol{g}^*_{\boldsymbol{z}^*}$ satisfy the conditions of (8.8). Consider two triples $\mathcal{G} = (G, \boldsymbol{z}, \boldsymbol{g}_{\boldsymbol{z}})$ and $\mathcal{G}^* = (G^*, \boldsymbol{z}^*, \boldsymbol{g}^*_{\boldsymbol{z}^*})$ as in Thm. 8.8. Assume $\boldsymbol{z}$ is a subset of $\boldsymbol{z}^*$. For this discussion, if $z' \in \boldsymbol{z}^* \setminus \boldsymbol{z}$ regard $\boldsymbol{g}_{\boldsymbol{z}}$ as having the identity element at z'. Also, assume there is a homomorphism $\alpha : G^* \to G$ taking $g^*_{z'}$ to $g_{z'}$ for $z' \in \boldsymbol{z}^*$. Regard $\alpha = \alpha_{\mathcal{G}^*,\mathcal{G}}$ as a map from $\mathcal{G}^*$ to $\mathcal{G}$.

Corollary 8.9. *The projective system $\{\mathcal{G}, \alpha_{\mathcal{G}^*,\mathcal{G}}\}$ of triples with maps has a limit consisting of a group $\mathcal{G}^{\mathrm{ab}}$ and elements $g^{\mathrm{ab}}_{z'}$ running over $z' \in \mathbb{P}^1_z$. Then, $\mathcal{G}^{\mathrm{ab}}$ identifies with the maximal abelian quotient of the absolute Galois group of $\mathbb{C}(z)$. Also, $g^{\mathrm{ab}}_{z'}$ acts trivially on any abelian algebraic function in $\mathcal{L}_{z'}$ and identifies with a generator of the automorphisms of $\mathcal{P}_{z'}/\mathcal{L}_{z'}$ in its restriction to the abelian algebraic functions in $\mathcal{P}_{z'}$ (Cor. 7.5).*

Call the group $\mathcal{G}^{\mathrm{ab}}$, the Galois group of the *maximal abelian extension* of $\mathbb{C}(z)$. A collection $\{g^{\mathrm{ab}}_{z'}\}_{z' \in \mathbb{P}^1_z}$ will be a *canonical system of generators* of $\mathcal{G}^{\mathrm{ab}}$. Any $g \in \mathcal{G}^{\mathrm{ab}}$ acts on the abelian algebraic functions in $\mathcal{P}_{z'}$ for any z'. This action is also the restriction of an automorphism of $\mathcal{P}_{z'}/\mathcal{L}_{z'}$. So monodromy action on a branch of $\log(z-z')$ determines this restriction element as a multiple of $g^{\mathrm{ab}}_{z'} \in \hat{\mathbb{Z}}$.

9. Exercises

Some exercises remind of basic Galois Theory. Use $\mathrm{char}(K)$ to denote the characteristic of a field K: The minimal positive integer n for which n times the identity in K is 0 (if such an integer exists, or 0 otherwise).

9.1. Substitutions and the chain rule. Consider more on (2.7c) as the defining property of analyticity.

(9.1a) For a path $\lambda : [a,b] \to \mathbb{C}$, compose it with any analytic function $h : \mathbb{C} \to \mathbb{C}$ to give $h \circ \lambda : [a,b] \to \mathbb{C}$, another path. If g and h satisfy (2.7c), show

$$\begin{aligned}\tfrac{d}{dt}(g\circ h)(\lambda(t_0)) = \tfrac{d}{dt}(g(h(\lambda)))(t_0) &= \tfrac{dg}{dw}_{|w=h(\lambda(t_0))}\tfrac{d}{dt}(h\circ\lambda)_{|t=t_0} \\ &= \tfrac{dg}{dw}(h\circ\lambda_{|t=t_0})\tfrac{dh}{dz}(\lambda(t_0))\tfrac{d\lambda}{dt}(t_0).\end{aligned}$$

(9.1b) Show: Existence of $f'(z_0)$ requires only checking (2.5) for $\lambda : [-1,1] \to D$ by $t \mapsto z_0 + tv$ with $v \neq 0$. That is, check directional derivative rule (2.7b).

(9.1c) Conclude, if in (2.7c) two of $g \circ h, g, h$ are analytic, then so is the third.

With $m(z,w) = w^k - h(z)$ and $w(t)$ and $z(t)$ (nonconstant) rational functions with $w(t)^k \equiv h(z(t))$ for all t, consider indefinite integrals for $I(z) = \int h(z)^{\frac{1}{k}}\,dz$.

(9.2a) Substitute $z(t)$ for t. Rewrite $I(z)$ as an antiderivative for $\frac{dz(t)}{dt}/w(t)$. Apply this with $k = 2$ and $h(z) = z^2 + az + b$ using [9.3d].

(9.2b) Ex. [9.10f] shows [9.2a] won't work often, not even with $k = 2$ and $\deg(h) = 3$ having no repeated roots. Show it does work for any h with at most two distinct zeros, but arbitrary degree.

(9.2c) Calculus uses a different substitution: $w(t)$ and $z(t)$ are trigonometric in t with $w(t)^2 = z(t)^2 + az(t) + b$. Result: The square root expression disappears; replaced by a function. Why choose transcendental over rational functions? Hint: Consider the antiderivative as a function of z.

9.2. Rational functions and field theory. Suppose K is any field. Consider $u(z) = P_1(z)/P_2(z)$ in $K(z)$. Follow the notation of §1.2.1.

(9.3a) Show $P_1(w) - zP_2(w)$ is irreducible. Hint: Factor it as $m_1(z,w)m_2(z,w)$. Then compute the degree in z of each factor.

(9.3b) Suppose $m \in K[z,w]$, $\deg_z(m) = 1$ and $m(z, f(z)) \equiv 0$ for some $f(z)$ analytic on a domain D. Show $K(z, f(z)) = K(f(z))$.

(9.3c) If $M \le L_1 \le L_2$ is a chain of fields, *transitivity for degrees* says $[L_2 : M] = [L_1 : M][L_2 : L_1]$. Use it to show $\deg(u_1(u_2(z))) = \deg(u_1)\deg(u_2)$ for $u_1, u_2 \in K(z) \setminus \{0\}$.

(9.3d) Suppose M is a field and $\mathrm{char}(K) \neq 2$. Assume $m(z,w) \in K[z,w]$ of total degree 2 is irreducible, $z_0, w_0 \in K$, $m(z_0,w_0) = 0$ and w' is a zero of $m(z,w)$ in $\overline{K(z)}$. Show $K(z)(w')$ is isomorphic to $K(t)$ for some $t \in K(z)(w')$. Hint: With t and s variables, let $z_0 + s = z$ and $w' = w_0 + ts$. Solve for s as a function of t in $m(z,'w) = 0$.

(9.3e) Show $z_0, w_0 \in K$ is necessary for the existence of t in (9.3d).

(9.3f) The fundamental theorem of algebra follows from knowing a function $f(z)$ bounded and analytic on $\mathbb{C}$ is constant. How does this imply every analytic function $P : \mathbb{P}^1_z \to \mathbb{P}^1_w$ (§4.6) by $z \mapsto P(z)$ is an element of $\mathbb{C}(z)$?

Now consider parametrizations by rational function curves. Use §6.1.2 with $f = f_1/f_2 \in \mathbb{C}(w)$ and $(f_1, f_2) = 1$. Parametrize X_f near (z_0, w_0) if w_0 is not a zero of the *Wronskian* $f_1(w)f_2'(w) - f_2(w)f_1'(w)$ of f_1, f_2 and $f_2(w_0) \neq 0$.

(9.4a) Use Def. 4.14 to show this includes when w_0 is a zero of f_2 ($z_0 = \infty$).

(9.4b) Extend a) to $w_0 = \infty$. Show an analytic parametrization of a neighborhood by $(z, g(z))$ exists if and only if $|\deg(f_1) - \deg(f_2)| \le 1$.

(9.4c) Suppose $f(g(z)) \equiv z$ for $g(z)$ analytic in a neighborhood of z_0. With these extensions, show the maximal number of branch points for $\mathbb{C}(z, g(z))$ (§6.2) is $2(\deg(f) - 1)$ with equality occurring for some rational functions f of degree n for any positive integer n.

(9.4d) Suppose w_0 is a zero of $f_1(w)f_2'(w) - f_2(w)f_1'(w)$ of multiplicity $e_{w_0} - 1$ and $f(w_0) = z_0$. Apply the Cor. 7.5 proof to find e_{w_0} distinct functions $g(u)$ analytic around 0 with $g(0) = w_0$ and $f(g(u)) - z_0 - u^{e_{w_0}} \equiv 0$?

(9.4e) Extend d) to have either z_0 or w_0 is ∞. Conclude for $f \in \mathbb{C}(z) \setminus \mathbb{C}$:

$$2(\deg(f) - 1) = \sum e_{w_0} - 1.$$

9.3. Galois theory of composite fields and using group theory. Suppose L_1/K and L_2/K are two field extensions. Given a field L containing both L_1 and L_2, there is an immediate *minimal field* $L_1 \cdot L_2$ in L containing them both [**Isa94**, Chap. 18].

(9.5a) Suppose M/K is *Galois*: Its group of automorphisms $G(M/K) = G$ fixed on K has order $[M : K]$. Consider $K < L < M$, a chain of fields. Suppose $L = L_1, \dots, L_n$ are the fields conjugate to L/K. Show $L_1 \cdot L_i = L_1$, $i = 1, \dots, n$, if and only if L/K is Galois ($G(M/L)$ is a normal subgroup; closed under conjugation from G).

(9.5b) Let $T : G \to S_n$ be the permutation representation of G on cosets of $G(M/L)$ (as in a). Show there is $j \neq 1$ with $L_1 = L_1 \cdot L_j$ if and only if $(1)T(g) = 1 \Leftrightarrow (j)T(g) = j$ for each $g \in G$.

(9.5c) The following notation holds for the next two subexercises. Suppose M_i/K is Galois with group G_i, $i = 1, 2$. Consider the group G defined as follows:

$$\{g = (g_1, g_2) \in G_1 \times G_2 \mid g_1(\alpha) = g_2(\alpha), \alpha \in M_1 \cap M_2\}.$$

Show G acts as automorphisms of $M_1 \cdot M_2$.

(9.5d) Show $|G| = [M_1 \cdot M_2 : K]$, and so $M_1 \cdot M_2/K$ is Galois with group G. Hint: Apply the Fundamental Theorem of Galois Theory [**Isa94**, Thm. 18.21] to the fixed field of G.

(9.5e) Conclude $M_1 \cdot M_2$ doesn't depend (up to isomorphism over K) on what field they both sit inside if both extensions are Galois.

(9.5f) Assume $\mathrm{char}(K)$ is p (a prime or 0). Suppose K has at most one extension of degree n for any integer $n > 0$ (or if $p > 0$, prime to p). Show extensions of K of degree prime to p are Galois with cyclic group.

We warmup in interpreting field theory with group theory. Let $K = \mathbb{C}(z)$. If f is algebraic over K denote $K(f)$ by L_f, and the Galois closure of L_f/K by $\hat{L}_f$. Suppose $m_i \in \mathbb{C}[z, w]$, of degree n_i in w, is the irreducible polynomial for a function f_i (algebraic according to (1.2)) over K, $i = 1, 2$. Denote $G(\hat{L}_{f_i}/K)$ by G_i, $i = 1, 2$. As in [9.5d], regard $G \overset{\text{def}}{=} G(\hat{L}_{f_1} \cdot \hat{L}_{f_2}/K)$ as a subgroup of $S_{n_1} \times S_{n_2}$. Let $\pi_i : G_1 \times G_2 \to G_i$ be projection on the ith factor.

(9.6a) For H a subgroup of $G_1 \times G_2$, let $\ker(\pi_i(H))$ be the kernel of projection of H on G_i. For $H \leq G_1 \times G_2$ with $\pi_i(H) = G_i$, $i = 1, 2$, let A_H be $\langle \ker(\pi_1(H)), \ker(\pi_2(H)) \rangle$. Show $H = \{(g_1, g_2) \mid \psi_1(g_1) = \psi_2(g_2)\}$ with $\psi_i : G_i \to G_1 \times G_2/A_H = G_H$: H is the *fiber product* of ψ_1 and ψ_2.

(9.6b) Consider L/F and F/M algebraic field extensions, with $\psi : F \to \bar{M}$ an embedding of F in the algebraic closure of M. Galois theory depends on the *Extension Theorem* [**Isa94**, Thm. 17.30]: There exists an embedding $\psi' : L \to \bar{M}$ extending ψ. Explain why this shows $\pi_i(G) = G_i$, $i = 1, 2$.

(9.6c) Let $G_2(1) = G(\hat{L}_{f_2}/L_{f_2})$. Consider $\pi_2^{-1}(G_2(1))$, the biggest subgroup of G projecting to $G_2(1)$. Show m_1 is irreducible over L_{f_2} if and only if $\pi_1(\pi_2^{-1}(G_2(1)))$ is transitive.

(9.6d) Let $f^{(1)}, \dots, f^{(n)}$ be the conjugates of $f^{(1)} = f$ with f algebraic over K of degree n. Denote $G(\hat{L}_f/K(f^{(i)}))$ by $G(i)$. Show: $K(f^{(1)})$ contains $f^{(i)}$ if and only if $G(1) = G(i)$.

9.4. Branch of log **and Puiseux expansions.** Assume $D \subset \mathbb{C}^*$ is a domain.

(9.7a) A classical domain D supporting a branch of log on D is any (subdomain of a) sector: $S_{\theta_1,\theta_2} = \{re^{i\,\theta} \mid \theta_1 < \theta < \theta_2\}$ under the condition $\theta_2 - \theta_1 \le 2\pi$. Give the branches of log on S_{θ_1,θ_2}.

(9.7b) If $H_1(z)$ and $H_2(z)$ are two branches of log in D and $H_1(z_0) = H_2(z_0)$ for $z_0 \in D$, show $H_1(z) = H_2(z)$ for $z \in D$.

(9.7c) Prop. 3.2 shows there exists a branch g_λ of log along any path in D. If for any $\lambda \in \Pi_1(D, z_0)$, $g_\lambda(1) = g_\lambda(0)$, show there is a branch of log on D. Hint: Let $G(z)$ be $g_\lambda(b)$ with $\lambda : [a, b] \to D$ so $\lambda(a) = z_0$, $\lambda(b) = z$ and g_λ is a branch of log along λ with $g_\lambda(a) = w_0$ (fixed). Apply Lem. 4.11.

(9.7d) Show there is a branch of log in a domain D if and only if each closed path in D has winding number 0 about the origin.

(9.7e) Consider $\gamma_1, \gamma_2; [0, 1] \to \mathbb{P}^1_z$ with these properties: $\gamma_1(0) = \gamma_2(0) = 0$, $\gamma_1(1) = \gamma_2(1) = \infty$, and for $t \in (0, 1)$ $\gamma_1(t) \ne \gamma_2(t)$, and $\gamma_i(t) \in \mathbb{C}^*$, $i = 1, 2$. Let D be any component ([9.17e]: there are two) of $\mathbb{C}^* \setminus \{\gamma_1, \gamma_2\}$. Show there is a branch of log in D.

Assume $f(z)$ is analytic near z_0 and algebraic according to (1.2): $m(z, f(z)) \equiv 0$ for some nonzero $m \in \mathbb{C}[z, w]$.

(9.8a) Why can we assume $m(z, w)$ is *irreducible* in the ring $\mathbb{C}[z, w]$? How does this same observation show the ring of analytic functions on a (connected) domain D is an integral domain. Hint: $h(z)$ analytic on D and zero at a set with a limit point in D is identically zero [**Ahl79**, p. 127].

(9.8b) Assume (f, D, z_0) is extensible. As in (1.1), why does $h(z) \in \mathcal{A}_f(D)$ also satisfy $m(z, h(z)) \equiv 0$. Conclude: $f(z)$ satisfies (1.1b).

(9.8c) Note in b) for given D, the conclusion requires only that $m(z, w)$ has coefficients meromorphic on D (not necessarily on $\mathbb{P}^1_z$).

(9.8d) Use §6.1 to complete showing $f(z)$ satisfies (1.1).

(9.8e) Suppose $f(z)$ is a branch of log on D. Show it satisfies neither of the properties (1.1a) or (1.1b). Yet, it does satisfy (1.1c).

(9.8f) If $g : D_1 \to D$ is analytic and $f(g(z)) \equiv z$, show $g(z)$ satisfies (1.2).

(9.8g) Suppose $f \in \mathcal{H}(\mathbb{C})$. Let $\boldsymbol{z} = \{\infty\}$. Then, f satisfies (1.1a) and (1.1b). Suppose f is not a polynomial function. Show it doesn't satisfy (1.1c). Hint: Apply the Caseroti-Weierstrass theorem [**Con78**, p. 109].

Consider how branches of log closely tie to Puiseux expansions. Use notation of §1.3 for the field $\mathcal{L}_{z'}$ around z'. For integer $e > 1$ create a copy $\mathcal{P}_{z',e}$ of $\mathcal{L}_{z'}$ by replacing $z - z'$ by a new variable u_e. Set $e^{2\pi i/e} = \zeta_e$.

(9.9a) Why is $\mathcal{L}_{z'}$ a field?

(9.9b) Suppose $e \,|\, e^*$: $t = e^*/e$. Map $\mathcal{P}_{z',e}$ to $\mathcal{P}_{z',e^*}$ by substituting $u^t_{e^*}$ for u_e. Show this map extends to a field homomorphism.

(9.9c) Identify $\mathcal{P}_{z',e}$ with its image in $\mathcal{P}_{z',e^*}$. Form the union, the ring of *Puiseux expansions* $\mathcal{P}_{z'}$, over all e. Why is it a field?

(9.9d) Show $\mathcal{P}_{z',e}$ is a Galois extension of $\mathcal{L}_{z'}$ with group $\mathbb{Z}/(e)$. Hint: A generator acts by $u_e \mapsto \zeta_e u_e$.

(9.9e) Suppose $z_0 \neq z'$. Let $h(z)$ be a branch of $\log(z - z')$ in a neighborhood D of z_0. Show $f_e(z) = e^{h(z)/e}$ is a branch of solutions of $w^e = z - z'$. So $f(z)$ is an algebraic function.

(9.9f) If $e > 1$, show $f_e(z)$ is not the analytic continuation of a function in $\mathcal{L}_{z'}$.

(9.9g) Consider $\varphi : \mathbb{P}^1_w \to \mathbb{P}^1_z$ by $w \mapsto w^e + z'$. Form $g(w) = f_e \circ \varphi$ and show it is an analytic continuation of some function (of w) around 0.

We may equally consider Puiseux expansions at ∞. Denote the Laurent series around ∞ by $\mathcal{L}_\infty$: expressions $(1/z)^n h(1/z)$ with n an integer and $h(z)$ convergent near $z = 0$. As in [9.9], form a copy $\mathcal{P}_{\infty,e}$ of $\mathcal{L}_\infty$ by replacing $1/z$ by u_e.

(9.10a) Follow [9.9] to form $\mathcal{P}_\infty$, the analog of $\mathcal{P}_{z'}$. Analytically continue a branch of $z^{1/e}$ *counterclockwise* on a circle around ∞. Hint: Apply $z \mapsto 1/z$; it is the same as continuing $z^{-1/e}$ clockwise around the origin.

(9.10b) For $f(w) \in \mathbb{C}[w]$ of degree n with leading coefficient 1, write $f(w) = w^n + a_{n-1}w^{n-1} + \cdots + a_0$, let $m(z,w) = f(w) - z$. Show there is $g(z) \in \mathcal{P}_\infty$ of form $z^{\frac{1}{n}} + \sum_{j=0}^{\infty} b_j z^{-\frac{j}{n}}$ with $f(g(z)) \equiv z$.

(9.10c) Let L_f be $\mathbb{C}(z, g(z))$, g from b). Let $\hat{L}_f/\mathbb{C}(z)$ be the splitting field of $L_f/\mathbb{C}(z)$. Show there is $g \in G(\hat{L}_f/\mathbb{C}(z))$ acting as an n-cycle on conjugates of $g(z)$. Hint: Apply $1/z^{\frac{1}{n}} \mapsto \zeta_n 1/z^{\frac{1}{n}}$.

(9.10d) Consider $f, h \in \mathbb{C}[w]$ with $\deg(h) = m$. Apply [9.6c] to $\hat{L}_f$ and $\hat{L}_h$. Show the group of $\hat{L}_f \cdot \hat{L}_h/\mathbb{C}(z)$ contains σ of order $nm/\gcd(n,m)$ with restriction of σ to $\hat{L}_f$ an n-cycle and its restriction to $\hat{L}_h$ an m-cycle.

(9.10e) If $(\deg(f), \deg(h)) = 1$, show $f(w) - h(u)$ is irreducible. Hint: Irreducibility is equivalent to $[K(w) : K] = \deg(w)$ with $K = \mathbb{C}(u)$. Use that d) shows $[K'(w) : K'] = \deg(w)$ with $K' = \mathbb{C}((1/u))$.

(9.10f) Suppose in d) (with $(\deg(f), \deg(h)) = 1$), $L_f \cdot L_h$ is pure transcendental (equals $\mathbb{C}(t)$). Show for some choice of t there are polynomials $g(t), k(t)$ of respective degrees m and n with $f(g(t)) = h(k(t))$.

(9.10g) Apply f) to $f(w) = w^2$ and $h(u) = u^3 - au - b$ where h has distinct zeros. Show $L_f \cdot L_h$ is not pure transcendental. Hint: Zeros of $g(t)^2$ are multiple.

Critical points over $z \in \mathbb{C}$ appear in (6.8). Now consider $z = \infty$. With $m \in \mathbb{C}[z,w]$ of degree n and $m = h_0(z)w^n + h_1(z)w^{n-1} + \cdots + h_n(z)$, assume h_0 has z_0 as multiplicity t zero. When h_0 is constant call m *integral* (over z).

(9.11a) Write $t = kn + t_0$ with $0 \le t_0 < n$. Show there is an integral polynomial $m_1(z,w) \in \mathbb{C}[z,w]$ satisfying $m_1(z, (z-z_0)^{k+1}w) \equiv (z-z_0)^{n-t_0} m(z,w)$.

(9.11b) Suppose K is a field and $P_1, P_2 \in K[w]$. The *Euclidean algorithm* gives the greatest common divisor of P_1 and P_2. Write $P_1 = R_0, P_2 = R_1$. Form the remainder R_2 of the division $R_1 \overline{)R_0}$. Inductively form successive remainders, $R_3, \ldots, R_u$, until the next stage remainder is 0. Do an induction to produce $A(w), B(w) \in K[w]$ with $A(w)P_1(w) + B(w)P_2(w) = R_u(w)$.

(9.11c) Continue b): Use that $\mathbb{C}[z]$ has unique factorization to clear denominators on $A(w)P_1(w) + B(w)P_2(w) = R_u(w)$. Suppose $P_i = P_i(z,w) \in \mathbb{C}[z,w]$, $i = 1,2$, have no common factor in w. Find $A(z,w), B(z,w) \in \mathbb{C}[z,w]$ and $M(z) \in \mathbb{C}[z] \setminus \{0\}$ with $A(z,w)P_1(z,w) + B(z,w)P_2(z,w) = M(z)$.

(9.11d) Result c) applies with any unique factorization domain replacing $\mathbb{C}[z]$. Comment on how it applies to $K = \mathcal{L}_{z'}$.

(9.11e) We outline examples where critical points of (m,w) $(m(z,f(z))\equiv 0)$ properly contain critical points of $\mathbb{C}(z,f)/\mathbb{C}(z)$. Let g_{z_0} be the conjugacy class of the branch cycle for m at z_0. Suppose $e=e_{z_0}$ is the order of g_{z_0}. Show, if $m(u^e+z_0,w)=m_1(u,w)$ is irreducible, then $u=0$ is a branch point of $m_1(u,w)$ but not a branch point of $\mathbb{C}(u,f(u^e+z_0))$.

(9.11f) Apply [9.10e] to give examples of e) by taking $h\in\mathbb{C}[w]$ of degree prime to e, so $h(w)-u^e$ is irreducible.

9.5. Elementary permutations from $\Pi_1(D,z_0)$. Let $\Delta_{z'}$ be a disk about z' and $\Delta^0_{z'}=\Delta_{z'}\setminus\{z'\}$. Choose $z_0\in\Delta^0_{z'}$.

(9.12a) Suppose $h(t)$ is a branch of $\log(z-z')$ along $\lambda:[a,b]\to\mathbb{C}-\setminus\{z'\}$. Then, what path is $h(t)$ a branch of log along?

(9.12b) Suppose $f(z)=(z-z')h(z)$ is analytic in $\Delta_{z'}$ with $h(z)\neq 0$ for any point in $\Delta_{z'}$. Show a branch $F(z)$ of $f(z)^{\frac{1}{e}}$ exists at any point in $\Delta^*_{z'}$. Further, show there is an embedding of the field $\mathbb{C}(z,F(z))$ into $\mathcal{P}_{z',e}$.

(9.12c) Let $g_j(z)$ be a branch of $(z-z_j)^{1/e_j}$, $j=1,\dots,r$ analytic in a neighborhood of z_0. With $f(z)=\prod_{j=1}^r g_j$ and $\lambda:[a,b]\to\mathbb{C}$ a path with winding number m_j around z_j, explicitly relate $f(z)$ and $f_\lambda(z)$.

Consider how analytic continuation easily forces us into groups that are not abelian. Follow Thm. 5.6 notation.

(9.13a) Show the conclusion of the case $\infty\in D$ as in §5.4.4 follows.

(9.13b) Recall the semi-direct product $M\times^s H$ of groups of H and M with $\psi: H\to\operatorname{Aut}(M)$ a homomorphism into the automorphisms of M. Then, $(m,h)\cdot(m',h')\stackrel{\text{def}}{=}(m\cdot\psi(h)(m),h\cdot h')$ defines multiplication on $M\times H$. Consider $M_0=\mathbb{Z}^3$, and $H_0=\mathbb{Z}/3$ where $1\in H_0$ maps $(m_1,m_2,m_3)\in M_0$ to (m_2,m_3,m_1). Show $M_0\times^s H_0$ is not abelian.

(9.13c) Let $f(z)$ be a branch of $z^{1/3}$ around $z_0\neq 0$. For $a\notin\{0,\infty,z_0\}$, consider $h=\frac{1}{f(z)^2(f(z)-a)}\in\mathbb{C}(z,f(z))$. Find $\boldsymbol{z}\subset\mathbb{P}^1_z$ so $(h,U_{\boldsymbol{z}})$ is extensible. Find the image of the permutation representation of $\Pi_1(U_{\boldsymbol{z}},z_0)$ on $\mathcal{A}_h(U_{\boldsymbol{z}})$.

(9.13d) Let $H(z)$ be a primitive for h (in d)) around z_0. Show the image of the permutation representation of $\Pi_1(U_{\boldsymbol{z}},z_0)$ on $\mathcal{A}_H(U_{\boldsymbol{z}})$ is $M_0\times^s H_0$ from b). Hint: Substitute w with $w^3=z$.

9.6. Fractional transformations and the elementary divisor theorem. Recall: For any ring R and integer $n\geq 1$, $\mathrm{PGL}_n(R)$ is $\mathrm{GL}_n(R)/\langle R^*I_n\rangle$ and $\mathrm{PSL}_n(R)=\mathrm{SL}_n(R)/\mathrm{SL}_n(R)\cap\langle R^*I_n\rangle$. Several nonabelian subgroups of $\mathrm{PGL}_2(\mathbb{C})$, like $\mathrm{PGL}(\mathbb{R})$ and $\mathrm{PSL}_2(\mathbb{Z})$ appear often in complex variables. We contrast their different appearances. Let $\mathcal{T}$ be the translations $\{\alpha\in\mathrm{PGL}_2(\mathbb{C})\mid\alpha(z)=z+a,\ a\in\mathbb{C}\}$. Let $\mathcal{M}$ be the multiplications $\{\alpha\in\mathrm{PGL}_2(\mathbb{C})\mid\alpha(z)=bz,\ a\in\mathbb{C}^*\}$. Finally, consider $\tau: z\mapsto 1/z$.

(9.14a) Show each $\alpha\in\mathrm{PGL}_2(\mathbb{C})$ has is one of $a'(z-z_1)$, $a'(z-z_1)/(z-z_2)=a'(1+(z_2-z_1)/(z-z_2))$, or $a'/(z-z_2)$. Why is $\alpha\in\mathrm{PGL}_2(\mathbb{C})$ a composition of elements from $\mathcal{M}$, $\mathcal{T}$ and τ: $\mathcal{M}$, $\mathcal{T}$ and γ generate $\mathrm{PGL}_2(\mathbb{C})$.

(9.14b) Give an $\alpha\in\mathrm{PGL}_2(\mathbb{C})$ mapping $\mathbb{R}$ to the boundary of the unit circle.

(9.14c) Elements of $\mathrm{PGL}_2(\mathbb{C})$ mapping $\mathbb{R}\cup\{\infty\}$ to itself are in $\mathrm{PGL}_2(\mathbb{R})$. What is the subgroup of these mapping the upper half plane $\mathbb{H}$ (Chap. 3 §3.2.2) into itself? Hint: $z\mapsto 1/z$ does not.

(9.14d) Combine with b) to describe elements of $\mathrm{PGL}_2(\mathbb{C})$ mapping $\mathbb{R}\cup\{\infty\}$ to the unit circle. Which map $\mathbb{H}$ to the inside of the circle?

(9.14e) Which $f\in\mathbb{C}(z)$ map the unit circle into the unit circle. Hint: $f\in\mathbb{C}(z)$ mapping $\mathbb{R}\to\mathbb{R}$ has zero and pole set closed under complex conjugation.

Let R be a principal ideal domain, M a finitely generated free R module, and N an R submodule of M. The Elementary Divisor Theorem (EDT [**Jac85**, p. 192]): There is a basis $\boldsymbol{v}_1,\dots,\boldsymbol{v}_m$ of M and elements $a_1,\dots,a_m\in R$ with nonzero elements of $a_1\boldsymbol{v}_1,\dots,a_m\boldsymbol{v}_m$ a basis of N. If $a_1,\dots,a_t$ are the nonzero a_i s, then we may choose $a_1,\dots,a_t$ so $a_i|a_{i+1}$, $i=1,\dots,t$.

(9.15a) Consider an abelian group quotient A of $\mathbb{Z}^n$. Apply EDT to show A is isomorphic to $\oplus_{i=1}^n\mathbb{Z}/(a_i)$ for some integers $a_1,\dots,a_n\in\mathbb{Z}$.

(9.15b) Show in a), if A is a finite group and $a_1|a_2|\cdots|a_m$ are positive integers, then the $a_1,\dots,a_n$ are unique.

(9.15c) $\mathrm{SL}_2(\mathbb{Z})$ (2×2 matrices over $\mathbb{Z}$ of determinant 1) acts on $M_2=\mathbb{Z}^2$ taking one basis to another. If N is a subgroup of M_2 of index n, then $\mathrm{SL}_2(\mathbb{Z})$ maps it in an orbit of index n subgroups. Apply EDT to count $N\le M_2$ of index $n=p^k$ (p a prime). Hint: Start with N for which M/N is cyclic.

(9.15d) Each N from c) defines a subgroup Γ_N of $\mathrm{PSL}_2(\mathbb{Z})$: the image of the stabilizer in $\mathrm{SL}_2(\mathbb{Z})$ of N. If $n=p$ is a prime, and U is the biggest normal subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ in Γ_N, show $\mathrm{PSL}_2(\mathbb{Z})/U=\mathrm{PSL}_2(\mathbb{Z}/p)$.

Let Δ be the open unit circle. Denote the linear fractional transformations that map $\Delta\to\Delta$ by $\mathrm{PGL}_2(\Delta)$. Form

$$(w_3-w_1)(w-w_2)/(w_2-w_1)(w-w_3)=L(w)=L(w_1,w_2,w_3,w)$$

for $w_1,w_2,w_3\in\mathbb{C}$. This problem follows a treatment from [**Spr57**, §9.2]

(9.16a) Use [9.14]. Show $\mathrm{PGL}_2(\mathbb{C})$ fixes $L(w)$:

$$L(w_1,w_2,w_3,w)=L(\alpha(w_1),\alpha(w_2),\alpha(w_3),\alpha(w)),\ \text{for } \alpha\in\mathrm{PGL}_2(\mathbb{C}).$$

(9.16b) Suppose $w_1,\dots,w_4\in\mathbb{C}$ are on a circle in that order. Show: $L(w_4)>1$. Conclude: With w_1,w_2,w_3 fixed, $w\mapsto L(w)$ maps the interior of the disk bounded counterclockwise by w_1,w_2,w_3 to the upper half plane $\mathbb{H}$.

(9.16c) Suppose $w_2,w_3\in\Delta$. Let C_{w_2,w_3} be the unique circle containing w_2 and w_3 meeting the unit circle at right angles (at two points). Why is C_{w_2,w_3} unique? Hint: Use $\alpha\in\mathrm{PGL}_2(\mathbb{C})$ taking the unit circle to the real line.

(9.16d) Let w_1 be the point on $C_{w_2,w_3}\cap\partial\Delta$ closest to w_2. Similarly, w_4 is the other point of intersection closest to w_3. Define the distance $d(w_2,w_3)$ to be $\frac{1}{2}\log(L(w_1,w_2,w_3,w_4))$. When $w_2=0$ and $w_3=re^{i\theta}$ express this as a function of r.

(9.16e) Notice $\beta_{w_2}(w)=\frac{w-w_2}{1-\bar{w}_2w}$ is in $\mathrm{PGL}_2(\Delta)$ and it maps $w_2\mapsto 0$. Use this to express $d(w_2,w_3)$ as $\frac{1}{2}\log\big(\frac{1+|\beta_{w_2}(w_3)|}{1-|\beta_{w_2}(w_3)|}\big)$.

9.7. Metrics on $\mathbb{P}^1_z$, Δ and more generally. The metric topology on $\mathbb{P}^1_z$ identifies it with the sphere around the origin in $\mathbb{R}^3$. Use coordinates (r,u,v): $z_0\in\mathbb{P}^1_z\mapsto(r_0,u_0,v_0)\in\mathbb{R}^3$. The unit sphere has this analytical description: $\{(r,u,v)\mid r^2+u^2+v^2=1\}=S$.

(9.17a) From vector calculus, this *implicit* description of S gives a unit normal direction to S at (r_0,u_0,v_0). It is a unit vector $\mathbb{N}_{(r_0,u_0,v_0)}$, (from the origin) in the direction of the gradient of $f(r,u,v) = r^2 + u^2 + v^2$. Compute two such vectors. Which suits the definition of *outward* normal vector?

(9.17b) Let $\mathbb{T}_{(r_0,u_0,v_0)}$ be points on the plane through (r_0,u_0,v_0) tangent to the sphere. There are two possible definitions of $\mathbb{T}_{(r_0,u_0,v_0)}$. Suppose the range of $(x,y) \mapsto (r(x,y),u(x,y),v(x,y)) = H(x,y)$ is a neighborhood of (r_0,u_0,v_0); H is differentiable in a neighborhood of the origin and $H(0,0) = (r_0,u_0,v_0)$, and $\frac{\partial H}{\partial x}(0,0)$ and $\frac{\partial H}{\partial y}(0,0)$ are linearly independent vectors in $\mathbb{R}^3$. Apply the chain rule to show

$$\mathbb{T}^{\dagger}_{(r_0,u_0,v_0)} \stackrel{\text{def}}{=} \{(r_0,u_0,v_0) + x\frac{\partial H}{\partial x}(0,0) + y\frac{\partial H}{\partial y}(0,0) \mid (x,y) \in \mathbb{R}^2\}$$

is independent of the choice of H.

(9.17c) The second definition of $\mathbb{T}_{(r_0,u_0,v_0)}$ is

$$\mathbb{T}^{\dagger\dagger}_{(r_0,u_0,v_0)} \stackrel{\text{def}}{=} \{(r,u,v) \mid ((r,u,v) - (r_0,u_0,v_0)) \cdot \mathbb{N}_{(r_0,u_0,v_0)} = 0\}.$$

Use the expression $f(H(x,y)) \equiv 0$ to show $\mathbb{T}^{\dagger\dagger}_{(r_0,u_0,v_0)} = \mathbb{T}^{\dagger}_{(r_0,u_0,v_0)}$.

(9.17d) Let $\gamma : [a,b] \to S$ be a simple closed path. Suppose $\frac{d\gamma}{dt}$ exists and is nonzero at $t_0 \in [a,b]$. Define the direction to the *left* of γ at t_0 to be the unit vector $\boldsymbol{u}_1$ for which $\det(\boldsymbol{u}_1 \,|\, \mathbb{N}_{\gamma(t_0)} \,|\, \frac{d\gamma}{dt}(t_0))$ is positive.

(9.17e) The complement $S \setminus \gamma$ of a simple closed path has two components U_1 and U_2: The *Jordan curve Theorem*. For simplicial γ this is easy (Chap. 4 [11.3]). Assume t_0 as in d). Give meaning to this: γ has positive orientation relative to U_1. Hint: Interpret $\boldsymbol{u}_1$ being parallel to U_1.

We explore $d(w_2,w_3)$ from [9.16], to prove the triangle inequality and to find its differential distance tensor. Use $U(z) = \frac{1+|z|}{1-|z|}$.

(9.18a) Use [9.16e] and find $\beta(w) \in \mathrm{PGL}_2(\Delta)$ with $\beta(w_2) = 0$, $\beta(w_1) = a > 0$ to reduce $d(w_1,w_3) \le d(w_1,w_2) + d(w_2,w_3)$, $w_1,w_2,w_3 \in \Delta$ to showing $U(\frac{z-a}{1-az}) \le U(a) \cdot U(z)$ with $a \in [0,1)$ and $z \in \Delta$.

(9.18b) Write $z = be^{i\theta}$. Show $U(\frac{z-a}{1-az})$ is maximum in θ when z is real. Conclude the inequality of a). Hint: $U(w)$ is increasing in $|w|$ and $\frac{z-a}{1-az}$ maps the circle of radius b on a circle with real center.

(9.18c) Use [9.16e] to compute the differential distance $S(x,y,dx,dy)$ by considering $w_1 = x + i\,y$ close to w_2. Show $S(x,y,dx,dy)$ to be $|\frac{dx+idy}{1-(x^2+y^2)}|$.

(9.18d) Apply $\alpha \in \mathrm{PGL}_2(\mathbb{C})$ mapping the upper half plane $\mathbb{H}$ to Δ. Define a distance on $\mathbb{H}$ by pulling back two points and using the value of the distance on Δ. Show this depend on the particular choice of α. Show geodesics on $\mathbb{H}$ are half-circles perpendicular to the real axis.

(9.18e) Use d) to show the metric on $\mathbb{H}$ has differential distance element $\frac{|dx+idy|}{y}$.

Consider [9.18] from the differential distance tensor view:

$$F_\Delta = |\frac{dx+idy}{1-(x^2+y^2)}| = h(x,y)\sqrt{dx^2+dy^2}$$

with $h(x,y) = |1-(x^2+y^2)|^{-1/2}$. Recover this metric's geodesics, circles perpendicular to the boundary of Δ, by applying the *Euler-Lagrange variational principle* from f). Consider $F^2 = \boldsymbol{y} \cdot Q(\boldsymbol{x})(\boldsymbol{y})$ in (2.3a): $Q(\boldsymbol{x})$ is an $n \times n$ positive definite symmetric matrix. Tensor notation replaces $\boldsymbol{y}$ by $dx_1,\dots,dx_n$. Classically, $F^2 = \sum_{1 \le i,j \le n} q_{i,j}(\boldsymbol{x})\, dx_i \otimes dx_j$ (with $q_{i,j} = q_{j,i}$) for a 2-tensor.

(9.19a) Suppose γ and λ are a pair of paths with $\gamma(t_0) = \lambda_2(t_0) = \boldsymbol{x}_0$. Define:

$$F^2(\frac{d\gamma}{dt}(t_0), \frac{d\lambda}{dt}(t_0)) = \sum_{i,j} q_{i,j}(\boldsymbol{x}_0)\, \frac{d\gamma_i}{dt}\frac{d\lambda_j}{dt}.$$

Show $\frac{F^2(\frac{d\gamma}{dt}(t_0), \frac{d\lambda}{dt}(t_0))}{F(\frac{d\gamma}{dt}(t_0), \frac{d\gamma}{dt}(t_0))F(\frac{d\lambda}{dt}(t_0), \frac{d\lambda}{dt}(t_0))}$ has absolute value at most 1. So, it has the form $\cos(\theta(\gamma, \lambda))$. Show $\theta(\gamma, \lambda)$, the *angle* between γ and λ at $\boldsymbol{x}_0$, is independent of their parametrizations.

(9.19b) Apply Ex. 9.1. Show $\sum_{i,j} \int_a^b \sqrt{q_{i,j}(\gamma(t))\, \frac{d\gamma_i}{dt}\frac{d\gamma_j}{dt}}\, dt$ is independent of how we parametrize the range of γ assuming $\gamma : [a, b] \to \mathbb{R}^n$ is one-one.

(9.19c) Let $H : \mathbb{R}^2 \to \mathbb{R}^n$ by $(u_1, u_2) \mapsto (h_1(u_1, u_2), \dots, h_n(u_1, u_2)) = \boldsymbol{h}(\boldsymbol{u})$ be a one-one (differentiable) map. Define $H^*(F^2)$, pullback of F^2 on the range of H, as $\sum_{1 \le i,j \le n} q_{i,j}(\boldsymbol{h}(\boldsymbol{u}))\, dh_i \otimes dh_j$: $dh_i(\boldsymbol{u}) = \frac{\partial h_i}{\partial u_1}\, du_1 + \frac{\partial h_i}{\partial u_2}\, du_2$. Suppose $\gamma : [a, b] \to H(\mathbb{R}^2)$. Show $\int_\gamma F = \int_{H^{-1} \circ \gamma} \sqrt{H^*(F^2)}$ from b).

(9.19d) Consider $H^*(F^2)$ in c) when $n = 2$. Call H *isothermal coordinates* if $H^*(F^2)$ is $h(u_1, u_2)(du_1 \otimes du_1 + du_2 \otimes du_2)$. Use $n = 2$ to factor F^2 to

$$(A(\boldsymbol{x})\, dx_1 + B(\boldsymbol{x})\, dx_2) \otimes (A(\boldsymbol{x})\, dx_1 + \bar{B}(\boldsymbol{x})\, dx_2)$$

($\bar{B}(\boldsymbol{x})$ is the complex conjugation of $B(\boldsymbol{x})$). Suppose $k(\boldsymbol{x})$ (complex valued) gives $k(\boldsymbol{x})(A(\boldsymbol{x})\, dx_1 + B(\boldsymbol{x})\, dx_2)$ with the form $du_1 + i du_2$. Show $(u_1(\boldsymbol{x}), u_2(\boldsymbol{x}))$ gives isothermal coordinates.

(9.19e) Produce $k(\boldsymbol{x})$ near any (x_1^0, x_2^0), as in c). Outline: Take real and imaginary parts. Rewrite: $du_i = \frac{\partial u_i}{\partial x_1}\, dx_1 + \frac{\partial u_i}{\partial x_2}\, dx_2$. Finding k comes to this. Suppose $M_1(\boldsymbol{x}), M_2(\boldsymbol{x})$ are real valued and differentiable. Then, there is $k_1(\boldsymbol{x})$ and $M^*(\boldsymbol{x})$ with $k_1(M_1(\boldsymbol{x})\, dx_1 + M_2(\boldsymbol{x})\, dx_2)$ of form $dM^*(\boldsymbol{x})$. Then, $M_1(\boldsymbol{x})\, dx_1 + M_2(\boldsymbol{x})\, dx_2 = 0$ defines $\{(x_1, x_2 \mid M^*(x_1, x_2) = 0\}$, an implicit surface, near (x_1^0, x_2^0). Find k_1.

(9.19f) We assume the situation of [9.18]. Let $\gamma = \gamma_1 + i\,\gamma_2 : [0, 1] \to \Delta$ be a path from z_0 to z_0'. Minimize $\int_\gamma F_\Delta = \int_0^1 S(\gamma_1(t), \gamma_2(t), \frac{d\gamma_1}{dt}, \frac{d\gamma_2}{dt})\, dt$ over all such γ. The Euler-Lagrange variation produces two partial differential equations, one for x, $\frac{d}{dt}\frac{\partial S}{\partial \dot{x}} = \frac{\partial S}{\partial x}$, and a similar one for y. Solve to show F_Δ geodesics are circles perpendicular to the boundary of Δ.

CHAPTER 3

COMPLEX MANIFOLDS AND COVERS

Chap. 4 replaces the field $\mathbb{C}(z, f(z))$ generated by an algebraic function $f(z)$ over $\mathbb{C}(z)$ by a geometric object, a 1-dimensional complex manifold (*Riemann surface*) that maps to the Riemann sphere $\mathbb{P}^1_z$. To prepare for this idea requires building some manifolds, and developing intuition for basic examples. We use fundamental groups to create new 1-dimensional complex manifolds from the space $U_{\boldsymbol{z}}$ with $\boldsymbol{z}$ a finite subset of $\mathbb{P}^1_z$.

Chap. 5 collects various Riemann surfaces into families. The parameter spaces for these families — one point in the space for each member of the family — are manifolds called *moduli spaces*. Chap. 4 has a prelude, the moduli space classically called the *j-line*: $\mathbb{P}^1_j \setminus \{\infty\}$. We use it for more general families than do classical texts on Riemann surfaces. Our moduli spaces may have arbitrarily high complex dimension. Still, their construction uses covering spaces (coming from fundamental groups) of open subsets of projective spaces. This chapter builds an intuition for using group theory to construct these spaces.

1. Fiber products and relative topologies

There is so much topology and we have so little space for it despite the need for some special constructions. The treatment is expedient and not completely classical to emphasize some subtle properties of manifolds.

1.1. Set theory constructions. For X and Y sets, the *cartesian product* of X and Y is the set

$$X \times Y = \{(x, y) \mid x \in X,\ y \in Y\}.$$

Let $\{X_\alpha\}_{\alpha\in I}$ be a collection of subsets of the set X *indexed* by the set I. The *union* of $\{X_\alpha\}_{\alpha\in I}$ is the set of $x \in X$ for which $x \in X_\alpha$ for some $\alpha \in I$. Denote this $\bigcup_{\alpha\in I} X_\alpha$. The *complement* of X_α in X, $X \setminus X_\alpha$, is $\{x \in X \mid x \notin \bigcup_{\alpha\in I} X_\alpha\}$. The *intersection* of $\{X_\alpha\}_{\alpha\in I}$ is the set of $x \in X$ with $x \in X_\alpha$ for each $\alpha \in I$. Denote this $\bigcap_{\alpha\in I} X_\alpha$.

Definition 1.1. For X_1 and X_2 sets, $Y_i \subset X_i$, $i = 1, 2$, let $f : Y_1 \to Y_2$ be a one-one onto function. The *sum* of X_1 *and* X_2 *along* f is the disjoint union of $X_1\setminus Y_1$, Y_2, and $X_2\setminus Y_2$. Denote this $X_1 \bigcup_f X_2$. Along with this, we have maps $f_i : X_i \to X_1 \bigcup_f X_2$, $i = 1, 2$: with $f_2(x_2) = x_2$ for $x_2 \in X_2$, $f_1(x_1) = x_1$ if $x_1 \in X_1\setminus Y_1$, and $f_1(x_1) = f(x_1)$ for $x_1 \in Y_1$. Call f_1 and f_2 the *canonical* maps.

Example 1.2 (The set behind a non-Hausdorff space). Consider

$$X_i = \{(t, i) \in \mathbb{R}^2 \mid -1 < t < 1\},\ i = 1, 2,\ \text{with}$$

$Y_i = X_i\setminus\{(0, i)\}$, $i = 1, 2$, and $f : Y_1 \to Y_2$ by $f(t, 1) = (t, 2)$ for $(t, 1) \in Y_1$. Then, $X_1 \bigcup_f X_2$ is the disjoint union of X_2 and the point $(0, 1)$ (see Def. 1.4 and Ex. 2.4).

Definition 1.3 (Set theoretic fiber products). Let $f_i : X_i \to Z$ be two functions with range Z, $i = 1, 2$. The *fiber product* $X_1 \times_Z X_2$ consists of

$$\{(x_1, x_2) \in X_1 \times X_2 \mid f_1(x_1) = f_2(x_2)\}.$$

Denote the natural map back to Z by $f_1 \times_Z f_2$. Suppose $X_i \subset Z$ and $f_i : X_i \to Z$ is inclusion, $i = 1, 2$. Then, identify $X_1 \times_Z X_2$ with $X_1 \cap X_2$.

Suppose $X_1 = X_2 = Z = \mathbb{C}$, and f_1 and f_2 are polynomials. Then, $X_1 \times_Z X_2$ is the subset of $(x_1, x_2) \in \mathbb{C}^2$ defined by $f_1(x_1) = f_2(x_2)$. Define the ith *projection map*, $\mathrm{pr}_i : X_1 \times_Z X_2 \to X_i$ by $\mathrm{pr}_i(x_1, x_2) \mapsto x_i$, $i = 1, 2$.

The fiber product is an *implicit* set: an equation describes it.

The ball of radius r about $\boldsymbol{x}_0 \in \mathbb{R}^n$ is the *basic open set* $\{\boldsymbol{x} \in \mathbb{R}^n \mid |\boldsymbol{x} - \boldsymbol{x}_0| < r\}$. When necessary denote this $B(\boldsymbol{x}_0, r)$. *Open sets* of $\mathbb{R}^n$ are either empty or are (arbitrary) unions of basic open sets. *Closed sets* are complements (in $\mathbb{R}^n$) of open sets. *Bounded sets* are those contained in some basic open set. The collection of open sets, $\mathcal{U}$, in $\mathbb{R}^n$ therefore satisfies the axioms for a *topology*: $\mathcal{U}$ contains the empty set and the whole space, and it is closed under taking arbitrary unions and finite intersections.

Definition 1.4 (Relative topology I). Let X be a subset of $\mathbb{R}^n$. Denote the collection of sets $X \cap U$ for U open subset in $\mathbb{R}^n$ by $\mathcal{U}_X$. Then $\mathcal{U}_X$ gives the *relative topology* on X. For $x_1, x_2 \in X$, two distinct points, $B(x_1, r/3) \cap X$ and $B(x_2, r/3) \cap X$ are disjoint open *neighborhoods* of the respective points x_1 and x_2 if $r = |x_1 - x_2|$. Thus, in this relative topology, X is a *Hausdorff space*.

Suppose X (resp. Y) is a topological space with open sets $\mathcal{U}_X$ (resp., $\mathcal{U}_Y$). Let $f : X \to Y$ be a function with *domain* a subset of X. Then f is *continuous* (for the *relative topology*) if for each $U \in \mathcal{U}_Y$,

$$f^{-1}(U) = \{x \text{ in the domain of } f \mid f(x) \in U\} \text{ is in } \mathcal{U}_X.$$

For U open in Y, denote *restriction* of f to $f^{-1}(U)$ by $f_U : f^{-1}(U) \to U$. If f is continuous, so is f_U.

The concept of relative topology generalizes to data $\{(X_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ on a set X with the following properties: $\bigcup_{\alpha \in I} X_\alpha = X$; $\varphi_\alpha : X_\alpha \to \mathbb{R}^n$ is a one-one map into $\mathbb{R}^n$; and $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(X_\alpha \cap X_\beta) \to \varphi_\beta(X_\alpha \cap X_\beta)$ is a continuous function for each $\alpha, \beta \in I$. We call the functions $\{\varphi_\beta \circ \varphi_\alpha^{-1}\}_{\alpha, \beta \in I}$ *transition functions*.

Definition 1.5 (Relative topology II). Let X and $\{(X_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ be as above. Consider subsets of X that are unions of $\varphi_\alpha^{-1}(U)$ with U running over open sets of $\varphi_\alpha(X_\alpha)$, $\alpha \in I$. Denote this collection of sets by $\mathcal{U}_X$. The topology on X from $\mathcal{U}_X$ is the *relative topology on* X induced from the topologizing data $\{(X_\alpha, \varphi_\alpha)\}_{\alpha \in I}$. For $x \in X$ and U an open set containing x, U is a *neighborhood* of x.

1.2. Extending topologies from $\mathbb{R}^n$. Two sets of topologizing data on X, $\{(X'_{\alpha'}, \varphi'_{\alpha'})\}_{\alpha' \in I'}$ and $\{(X_\alpha, \varphi_\alpha)\}_{\alpha \in I}$, are *equivalent* (the *same*, or *give the same topology*) if each defines the same open sets on X.

Consider X and Y, topological spaces with respective data $\{(X_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ and $\{(Y_\beta, \psi_\beta)\}_{\beta \in J}$. A one-one map $f : X \to Y$ is a (*topological*) *embedding* if the topologizing data from $\{(f^{-1}(Y_\beta), \psi_\beta \circ f)\}_{\beta \in J}$ is equivalent to $\{(X_\alpha, \varphi_\alpha)\}_{\alpha \in I}$. Note: Ex. 2.4 has a space with no embedding in $\mathbb{R}^n$ (for any n). It isn't Hausdorff. Yet, each point has a neighborhood embeddable as an open interval in $\mathbb{R}^1$.

Associate to each subset Y of a topological space X the *closure* $\bar{Y}$ of Y in X: $\bar{Y}$ (a closed set) is the points $x \in X$ with each neighborhood of x containing at least one point of Y. If each neighborhood of x contains a point of Y distinct from x, then x is a *limit point of* Y.

Compact subsets of $\mathbb{R}^n$ are those both closed and bounded. The Heine-Borel covering theorem [**Rud76**, p. 40] characterizes these sets through the concept of an *open covering*. A collection $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of open subsets of $\mathbb{R}^n$ is an open cover of Y if $Y \subseteq \bigcup_{\alpha \in I} U_\alpha$. Then Y has the *finite covering property* if for *each* open cover $\mathcal{U}$ there is a *finite* collection $\{U_{\alpha_i}\}_{i=1}^t$, $\alpha_1, \ldots, \alpha_t \in I$, covering Y.

THEOREM 1.6 (Heine-Borel). *The finite covering property is equivalent to compactness for subsets of* $\mathbb{R}^n$.

Thus, for any topological space X, without reference to the concept of bounded set, one says a subset Y is *compact* if it has the finite covering property.

A subset Y of a topological space X is *disconnected* if there are two nonempty open sets U_1 and U_2 of Y (in the relative topology) with $U_1 \cap U_2$ empty and $U_1 \bigcup U_2 = Y$. If Y is not disconnected call it *connected* (in X). For any $x \in X$, there is a maximal connected set U_x containing x. So, each topological space decomposes into a union of disjoint *connected components*. If $f : Y \to X$ is continuous, the image of any connected subset of Y is a connected subset of X.

2. Functions on X from functions on $\mathbb{R}^n$

There are several points to make about Def. 1.5. First it includes many topologies as our next example illustrates.

EXAMPLE 2.1. Let X be any set whose points, x_α, are indexed by $\alpha \in I$. Let $X_\alpha = \{x_\alpha\}$ and $\varphi_\alpha : \{x_\alpha\} \to \{\mathbf{0}\}$, $\alpha \in I$, where $\mathbf{0}$ is the origin of $\mathbb{R}^n$. The relative topology on X is the *discrete topology*.

By using another target space Y with a well-known topology on it (like the p-adic numbers $\mathbb{Z}_p$, replacing $\mathbb{R}^n$), we could include p-adic topologies, too. Still, it does not include all the topologies significant to modern mathematics even for spaces we consider as manifolds. Later we will extend it to *Grothendieck topologies*. It is appropriate for that example to notice we don't need a topology on X to start the process (§2.1).

Further, the point of topologizing data is to pull back functions (differentials, and other objects) from $\mathbb{R}^n$ so X has local functions (differentials, etc.) just like those of $\mathbb{R}^n$. Since $\mathbb{R}^n$ also has the notion of *real analytic*, *differentiable* and *harmonic* functions, transition functions also allow us to pull those back, to identify such functions on X. For these definitions, however, to be meaningful, they must be locally independent of which function we use for pullback. This requires the transition functions also have these respective properties (§3).

When $n = 2m$ is even, suppose the following two conditions hold.

(2.1a) We have chosen a fixed $\mathbb{R}$ linear map $L = L_n : \mathbb{R}^n \to \mathbb{C}^m$.
(2.1b) Using L, the transition functions are analytic from $\mathbb{C}^m \to \mathbb{C}^m$.

These conditions allow identifying a set of functions in a neighborhood of any point on X as analytic (§3.1.2).

Finally, there is a warning. Local function theory immediately challenges us to identify global functions and differentials on X through their local definitions. There is an immediate first problem to assure a simple property we expect from functions in $\mathbb{R}^n$. If a function f in a neighborhood of $x \in X$ has good behaviour as $x' \in X$ approaches x, then it should have a unique limit value (see §2.2 on the Hausdorff property).

2.1. Defining a topological space from its atlas. Def. 1.5 shows we don't need X to start with a topology. It inherits one from its topologizing data. So, it is reasonable to ask if we need an a priori space X at all.

2.1.1. *Equivalence relations define topological spaces.* For example, suppose $\{U_\alpha\}_{\alpha \in I}$ is a collection of open sets in $\mathbb{R}^n$, and for some subset $(\beta, \alpha) \in I \times I$, there are invertible continuous maps $\psi_{\beta,\alpha} : V^\alpha_\beta \to V^\beta_\alpha$, with V^β_α open in U_α (resp. V^α_β open in U_α). Can we form an X so that $\{\psi_{\beta,\alpha}\}_{\alpha,\beta \in I}$ are the transition functions for its topological structure? Almost!

Let X be the disjoint union $\dot\cup_{\alpha \in I} U_\alpha$ modulo the relation R_I on this union defined by $x \in U_\alpha \sim x' \in U_\beta$ if $\psi_{\beta,\alpha}(x) = x'$. If R_I is an equivalence relation, then the equivalence classes form a set X and on it a topological structure. On this space, of course, the open sets do look like those of $\mathbb{R}^n$ (in contrast to Ex. 2.1). The following lemma keeps track of the definitions.

LEMMA 2.2. *The relation R_I is an equivalence relation if and only if the following properties hold:*

(2.2a) $\psi_{\alpha,\alpha}$ *is the identity map;* $\psi_{\alpha,\beta} = \psi^{-1}_{\beta,\alpha}$*; and*
(2.2b) $\psi_{\gamma,\beta} \circ \psi_{\beta,\alpha} = \psi_{\gamma,\alpha}$ *wherever any two of the maps are defined.*

Suppose R_I is an equivalence relation. Then the inverse of the natural inclusion maps $U_\alpha \to X$ are functions φ_α giving transition functions $\varphi_\beta \circ \varphi_\alpha^{-1} = \psi_{\beta,\alpha}$.

2.1.2. *Quotient topologies.* Suppose X is a topological space with topologizing data $\{(X_\alpha, \varphi_\alpha)\}_{\alpha \in I}$. Let $f : X \to Y$ be any surjective map. Then, there is a topology on Y with open sets $\mathcal{U}_Y$ the images by f of all sets in $\mathcal{U}_X$. We can't, however, expect topologizing data on Y by pushing down the functions φ_α without extra conditions. It usually makes sense to write f for restriction of f to any subset $V \subset X$. The argument here, however, requires tracking the domain, and so we write f_V.

Let J be the subset of I for which $f_{X_\beta} : X_\beta \to Y$ is one-one for $\beta \in J$. Let $\mathcal{U}_{X,Y}$ be $\{X_\beta\}_{\beta \in J}$ and assume $\mathcal{U}_{X,Y}$ is a cover of X. With no loss assume the coordinate chart for X contains only sets from $\mathcal{U}_{X,Y}$. The hypothesis provides coordinate functions $\psi_\alpha : f(X_\alpha) \to \mathbb{R}^n$ by setting $\psi_\alpha = \varphi_\alpha \circ f^{-1}_{X_\alpha}$ on $f(X_\alpha)$.

From Lem. 2.2 we want an equivalence relation on $\dot\cup_{\alpha \in J} \psi_\alpha(f(X_\alpha))$ that reproduces the set Y as equivalence classes: $y \in \psi_\alpha(f(X_\alpha)) \sim y' \in \psi_\beta(f(X_\beta))$ if $\psi_{\beta,\alpha}(y) = y'$. So, the problem is to define $\psi_{\beta,\alpha}$, using that $f^{-1}_{X_\alpha}$ is different from $f^{-1}_{X_\beta}$ on $f(X_\alpha) \cap f(X_\beta)$. If $f(X_\alpha \cap X_\beta) = f(X_\alpha) \cap f(X_\beta)$, then it is consistent to define $\psi_{\beta,\alpha}$ as $\psi_\beta \circ \psi^{-1}_\alpha = \varphi_\beta \circ \varphi^{-1}_\alpha$. More generally, an additional hypothesis is essentially necessary and sufficient if we use the full set $\mathcal{U}_{X,Y}$.

LEMMA 2.3. *Suppose in addition to the above, for each pair $X_\alpha, X_\beta \in \mathcal{U}_{X,Y}$ with $f(X_\alpha) \cap f(X_\beta) \neq \emptyset$, there exists $X_{\beta'} \in \mathcal{U}_{X,Y}$ with*

(2.3) $f(X_{\beta'}) = f(X_\beta)$ *and* $f(X_\alpha \cap X_{\beta'}) = f(X_\alpha) \cap f(X_{\beta'})$.

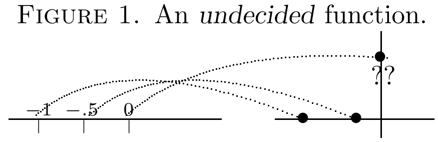
Then, the topologizing data on X provides topologizing data on Y.

PROOF. Apply f to $\mathcal{U}_{X,Y}$ get $\mathcal{U}_Y$. Suppose $f(X_\alpha) \cap f(X_\beta) \neq \emptyset$. Then, choose $(X_{\beta'}, \varphi_{\beta'})$ and form $\psi_{\beta,\alpha}$ by replacing f^{-1}_β by $f^{-1}_{\beta'}$. □

2.2. $\mathbb{R}^n$-like behavior requires Hausdorffness. Here is the problem with a space that isn't Hausdorff. Suppose $f : [0, 1) \to X$ is a continuous function, everything of a path except the end point. Manifolds in this book appear as extensions of open subsets of $\mathbb{R}^n$. So, the only thing that should prevent us from extending our path (continuously) to $f^* : [0, 1] \to X$ is that there is no point $f^*(1) \in X$ giving a continuous f^*. If there are several possible choices $f^*(1)$ giving a continuous function f^*, these extending points would have more exotic neighborhoods than do points in $\mathbb{R}^n$. In practice, the use of Hausdorff is to assure in theorems of Chap. Chap. 4 that there is a unique manifold solution to many existence problems.

EXAMPLE 2.4 (Continuation of Ex. 1.2). As in Ex. 1.2, let $\varphi_i : X_i \to \mathbb{R}^1$ by $\varphi_i(t, i) = t$, $i = 1, 2$. The relative topology on $X_1 \bigcup_f X_2$ is not Hausdorff [9.1].

FIGURE 1. An *undecided* function.



There is a topological formulation of the possibility that we could end a path in two different points. That is, $(f, f) : [0, 1) \to X \times X$ has topological closure not in the diagonal $\Delta_X = \{(x, x) \mid X] \times X\}$. That is, if $f^*(1)$ and $f^\dagger(1)$ are two different ways to extend f to a path on $[0, 1]$, then $(f^*(1), f^\dagger(1))$ is in the closure of Δ_X. Conveniently, the exact property that prevents this situation is that X is Hausdorff [9.1b].

LEMMA 2.5. *X is Hausdorff if and only if Δ_X is closed in $X \times X$ [9.1d].*

Here is a classical fact. If $f : X \to Y$ is continuous and one-one and Y is Hausdorff, then the restriction of f to any compact subset of X is a homeomorphism onto its image. This uses that the image of a compact set is compact; then Hausdorff assures that the image of the compact set (and all closed subsets of it) is closed. It is, however, common to have such an f where the *inverse image* of some compact sets are not compact. For example, let $f : \mathbb{C}_z^* \to \mathbb{C}_z$ be the identity map. Then, the inverse image of the unit disk is not compact (compare with [9.1e]). Call a map $f : X \to Y$ *proper* if the inverse image of compact sets is compact.

3. Manifolds: differentiable and complex

Let X be a topological space with topologizing data $\{(X_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ (relative to $\mathbb{R}^n$). We add conditions to define differentiable and complex manifolds. Classical cases of the latter include the Riemann sphere, the complex torus and algebraic sets defined by $m \in \mathbb{C}[z, w]$ with nonzero gradient everywhere.

DEFINITION 3.1. Let X be a Hausdorff space with $\{(X_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ as topologizing data. Assume φ_α maps U_α to an open connected subset of $\mathbb{R}^n$ for each $\alpha \in I$. Call X an *n-dimensional* (*topological*) *manifold*.

In this case, replace the open sets X_α by the notation U_α. Call $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ a *coordinate system* or *atlas*. An individual member $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ is a *(coordinate) chart*. Ex. 2.4 shows the Hausdorff condition isn't automatic.

3.1. Manifold structures. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function defined on an open set U. For $\boldsymbol{x}_0 \in U$ and $\boldsymbol{v} \in \mathbb{R}^n$, the *directional derivative* of f at $\boldsymbol{x}_0$ in the direction $\boldsymbol{v}$ is the limit

$$\lim_{t\to 0} \frac{f(\boldsymbol{x}_0 + t\boldsymbol{v}) - f(\boldsymbol{x}_0))}{t} \stackrel{\text{def}}{=} \frac{\partial f}{\partial \boldsymbol{v}}(\boldsymbol{x}_0),$$

if it exists. If $\boldsymbol{e}_i = \boldsymbol{v}$ is the vector with 1 in the ith coordinate and 0 in the other coordinates, denote the directional derivative by $\frac{\partial f}{\partial x_i}(\boldsymbol{x}_0)$. Then

$$\nabla f(\boldsymbol{x}_0) \stackrel{\text{def}}{=} (\frac{\partial f}{\partial x_1}(\boldsymbol{x}_0), \dots, \frac{\partial f}{\partial x_n}(\boldsymbol{x}_0))$$

is the *gradient* of f at $\boldsymbol{x}_0$.

LEMMA 3.2. [**Rud76**, p. 218] *Suppose $\frac{\partial f}{\partial x_i}$ exists and is continuous near $\boldsymbol{x}_0$ for $i = 1, \dots, n$. Then, for each vector $\boldsymbol{v}$, $\frac{\partial f}{\partial \boldsymbol{v}}(\boldsymbol{x}_0)$ exists and equals $\nabla f(\boldsymbol{x}_0) \cdot \boldsymbol{v}$.*

Call a function satisfying the hypotheses of Lemma 3.2 *differentiable* at $\boldsymbol{x}_0$. A function $\boldsymbol{f} = (f_1(\boldsymbol{x}), \dots, f_m(\boldsymbol{x}))$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ is differentiable at $\boldsymbol{x}_0$ if each of the coordinate functions $f_i(\boldsymbol{x})$ is differentiable at $\boldsymbol{x}_0$. While it is not absolutely necessary, our manifolds often have transition functions with continuous partial derivatives of all orders: *smoothly* differentiable.

Assume $g : \mathbb{R}^m \to \mathbb{R}$ is a composite of $\mathbb{R}^m \xrightarrow{H} \mathbb{R}^n \xrightarrow{f} \mathbb{R}$. Let $\boldsymbol{y}_0 \in \mathbb{R}^m$. Suppose each coordinate function from $H(\boldsymbol{y}) = (h_1(\boldsymbol{y}), \dots, h_n(\boldsymbol{y}))$ of H is differentiable at $\boldsymbol{y}_0$ and f is differentiable at $H(\boldsymbol{y}_0)$. Write $J(H)(\boldsymbol{y}_0)$ for the matrix whose ith row is $\nabla h_i(\boldsymbol{y}_0)$. As a slight generalization of Lem. 3.2, $\nabla g(\boldsymbol{y}_0)$ exists and equals

$$= \nabla f(H(\boldsymbol{y}_0)) \cdot J(H)(\boldsymbol{y}_0). \tag{3.1}$$

3.1.1. *Differentiable functions.* Let X be an n-dimensional manifold. Denote an atlas for it by $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$.

DEFINITION 3.3. Call X a *differentiable manifold* if each transition function $\varphi_\beta \circ \varphi_\alpha^{-1}$ is smoothly differentiable on its domain of definition.

For any $x \in U_\alpha$ on a chart of a differentiable manifold X, define the (smoothly) differentiable functions on U_α to be $C^\infty(U_\alpha) = \{f \circ \varphi_\alpha \mid f \in \mathrm{C}^\infty(\varphi_\alpha(U_\alpha))\}$. This definition should be independent of the chart: We declare that restricting a differentiable function to an open subset of U_α still gives a differentiable function. This, however, must be compatible with the definition of differentiable using any other coordinate chart (U_β, φ_β) which also contains x.

LEMMA 3.4. *Suppose $x \in U_\alpha \cap U_\beta$, and $f \circ \varphi_\alpha$ is restriction of a differentiable function to an open neighborhood W of x in $U_\alpha \cap U_\beta$. Then, $f \circ \varphi_\alpha = g \circ \varphi_\beta$ for some differentiable function g defined on $\varphi_\beta(W)$.*

PROOF. Write $f \circ \varphi_\alpha$ as $f \circ \varphi_\alpha \circ \varphi_\beta^{-1} \circ \varphi_\beta$ and take g as $f \circ \varphi_\alpha \circ \varphi_\beta^{-1}$. This is defined on $\varphi_\beta(W)$. As the composite of two differentiable functions f and $\varphi_\alpha \circ \varphi_\beta^{-1}$, g is differentiable from (3.1). □

DEFINITION 3.5 (Global differentiable functions on X). If X is a differentiable manifold, then a function $f : X \to \mathbb{R}$ is differentiable if its restriction to each U_α in a coordinate chart is differentiable.

3.1.2. *Complex functions.* Decompose a complex number z_i into its real and complex parts as $x_i + i\,y_i$. This produces (as in (2.1)) a natural one-one map:

$$L = L_n : \mathbb{R}^{2n} \to \mathbb{C}^n \text{ by } (x_1, y_1, \dots, x_n, y_n) \mapsto (z_1, \dots, z_n).$$

Topologize $\mathbb{C}^n$ so L (and its inverse) are continuous. Identify $\mathbb{C}^n$ and $\mathbb{R}^{2n}$ to consider any differentiable function: $g : \mathbb{R}^{2n} \to \mathbb{R}$ as a function $g \circ L^{-1} : \mathbb{C}^n \to \mathbb{R}$. Further, a pair u and v of differentiable functions with a common domain U from $\mathbb{R}^{2n} \to \mathbb{R}$ produces a differentiable function $f : \mathbb{C}^n \to \mathbb{C}$ on U:

$$\boldsymbol{z} \mapsto u \circ L^{-1}(\boldsymbol{z}) + iv \circ L^{-1}(\boldsymbol{z}).$$

Call $f : \mathbb{C}^n \to \mathbb{C}$ *analytic* at $\boldsymbol{z}_o = (z_{1,0}, \dots, z_{n,0})$ if each complex partial derivative

$$\frac{\partial f}{\partial z_i}(\boldsymbol{z}') = \lim_{z_i \to z_i'} \frac{(f(z_1', \dots, z_{i-1}', z_i, z_{i+1}', \dots, z_n') - f(\boldsymbol{z}'))}{z_i - z_i'}$$

exists and is continuous, $i = 1, \dots, n$, with $\boldsymbol{z}'$ near $\boldsymbol{z}_0$. We say $\boldsymbol{f} = (f_1(\boldsymbol{z}), \dots, f_m(\boldsymbol{z}))$ from $\mathbb{C}^n$ to $\mathbb{C}^m$ is *analytic* at $\boldsymbol{z}_0$ if each *coordinate function* $f_i(\boldsymbol{z})$ is analytic at $\boldsymbol{z}_0$. Analytic functions behave for differentiation (or integration) as if each z_i ranging over a 2-dimensional set were a single real variable. [9.4] explores how changing the particular linear identification L_n affects this definition. In the first half of the 1800's, researchers realized the geometry underlying this definition could characterize special recurring collections of integrals. A motivating problem (Chap. 4) was whether the integrals of these functions were serious new functions. By, however, defining —as in Def. 3.6 —analytic manifolds, Riemann replaced complicated sets of functions by geometric properties.

To match with previous notation, if U be an open connected subset of $\mathbb{C}^n$, denote the analytic functions on U by $\mathcal{H}(U)$. The natural quotient field $\mathcal{M}(U)$ of $\mathcal{H}(U)$ (Lem. 3.9), the field of meromorphic functions on U, consists of ratios from $\mathcal{H}(U)$ with nonzero denominators. When $n = 1$, at each point of U any meromorphic function takes a well-defined value in $\mathbb{P}^1_z$. Simple examples like $\frac{z_1}{z_2}$ at $(0,0)$ show this is not true for $n \ge 2$ [9.11e].

DEFINITION 3.6. Let X be a $2n$-dimensional manifold with atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ where $\varphi_\alpha : U_\alpha \to \mathbb{C}^n$. Call X an *analytic* (or *complex*) *n-dimensional manifold* if each transition function $\psi_{\beta,\alpha} = \varphi_\beta \circ \varphi_\alpha^{-1}$ is analytic on $\varphi_\alpha(U_\alpha \cap U_\beta)$. So, an analytic manifold is differentiable. A *Riemann surface* is a 1-dimensional complex manifold.

For any $x \in U_\alpha$ on a chart $\mathcal{U}$ of an analytic manifold X, define analytic (resp. meromorphic) functions on U_α to be $\mathcal{H}_{\mathcal{U}}(U_\alpha) = \{f \circ \varphi_\alpha \mid f \in \mathcal{H}(\varphi_\alpha(U_\alpha))\}$ (resp. $\mathcal{M}_{\mathcal{U}}(U_\alpha)$ where we replace f analytic by f meromorphic). Exactly as previously, Lem. 3.4 has a version for analytic or meromorphic functions. What changes if we adjust the atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ in simple ways?

DEFINITION 3.7. Assume $X = X_{\mathcal{U}}$ is an n-dimensional analytic manifold, and $h_\alpha : \mathbb{C}^n \to \mathbb{C}^n$ is one-one, differentiable, but not necessarily analytic, on $\varphi_\alpha(U_\alpha)$ for each $\alpha \in I$. Topologies of X from $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I} = \mathcal{U}$ or $\{(U_\alpha, h_\alpha \circ \varphi_\alpha)\}_{\alpha \in I} = \mathcal{U}_{\boldsymbol{h}}$ are the same. Call $\boldsymbol{h}$ a *coordinate adjustment* and $\mathcal{U}_{\boldsymbol{h}}$ the *adjustment* of $\mathcal{U}$ by $\boldsymbol{h}$. Then, $\boldsymbol{h}$ is an *analytic adjustment* if transition functions for $\mathcal{U}_{\boldsymbol{h}}$ are analytic.

Only special coordinate adjustments are analytic. Even if $\boldsymbol{h}$ is an analytic adjustment, unless all the h_α s are analytic themselves, the functions we call analytic (or meromorphic) on an open set U_α of $X_{\mathcal{U}}$ are usually different from those on the same open set of $X_{\mathcal{U}_{\boldsymbol{h}}}$. For example, suppose $I = \{\alpha\}$ and $U_\alpha = D$ is an open set in

$\mathbb{C}$. Then, the functions $\mathcal{H}_{(D,h)}(D) = \{f \circ h \mid f \in \mathcal{H}(D)$ we call analytic on $\{(D,h)\}$ are the same as $\mathcal{H}(D)$ if and only if h is analyic.

If D is simply connected (and not all of $\mathbb{C}_z$), then Riemann's Mapping Theorem says $\mathcal{H}(D,h)$ is isomorphic as a ring to the convergent power series on the unit disk in $\mathbb{C}_z$. [**Ahl79**, p. 230] says this if h is the identity, though composing with h^{-1} for any diffeomorphisms is a ring isomorphism. A nontrivial case of adjustments is where all the h_α s are the same (see [9.4c]). We explore this further in Chap. 4 §**??**. In the next observation (see §5.2.1 for the definition of $\frac{\partial}{\partial \bar z}$) denote range variables for $\varphi_\alpha : U_\alpha \to \mathbb{C}^n$ by $z_{\alpha,1}, \dots, z_{\alpha,n}$.

LEMMA 3.8. *That $X_{\mathcal{U}_{\boldsymbol{h}}}$ is an analytic manifold is equivalent to*

(3.2) $h_\beta \circ \varphi_\beta \circ \varphi_\alpha^{-1} \circ h_\alpha^{-1}$ *is analytic on* $h_\alpha \circ \varphi_\alpha(U_\alpha \cap U_\beta)$ *for all* $(\alpha,\beta) \in I^2$: $\frac{\partial}{\partial \bar z_{\alpha,i}}$ *applied to each of its matrix entries is 0,* $i = 1, \dots, n$.

If the $\{h_\alpha\}_{\alpha \in I}$ *are all analytic, then* $\mathcal{H}_{\mathcal{U}}(U_\alpha) = \mathcal{H}_{\mathcal{U}_{\boldsymbol{h}}}(U_\alpha)$ *for all* $\alpha \in I$.

Suppose $X_{\mathcal{U}}$ and $X_{\mathcal{U}_{\boldsymbol{h}}}$ are both analytic manifolds. Lem. 3.8 shows the local analytic functions change unless $\boldsymbol{h}$ consists of analytic functions. We regard the complex structures as the same if and only if both $X_{\mathcal{U}}$ and $X_{\mathcal{U}_{\boldsymbol{h}}}$ have the same analytic functions in a neighborhood of each point. A special case appears often in the theory of complex manifolds. It is when all the functions h_α are complex conjugation (Chap. 4 Lem. **??**). Notice: Complex conjugation reverses orientation in $\mathbb{C}$ by mapping clockwise paths around the origin to counterclockwise paths.

3.1.3. *A tentative definition of algebraic manifold.* For complex manifolds, a coordinate chart allows us to define global meromorphic functions as a collection $g_\alpha \in \mathcal{M}(U_\alpha)$ for which $g_\alpha = g_\beta$ on any points of $U_\alpha \cap U_\beta$ where both make sense. Our major study treats families of compact Riemann surfaces. Often each family member appears explicitly with a finite set of points removed, using Riemann's Existence Theorem to produce such surfaces as covers of $U_{\boldsymbol{z}}$. Meromorphic functions mean for us functions meromorphic on some compactification of this manifold. This includes that the functions are ratios of holomorphic functions at those points that might not be included in the initial presentation. For example, global meromorphic functions on $U_{\boldsymbol{z}}$ refer to elements of $\mathbb{C}(z)$. They are among the ratios of algebraic functions on $U_{\boldsymbol{z}}$, so they have no essential singularities as we approach $\boldsymbol{z}$.

Understanding manifolds which have a coordinate description is important to the goals of this book. When we deal with compact complex manifolds, global coordinate functions live inside the field of global meromorphic functions. Our first tentative definition of algebraic excludes some manifolds that everyone considers algebraic. Still, it is simple, close to the general meaning of algebraic and it leads naturally to that definition.

LEMMA 3.9. *Suppose $X_{\mathcal{U}}$ is a connected topological space and an analytic manifold. Then, the (global) meromorphic functions on $X = X_{\mathcal{U}}$ form a field, $\mathbb{C}(X)$.*

PROOF. Add (resp. multiply) functions of form $f_1(\varphi_\alpha)$ and $f_2(\varphi_\alpha)$ by computing the value at $x \in U_\alpha$ as $f_1(\varphi_\alpha(x)) + f_2(\varphi_\alpha(x))$ (resp. $f_1(\varphi_\alpha(x))f_2(\varphi_\alpha(x))$). Quotients, too, are obvious for they will also be ratios of holomorphic functions at each point. We need only to see that $\mathbb{C}(X)$ is an integral domain. If, however, $f_1(\varphi_\alpha(x))f_2(\varphi_\alpha(x)) = 0$ for $x \in U_\alpha$, then $f_1(\boldsymbol{z})f_2(\boldsymbol{z}) = 0$ for $\boldsymbol{z}$ on the open set $\varphi_\alpha(U_\alpha)$. Chap. 2 [9.8a] shows either $f_1(\varphi_\alpha)$ or $f_2(\varphi_\alpha)$ is 0 on U_α. □

A goal for compact Riemann surfaces is to understand adjustments well enough to be able to list the isomorphism classes of fields $\mathbb{C}(X_{\mathcal{U}_{\boldsymbol{h}}})$, the *function field* of

$X_{\mathcal{U}_{\boldsymbol{h}}}$, as $\boldsymbol{h}$ varies. How can we describe the complete set of function fields up to isomorphism? This book shows how to apply various answers to many seemingly unrelated problems.

Suppose $x_1, x_2 \in X$ and $f \in \mathbb{C}(X_{\mathcal{U}})$ are holomorphic in a neighborhood of x_1 and x_2 and takes different values there. We say f *separates* x_1, x_2. If for each pair of distinct points $x_1, x_2 \in X$ there is an $f \in \mathbb{C}(X_{\mathcal{U}})$ separating them, we say $\mathbb{C}(X_{\mathcal{U}})$ separates points. Suppose $X_{\mathcal{U}}$ has complex dimension n, $x \in X_{\mathcal{U}}$ is in a coordinate chart $\varphi_\alpha : U_\alpha \to \mathbb{C}^n$ and there are n functions $f_1, \dots, f_n \in \mathbb{C}(X_{\mathcal{U}})$ all holomorphic in a neighborhood of x. If the Jacobian of $f_1, \dots, f_n$ —determinant of the matrix with (i,j)-entry of $\frac{\partial f_i \circ \varphi_\alpha^{-1}}{\partial z_j}$, $i = 1, \dots, n$, $j = 1, \dots, n$ —is nonzero at $\varphi_\alpha(x)$, we say $f_1, \dots, f_n$ *separate tangents* at x.

Definition 3.10. An n-dimensional compact complex manifold X (with topologizing data $\mathcal{U}$) is $\mathbb{P}^1$-*algebraic* if there is a collection $f_1, \dots, f_N \in \mathbb{C}(X_{\mathcal{U}})$ so the following conditions hold.

(3.3a) For each $x \in X_{\mathcal{U}}$, there is a collection $\epsilon_1, \dots, \epsilon_N \in \{\pm 1\}$ (dependent on x) so that $f_1^{\epsilon_1}, \dots, f_N^{\epsilon_N}$ are all holomorphic at x.
(3.3b) Among $f_1^{\epsilon_1}, \dots, f_N^{\epsilon_N}$ there are n that separate tangents at x.
(3.3c) Given distinct $x_1, x_2 \in X$, one from $f_1, \dots, f_N$ separates x_1 and x_2.

Note: In (3.3c), if f_i is holomorphic at x, and $f_i(x) = 0$, we include ∞ as the value of $1/f_i(x)$. Algebraic manifolds are the analytic manifolds $X_{\mathcal{U}}$ most significant to us ($\mathbb{P}^1$-algebraic manifolds are a special case; see §4.1.2). There are 2-dimensional analytic manifolds with function fields consisting only of constant functions. Our examples will be complex torii. The phrase *abelian variety* (Chap. 4§**??**; usually with a extra structure called a *polarization*) is the name for a complex torus that is algebraic. Chap. 4 analyzes all analytic structures on a dimension one complex torus by corresponding them precisely to the isomorphism class of their function fields. This topic starts in § 3.2.2.

There are two distinct generalizations: To compact Riemann surfaces and to abelian varieties. The former are $\mathbb{P}^1$-algebraic while the latter are not in general.

3.2. Classical examples. We discuss two natural first cases of compact complex manifolds.

3.2.1. *The Riemann sphere* $\mathbb{P}^1_z$. Let X be the disjoint union of the complex plane $\mathbb{C}$ and a point labeled ∞. Here is a coordinate chart:

$$\begin{aligned} U_1 = \mathbb{C}, &\quad \varphi_1 : U_1 \to \mathbb{C} \text{ by } \varphi_1(z) = z; \text{ and} \\ U_2 = (\mathbb{C}\setminus\{0\}) \cup \{\infty\}, &\quad \varphi_2 : U_2 \to \mathbb{C} \text{ by } \varphi_2(\infty) = 0 \text{ and} \\ &\quad \varphi_2(z) = \tfrac{1}{z} \text{ for } z \in \mathbb{C}\setminus\{0\}. \end{aligned}$$

Chap. 2 used the Riemann sphere. It embeds in $\mathbb{R}^3$. So it is Hausdorff. Then, X is a complex manifold: $\varphi_2 \circ \varphi_1^{-1}(z) = \varphi_1 \circ \varphi_2^{-1}(z) = \frac{1}{z}$ on $\mathbb{C}\setminus\{0\}$ are analytic.

If a complex manifold is compact, some atlas for it contains only finitely many elements. The Riemann sphere required only two (one wouldn't do, would it?).

3.2.2. *Complex torus.* An atlas for our next example will require four open sets. Let ω_1 and ω_2 be two nonzero complex numbers satisfying the *lattice condition*: $\frac{\omega_2}{\omega_1}$ is not real. Consider the *lattice* ω_1 and ω_2 generate:

$$L(\omega_1, \omega_2) = \{m_1\omega_1 + m_2\omega_2 \mid m_1, m_2 \in \mathbb{Z}\}. \tag{3.4}$$

The lattice condition guarantees the natural quotient map $\mathbb{C} \to \mathbb{C}/L(\omega_1, \omega_2)$ has open sets that are like open sets in $\mathbb{C}$ [9.6c]. According to Lem. 2.3, the manifold structure on $\mathbb{C}$ automatically gives the manifold structure on $\mathbb{C}/L(\omega_1,\omega_2)$. Use the chart $\{(U'_i,\varphi'_i)\}_{i\in\{0,1,2,3\}}$ of Fig. 3 with φ'_i the inclusion of U'_i in $\mathbb{C}$. This assures satisfying the Lem. 2.3 condition: Each $z\in\mathbb{C}$ has an $i=i_z$ for which $z\in U'_i$ and the natural map $\mathbb{C}\to\mathbb{C}/L(\omega_1,\omega_2)$ is one-one on U_i.

The resulting complex manifold $\mathbb{C}/L(\omega_1,\omega_2)$ depends only on $L(\omega_1,\omega_2)$. Among the many choices we can make of ω_1,ω_2 generating this lattice, it is traditional to choose them satisfying special conditions. Elements of the group $\mathrm{SL}_2(\mathbb{Z})$ act on ω_1,ω_2 to give all pairs of basis elements Chap. 2 [9.15c]. Further, for $a\in\mathbb{C}^*$ the scaling $\mathbb{C}\to\mathbb{C}$ by $z\mapsto az$ induces a homomorphism $\mathbb{C}/L(\omega_1,\omega_2)\to\mathbb{C}/L(a\omega_1,a\omega_2)$ of abelian groups. At the level of coordinate charts, the same scaling gives the map. So, it induces an analytic isomorphism (for precision use Def. 4.1). With no loss take $a=1/\omega_1$, to change the basis of the lattice to $1,\omega_2/\omega_1$. The ratio $\omega_2/\omega_1=\tau$ aptly indicates the shape of the parallelogram (3.6). This starts a typical normalization for the complex structure. If we could uniquely indicate the complex structure by τ, that would be an excellent way to parametrize them. The problem is that the complex structure depends only on the lattice $L(1,\tau)$ generated by 1 and τ. Many values of τ giving the same $L(1,\tau)$. For example, here are three obvious changes:

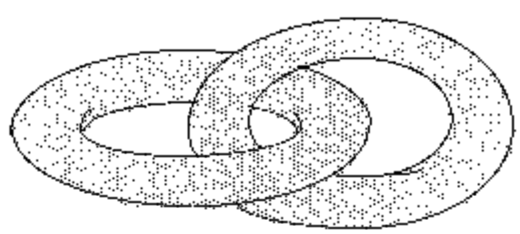
(3.5a) If necessary, replace $\{1,\tau\}$ by $\{1,-\tau\}$ to assume $\Im(\tau)$ is in the *upper half plane* $\mathbb{H}\stackrel{\text{def}}{=}\{\tau\in\mathbb{C}\mid\Im(\tau)>0\}$; or
(3.5b) replace $\{1,\tau\}$ by $\{1,\tau+n\}$ for some integer n to assume $0\le\Re(\tau)<1$; or
(3.5c) scale by $-1/\tau$ to replace $\{1,\tau\}$ by $\{1,-1/\tau\}$.

Changes from (3.5) generate a group, $\mathrm{PSL}_2(\mathbb{Z})$ ($<\mathrm{PSL}_2(\mathbb{R})$; §8.2), acting on $\tau\in\mathbb{H}$.

LEMMA 3.11. *Together,* (3.5) *permits restricting a* τ *representing a given complex torus (up to isomorphism) to the narrow strip in* $\mathbb{H}$ *over the closed interval* $[0,1)\subset\mathbb{R}$ *lying within the closed unit circle around the origin.*

Transition functions restrict on each connected component of an intersection of charts to be translation in the complex plane. Topologically this is the same as a *torus* in $\mathbb{R}^3$. Topologists deal with torii, too, though they concentrate especially on the topological space in which the torii sit (see [9.5] for the point of Fig. 2). We care most about this additional complex structure, while they rarely distinguish between one complex torus and another. See §7.2.3 for additional comments on attempts to draw pictures in $\mathbb{R}^3$.

FIGURE 2. These two torii could unknot in $\mathbb{R}^4$.

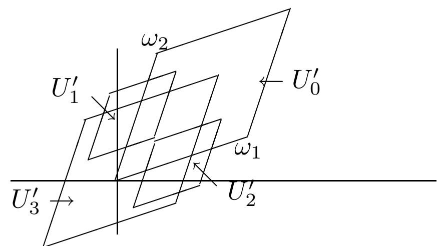


Here is the set behind the manifold:

$$X=\{t_1\omega_1+t_2\omega_2\mid 0\le t_i<1, i=1,2\}. \tag{3.6}$$

Standard open parallelograms in $\mathbb{C}$ represent each of four coordinate charts in Fig. 3, U_i, $i = 0, 1, 2, 3$, that do lie in X.

FIGURE 3. Four open sets sort of covering a torus



Let $U_0 = \{t_1\omega_1 + t_2\omega_2 \mid 0 < t_i < 1, i = 1, 2\}$, with $\varphi_0 : U_0 \to \mathbb{C}$ the identity map. The corresponding U_0' is equal to U_0 in Fig. 3. On the other hand, consider

$$U_1 = \{t_1\omega_1 + t_2\omega_2 \mid \frac{1}{3} < t_2 < \frac{2}{3} \text{ and either } 0 \le t_1 < \frac{1}{3} \text{ or } \frac{2}{3} < t_1 < 1\},$$

and $\varphi_1 : U_1 \to \mathbb{C}$ by

$$\varphi_1(t_1\omega_1 + t_2\omega_2) = \begin{cases} t_1\omega_1 + t_2\omega_2 & \text{for } 0 \le t_1 < \frac{1}{3} \\ (t_1 - 1)\omega_1 + t_2\omega_2 & \text{for } \frac{2}{3} < t_1 < 1. \end{cases} .$$

Form the corresponding U_1' by translating a pieces of the range of φ_1.

The remaining charts are similar (though slightly more complicated):

$$U_2 = \{t_1\omega_1 + t_2\omega_2 \mid \tfrac{1}{3} < t_1 < \tfrac{2}{3} \text{ and either } 0 \le t_2 < \tfrac{1}{3} \text{ or } \tfrac{2}{3} < t_1 < 1\},$$
$$\varphi_2(t_1\omega_1 + t_2\omega_2) = \begin{cases} t_1\omega_1 + t_2\omega_2 & \text{for } 0 \le t_2 < \frac{1}{3} \\ t_1\omega_1 + (t_2 - 1)\omega_2 & \text{for } \frac{2}{3} < t_2 < 1. \end{cases}$$

$$U_3 = \{t_1\omega_1 + t_2\omega_2 \mid 0 \le t_1 < \tfrac{1}{2} \text{ or } \tfrac{1}{2} < t_1 < 1,\ 0 \le t_2 < \tfrac{1}{2} \text{ or } \tfrac{1}{2} < t_2 < 1\}, \text{ and}$$
$$\varphi_3(t_1\omega_1 + t_2\omega_2) = \begin{cases} t_1\omega_1 + t_2\omega_2 & \text{for } 0 \le t_1, t_2 < \frac{1}{2}, \\ (t_1 - 1)\omega_1 + t_2\omega_2 & \text{for } \frac{1}{2} < t_1 < 1, 0 \le t_2 < \frac{1}{2}, \\ t_1\omega_1 + (t_2 - 1)\omega_2 & \text{for } 0 \le t_1 < \frac{1}{2}, \frac{1}{2} < t_2 < 1, \\ (t_1 - 1)\omega_1 + (t_2 - 1)\omega_2 & \text{for } \frac{1}{2} < t_1, t_2 < 1. \end{cases}$$

To see X is a 1-dimensional complex manifold check the transition functions $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$. For each i and j, $\varphi_i(U_i \cap U_j)$ is the union of a finite number of connected open sets. For example,

$$\varphi_0(U_0 \cap U_1) = U_1' \setminus \Big\{t_2\omega_2 \mid \frac{1}{3} < t_2 < \frac{2}{3}\Big\}.$$

On each connected component of $\varphi_i(U_i \cap U_j)$, $\varphi_j \circ \varphi_i^{-1}$ is translation by one of the complex numbers $\delta_1\omega_1 + \delta_2\omega_2$ where δ_k is 0 or ± 1, $k = 1, 2$.

With this manifold structure, X is the *complex torus with periods* ω_1 *and* ω_2.

3.3. Manifolds from algebraic functions. Let $m \in \mathbb{C}[z, w]$ be an irreducible polynomial. Denote the branch points of m by $\boldsymbol{z}$ with $z_0 \in U_{\boldsymbol{z}} = \mathbb{P}^1_z \setminus \boldsymbol{z}$ as in Chap. 2 Def. 6.3. Assume $f(z)$ is analytic in a neighborhood of z_0 and it satisfies $m(z, f(z)) \equiv 0$. Chap. 2 started with two definitions of algebraic functions Def. 1.1 and Def. 1.2. They characterize the same set of functions (Chap. 2 Prop. 7.3).

Riemann's Existence Theorem starts by attaching to each algebraic function a unique (up to analytic isomorphism) compact complex manifold of dimension 1. The next two examples are the first step in that construction, producing an open subset of the final manifold. We introduce some algebraic geometry using as an excuse showing how to construct explicit manifold compactifications in special cases. We expect coordinates for the abstract compactification of a general Riemann surface to be somewhat mysterious.

3.3.1. *An unramified cover of $U_{\boldsymbol{z}}$.* Consider first the set

$$X^{[0]} = X_f^0 = \{(z,w) \in \mathbb{C} \times \mathbb{C} \mid z \notin \boldsymbol{z},\ m(z,w) = 0\}.$$

PROPOSITION 3.12. *The projection map* $\mathrm{pr}_z : X^{[0]} \to U_{\boldsymbol{z}}$ *by* $(z,w) \mapsto z$ *produces a natural atlas on* $X^{[0]}$ *making it a connected complex manifold. For* $\lambda \in \Pi_1(U_{\boldsymbol{z}}, z_0, z_1)$ *(*Chap. 2 §*1.1), naturally identify the manifolds* X_f^0 *and* $X_{f_\lambda}^0$.

PROOF. To simplify the construction, assume $\infty \in \boldsymbol{z}$. As usual, apply an element of $\mathrm{PGL}_2(\mathbb{C})$ to $\boldsymbol{z}$ to arrange that situation (Chap. 2 §5.2.1; see Lem. 4.3).

Use the implicit function theorem (Chap. 2 §6.2) as follows. For $(z',w') \in X^{[0]}$, let $\Delta_{z'}$ be the open disk centered at z' of radius the minimum distance from z' to a point of $\boldsymbol{z}$. Then, for some one-one analytic function $f_{z',w'}(z)$ the following holds.

(3.7) The points $(z, f_{z',w'}(z))$ are on $X^{[0]}$ and $f_{z',w'}(z') = w'$.

For each (z',w') let $U_{z',w'}$ be the range of $z \mapsto F_{z'}(z) \stackrel{\text{def}}{=} (z, f_{z',w'}(z))$ on $\Delta_{z'}$. The inverse of $F_{z'}$ is pr_z, projection of a pair (z,w) onto its z-coordinate. Compatible with the definition of manifold, here denote pr_z by $\varphi_{z',w'}$. Then, $F_{z'}$ *parametrizes* the neighborhood $U_{z',w'}$ of (z',w') and $\varphi_{z',w'}$ maps it into $\mathbb{C}_z$. If $V = U_{z',w'} \cap U_{z'',w''}$ is nonempty, then $\varphi_{z'',w''} \circ \varphi_{z',w'}^{-1}$ is the identity map on the overlap of $\Delta_{z'} \cap \Delta_{z''}$.

That gives an atlas. As it is a subspace of the Hausdorff space $\mathbb{C} \times \mathbb{C}$, $X^{[0]}$ is Hausdorff. So, it is a connected (from Chap. 2 §6.4) complex manifold. Let λ be a path as in the statement of the proposition. The point set of X_f^0 consists of pairs $(z',x') \in \mathbb{C} \times \mathbb{C}$ of the form $(z', f_\gamma(z'))$ with $\gamma : [a,b] \to U_{\boldsymbol{z}}$ with $\gamma(a) = z_0$ and $\gamma(b) = z'$. As $X_{f_\lambda}^0$ is connected, we can write any point on it as the endpoint of $(z, f_{\lambda\cdot\gamma})$ for some λ. So, $X_{f_\lambda}^0$ is the same subset of points in $\mathbb{C} \times \mathbb{C}$. □

Note: Each $z' \in U_{\boldsymbol{z}}$ has a neighborhood $\Delta_{z'}$ with this property.

(3.8) pr_z restricted to each connected component $U_{z',w'}$ of $\mathrm{pr}_z^{-1}(\Delta_{z'})$ is a homeomorphism with $\Delta_{z'}$.

This is a stronger property than pr_z being an immersion. It means $\mathrm{pr}_z : X^{[0]} \to U_{\boldsymbol{z}}$ is an (unramified) cover according to Def. 7.12. The inverse image by pr_z of small closed disks around z' are closed disks around points lying over z'. That is, the preimage of a compact set is compact, and pr_z is a *proper* map [9.1d].

REMARK 3.13 (Finite atlas). The atlas of Prop. 3.12 contains an infinite number of elements. For a manifold that adds one complication §3.2.1 and §3.2.2 don't have. This came about to include a deleted neighborhood of (z_i, w') with $z_i \in \boldsymbol{z}$ and w' a solution of $m(z_i, w')$. That's because we chose disks on $\mathbb{C}_z$ as the domain for the $F_{z'}$ parametrization. To remedy this choose other simply connected sets, including traditional slit disks given by scaling, translating and rotating

$$\{z \in \mathbb{C} \mid |z| < 1\} \setminus \{0 \le \Re(z) < 1\}.$$

Chap. 4 §2.4 has further justification for these charts.

3.3.2. *Further compactification and use of equations.* Chap. 4 Thm. 2.6 shows there is a unique compact complex manifold, up to analytic isomorphism (Def. 4.1), extending $X^{[0]}$ (and also the analytic map to $\mathbb{P}^1_z$). To get it we must compatibly add points and analytic disk neighborhoods to match with the analytic structure on $X^{[0]}$. Using the equation $m(z, f(z)) \equiv 0$ often allows adding further points (z_i, w') to $X^{[0]}$ and their local analytic functions to extend the complex manifold structure. The simplest such extension includes those points (z_i, w') where, even though z_i is a branch point, $\frac{\partial m}{\partial w}(z_i, w') \neq 0$. That is, consider

$$X^{[1]} = \Big\{(z, w) \in \mathbb{C} \times \mathbb{C} \mid m(z, w) = 0,\ \frac{\partial m}{\partial w}(z, w) \neq 0\Big\}.$$

The variable for a local chart around w' is w. Prop. 3.15 gives the details.

EXAMPLE 3.14. Suppose $h \in \mathbb{C}[w]$ of degree $n > 1$ produces $h : \mathbb{P}^1_w \to \mathbb{P}^1_z$. Let z_i be a branch point of $m(z, w) = h(w) - z$ and let $g_{z_i} \in S_n$ be a representative of the conjugacy class attached to z_i (Chap. 2 Lem. 7.9). Then, there is a one-one correspondence between the following sets. Chap. 2 [9.4]:

(3.9a) Points (z_i, w') over z_i for which $z \mapsto (z, f_{z_i,w'}(z))$ (3.7) parametrizes a neighborhood of (z_i, w').

(3.9b) Disjoint cycles of length 1 in g_{z_i}.

Example: Consider $h_1(w) = w(w-1)(w-2)$. Use notation from Chap. 2 Lem. 7.9. The group attached to an algebraic $f_1(z)$ satisfying $h_1(f_1(z)) - z \equiv 0$ is S_3.

Branch cycles g_{z_1} and g_{z_2} at the two branch points z_1, z_2 have the shape (1)(2) (§7.1.1): disjoint cycles of length 1 and 2. So each branch point has two points above it. Then, for each z_i there are two solutions $w_{i,1}$ and $w_{i,2}$ of $h(w) - z_i$. Select $w_{i,1}$ so that $\frac{dh}{dw}(w_{i,1}) \neq 0$ and $\frac{dh}{dw}(w_{i,2}) = 0$, $i = 1, 2$. Adding $(z_i, w_{i,1})$ to $X^{[0]}$ produces an open set on which pr_z maps one-one to $\mathbb{P}^1_z$. This does not hold for the point $(z_i, w_{i,2})$. So, $X^{[1]}$ has exactly one point on it over each of z_1 and z_2.

For any $h(w)$ in Ex. 3.14, $X^{[1]}$ will have missing points in that the map pr_z is not proper over some points $z_i \in \boldsymbol{z}$ (§2.2). For f analytic in several variables $z_1, \dots, z_n$ in a neighborhood of a point $\boldsymbol{z}_0$, we call

$$\nabla f(\boldsymbol{z}_0) \stackrel{\text{def}}{=} \Big(\frac{\partial f}{\partial z_1}(\boldsymbol{z}_0), \dots, \frac{\partial f}{\partial z_n}(\boldsymbol{z}_0)\Big)$$

the *complex gradient* of f at $\boldsymbol{z}_0$. Now consider a set (usually) larger than $X^{[1]}$:

$$X^{[2]} = \{(z, w) \in \mathbb{C} \times \mathbb{C} \mid m(z, w) = 0,\ \nabla(m)(z, w) \neq 0\}.$$

PROPOSITION 3.15. *A natural atlas makes $X^{[2]}$ into a complex manifold.*

PROOF. Since $X^{[2]}$ is a subspace of $\mathbb{C} \times \mathbb{C}$ it is Hausdorff. From Prop. 3.12 we have only to add (z_i, w') lying over $z_i \in \boldsymbol{z}$ sitting in $X^{[2]}$ to their neighborhoods in $X^{[1]}$. Change the w' coordinate by an element of $\mathrm{PGL}_2(\mathbb{C})$ to assume none of the finitely many w' s is ∞.

By assumption $\nabla(m)(z_i, w') \neq 0$, though by definition $\frac{\partial m}{\partial w}(z_i, w') = 0$. Therefore, $\frac{\partial m}{\partial z}(z_i, w') \neq 0$. Apply the implicit function theorem to find a disk $\Delta_{w'} \subset \mathbb{C}_w$ and $h_{z_i,w'}(w)$ analytic on $\Delta_{w'}$ with the following properties.

(3.10a) The points $(h_{z_i,w'}(w), w)$ are on $X^{[1]}$.

(3.10b) The radius of $\Delta_{w'}$ is the minimum distance from w' to any branch point of $m^*(w, z) \stackrel{\text{def}}{=} m(z, w)$ (switch the variables z and w).

Similar to the proof of Prop. 3.12, let $V_{z_i,w'}$ be the range of $w \mapsto (h_{z_i,w'}(w), w)$ on $\Delta_{w'}$. Then the coordinate map at (z_i, w') is pr_w by $(z, w) \mapsto w$.

The essence of producing the manifold structure is to check the transition functions. The key check occurs when the intersection a neighborhood of (z_i, w') meets a neighborhood of (z'', w'') with $z'' \notin \boldsymbol{z}$. For example:

$$\mathrm{pr}_w \circ \mathrm{pr}_z^{-1} : z \mapsto (z, f_{z'',w''}(z)) \mapsto f_{z'',w''}(z)$$

is analytic. Similarly, so is

$$\mathrm{pr}_z \circ \mathrm{pr}_w^{-1} : w \mapsto (h_{z_i,w'}(w), w) \mapsto h_{z_i,w'}(w).$$

That concludes the proof of the lemma. □

4. Coordinates and meromorphic functions

Here we define analytic maps between complex manifolds. In many areas of mathematics, being able to compare all objects of study with a core of special cases can help. For example, it is helpful to know that all finite groups have a Jordan-Hölder series of finite simple groups and that this collection of finite simple groups (including their multiplicities) is an invariant of the group. Still, even an expert on the classification of finite simple groups can't be confident of a complete understanding of the finite group from knowing its Jordan-Hölder series.

For certain compact complex manifolds, knowing how to use their meromorphic functions can help decide how such a manifold fits among all related manifolds. That is a rough statement of how we use coordinates on compact complex manifolds. This subsection uses explicit (though only partial) compactification of Riemann surfaces of algebraic functions to illustrate how coordinates give defining equations.

4.1. Comparing analytic spaces. We define maps between analytic spaces, and then emphasize the significance of such maps to $\mathbb{P}^1$.

4.1.1. *Maps between spaces.* Let X_i be a differentiable (resp., complex) manifold of dimension n_i with topologizing data $\{(U_{\alpha_i}, \varphi_{\alpha_i})\}_{\alpha_i \in I_i}$. Consider a function $f : X_1 \to X_2$ and the functions

$$\varphi_{\alpha_2} \circ f \circ \varphi_{\alpha_1}^{-1} : \varphi_{\alpha_1}(U_{\alpha_1} \cap f^{-1}(U_{\alpha_2})) \to \varphi_{\alpha_2}(f(U_{\alpha_1}) \cap U_{\alpha_2}) \tag{4.1}$$

for $(\alpha_1, \alpha_2) \in I_1 \times I_2$.

Definition 4.1 (Analytic map). Call f differentiable (resp. analytic) if the functions of (4.1) are differentiable (resp. analytic) on their domains. For $X_1 \subseteq \mathbb{R}^n$ and $X_2 \subseteq \mathbb{R}^m$, this is equivalent to f being differentiable as usual. If f is one-one and onto, call f a differentiable (resp. analytic) isomorphism between X_1 and X_2.

The phrase *isomorphism* in Def. 4.1 implies there is a differentiable (resp. analytic) $g : X_2 \to X_1$ inverse to f. That is the gist of our next statement.

Lemma 4.2. *Let X and Y be differentiable manifolds. Assume $f : Y \to X$ is a differentiable map, and in a neighborhood U_y of some point $y \in Y$, one-one. Then, there exists differentiable $g : f(U_y) \to U_y$ that is an inverse to f. So, if f is one-one and onto, it has differentiable inverse. If we replace the word differentiable by analytic, there is an analogous result.*

Proof. Both statements are consequences of the *inverse function theorem*. This says that a local inverse exists and is differentiable. There is an inverse function to a one-one onto map (§2.2), so the differentiability is all we need. The definition of differentiable (or analytic) function reverts this result to one about $f : \mathbb{R}^n \to \mathbb{R}^n$ (or for $f : \mathbb{C}^n \to \mathbb{C}^n$) for some integer n.

Chap. 2 §6.1 discusses the inverse function theorem for one complex variable. The full inverse function theorem is an inductive procedure for several complex variables. See [**C89**, p. 72] or [**Rud76**, p. 224] for the general case. For differentiable functions, equation (3.1) says that an inverse g to $f : \mathbb{R}^n \to \mathbb{R}^n$ would have Jacobian matrix $J(g)(\boldsymbol{y}) = J(f(\boldsymbol{x}))^{-1}$ at $\boldsymbol{y} = f(\boldsymbol{x})$. This is a differential equation for $g = (g_1(\boldsymbol{y}), \dots, g_n(\boldsymbol{y}))$, given f. The case when f is real analytic is much more likely for our use, and that has easier proofs in the literature. □

4.1.2. $\mathbb{P}^1$-*algebraic spaces.* Let $\varphi : X \to Y$ be an analytic map of complex manifolds. If U is an open subset of Y, denote the *restriction* of φ over U by $\varphi_U : \varphi^{-1}(U) \to U$. Then, composing holomorphic functions on an open set $U \subset Y$ with φ produces a map $\varphi^* : \mathcal{H}(U) \to \mathcal{H}(f^{-1}(U))$. In particular, if both spaces are connected, and φ is onto, this induces an injection $\varphi^* : \mathbb{C}(X) \to \mathbb{C}(Y)$, an embedding of the function field of Y into that of X.

Chap. 2 Def. 4.14 includes the definition of analytic maps from a domain on $\mathbb{P}^1_w$ to $\mathbb{P}^1_z$, a special case of Def. 4.1. More generally, for any complex manifold X, a nonconstant analytic map $\varphi : X \to \mathbb{P}^1_z$ is a meromorphic function on X (represented by z). Chap. 2 Lem. 2.1 guarantees a nonconstant map of compact Riemann surfaces is surjective. This also applies to φ, even if X (compact) has larger dimension, for again these functions come locally from power series expressions and so give an open map. Further, if X is a compact Riemann surface, Chap. 4 Thm. 2.6 shows any meromorphic function on X extends to give an analytic map from X to projective 1-space. Chap. 4 Lem. 2.1 shows the following points. If X is compact (and φ is nonconstant), then φ has a degree, $|\varphi^{-1}(z')|$ for $z' \in \mathbb{P}^1_z$ not in a finite set of values where this cardinality is a smaller number. Further, if we count points in $\varphi^{-1}(z')$ with *appropriate multiplicity* for their appearance in the fiber, the degree is independent of $z' \in \mathbb{P}^1_z$.

Many compact complex manifolds of dimension at least 2 (example: $\mathbb{P}^n$, $n \geq 2$, [9.11e]), have the following property. Though they have many nonconstant meromorphic functions, none are represented by an analytic map to $\mathbb{P}^1_z$. The compact complex manifolds that are $\mathbb{P}^1$-algebraic are exactly those that embed in $(\mathbb{P}^1)^N$ for some integer N. That is, they have sufficiently many functions represented by an analytic map to $\mathbb{P}^1$, the gist of condition (3.3a).

A virtue of the definition $\mathbb{P}^1$-algebraic is its simplicity, this use of special elements of the function field giving maps to $\mathbb{P}^1$. Still, Chap. 4 §6.1.1 extends this, as is traditional, to say a manifold is *algebraic* if it embeds in $\mathbb{P}^N$ for some N. The effect of that is to show why a set of basic principles forces extending $\mathbb{P}^1$-algebraic manifolds to include $\mathbb{P}^N$ as algebraic. We hope this adds historical perspective on what was less than a century ago a complicated issue. Witness this [**Mu66**, p. 15] quote on going directly from affine space to projective space:

> Among others, Poncelet realized that an immense simplication could be introduced in many questions by by considering "projective" algebraic sets (cf. Felix Klein, *Die Enwicklund der Mathematik*, Part I, p. 80–82). Even to this day, . . . projective algebraic sets play a central role in algebro-geometric questions: therfore we shall define them as soon as possible.

Mumford's quote, and the total acceptance of it in [**Har77**], shouldn't deny the natural way that $\mathbb{P}^1$-algebraic spaces and fiber products illuminate special meromorphic functions arise in providing coordinates.

In practice, on many intensely studied algebraic manifolds, you can choose a finite set, $f_1, \dots, f_m$, of global meromorphic functions to construct the manifold, whose points we can then *see* as given by the values of $f_1, \dots, f_m$ at the given point. From these, it is theoretically possible to construct anything else you would expect attached to the manifold from $f_1, \dots, f_m$. Still, much classical algebraic geometry spends great time on using coordinates (embeddings in projective space) of special types to make these constructions. For many applications, however, this is a too-detailed reliance on specific use of coordinates. We hope discussions in this chapter help the reader see why coordinates are necessary, though one shouldn't insist on seeing them explicitly at all stages.

We especially study families of compact Riemann surfaces with each family member appearing with an attached equivalence class of maps to $\mathbb{P}^1_z$. What, however, is the analogy, so important to individual measurements, for comparing different function fields (Lem. 3.9) associated to different complex manifolds? Where would we expect such comparisons to arise? Comparing Riemann surfaces is possible if there is an efficient labeling of function field generators. The easiest event is if all these Riemann surfaces embed naturally in a space with global coordinates that restrict to give coordinates on the individual surfaces. §4.2 gives examples of how coordinates can help compactify some Riemann surfaces.

An easy way to get new analytic maps from old appears if $\varphi : X \to \mathbb{P}^1_z$ is a meromorphic function. Let $\alpha \in \mathrm{PGL}_2(\mathbb{C})$. Then $\alpha \circ \varphi : X \to \mathbb{P}^1_z$ is a new meromorphic function.

For Ex. 3.14, Prop. 3.15, produces $X^{[2]}$ analytically isomorphic to $\mathbb{C}_w$. We already knew this was a manifold. The proof of Props. 3.12 and 3.15 simplifies because $\infty \in \boldsymbol{z}$. The following lemma removes that assumption [9.1b].

LEMMA 4.3. *Let $U_i \subset \mathbb{P}^1_z$, $i = 1, 2$ be domains. Let $\varphi : X \to U_1 \cup U_2$ denote projection of a manifold for an algebraic function onto the z coordinate. With $\alpha_i \in \mathrm{PGL}_2(\mathbb{C})$, $i = 1, 2$, assume $\alpha_i \circ \varphi_{U_i} : \varphi^{-1}(U_i) \to \alpha_i^{-1}(U_i)$ is a manifold from the construction of Prop. 3.15, $i = 1, 2$. Then, X is a complex manifold extending the manifold structure on $\varphi^{-1}(U_i)$.*

Assume X is a manifold from Prop. 3.15. Let $\varphi : X \to U \subset \mathbb{P}^1_z$ be the algebraic function giving projection onto the z coordinate. Riemann's Existence Theorem (Chap. 4) produces a unique compact complex manifold $\bar{X}$ containing X as an open subset. We do this by extending φ to an analytic map $\bar{\varphi} : \bar{X} \to \mathbb{P}^1_z$. This is an abstract approach to *compactification*. It will help to see preliminary examples that relate compactifications and coordinates. In §4.2 we give these.

4.2. Compactifications and fiber products. Continue the notation for m and its branch points $\boldsymbol{z}$ from §3.3. Denote

$$\{w' \in \mathbb{P}^1_w \mid (z', w') \in X^{[0]},\ z' \notin \boldsymbol{z}\} \text{ by } U_{\mathrm{pr}_z^{-1}(\boldsymbol{z})}.$$

To further compactify we might embed the subset $X^{[0]}$ of $U_{\boldsymbol{z}} \times U_{\mathrm{pr}_z^{-1}(\boldsymbol{z})}$ into a compact space Z; then take the closure X of $X^{[0]}$ in Z. (Or apply to the already extended spaces $X^{[1]}$ or $X^{[2]}$.) As a closed subspace of compact space, X is compact.

4.2.1. *Local holomorphic functions from equations.* We note especially that equations give more than an (*implicit*) description of a point set. Using the implicit function theorem, they often give local parametrizing functions. In this section we use spaces Z to compactify that give natural local equations around points of the closure of $X^{[0]}$. Such equations help decide which points of the closure have extensions to the analytic structure on $X^{[0]}$ (or just manifold structure). This is an aspect of saying such Z provide *global coordinates*.

We need a notation for holomorphic functions compatible with §1.3 for the Laurent field $\mathcal{L}_{z'}$. We use $\mathcal{L}^h_{z'}$ for the ring of functions, with each holomorphic in some disk (dependent on the function) about z': power series $\sum_{n=0}^{\infty} a_n(z-z')^n$, convergent in some neighborhood of z'. For a general space X and point $x \in X$, the notation would be $\mathcal{L}^h_{X,x}$. For the holomorphic elements of $\mathcal{P}_{z',e}$ use $\mathcal{P}^h_{z',e}$.

We've been giving examples of point sets $\{(z,w) \mid m(z,w)=0\}$ in $\mathbb{C}\times\mathbb{C}$ using just one equation. Defining algebraic functions $f(z_1,\dots,z_n)$ in several variables is easy: Consider $X_m = \{(z_1,\dots,z_n,w) \mid m(z_1,\dots,z_n,w)=0\}$, and we say m algebraically defines $f(z_1,\dots,z_n)$, holomorphic in the variables $z_1,\dots,z_n$, if $m(z_1,\dots,z_n,f(z_1,\dots,z_n)) \equiv 0$. Also, the notation above extends to consider $\mathcal{L}^h_{z'_1,\dots,z'_n}$. Suppose $m(z'_1,\dots,z'_n,w')=0$, for $(z'_1,\dots,z'_n,w') \in \mathbb{C}^{n+1}$. Assume also that m defines $f(z_1,\dots,z_n)$ algebraically, and $f(z'_1,\dots,z'_n)=w'$. Then, we say the local holomorphic (or analytic) functions around $(z'_1,\dots,z'_n,w')$ consists of elements of the ring $\mathcal{L}^h_{z'_1,\dots,z'_n}$. This ring is invariant under analytic change of variables.

The next definition extends this to consider local holomorphic functions even with no a priori algebraic function f satisfying m. Recall the residue class map $\mathrm{rc}_{z'_1,\dots,z'_n,w'} : \mathbb{C}[z_1,\dots,z_n,w] \to \mathbb{C}$ by $(z_1,\dots,z_n,w) \mapsto (z'_1,\dots,z'_n,w')$. This is a ring homomorphism, and we record this in the form of the following. The completion of the ring $\mathbb{C}[z_1,\dots,z_n,w]/(m)$ at $(z'_1,\dots,z'_n)$ is

$$\mathcal{L}^h_{z'_1,\dots,z'_n}[z_1,\dots,z_n,w]/(m(z_1,\dots,z_n,w)) \stackrel{\text{def}}{=} \mathcal{L}^h_{X_m,z'_1,\dots,z'_n}.$$

Definition 4.4. Analytic functions on X_m around $(z'_1,\dots,z'_n,w')$ are elements of the localization of $\mathcal{L}^h_{X_m,z'_1,\dots,z'_n}$ at $w=w'$:

$$\mathcal{L}^h_{X_m,z'_1,\dots,z'_n,w'} \stackrel{\text{def}}{=} \{u/v \mid u \in \mathcal{L}^h_{X_m,z'_1,\dots,z'_n},\quad v \in \mathbb{C}[z_1,\dots,z_n,w], \\ \text{with } u(z'_1,\dots,z'_n,w') \neq 0.$$

Lemma 4.5. *Assume $z'_1,\dots,z'_n,w'$ is on X_m. Then $\mathrm{rc}_{z'_1,\dots,z'_n,w'}$ factors through $\mathbb{C}[z_1,\dots,z_n,w]/(m)$; even through $\mathcal{L}^h_{X_m,z'_1,\dots,z'_n,w'}$. This defines the value, $\mathrm{rc}_(s)$, of $s \in \mathcal{L}^h_{X_m,z'_1,\dots,z'_n,w'}$ at $(z'_1,\dots,z'_n,w')$. Assume the lead coefficient of $m(z,w)$ is invertible in $\mathcal{L}^h_{z'_1,\dots,z'_n}$. Use W_m for the set of distinct $w'=w$ solving $m(z'_1,\dots,z'_n,w)=0$ (allowing multiple zeros). There is a natural injective homomorphism*

$$\mathcal{L}^h_{X_m,z'_1,\dots,z'_n} \to \oplus_{w'\in W_m}\mathcal{L}^h_{X_m,z'_1,\dots,z'_n,w'}.$$

Definition 4.6 (Local holomorphic functions). Suppose $m(z'_1,\dots,z'_n,w')=0$, for $(z'_1,\dots,z'_n,w') \in \mathbb{C}^{n+1}$ and there are but finitely many solutions w to $m(z'_1,\dots,z'_n,w')=0$. Then, the local holomorphic (or analytic) functions that m defines consist of elements of $\mathcal{L}^h_{X_m,z'_1,\dots,z'_n,w'}[z_1,\dots,z_n,w]/(m(z_1,\dots,z_n,w)) = R$. We say this defines a manifold neighborhood if R is isomorphic to the convergent power series around a point of $\mathbb{C}^n$.

It is appropriate to say R is the *restriction* of local holomorphic functions on $\mathbb{C}^{n+1}$ to the set X_m around $(z'_1,\dots,z'_n,w')$. Further, the definition works as well if several equations, $m_1,\dots,m_u$, instead of just one, define the set.

4.2.2. $\mathbb{P}^1_z\times\mathbb{P}^1_w$ *compactification.* Since $Z=\mathbb{P}^1_z\times\mathbb{P}^1_w$ is a product of compact spaces, it is compact. Further, the compactification of $X^{[0]}$, if it is a manifold, suits the definition for $\mathbb{P}^1$-algebraic in (3.3).

The natural manifold structure on Z has four open sets in its atlas following Ex. 3.2.1. Label these $U_{i,z}\times U_{j,w}$, $1\le i,j\le 2$: $U_{1,z}=\mathbb{C}_z$ and $U_{2,z}=\mathbb{C}^*_z\cup\{\infty\}$, etc. The atlas gives an isomorphism of each of the four opens sets $U_{i,z}\times U_{j,w}$ with $\mathbb{C}\times\mathbb{C}$, by a map we call $\varphi_{i,j}$. Let $\bar X$ be the closure of $X^{[0]}$ in Z. We describe the part of $\bar X$ lying inside $U_{i,z}\times U_{j,w}$ by an algebraic equation. Then a previous procedure allows checking points at which X has a manifold structure.

Start with $\bar X\cap U_{2,z}\times U_{2,w}$, and leave the other open sets as analogous. On the open subset $\mathbb{C}^*\times\mathbb{C}^*\subset U_{2,z}\times U_{2,w}$, $\varphi_{2,2}$ acts as

$$(z,w)\mapsto(1/z,1/w)=(z',w').$$

An equation in (z',w') describes $\varphi_{2,2}$ applied to $X\cap(U_{2,z}\times U_{2,w})=X_{2,2}$: $\varphi_{2,2}(X_{2,2})$ is the closure of $\{(z',w')\mid m(1/z',1/w')=0\}$ in $\mathbb{C}_{z'}\times\mathbb{C}_{w'}$. Get the closure points by allowing z' or w' to go to 0. To include those limit values, multiply $m(1/z',1/w')$ by the minimal powers of z' and w' to clear the denominators.

EXAMPLE 4.7 (Continuation of Ex. 3.14). Continue with $m(z,w)=h(w)-z$ and $\deg(h)=n$. The set $\varphi_{2,2}(X_{2,2})$ is $\{(z',w')\mid z'h^*(w')-(w')^n=0\}$ where $h^*(w')=h(1/w')(w')^n$. Check that $X_{1,2}$ and $X_{2,1}$ have no new points beyond those already in $X_{1,1}$. Still, $X_{2,2}$ has a new point, corresponding to $(z',w')=(0,0)$. The gradient of $z'h^*(w')-(w')^n$ at zero is $(h^*(0),0)\ne(0,0)$. So, there is a manifold neighborhood of this point [9.10a].

4.2.3. *Tensor products and fiber products of* $\mathbb{P}^1$ *covers.* We combine two cases of Ex. 4.7. Suppose $m(z,w)=h(w)-g(z)$, a *variables separated* equation. Rename z to a variable w', and use z for the value $h(w)$. Rewrite $m(z,w)$ as $m(w',w)$.

Consider $(w',w)\in\mathbb{C}_{w'}\times\mathbb{C}_w$ satisfying $m(w',w)=0$. Call this X_m. Denote the Riemann surface for a function $w'(z)$ (resp. $w(z)$, as in Ex. 4.7) of z satisfying $h(w'(z))\equiv z$ (resp. $g(w)=z$) by $X_{w'}$ (resp. X_w). There is a map $\varphi_{w'}:X_{w'}\to\mathbb{P}^1_z$ by $w'\mapsto h(w')=z$. Similarly for a map φ_w.

Compare with Def. 1.3: X_m as a set is the same as the fiber product of these two maps. Now apply the $\mathbb{P}^1_{w'}\times\mathbb{P}^1_w$ compactification to $m(w',w)$. The resulting set is $\bar X_{w'}\times_{\mathbb{P}^1_z}\bar X_w=\bar X_m$. (In our example, $\bar X_{w'}=\mathbb{P}^1_{w'}$ and $\bar X_w=\mathbb{P}^1_w$.) This is the fiber product (over $\mathbb{P}^1_z$) of the compactifications of $X_{w'}$ and X_w from Ex. 4.7.

Now consider points of $\bar X_m$ to decide what are the natural local analytic functions in a neighborhood within one of the four charts for $\mathbb{P}^1\times\mathbb{P}^1$:

$$X_{i,j}=U_{i,z}\times U_{j,w},\ 1\le i,j\le 2. \tag{4.2}$$

For $(w'_0,w_0)\in\bar X_m$. Let $e_{w'_0}$ (resp. e_{w_0}) be the ramification index (Chap. 2 Def. 7.6) of w'_0 over $h(w'_0)=z_0$ (resp. w_0 over $g(w_0)=z_0$). New cases are with $e_{w'_0}=e'>1$ *and* $e_{w_0}=e>1$.

Local holomorphic functions in a neighborhood of (w'_0,w_0) that come from the coordinates w' and w are analytic in the solutions w' of $h(w')=z$ expanded about w'_0 and in the solutions w of $g(w)=z$ expanded about w_0. As usual, use ζ_d for the complex number $e^{2\pi i/d}$. Assume R is a ring, and S_1 and S_2 are two R algebras.

Then the tensor product $S_1 \otimes_R S_2$ is the natural direct sum of R algebras. That is, it is an R algebra T with R algebra homomorphisms $\psi_i : S_i \to T$, $i = 1,2$ ($\psi_1 : s_1 \in S_1 \mapsto s_1 \otimes 1$, etc.) and any such homomorphism will naturally factor through the map to $S_1 \otimes_R S_2$. As in Chap. 2 Cor. 7.5: $[e_1, e_2]$ is the least common multiple of e_1 and e_2; $u(z) = (z - z')^{1/[e_1,e_2]}$ is a choice of $[e_1, e_2]$th root of $z - z'$ (a generator of $\mathcal{P}_{z',[e_1,e_2]}$); and $\zeta_d = e^{2\pi i/d}$. Our first lemma is a famous consequence of the Euclidean algorithm.

LEMMA 4.8. *Assume K is a characteristic 0 field and $f \in K[x]$ is $\prod_{i=1}^{u} g_i(x)^{r_i}$ with $g_1, \dots, g_u$ irreducible and distinct monic polynomials over K. Then the natural map $\mu : K[x]/(f(x)) \to \oplus_{i=1}^{u} K[x]/(g_i^{e_i})$ by $h(x) \mapsto (h \mod (g_i^{e_1}), \dots, h \mod (g_i^{e_1})$ is an isomorphism.*

PROOF. Check that the kernel of μ trivial. So this linear vector space map, injects a space of dimension $\deg(f)$ into one of the same dimension $\sum_{i=1}^{u} e_i \deg(g_i)$. Conclude: μ is onto. ☐

PROPOSITION 4.9. *Suppose U is an open subset of $\mathbb{P}^1_z$, and $\varphi : X \to U$ is an analytic map of Riemann surfaces. For x' over $\varphi(x') = z'$ with ramification index $e_{x'/z'} = e$, $\mathcal{L}^h_{X,x'}$ is a natural $\mathcal{L}^h_{\mathbb{P}^1_z,z'}$ algebra that identifies with $\mathcal{P}^h_{z',e}$.*

Let $\varphi_i : X_i \to U$ be two such maps, with $x'_i \in X_i$ over z' having ramification index e_i, $i = 1,2$. Let $d = (e_1, e_2)$. Then, the ring of local holomorphic functions about (x_1, x_2) on $X_1 \times_{\mathbb{P}^1_z} X_2 = Y$ is $\mathcal{L}^h_{X_1,x'_1} \otimes_{\mathcal{L}^h_{\mathbb{P}^1_z}z'} \mathcal{L}^h_{X_2,x'_2}$. So $u^{e_1/d} = u_2$ (resp. $u^{e_2/d} = u_1$) is an e_2th (e_1th) root of $(z - z')$. Then, $\mathcal{L}^h_{Y,(x_1,x_2)}$ naturally identifies with $\mathcal{L}^h_{\mathbb{P}^1_z,z'}[u_1 \otimes 1, 1 \otimes u_2] = R$ (with $(u_1 \otimes 1)^{e_1} = z \otimes 1 = (1 \otimes u_2)^{e_2}$ according to the rules of tensoring over $\mathcal{L}^h_{\mathbb{P}^1_z}z'$). This ring has a single maximal ideal. There is an injective homomorphism

$$\mu : \mathcal{L}^h_{\mathbb{P}^1_z,z'}[u_1 \otimes 1, 1 \otimes u_2] \to \oplus_{j+1}^{d} \mathcal{L}^h_{\mathbb{P}^1_z,z'}[x,y]/(x^{e_1/d} - \zeta_d^j y^{e_2/d}, z = y^{e_2})$$

by $u_1 \otimes 1 \mapsto x$ and $1 \otimes u_2 \mapsto y$ in each coordinate. Each summand on the right of (4.9) *is an integral domain whose quotient field naturally identifies with $\mathcal{P}_{z',[e_1,e_2]}$.*

Then, R is an integral domain if and only if $d = 1$, and the image of μ in each summand is a proper subring of the summand unless one of e_i/d is 1. Conclude: Restricting local holomorphic functions on $X_1 \times X_2$ defines an analytic manifold structure around (x'_1, x'_2) if and only if one of the e_i s is 1. Yet, the image of μ generates the quotient field of each summand.

PROOF. According to Def. 4.1, by rewriting φ using local analytic coordinates $z_{x'}$ and $z_{z'}$ around x' and z', we get a very simple normal form. A local analytic change of variables identifies $z_{x'}$ with one of the solutions of $u^e = z_{z'}$. Chap. 2 Cor. 7.5 shows this when φ is given by an algebraic function. Chap. 4 (proof of Lem. 2.1) shows it is not dependent on a priori knowing φ is algebraic. That gives the first paragraph in the lemma.

Now consider φ_i, $i = 1,2$, in the statement of the lemma. From above, identify an analytic coordinate around $x_i(z)$ around x'_i with $(z - z')^{1/e_i}$ and the map φ_i with the e_ith power map, $i = 1,2$. The only relations among $u_1 \otimes 1$ and $1 \otimes u_2$ are generated by $(u_1 \otimes 1)^{e_1} = z \otimes 1 = (1 \otimes u_2)^{e_2}$ and the kernel of the map μ is in the ideal generated by this relation.

If $d > 1$, then $(u_1 \otimes 1)^{e_1/d} - \zeta_d^j (1 \otimes u_2)^{e_2/d}$ divides $(u_1 \otimes 1)^{e_1} - (1 \otimes u_2)^{e_2} = z$.

Replace $\mathcal{L}^h_{\mathbb{P}^1_z,z'}$ by $\mathcal{L}_{z'}(y) = K$, a field (leaving x as a variable). Then applying Lem. 4.8 to μ actually gives an isomorphism. The corresponding summands on the right side of (4.9) would be fields identified with the quotient fields of the summands on the right side of the actual (4.9). So, to finish the result we have only to show the quotient field of the summand $\mathcal{L}^h_{\mathbb{P}^1_z,z'}[x,y]/(x^{e_1/d} - \zeta_d^j y^{e_2/d}, z = y^{e_2})$ identifies with $\mathcal{P}_{z',[e_1,e_2]}$, though the summand itself is a proper subring of the locally holomorphic functions in $(z-z')^{1/[e_1,e_2]}$ [9.11b]. □

Now apply Prop. 4.9 to (4.2).

Corollary 4.10. *Restricting local holomorphic functions on* $\mathbb{P}^1_{w'} \times \mathbb{P}^1_w$ *to the fiber product* $\mathbb{P}^1_{w'} \times_{\mathbb{P}^1_z} \mathbb{P}^1_w$ *compactification gives an analytic manifold structure around* (w'_0, w_0) *if and only if* $(e'_{w_0}, e_{w_0}) = 1$.

Remark 4.11 (simplifying the use of Prop. 4.9). Riemann's Existence Theorem gives a unique compact manifold by completing a cover of $U_{\boldsymbol{z}}$. In so doing, it computes precisely what to expect when you take the fiber product of two ramified covers of $\mathbb{P}^1_z$ (over of any other Riemann surface). Chap. 4 §3.3.2 shows the combinatorial result of getting d distinct points on the correctly compactified fiber product (ramified of order $[e_1, e_2]$ over z') over the pair (x'_1, x'_2) is built transparently into the use of branch cycles. Since, however, fiber products (and tensor products) are so important, Prop. 4.9 gives a relatively simple example readers may return to for help with other examples.

4.3. $\mathbb{P}^n$ **compactifications.** Denote the origin in $\mathbb{C}^{n+1}$ by 0. There is an action of $\mathbb{C}^*$ on $\mathbb{C}^{n+1} \setminus \{0\}$. Given a nonzero vector $\boldsymbol{v} = (v_0, \dots, v_n) \in \mathbb{C}^{n+1}$ and $\alpha \in \mathbb{C}^*$ form the result of scalar multiplication $\alpha \cdot \boldsymbol{v} = (\alpha v_0, \dots, \alpha v_n)$. *Projective n-space* is a quotient definition like that of a complex torus: $\mathbb{P}^n = \mathbb{C}^{n+1} \setminus \{0\}/\mathbb{C}^*$. Mapping $\boldsymbol{v}$ to the set equivalent to $\boldsymbol{v}$ gives $\Gamma_n : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$.

4.3.1. *An atlas on* $\mathbb{P}^n$. In this form, it can be convenient (though cumbersome) to label $\mathbb{P}^n$ as either $\mathbb{P}^n_{v_1/v_0,\dots,v_n/v_0}$ (*inhomogeneous* coordinates) or $\mathbb{P}^n_{v_0,\dots,v_n}$ (*homogenous* coordinates). The extra notation means we have added data for a standard set of coordinate functions for $\mathbb{P}^n$. Algebraic geometry texts might refer to a manifold analytically isomorphic to this manifold as $\mathbb{P}^n$. Still, there is a significance to adding specific coordinates as Chap. 5 does. To practice this distinction try [9.11e]. Taking $n = 1$ and $v_1/v_0 = z$ gives the notation for $\mathbb{P}^1_z$ from Chap. 2.

Standard coordinates on $\mathbb{P}^n$ produce standard transition functions for its manifold structure. Typical of forming an object by an equivalence relation, each point of $\mathbb{P}^n$ is a set in $\mathbb{C}^{n+1}$. As some coordinate is not 0, such a point has a *representative* with some coordinate equal 1. If you tell which coordinate that is, the representative will be unique.

Let U_i be the points with representative having 1 in the ith position. Each point of $\mathbb{P}^n$ has a representative in U_i for some i. Projecting U_i onto coordinates different from the ith gives a coordinate chart $\varphi_i : U_i \to \mathbb{C}^n$, $i = 0, \dots, n$. If $\boldsymbol{v}$ is any other representative of a point in U_i, first scale it by $1/v_i$ before this projection.

Lemma 4.12. *The atlas* $\{U_i, \varphi_i\}_{i=1}^n$ *makes* $\mathbb{P}^n$ *a compact dimension n complex manifold. The map* $\mathbb{C}^{n+1} \setminus \{\mathbf{0}\} \to \mathbb{P}^n$ *is a map of analytic manifolds.*

Proof. An explicit computation of the transition function

$$\varphi_j \circ \varphi_i^{-1} : \mathbb{C}^n_{v_0,v_1,\dots,v_{i-1},v_{i+1},\dots,v_n} \to \mathbb{C}^n_{v_0,v_1,\dots,v_{j-1},v_{j+1},\dots,v_n}$$

is easy. If $i = j$ it is the indentity. Otherwise, it maps $(v_0, v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n)$ to $1/v_j(v_0, v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_n)$ (with $v_i = 1$). It is analytic on $\varphi_i(U_i \cap U_j)$.

To see $\mathbb{P}^n$ is compact, use the standard absolute value $|v|$ on $\mathbb{C}$. Let $\mathbb{C}_c^{n+1}$ be the vectors $\boldsymbol{v}$ with $\max_{i=0}^n(|v_i|) \le 1$. This is a closed bounded subset of $\mathbb{C}^{n+1}$. So, by the Heine-Borel compactness theorem, it is compact. Every point of $\mathbb{P}^n$ has a representative in $\mathbb{C}_c^{n+1}$: Scale it by the largest nonzero entry. Now use that the image of a compact set under a continuous map is compact. An alternate could use this characterization of compactness: Infinite sequences of points in a separable metric space have convergent subsequences [9.10b].

The diagonal in $\mathbb{P}^n \times \mathbb{P}^n$ is the image of a compact subset of the diagonal in $\mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$. Though the image is compact, until we know $\mathbb{P}^n$ is Hausdorff we can't invoke Lem. 2.5 to see the image is closed. Here, however, a direct argument can establish that $\mathbb{P}^n$ is Hausdorff. Suppose two points are in one of the U_i *s*, a copy of $\mathbb{C}^n$. As this is Hausdorff, separate the two points by open sets. So, given any two points it suffices to change coordinates to assure, in the new coordinates, these are both in one of the U_i s. Do that choosing a linear combination $L_{\boldsymbol{a}} = \sum_{i=0}^n a_i v_i$ so neither point lies on the zero set of $L_{\boldsymbol{a}}$. Use $L_{\boldsymbol{a}}$ in place of v_j as one of the new coordinates for any j for which $a_j \ne 0$.

Use $\Gamma_n^{-1}(U_i) = V_i \subset \mathbb{C}^{n+1}$ and the same transition functions for a coordinate chart on $\mathbb{C}^{n+1}$. This shows Γ_n is a map of complex manifolds. □

4.3.2. $\mathbb{P}^2_{z,w,u}$ *compactifications.* As in §4.2.2, let $Z' = \mathbb{P}^2_{z,w,u}$. Embed $\mathbb{C}_z \times \mathbb{C}_w$ in this by $\varphi_u^{-1} : (z, w) \mapsto (z, w, 1) \mod \mathbb{C}^* \in Z'$. Call the image U_u. Similarly, let U_w be points of $\mathbb{P}^2_{z,w,u}$ with a representative of form $(z, 1, u)$ and U_z points with a representative of form $(1, w, u)$. Take X' to be the closure of $\{(z, w) \mid m(z, w) = 0\}$ in the compact space Z'. To check points of X' for a manifold neighborhood requires an equation around each point of X'. It suffices to *define* this equation for points of $X' \cap U_z$ and $X' \cap U_w$. We do the former; the latter is similar.

Since φ_z identifies U_z with $\mathbb{C}_w \times \mathbb{C}_u$, it suffices to define the image of $X' \cap U_z$ under φ_z. With n' the total degree of m, it is

$$X'_z = \{(w, u) \mid u^{n'} m(1/u, w/u).$$

4.3.3. *Hyperelliptic curves.* Suppose $\varphi : X \to \mathbb{P}^1_z$ is a degree 2 map of compact Riemann surfaces. Let $\boldsymbol{z}$ be the finite set of branch points (as in Chap. 4 Lem. 2.1). The theme of Chap. 2 §8 is that we already know, from branches of log, what are the abelian covers of $U_{\boldsymbol{z}} = U$ (see Chap. 4 Prop. 2.11). That is, $\pi_U : X_U \to U_{\boldsymbol{z}}$ is equivalent to the cover defined by a branch of square root of $h(z) \in \mathbb{C}(z)$. Also, h has multiplicity one zeros and poles contained in $\boldsymbol{z}$ (Chap. 2 (6.2)): φ is a cover from a branch of solutions $f(z)$ of $m(z, w) = w^2 - h(z)$ with $h(z) = \frac{\prod_{i=1}^t (z-z_i)}{\prod_{j=t+1}^r (z-z_j)}$.

Suppose the z_i s are distinct, and all different from 0 or ∞ ($r = 2t$ so the degrees of the numerator and denominator are the same). Then, according to Prop. 4.9, this is an if and only if condition that for a manifold compactification given by the fiber product embedding in $\mathrm{pr}^1_z \times \mathbb{P}^1_w$. This is good, yet the standard normalization of hyperelliptic curves changes the variables so that h is a polynomial. Do this by multiplying both sides by the square of the denominator, then change the variable w to $w \prod_{j=t+1}^r (z - z_j)$. For simplicity we keep the name of the variables the same. So, now consider the equation $wr = h(z)$ where $h = \prod_{i=1}^r (z - z_i)$. Here r is even, and we assume it is at least 4. Another common normalization is make the changes

$z \mapsto z_1 + 1/z$ and $w \mapsto w/z$, thereby replacing h by a polynomial having odd degree $r \geq 3$. As it stands let us consider the $\mathbb{P}^2_{z,w,u}$ compactification.

Then, $X'_u = \{(z,w) \mid w^2 - h(z) = 0\}$ has a manifold neighborhood around each point: $\nabla(m) = 0$ implies $w = 0$ and $\frac{dh}{dz} = 0$ (z is a repeated root of h). From above,

$$
\begin{aligned}
X'_w = &\ \{(z,u) \mid u^{n-2} - u^n h(z/u) = m^{(w)}(z,u) = 0\} \text{ and}\\
X'_z = &\ \{(w,u) \mid u^{n-2}w^2 - u^n h(1/u) = m^{(z)}(w,u) = 0\}.
\end{aligned}
\tag{4.3}
$$

On X'_w new points (not already represented on X'_u) have $u = 0$ and $z = 0$. For $r > 3$, $\nabla(m^{(w)})(0,0) = 0$. So, it has no manifold neighborhood. Note this contrasts with the $\mathbb{P}^1_z \times \mathbb{P}^1_w$ compactification of m, in which all points have manifold neighborhoods when you use the right algebraic change of coordinates [9.11c]. For $r = 3$, however, the point $(0,0)$ has a manifold neighborhood in $\mathbb{P}^2$. There are no new points on X'_z; $u = 0$ gives no solution in w to $m^{(z)}(w,u) = 0$.

4.3.4. *Coordinates give meromorphic functions.* Let $\bar{X}$ be the $\mathbb{P}^1_z \times \mathbb{P}^1_w$ compactification (§4.2.2) of $X = \{(z,w) \mid m(z,w) = 0\}$ with $m \in \mathbb{C}[z,w]$. Assume every point of $\bar{X}$ has a manifold neighborhood in this compactification. Then, every point of $\bar{X}$ has the form $(z,w) \in \mathbb{P}^1_z \times \mathbb{P}^1_w$. Thus, projection of (z,w) onto z (or onto w) provides a meromorphic function on $\bar{X}$.

Similarly, suppose $\bar{X}$ is the $\mathbb{P}^2_{z,w,u}$ compactification (§4.3.2) of X and every point of $\bar{X}$ has manifold neighborhood. Then, many meromorphic functions come from this compactification. A linear form in (z,w,u) is a nonzero linear combination of z, w, u (like $L_{\boldsymbol{a}}$, used in the proof of Lem. 4.12). Assume $\bar{X}$ is not in the zero set of any linear form. For example, suppose $m(z,w)$ is irreducible and has total degree $n > 1$.

Proposition 4.13. *Let L_1 and L_2 be linear forms in (z,w,u), not multiples of one another. Let (z_0,w_0,u_0) represent the unique point of intersection of the zero sets of L_1 and L_2. Then, with $z' = L_1(z,w,u)/L_2(z,w,u)$, there is a natural (nonconstant) meromorphic function $\bar{\varphi} : \bar{X} \to \mathbb{P}^1_{z'}$. The degree of $\bar{\varphi}$ is n if $(z_0,w_0,u_0) \notin \bar{X}$ and $n-1$ otherwise.*

Proof. Give the map by $(z,w,u) \in \bar{X} \mapsto L_1(z,w,u)/L_2(z,w,u)$. We verify this map is well-defined. If $(z_0,w_0,u_0) \notin \bar{X}$, then meaningfully assign a value $z' \in \mathbb{C} \cup \{\infty\}$ to the evaluation of L_1/L_2 at any point of $\bar{X}$. Let $H_{z'_0}$ be the line in $\mathbb{P}^2$ given as the zero set of $L_{z'_0} = L_1 - z'_0 L_2$. To see the degree, check the number of points in the intersection of $H_{z'_0}$ and $\bar{X}$ if z'_0 is suitably general. This is n. These are exactly the points that go to z'_0.

On the other hand, suppose $(z_0,w_0,u_0) \in \bar{X}$. Then each $H_{z'_0}$ goes through (z_0,w_0,u_0). If z'_0 is general, $L_1(z,w,u)/L_2(z,w,u)$ has a clear ratio value at the $n-1$ points other than (z_0,w_0,u_0). So, this gives a map of degree $n-1$ of $\bar{X} \to \mathbb{P}^1_{z'}$. Check: For only one value z'_0 is $H_{z'_0}$ tangent to $\bar{X}$ at (z_0,w_0,u_0) because we assumed $\bar{X}$ is nonsingular [9.11f]. Interpret such a z'_0 as having (z_0,w_0,u_0) above it. $\square$

5. Paths, vectors and forms

Notation for paths started in Chap. 2 §2.2. Let X be a topological space. A path in X is a continuous $\gamma : [a,b] \to X$ for some choice of a and b with $a < b$. The points $\gamma(a)$ and $\gamma(b)$ are, respectively, the *initial* and *end* points of the path. The path γ is *closed* if $\gamma(a) = \gamma(b)$.

The idea a path being piecewise differentiable (simplicial) works if X is an n-dimensional differentiable manifold (or, more generally, a finite union of differentiable manifolds), with topologizing data $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in I}$. Then, γ is differentiable if $\frac{d}{dt}(\varphi_\alpha \circ \gamma(t)) = \boldsymbol{v}_\alpha(t)$ exists for each $t \in [a, b]$ (use one-sided limits at the endpoints) and each $\alpha \in I$ with $\gamma(t) \in U_\alpha$. The vector $\boldsymbol{v}_\alpha(t)$ is the *tangent vector* to γ at t with respect to $(U_\alpha, \varphi_\alpha)$. It depends only on γ close to t.

As in Chap. 2, simplicial paths support applications to integration, and to forming convenient analytic continuations of functions. Still, it is awkward to analyze *homotopy classes* of paths without allowing paths that are only continuous in the homotopy (see Prop. 6.10).

5.1. Tangent vectors. The above formulation presents a tangent vector as something attached to a path. We recognize a tangent vector at a point x_0 without having a path through the point. Let $\mathcal{C}^\infty_{x_0} = \mathcal{C}_{x_0,X}$ be functions, differentiable and complex valued, defined in some neighborhood of x_0.

DEFINITION 5.1. A (complex valued) tangent vector to a differentiable manifold X at a point x_0 is a linear map $\boldsymbol{v} : C^\infty_{x_0} \to C^\infty_{x_0}$ satisfying Leiznitz's rule:

$$\boldsymbol{v}(f_1 f_2)(x_0) = \boldsymbol{v}(f_1)(x_0) f_2(x_0) + (f_1)(x_0)\boldsymbol{v}(f_2)(x_0). \tag{5.1}$$

That is, $\boldsymbol{v}$ is a derivation of $\mathcal{C}_{x_0}$ defined at x_0.

5.1.1. *Tangent vectors and paths.* To relate to tangent vectors attached to a path, assume $x_0 \in U_\alpha$. A function f in a neighborhood of x_0 defines a function $f \circ \varphi_\alpha^{-1}$ on a neighborhood of $\varphi_\alpha(x_0) \in \mathbb{R}^n$. Denote the variables of $\mathbb{R}^n$ here by $\boldsymbol{y} = (y_1, \dots, y_n)$. Consider $F : \mathbb{R}^n \to \mathbb{R}^n$ by $\boldsymbol{y} \mapsto (F_1(\boldsymbol{y}), \dots, F_n(\boldsymbol{y}))$. Suppose each coordinate function $F_i(\boldsymbol{y})$ has continuous partial derivatives. The *Jacobian matrix* $J(F)$ of F is the $n \times n$ matrix with (i, j)-entry $\frac{\partial F_i}{\partial y_j}$ at the point $\boldsymbol{y}$.

LEMMA 5.2. [**Rud76**, p. 214] *Identify derivations of functions* $f \in \mathcal{C}_{\varphi_\alpha(x_0),\mathbb{R}^n}$ *with linear combinations* $T_{\boldsymbol{v}} = \sum_{i=1}^n v_i \frac{\partial}{\partial y_i}$, $v_1, \dots, v_n \in \mathcal{C}_{\varphi_\alpha(x_0),\mathbb{R}^n}$.

So, $T_{\boldsymbol{v}}(f)(\varphi_\alpha(x_0))$ *is the directional derivative of* f *in the direction* $\boldsymbol{v}(\varphi_\alpha(x_0))$.

For $\gamma(t) \in U_\alpha \cap U_\beta$, *the chain rule relates* $\boldsymbol{v}_\alpha(t)$ *and* $\boldsymbol{v}_\beta(t)$:

$$\big(J(\varphi_\beta \circ \varphi_\alpha^{-1})_{|(\varphi_\alpha\circ\gamma)(t)}\big)(\boldsymbol{v}_\alpha(t)) = \boldsymbol{v}_\beta(t). \tag{5.2}$$

So, $\boldsymbol{v}_\alpha(t)$ *is nonzero if and only if* $\boldsymbol{v}_\beta(t)$ *is nonzero. To check if* γ *has a nonzero tangent vector doesn't depend on the choice of* $(U_\alpha, \varphi_\alpha)$.

5.1.2. *Vector fields.* A *vector field* T_U on an open set U in a (differentiable) manifold X is a differentiable assignment of derivations at each point of U. A formal definition shows the effect of transition functions from an atlas. Sometimes it is confusing to use $\boldsymbol{y}$ for variables of *all* copies of $\mathbb{R}^n$. So, we use $\boldsymbol{y}_\alpha = (y_{\alpha,1}, \dots, y_{\alpha,n})$ for variables in the range of φ_α.

DEFINITION 5.3. Assume $\{U_\alpha, \varphi_\alpha\}_{\alpha\in I}$ is an atlas for the differentiable manifold X. Then, T_U consists of giving $T_\alpha = \sum_{i=1}^n f_{\alpha,i} \frac{\partial}{\partial y_{\alpha,i}}$ with the $f_{\alpha,i}$ s differentiable functions on $V_\alpha = \varphi_\alpha(U_\alpha)$, for each $\alpha \in I$, subject to the following rule. Assume $U_\alpha \cap U_\beta$ is nonempty. Consider any differentiable function $f : U_\alpha \to \mathbb{R}^n$. Use the same notation T_α for the restriction of T_α to $\varphi_\alpha(U_\alpha \cap U_\beta)$. Here is a relation between T_α and T_β on $U_\alpha \cap U_\beta$:

$$T_\alpha(f \circ \varphi_\alpha^{-1}(y_{\alpha,1}, \dots, y_{\alpha,n})) = T_\beta(f \circ \varphi_\beta^{-1}(y_{\beta,1}, \dots, y_{\beta,n})). \tag{5.3}$$

Apply $(\frac{\partial}{\partial y_{\beta,1}}, \dots, \frac{\partial}{\partial y_{\beta,n}})$ to $f \circ \varphi_\alpha^{-1}(\boldsymbol{y}_\beta) = f_\alpha(\boldsymbol{y}_\alpha)$ to get a *gradient* vector of $(f_{\alpha 1}, \dots, f_{\alpha n})(\boldsymbol{y}_\alpha)$ functions. A traditional expression rewrites (5.3) as

$$J(\psi_{\boldsymbol{y}_\beta,\boldsymbol{y}_\alpha})^{-1}\big(\frac{\partial}{\partial y_{\alpha,1}}, \dots, \frac{\partial}{\partial y_{\alpha,n}}\big) = \big(\frac{\partial}{\partial y_{\beta,1}}, \dots, \frac{\partial}{\partial y_{\beta,n}}\big) \tag{5.4}$$

applied to $f(\psi_{\beta,\alpha}(\boldsymbol{y}_\alpha))$ [9.14c]. Thus, (5.3) translates to a linear relation between $(f_{\alpha,1}, \dots, f_{\alpha,n})(\boldsymbol{y}_\alpha))$ and $(f_{\beta,i}, \dots, f_{\beta,n})(\psi_{\beta,\alpha}(\boldsymbol{y}_\alpha))$ [9.14].

So, a chart produces a preferred basis for vector fields and a preferred basis for differential 1-forms from the coordinate functions for the chart.

Definition 5.4. As in Chap. 2 Def. 2.1, $\gamma : [a, b] \to X$ is *simplicial* if there is an integer n and $t_0 = a < t_1 < \cdots < t_{n-1} < t_n = b$ with $\gamma_{|[t_i,t_{i+1}]}$ differentiable, $i = 0, \dots, n-1$. Also, γ is *special simplicial* if either $\frac{d}{dt}(\gamma(t))$ is identically zero for $t \in (t_i, t_{i+1})$ or it is nonzero for *each* $t \in (t_i, t_{i+1})$, $i = 0, \dots, n-1$. A space X is *simplicially connected* if, for each pair $x_0, x_1 \in X$, there is a simplicial path $\gamma : [a, b] \to X$ with $\gamma(a) = x_0, \gamma(b) = x_1$.

Lemma 5.5 (Integrating vector fields). *Let T_U be a vector field on the open set U of the differentiable manifold X. For each $u_0 \in U$ there exists $\epsilon > 0$ and a unique differentiable path $\gamma : [-\epsilon, \epsilon] \to U$, with $\gamma(0) = u_0$, so the following holds. The derivation $T_{U,\gamma(t)}$ at $\gamma(t)$ is the directional derivative of γ at $t \in [-\epsilon, \epsilon]$.*

Proof. With no loss, assume u_0 is in an atlas element U_α. We summarize the meaning of the lemma using the previous notation $\varphi_\alpha : U_\alpha \to \mathbb{C}^n$.

Let $\boldsymbol{y}$ be coordinates on $\mathbb{R}^n \supset \varphi_\alpha(U_\alpha)$. Use the path $t \mapsto \varphi_\alpha \circ \gamma(t) = \gamma^*(t)$. By definition, T_{U_α} is an expression $\sum_{i=1}^n f_{\alpha,i}\frac{\partial}{\partial y_i}$. The lemma says there is $\gamma^*(t)$ so $\frac{d\gamma_i^*}{dt}(t) = f_{\alpha,i}(\gamma^*(t))$, $i = 1, \dots, n$.

Many books quote this result ([**Hi65**, p. 12], for example) by referring to the existence and uniqueness of solutions to ordinary differential equations. The path in U_α is then $\varphi_\alpha^{-1}(\gamma^*(t))$. All general proofs we've seen use fixed point arguments and involve considerable detail, as in the exercises of [**Rud76**, p. 118, #25–29, p. 170, #25-26] giving uniqess and existence under all conditions that would come up for us. Analytic dependence of the solutions on u_0 is considered more difficult (see [**Bo86**, p. 171-174, Thm. 4.1]). □

Suppose T_U is a vector field on U and $\gamma : [a, b] \to U$ is a differentiable path. Then, call γ an *integral* curve of T_U. With some assumptions there is a useful converse producing T_U from a path. [9.13].

5.2. Holomorphic vector fields and differential forms. Analogs of differentiable vector fields reflect the complex structure on a manifold X. The main example from Def. 5.3 has V_α as $T_\alpha = \sum_{i=1}^n f_{\alpha,i}\frac{\partial}{\partial z_{\alpha,i}}$ with the $f_{\alpha,i}$s holomorphic in the complex coordinates $z_{\alpha,i}$, $i = 1, \dots, n$. Though T_α initially only applies to functions analytic in $(z_{\alpha,1}, \dots, z_{\alpha,n})$, we may extend it to all differentiable functions taking complex values.

5.2.1. *Extend T differentiably.* Let $\boldsymbol{z} = (z_1, \dots, z_n)$ be the coordinate functions on $\mathbb{C}^n$. Write $z_j = x_j + iy_j$ and $\bar z_j = x_j - iy_j$, breaking the coordinates into their real and imaginary parts. Then, $x_j = \frac{1}{2}z_j + \bar z_j$ and $y_j = \frac{1}{2i}z_j - \bar z_j$. Define $\frac{\partial}{\partial z_j}$ on holomorphic functions $f(z_1, \dots, z_n)$ as the jth *partial* derivative with respect to the variables $z_1, \dots, z_n$. The partials $\frac{\partial}{\partial x_j}$ and $\frac{\partial}{\partial y_j}$ act on any differentiable functions of the variables $x_1, \dots, x_n, y_1, \dots, y_n$ (see Chap. 2 Lem. 2.6).

LEMMA 5.6. *The operator* $\frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j})$ *maps* z_j *to 1,* z_k *to 0 for* $k \neq j$*. Further, it maps* $\bar{z}_l$ *to 0 for all* l*. So, it extends* $\frac{\partial}{\partial z_j}$ *to act as previously on holomorphic functions, and to kill anti-holomorphic functions. Similarly,* $\frac{1}{2}(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j})$ *extends* $\frac{\partial}{\partial \bar{z}_j}$ *from anti-holomorphic functions to all differentiable functions.*

5.2.2. *Vector fields in local coordinates.* Suppose T_α and T_β are the expressions for a holomorphic vector field on two coordinate charts. Interpret the relation between the $f_{\alpha,i}$ s and $f_{\beta,j}$ s given by the complex version of the Jacobian of the transition functions. So, for X a 1-dimensional complex manifold, the equation relating $f_\alpha(z_\alpha)\frac{\partial}{\partial z_\alpha}$ and $f_\beta(z_\beta)\frac{\partial}{\partial z_\beta}$ comes from expecting the same value upon application of both to $z_\beta = \psi_{\beta,\alpha}(z_\alpha)$:

$$f_\beta(\psi_{\beta,\alpha}(z_\alpha)) = f_\alpha(z_\alpha)\frac{\partial \psi_{\beta,\alpha}}{\partial z_\alpha}. \tag{5.5}$$

5.2.3. *Differential 1-forms.* Now consider the collection of differential 1-forms Ω_U defined on an open set U in a differentiable manifold X. Use notation of §5.1.2 analogous to that for vector fields. As in §Chap. 2 2.3 our motivation is to form integrals of $\omega_U \in \Omega_U$ along any piecewise differentiable path in U.

DEFINITION 5.7. Such an ω_U comes by giving $\omega_\alpha = \sum_{i=1}^n g_{\alpha,i}\, dy_{\alpha,i}$ with the $g_{\alpha,i}$ s differentiable functions on $V_\alpha = \varphi_\alpha(U_\alpha \cap U)$, for each $\alpha \in I$, subject to the following rule. If $V_\alpha \cap V_\beta$ is nonempty, denote restriction of ω_α to $\varphi_\alpha(V_\alpha \cap V_\beta)$ also by ω_α and let $\gamma : [a,b] \to V_\alpha \cap V_\beta$ be a differentiable path. Then,

$$\int_{\varphi_\alpha \circ \gamma} \omega_\alpha = \int_{\varphi_\beta \circ \gamma} \omega_\beta. \tag{5.6}$$

Equation (5.6) translates to a linear relation between $(g_{\alpha,1}, \dots, g_{\alpha,n})(\boldsymbol{y}_\alpha)$ and $(g_{\beta,i}, \dots, g_{\beta,n})(\psi_{\beta,\alpha}(\boldsymbol{y}_\alpha))$. This formula applies with $\gamma_{[t,t+\epsilon]}$ (restriction of γ to $[t, t+\epsilon]$) replacing γ for any value of $t \in [a,b]$ and $\epsilon > 0$. So, it gives equality of the integrands as a function of t.

DEFINITION 5.8 (Contraction). Suppose T_U is a vector field defined on U. Use the previous notation for expressing T_U on V_α: $T_\alpha = \sum_{i=1}^n f_{\alpha,i}\frac{\partial}{\partial y_{\alpha,i}}$. The *contraction* of T_α and ω_α is the function $\sum_{i=1}^n f_{\alpha,i} g_{\alpha,i}$. Denote it by $\langle T_\alpha, \omega_\alpha\rangle$. More generally, the *contraction* $\langle T_U, \omega_U\rangle$ of T_U and ω_U is $F \in C_U^\infty$ with this property.

(5.7) $F \circ \varphi_\alpha^{-1}(\boldsymbol{y}_\alpha) = \langle T_\alpha, \omega_\alpha\rangle$ on $\varphi_\alpha(V_\alpha)$, for each $\alpha \in I$.

LEMMA 5.9. *As above,* $F \circ \varphi_\alpha^{-1}$ *at* $\varphi_\alpha(x)$ *does not depend on* α *and the contraction* $\langle T_U, \omega_U\rangle$ *is a differentiable function on* U*. Further, the vector of differentials* $(dy_{\beta,1}, \dots, dy_{\beta,n})$ *evaluated at* $\psi_{\beta,\alpha}(\boldsymbol{y}_\alpha)$ *is* $J(\psi_{\beta,\alpha})(dy_{\alpha,1}, \dots, dy_{\alpha,1})$*.*

PROOF. By explicit computation using Lemma 5.2, $f \circ \varphi_\alpha^{-1}$ is the integrand of the left of (5.6). The comment following (5.6) shows this equals the contraction for β evaluated at $\psi_{\beta,\alpha}(\boldsymbol{y}_\alpha)$. To conclude the proof use the vector field formula [9.14c]. Contract each side with the differentials $dy_{\beta,j}$ to see the transformation formula for differentials is inverse to that for vector fields. □

5.2.4. *Tensors.* Suppose $\{U_\alpha, \varphi_\alpha\}_{\alpha\in I}$ is an atlas for a differentiable manifold X. On each U_α let $\mathbb{T}^0_{U_\alpha}$ (resp. $\mathbb{D}^0_{U_\alpha}$) be the tensor algebra over $C^\infty(U_\alpha)$ generated by tangent vectors (resp. differential 1-forms) on U_α. By definition that means elements of $\mathbb{T}^0_{U_\alpha}$ are finite sums of terms $gT_1 \otimes T_2 \otimes \cdots \otimes T_k$ with k any nonnegative integer, $g \in C^\infty(U_\alpha)$ and $T_1, \dots, T_k$ tangent vectors on U_α. If $k = 0$, the element is just the function g.

Suppose $h_1, h_2 \in C^\infty$ and $T_i^{(1)}$ and $T_i^{(2)}$ are tangent vectors on U_α. Further, interpret the tensor sign $\otimes$ to be a formal symbol modulo the following relations. Replacing T_i by $h_1T_i^{(1)} + h_2T_i^{(2)}$ replaces $gT_1 \otimes \cdots \otimes T_i \otimes \cdots \otimes T_k$ by the sum

$$gh_1T_1 \otimes \cdots \otimes T_i^{(1)} \otimes \cdots \otimes T_k + gh_2T_1 \otimes \cdots \otimes T_i^{(2)} \otimes \cdots \otimes T_k.$$

There are two things to note:

(5.8a) Unless it follows from these allowed relations, we do not expect $T_1 \otimes T_2$ to equal $T_2 \otimes T_1$.

(5.8b) Declaring $T_1 \otimes \cdots \otimes T_k$ times $T'_1 \otimes \cdots \otimes T'_{k'}$ (in that order) to be $T_1 \otimes \cdots \otimes T_k \otimes T'_1 \otimes \cdots \otimes T'_{k'}$ generates an associative ring multiplication on $\mathbb{T}^0_{U_\alpha}$.

Similarly for $\mathbb{D}^0(U_\alpha)$. Both have $C^\infty(U_\alpha)$ as a subring acting by multiplication on each element of $\mathbb{T}^0_{U_\alpha}$ (or $\mathbb{D}^0_{U_\alpha}$): These are *associate algebras* over $C^\infty(U_\alpha)$. We may even tensor together elements of $\mathbb{T}^0_{U_\alpha}$ and $\mathbb{D}^0_{U_\alpha}$ for a bigger algebra $\mathbb{T}^0_{U_\alpha} \otimes \mathbb{D}^0_{U_\alpha}$. In this convention, however, we can distinguish between tangent vectors and differential forms, and typically we pass all the tangent vectors to the left.

A subtlety occurs in comparing elements $\omega_\alpha \in \mathbb{T}^0_{U_\alpha} \otimes \mathbb{D}^0_{U_\alpha}$ and $\omega_\beta \in \mathbb{T}^0_{U_\alpha} \otimes \mathbb{D}^0_{U_\alpha}$ on the intersection $U_\alpha \cap U_\beta$. Use the transition function $\varphi_\beta \circ \varphi_\alpha^{-1}$ to reexpress ω_β in the variables $y_{\alpha,1}, \dots, y_{\alpha,n}$ for $(U_\alpha, \varphi_\alpha)$ as previously for 1-forms (and vectors). Then, using the formal rules for $\otimes$, compare ω_α and ω_β upon their restriction to $U_\alpha \cap U_\beta$. Suppose the restriction of ω_α and ω_β (using the variables $y_{\alpha,1}, \dots, y_{\alpha,n}$) are the same on $U_\alpha \cap U_\beta$. Then, we declare them together as forming a general element ω of the tensor algebra on $U_\alpha \cup U_\beta$. The subtlety is that ω likely will not be in $\mathbb{T}^0_{U_\alpha\cup U_\beta} \otimes \mathbb{D}^0_{U_\alpha\cup U_\beta}$. Drop the 0 superscript for a more general algebra.

Definition 5.10. The (mixed) tensor algebra $\mathbb{T}_X \otimes \mathbb{D}_X$ on X consists of collections $\omega_{\alpha_i} \in \mathbb{T}^0_{U_{\alpha_i}} \otimes \mathbb{D}^0_{U_{\alpha_i}}$, $i = 1, \dots, t$, with $\cup_{i=1}^t U_{\alpha_i} = X$ and ω_{α_i} and ω_{α_j} restricting to equal elements in $\mathbb{T}^0_{U_{\alpha_i}\cap U_{\alpha_j}} \otimes \mathbb{D}^0_{U_{\alpha_i}\cap U_{\alpha_j}}$ for all allowed i and j.

Elements of $\mathbb{D}_X$ are *covariant* tensors. If everywhere locally $\omega \in \mathbb{D}_X$ is a sum of terms with each a tensor of exactly k differential 1-forms, then it is a k-covariant tensor. Generalize contraction (Def. 5.8) to define ω paired with k ordered tangent vectors $(T_1, \dots, T_k)$. Notice how this requires local expressions of ω as a sum of terms like $g\omega_1 \otimes \cdots \otimes \omega_k$, with each ω_i a local differential 1-form. This contraction, $\langle (T_1, \dots, T_k), \omega\rangle$, is a global C^∞ function on X. For $\omega = g\omega_1 \otimes \cdots \otimes \omega_k$ write it as as $g \prod_{i=1}^k \langle T_i, \omega_i\rangle$. Such an ω is *symmetric* if $\langle (T_1, \dots, T_k), \omega\rangle = \langle (T_{(1)\pi}, \dots, T_{(k)\pi}), \omega\rangle$ for any permutation $\pi \in S_k$. It is *alternating* (or a *differential k-form*) if

$$\langle (T_1, \dots, T_k), \omega\rangle = \mathrm{Det}(\pi)\langle (T_{(1)\pi}, \dots, T_{(k)\pi}), \omega\rangle \ \pi \in S_k \ (\S 7.1.4).$$

5.2.5. *Orientation of a differentiable manifold.* A traditional and fuller treatment of the tensor algebra appears in texts on Riemannian geometry like [**Hi65**, Chap. 4]. Riemannian geometry starts with a differentiable manifold and a given symmetric 2-tensor furnished for measuring distances and angles [9.19]. From that tensor appear others for measuring other quantities on the manifold. For example, if on a differentiable 2-manifold we can measure distances along parametrized paths, then we should also be able to define the area of an open subset. The problem here is that you aren't likely to find a single parametrization by $\mathbb{R}^2$ of the whole area, and you must parametrize it in pieces, then add up the resulting areas. This forces the notion of *orientation*. The only 2-manifolds that have a well-defined area are orientable, which does include all Riemann surfaces Chap. 4 [11.11].

An orientation on a 2-dimensional differentiable manifold X consists of a rule for continuously assigning a left and right direction at the transversal meeting of two paths on the manifold. Precisely: Suppose given $\gamma^i : [-1,1] \to X$, $i = 1,2$, differentiable paths for which $x\gamma_i(0) = x \in X$, $i = 1,2$, and $(U_\alpha, \varphi_\alpha)$ is a coordinate chart containing x. So, we start with oriented 1-dimensional differential manifolds meeting at a point. Assume also that $\frac{\varphi_i \circ \gamma_i}{dt}(0) = \boldsymbol{v}_i$, $i = 1,2$, are distinct nonzero vectors. View a *traveler* as moving along $\varphi_\alpha \circ \gamma_1(t)$, facing at time $t = 0$ the direction $\boldsymbol{v}_1$ in $\mathbb{R}^2$ regarded as the (x,y) plane in $\mathbb{R}^3$. Then, the parametric line $L_0 = \{\varphi_\alpha \circ \gamma_1(0) + s\boldsymbol{v}_1 \mid s \in \mathbb{R}^1\}$ cuts the plane so that $\boldsymbol{v}_2$ points in the direction of the left half or the right half.

Definition 5.11. Suppose there is a new $\{(V_\beta, \psi_\beta)\}_{\beta\in J}$ on X, compatible with the original atlas (usually taken as a subcollection of its coordinate charts) with this property. Independently of the choice of a coordinate chart in the new atlas containing x, the vector $\boldsymbol{v}_2$ lies consistently in the same half plane (left or right) defined by the corresponding L_0. Then, we say the new atlas defines an orientation at x. The atlas defines an orientation on X if it gives an orientation at each $x \in X$. Riemann surfaces are examples of oriented manifolds.

A generalizing definition inductively allows discussing an orientation of X defined by the oriented meeting of an oriented $n-1$ dimensional manifold meeting an oriented 1-dimensional manifold Chap. 4 [11.5c].

5.3. Meromorphic vector fields and differentials. The definition of vector fields and differential forms is formal. So for each chart, $(U_\alpha, \varphi_\alpha)$, it extends to objects of form $T_\alpha = \sum_{i=1}^n f_{\alpha,i} \frac{\partial}{\partial z_{\alpha,i}}$ or $\omega_\alpha = \sum_{i=1}^n f_{\alpha,i}\, dz_{\alpha,i}$ with the $f_{\alpha,i}$ s meromorphic in the complex coordinates $z_{\alpha,i}$, $i = 1,\dots,n$. Then, since the jacobian of transition functions (and its inverse) have holomorphic function entries, this assures it maps a vector of meromorphic functions to a vector of meromorphic functions.

Example 5.12 (Differential of a meromorphic function). Suppose X is a Riemann surface (not necessarily compact) and $\psi : X \to \mathbb{P}^1_z$ is a (nonconstant) meromorphic function on X. We produce a meromorphic differential from ψ and an atlas $\mathcal{U}_X = \{U_\alpha, \varphi_\alpha\}_{\alpha\in I}$ for X. Define $d\psi_\alpha$ to be $\frac{d\psi\circ\varphi_\alpha^{-1}}{dz_\alpha}\, dz_\alpha$. Check: This is a differential form satisfying transformation formula (5.6).

Finally, let ω be a meromorphic differential 1-form on the Riemann surface X. Let $x_0 \in X$ lie in U_α where ω has the expression $f_\alpha(z_\alpha)\, dz_\alpha$. Suppose $\varphi_\alpha(x_0) = 0$. Then, the *order* m_{x_0} of ω at x_0 is the order of f_α at 0. Transition functions have neither zeros nor poles. So this order doesn't change if we compute it from another coordinate chart through φ_β with $x_0 \in U_\beta$.

5.3.1. *Divisors.* Conclude: For a given ω, the formal sum $\sum_{x\in X} m_x x$ has meaning. Denote it (ω) or D_ω depending on the notational context. It is the *divisor* of ω. Similarly, for any meromorphic function and meromorphic tangent vector on X we may define its divisor (f) or D_f. Call any formal sum $D = \sum_{x\in X} m_x x$ a divisor, and m_x is its *support multiplicity* at x.

LEMMA 5.13. *On a connected Riemann surface X, let D be the divisor of a nonconstant meromorphic differential, function or tangent vector. Then, the points of nonzero support multiplicity for D have no accumulation point. So, if X is also compact, divisors of nonconstant meromorphic differentials, functions or tangent vectors have only a finite number of nonzero support multiplicities.*

PROOF. We do the case for differentials. The others are similar. Suppose $(\omega) = \sum_{x\in X} m_x x$ is the divisor of a differential and infinitely many of the m_x are nonzero. Then, this set of xs has an accumulation point, x_0. Let $(U_\alpha, \varphi_\alpha)$ be a coordinate chart containing x_0, so the statement is that on $\varphi_\alpha(U_\alpha)$ we have a meromorphic differential $f_\alpha(z_\alpha)\,dz_\alpha$ having an accumulation of zeros or poles at $\varphi_\alpha(x_0) = z'_\alpha$. As in Chap. 2 [9.8a], this implies f_α is identically zero (or ∞) and using connectedness, that the same holds for the differential, contrary to our assumption (for extra help, see the argument of Chap. 4 Lem. 2.1). □

If X is not compact, divisors as in Lem. 5.13 may have infinitely many nonzero support terms (as with a holomorphic nonpolynomial function in the complex plane $\mathbb{C}_z$). In fact, the next general result in the complex plane has a similar version for any noncompact Riemann surface attached to an algebraic function [**Ahl79**, p. 195].

PROPOSITION 5.14 (Weierstrass factorization). *Suppose $\{m_{x_i}\}_{i\in I}$ is any collection of nonzero integers attached to a sequence of distinct points $\{x_i \in \mathbb{C}_z\}_{i\in I}$ with no accumulation point in $\mathbb{C}_z$. Then, there is a holomorphic function $f(z)$ with $(f) = \sum_{i\in I} m_{x_i} x_i$. Also, $f(z)\,dz$ (resp. $f(z)\frac{\partial}{\partial z}$) is a holomorphic differential (resp. vector field) with exactly the same divisor.*

Still, our tool will be the investigation of differentials, functions, etc., that extend meromorphically to a natural compactification of X. So, we typically assume (unless otherwise said) that $m_x = 0$ except for finitely many $x \in X$. For such a divisor D, the sum $\sum_{x\in X} m_x$ is the *degree* $\deg(D)$ of D. A divisor D is *positive* (or $D \geq 0$) if all its support multiplicities are nonnegative. This definition gives a partial ordering on divisors: With $D = \sum_{x\in X} m_x x$ and $D' = \sum_{x\in X} m'_x x$, $D \geq D'$ if $m_x \geq m'_x$ for each $x \in X$. Equivalently, with the obvious subtraction of divisors, $D - D'$ is positive.

Multiplying two functions or a function and a differential gives an object with divisor having the sum of the constituent multiplicities: $(f\omega) = (f) + (\omega)$.

DEFINITION 5.15. Suppose X is a compact Riemann surface. We say two divisors D_1 and D_2 on X are linearly equivalent if $D_2 - D_1 = (f)$ for some meromorphic function $f : X \to \mathbb{P}^1_z$. This is an equivalence relation between divisors.

Our notation for the linear equivalence class of a divisor D on a compact Riemann surface will be $[D]$. On a compact Riemann surface, the divisor of a meromorphic function has degree 0 (Chap. 4 Lem. 2.1; see Ex. 5.17). Anticipating that, conclude there is a well-defined degree attached to a linear equivalence class of divisors. Finally, we have a crucial definition attached to a divisor for which the reader should practice the notation.

DEFINITION 5.16. For any divisor D on a Riemann surface, the linear system of D, $L(D)$, is the collection of meromorphic functions f for which $(f) + D \geq 0$.

5.3.2. *Relation between functions and differentials.* As in Ex. 5.12, any (nonconstant) meromorphic function on a Riemann surface X provides us a nontrivial meromorphic differential form. Further, assume ω_1, ω_2 are meromorphic differentials and ω_1 is not a constant multiple of ω_2. This produces a nonconstant meromorphic function $\psi : X \to \mathbb{P}^1_z$ by the formula

$$\psi \circ \varphi_\alpha^{-1}(z_\alpha) = \omega_{\alpha,1}/\omega_{\alpha,2}. \tag{5.9}$$

So, all nonconstant differentials are linearly equivalent, and (see Def. 5.15), on a compact Riemann surface, all have the same degree.

EXAMPLE 5.17. Consider the identity map $z : \mathbb{P}^1_z \to \mathbb{P}^1_z$ by $z \mapsto z$. Carefully consider what is $dz = \omega$ using Ex. 3.2.1. To clarify notation, denote φ_1 by φ_α and φ_2 by $\varphi_{\alpha'}$. Then, $\varphi_\alpha : \mathbb{C}_z \to \mathbb{C}_{z_\alpha}$ by $z \mapsto z$, and so $\omega_\alpha = \frac{dz_\alpha}{dz_\alpha} dz_\alpha = dz_\alpha$. Also, $\varphi_{\alpha'} : \mathbb{C}^*_z \cup \{\infty\} \to \mathbb{C}_{z_{\alpha'}}$ by $z \mapsto z^{-1}$. So,

$$\omega_{\alpha'} = \frac{dz_{\alpha'}^{-1}}{dz_{\alpha'}} dz_{\alpha'} = -z_{\alpha'}^{-2} dz_{\alpha'}. \tag{5.10}$$

The differential dz is meromorphic, not holomorphic, and it has degree -2. To see there are no nonconstant holomorphic differentials on $\mathbb{P}^1_z$, write such a differential as $g(z)\,dz$ with g a meromorphic function on $\mathbb{P}^1_z$. Liouville's Theorem says g has as many zeros as poles [**Ahl79**, p. 122]. So the degree of $g(z)\,dz$ also is -2, and $(g(z)\,dz)$ cannot be positive. A similar computation shows the vector space of holomorphic differentials on a complex torus has dimension 1 [9.8].

5.3.3. *Pulling back differentials.* Let $f : X_1 \to X_2$ be an analytic and surjective map between complex manifolds. Then, a meromorphic function $\psi : Y \to \mathbb{P}^1_z$ produces a meromorphic function $\psi \circ f \overset{\text{def}}{=} f^*(\psi) : X \to \mathbb{P}^1_z$ giving an embedding $\mathbb{C}(Y) \subset \mathbb{C}(X)$ (§4.1.2).

LEMMA 5.18. *We may extend f^* to embed meromorphic differentials $\mathcal{M}^1(Y)$ on Y into meromorphic differentials $\mathcal{M}^1(X)$. Further, this maps holomorphic differents $\Omega^1(Y)$ on Y into holomorphic differentials on X. Then φ^* has the following property. For $\omega \in \mathcal{M}^1(Y)$, suppose $\gamma \in \Pi_1(X, x_0)$ does not go through a pole of $\varphi^*(\omega)$. Then, $\int_\gamma \varphi^*(\omega) = \int_{\varphi_*(\gamma)} \omega$.*

PROOF. Use the notation of (4.1). To simplify we do this for the case of 1-dimensional complex manifolds, though the many variable case is just a slight addition to the notation. This is truely a local statement. Write ω as $h_{\alpha_2}(z_{\alpha_2})\,dz_{\alpha_2}$ on $\varphi_{\alpha_2}(f(U_{\alpha_1}) \cap U_{\alpha_2})$. Then, define $f^*(\omega)$ by

$$h_{\alpha_2}(\varphi_{\alpha_2} \circ f \circ \varphi_{\alpha_1}^{-1}(z_{\alpha_1}))\, d(\varphi_{\alpha_2} \circ f \circ \varphi_{\alpha_1}^{-1}(z_{\alpha_1})) \text{ on } U_{\alpha_1} \cap f^{-1}(U_{\alpha_2}).$$

The equality of the integrals is nothing more, after substituting for the coordinates of the path γ, than the change of variables formula Chap. 2 Lem. 2.3. □

5.4. Half-canonical differentials. Square-roots of differentials appear on a Riemann surface X when we seek a canonical choice of θ function attached to the surface. The case when X has genus 1 (Chap. 4 §7.5) will be our guide.

Riemann's θ functions often allow us to put coordinates (as in the initial discussion of §4) on such total familes. Whenever possible, we would like the construction of such coordinates to be canonical. Usually, however, constructing θ functions depends on choices. So, we are careful to note, for curves in families, how the construction varies with the points parametrizing the family members.

Riemann used θ functions to give coordinates for constructing objects, like differentials and functions on a Riemann surface. When the Riemann surface has genus 1 (or 0) there are natural choices for working with Riemann's coordinates. When, however, the genus exceeds 1, and the surface is not special, there are several $(2^{2g-1}-2^{2g-2})$ potential choices of the *odd* θ function Riemann required to generalize Abel's Theorem. We will see that half-canonical differentials precisely differentiate between these choices.

5.4.1. *Cocycles.* For X an n-dimensional complex manifold, let $\{U_\alpha, \varphi_\alpha\}_{\alpha\in I}$ be the coordinate chart, and $\{\psi_{\beta,\alpha} = \varphi_\beta \circ \varphi_\alpha^{-1}\}_{\alpha,\beta\in I}$ the corresponding collection of transition functions (as in Def. 3.6). Each $\psi_{\beta,\alpha}$ then is a one-one analytic function on an open subset of $\mathbb{C}^n$ whose coordinates we label $z_{\alpha,1},\dots,z_{\alpha,n}$. Denote the $n\times n$ *complex Jacobian matrix* for $\psi_{\beta,\alpha}$ by $J(\psi_{\beta,\alpha})$. Call the matrices $\{J(\psi_{\beta,\alpha})\}_{\alpha,\beta\in I}$ the (transformation) *cocycle* attached to meromorphic differentials. Similarly $\{J(\psi_{\beta,\alpha})^{-1}\}_{\alpha,\beta\in I}$ is the cocycle attached to meromorphic tangent vectors. Recall the notation for $n\times n$ matrices, $\mathbb{M}_n(R)$ with entries in an integral domain R and for the *invertible* matrices $\mathrm{GL}_n(R)$ with entries in R under multiplication. Cramer's rule says for each $A\in\mathbb{M}_n(R)$ there is an adjoint matrix A^* so that AA^* is the scalar matrix $\det(A)I_n$ given by the determinant of A. This shows the invertibility of $A\in\mathbb{M}_n(R)$ is equivalent to $\det(A)$ being a *unit* (in the multiplicatively invertible elements R^*) of R. Denote the $n\times n$ identity matrix (resp. zero matrix) in $\mathrm{GL}_n(R)$ by I_n (resp. $\mathbf{0}_n$).

Definition 5.19 (1-cycocle). Suppose $g_{\beta,\alpha}\in \mathrm{GL}_n(\mathcal{H}(U_\alpha\cap U_\beta))$, $\alpha,\beta\in I$. Assume also that $g_{\gamma,\beta}g_{\beta,\alpha}=g_{\gamma,\alpha}$ for all $\alpha,\beta,\gamma\in I$ on $U_\alpha\cap U_\beta\cap U_\gamma$ (if this is nonempty). Then, $\{g_{\beta,\alpha}\}_{\alpha,\beta\in I}$ is a multiplicative *1-cocycle with values in* $\mathcal{GL}_{n,X}$. Similarly, suppose $g_{\beta,\alpha}\in\mathbb{M}_n(\mathcal{H}(U_\alpha\cap U_\beta))$, $\alpha,\beta\in I$. Suppose $g_{\gamma,\beta}+g_{\beta,\alpha}=g_{\gamma,\alpha}$ for all $\alpha,\beta,\gamma\in I$ on $U_\alpha\cap U_\beta\cap U_\gamma$. Then, $\{g_{\beta,\alpha}\}_{\alpha,\beta\in I}$ is an additive *1-cocycle with values in* $\mathcal{GL}_{n,X}$.

We also name (1-)cocycles for collections of subgroups in $\mathcal{GL}_{n,X}$ (resp. $\mathcal{M}_{n,X}$) for which it makes sense to multiply (resp. add) $g_{\gamma,\beta}$ and $g_{\beta,\alpha}$. So, for example, we may consider a multiplicative cocycle with values in $\{\pm I_n\}$ or an additive cocycle with values in $\mathbb{Z}I_n$. When there are 1-cocycles, there are also 0-chains and their associated 1-boundaries. We write the definition for GL_n, recognizing there are analogous versions for all other types of cocycles.

Definition 5.20 (1-boundary). With $u_\alpha\in\mathrm{GL}_n(\mathcal{H}(U_\alpha))$, $\alpha\in I$, suppose $g_{\beta,\alpha}=u_\beta(u_\alpha)^{-1}$ for all $\alpha,\beta,\gamma\in I$ in $U_\alpha\cap U_\beta$ (if nonempty). Then, $\{g_{\beta,\alpha}\}_{\alpha,\beta\in I}$ is a 1-cocycle, called a *1-boundary* with values in $\mathcal{GL}_{n,X}$. Call the set $\{u_\alpha\}_{\alpha\in I}$ a *0-chain* with values in $\mathcal{GL}_{n,X}$.

5.4.2. *Half-canonical divisors.* Suppose ω is a meromorphic differential on a Riemann surface X, written locally as $f_\alpha(z_\alpha)dz_\alpha$ on simply connected domains U_α (Chap. 2 §8.3). Assume also the *square hypothesis*:

(5.11) The divisor of $f_\alpha(z_\alpha)$ has the form $2D_\alpha$ for U_α running over a subchart covering X.

Then, there is a branch $h_\alpha(z_\alpha)$ of square root (of $f_\alpha(z_\alpha)$) on U_α (Chap. 2 (6.2)). Of course, there are two of these; our notation means we have chosen one. Call the symbol $\tau_\alpha = h_\alpha(z_\alpha)\sqrt{dz_\alpha}$, a *half-canonical divisor* on U_α. The squares of these form a global differential on X. Denote the collection $\{h_\alpha(z_\alpha)\}_{\alpha\in I}$, by $\boldsymbol{h}$ and refer to it as a square-root of ω.

Lemma 5.21 (Half-canonical divisor). *The collection of divisors* $\{(h_\alpha(z_\alpha))\}_{\alpha\in I}$ *from a square root of* ω *give a well-defined divisor: a half-canonical divisor on* X.

Proof. Let $D = (\omega)$ be the divisor of ω. Since, $h_\alpha^2 = f_{i,\alpha}$, the support multiplicities of D are all even integers. So, a square-root of ω defines $D_{1/2} = (\omega)/2$, a divisor uniquely given by the zeros and poles of the h_α s. □

Now consider how to decide, based on a square-root of ω, if there is an object $\omega_{1/2}$ with values at points on X whose divisor is $D_{1/2} = (\omega)/2$. Continue the transition function notation $\psi_{\beta,\alpha}$ from §5.4.1. This requires us to make sense, on $U_\alpha \cap U_\beta$, of equality between

$$\tau_\alpha(z_\alpha) = h_\alpha(z_\alpha)\sqrt{dz_\alpha} \text{ and } \tau_\beta(\psi_{\beta,\alpha}(z_\alpha)) = h_\beta(\psi_{\beta,\alpha}(z_\alpha))\sqrt{d\psi_{\beta,\alpha}(z_\alpha)}. \tag{5.12}$$

Proposition 5.22. *Assume each component of $U_\alpha \cap U_\beta$, $(\alpha,\beta) \in I\times I$ is simply connected and for such, we have made a choice of $\sqrt{J(\psi_{\beta,\alpha})} = g_{\beta,\alpha}$ on $U_\alpha \cap U_\beta$. Then, independent of α with $x' \in U_\alpha$, setting the value of τ_α to $h_\alpha(\varphi_\alpha(x'))$ is well-defined if and only if $\{g_{\beta,\alpha}\}_{(\alpha,\beta)\in I\times I} = \boldsymbol{g}$ is a 1-cocycle. If there is a $\boldsymbol{g}$ that is a 1-cocycle, call the resulting half-canonical differential $\omega_{1/2,\boldsymbol{h},\boldsymbol{g}}$. Then, with $\boldsymbol{g}$ fixed, but $\boldsymbol{h}'$ varying over square-roots of ω, any pair of $\omega_{1/2,\boldsymbol{h}',\boldsymbol{g}}$ differ by a 1-boundary with values in $\{\pm 1\}$.*

Proof. We need only add that the cocycle condition on $\boldsymbol{g}$ is necessary and sufficient for (5.12). For this check that if $x' \in U_\alpha \cap U_\beta \cap U_\gamma$, then all the values $h_\alpha(\varphi_\alpha(x'))$, $h_\beta(\varphi_\alpha(x'))$ and $h_\gamma(\varphi_\gamma(x'))$ at x' match up using $\boldsymbol{g}$. Comparing (5.12) for each of the pairs (α,β), (β,γ) and (α,γ) gives the cocycle condition. □

5.4.3. *Square-hypothesis for hyperelliptic curves.* Suppose the affine part of a hyperelliptic curve X, with compactification from Ex. 4.2.3, is $\{(z,w) \mid w^2 = h(z)\}$. We explicitly display differentials ω satisfying the square hypothesis of (5.11). For simplicity, assume h has odd degree and distinct zeros $z_1,\dots,z_{r-1}$ (with $z_r = \infty$). Denote the point on X over z_i by x_i, with x_∞ lying over $z = \infty$. As in [**Mum76**, p. 7], form the differentials

$$\omega_i = \frac{(z-z_i)^{\frac12}}{(\prod_{j\neq i} z-z_j)^{\frac12}}\,dz,\ i = 1,\dots,r-1.$$

Since $w = \sqrt{h(z)}$, the factor in front of the dz in ω_i is just $\frac{z-z_i}{w}$, a meromorphic function on X. The divisor of ω_i is therefore $2x_i - 2x_\infty = D_i$. For the check at a neighborhood of x_∞ over $z = \infty$, use $t = 1/\sqrt{z}$ as the uniformizing parameter on X. Consider the case $\deg(h) = 3$. Then, $(t^{-1} - z_i)(-2wt^3)\,dt$ has $t = 0$ as a pole of order 2. So, D_i is the same divisor as $(z - z_i)$.

Now consider the case $\deg(h) = r - 1$, $r \geq 6$ an even integer. Similarly, $(\omega_i) = 2x_i + 2(r/2-3)x_\infty$, as $\frac{z-z_i}{-2wt^3}\,dt$ has $t = 0$ as a zero of multiplicity $2(r/2-3)$.

6. Homotopy, monodromy and fundamental groups

Complex structure provides the notion of analytic continuation. We detect the effects of analytic continuation through monodromy action, a representation of some fundamental group. In practice this can be a permutation representation, a representation as automorphisms of a vector space or a representation into automorphisms of a more general group. The prototype use of monodromy is Riemann's Existence Theorem: We replace constructing a compact Riemann surface using charts with permutation representations of a fundamental group. For example, using classical generators (Chap. 4 Fig. 3) for the fundamental group of

$U_{\boldsymbol{z}} = \mathbb{P}^1_z \setminus \{\boldsymbol{z}\}$ gives an effective listing of Riemann surface covers (and their corresponding algebraic functions; Chap. 4 Cor. 2.9).

6.1. Homotopy of paths. Let $\gamma_i : [a_i, b_i] \to X$, $i = 1, 2$, be two one-one simplicial paths in X with the same range, initial, and end points. The function $f(t) = \gamma_2^{-1} \circ \gamma_1$ is a simplicial path $f : [a_1, b_1] \to [a_2, b_2]$ for which $\frac{d}{dt}(f(t)) \geq 0$ (where the derivative is defined) and $\gamma_2(f(t)) = \gamma_1$. (Use the chain rule.) We give a more general statement.

Definition 6.1 (Image equivalent paths). Let $\gamma : [a_1, b_1] \to X$ be a simplicial path in X, and let $f_1 : [a_2, b_2] \to [a_1, b_1]$ and $f_2 : [a_1, b_1] \to [a_2, b_2]$ be simplicial paths with $\frac{d}{dt}(f_i(t)) \geq 0$ where it is defined, $i = 1, 2$. Assume also $\gamma \circ f_1 \circ f_2(t) = \gamma(t)$ for $t \in [a_1, b_1]$. Call γ and $\gamma \circ f_1$ *image equivalent* paths. It is a simple exercise to show each path is image equivalent to a path $\gamma : [0, 1] \to X$.

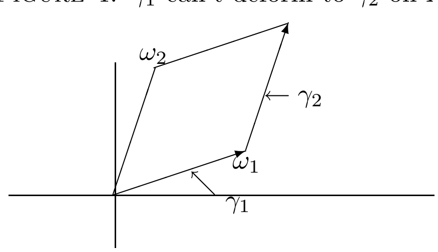
Definition 6.2 (Homotopically equivalent paths). Consider a continuous map $F : [a, b] \times [0, 1] \to X$, and points $x_a, x_b \in X$, with the following properties: $F(t, s) = \gamma_s(t)$ is a path for each $s \in [0, 1]$ with initial point x_a and end point x_b. Call F a *homotopy* between γ_0 and γ_1 (or γ_0 and γ_1 are *homotopic*).

Remark 6.3 (Warnings!). The end points of the paths γ_s remain fixed throughout a homotopy, or else all paths in a connected space would be homotopic.

Even if γ_0 and γ_1 are simplicial paths, we do not initially assume γ_s is also simplicial. Still, the argument of Chap. 2 Lem. 4.3 generalizes easily to any (union of) differentiable manifold(s) to say that any continuous path is homotopic to a simplicial path. Further, it is then image equivalent to a product of simplicial paths that are either constant or have nonzero derivative, and if it is a nonconstant path, you can toss out —up to equivalence —the constant paths. We use this statement freely [9.12]. It is common to think of both s and t as time parameters. It is compatible to consider the range of γ_0 as a physical object layed down parametrically. As a function of time, each point $\gamma_0(t)$ of the range of γ_0 moves to a different position $\gamma_s(t)$. So, F represents deforming an initial path, perhaps along which it is more efficient to accrue similar information from traversing γ_0.

In Fig. 4 the space X is the same as Fig. 3. Note: γ_1 and γ_2 are closed, beginning and ending at $0 \mod L \in \mathbb{C}/L$.

Figure 4. γ_1 can't deform to γ_2 on X



Definition 6.4. Extend the definition of *homotopic paths*. We say two paths $\gamma_i : [a_i, b_i] \to X$, $i = 1, 2$, with $\gamma_1(a_1) = \gamma_2(a_2)$ and $\gamma_1(b_1) = \gamma_2(b_2)$ are *equivalent* (or *homotopic*) if γ_1 and γ_2 are image equivalent, respectively, to homotopic paths $\gamma_i^* : [a, b] \to X$, $i = 1, 2$, for some $a < b$. This is an equivalence relation.

6.2. Analytic continuation on a manifold. Suppose $f \in \mathcal{E}(D, z_0)$ is extensible in a domain D and $\gamma[a,b] \to D$ is a path. Chap. 2 Rem. 4.4 notes the production of a simplicial path γ^* in D for which the analytic continuations f_γ and f_{γ^*} are the same. Further, assume f is extensible as a holomorphic (rather than just meromorphic) function in D. Then, define F_γ for any antiderivative F of f (around z_0) as the analytic continuation F_{γ^*}. Chap. 2 Lem. 4.3 produces γ^* from γ by a succession of homotopies, between a piece of path on γ contained in a disk and a line segment joining two points on the boundary of the disk. Disks are a crucial case of the following definition. The simple lemma following it, hidden in the construction of γ^*, appears in most arguments about homotopy classes.

DEFINITION 6.5. Call a topological space X *contractible* (to $x_0 \in X$) if there is a continuous function $f : X \times [0,1] \to X$ satisfying $f(x,0) = x$ and $f(x,1) = x_0$ for each $x \in X$.

LEMMA 6.6. *A closed or open ball (or anything homeomorphic to such) in $\mathbb{R}^n$ is contractible. If X is contractible, then any two paths with the same endpoints are homotopic [9.12b].*

Analytic continuation of a meromorphic function (Chap. 2 Def. 4.1) extends to manifolds by imitating the other extensions to manifolds. Suppose X is a complex manifold with coordinate chart $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in I}$. Consider any path $\gamma : [a,b] \to X$. Our notation follows the case for a dimension 1 complex manifold, though it extends easily to the general case.

By a disk (or ball) D on X we mean an open set in X which lies in one coordinate neighborhood U_α where $\varphi_\alpha(D)$ is a disk (or ball) in $\varphi_\alpha(U_\alpha) = V_\alpha$.

6.2.1. *Extensible functions on X.* Follow Chap. 2 §4.1 to extend analytic continuation of a function along a path to where the path is in a complex manifold.

DEFINITION 6.7 (Analytic continuation along a path). Suppose f is meromorphic in a neighborhood $U_{x_0} \subset X$ of $x_0 \in X$ and $\gamma : [a,b] \to X$ is a path based at x_0. Let $f^* : [a,b] \to \mathbb{P}^1_z$ be a continuous function with the following properties.

(6.1a) $f^*(t) = f(\gamma(t))$ for t close to a (in $[a,b]$).

(6.1b) For each $t' \in [a,b]$, there is a neighborhood $U_{\gamma(t')}$ of $\gamma(t')$ and an analytic function $h_{t'} : U_{\gamma(t')} \to \mathbb{P}^1_z$ with $h_{t'}(\gamma(t)) = f^*(t)$ for t near t' (in $[a,b]$).

As before, $h_{t'}$ is the analytic continuation of f *to* t'. It is an analytic function in some neighborhood of $\gamma(t')$. Reference is usually to the *end* function $h_b = f_\gamma$, analytic in a neighborhood of $\gamma(b)$. This is the analytic continuation of f (along γ). As with analytic continuation along a path in $\mathbb{P}^1_z$, $f^*(t)$ determines all data for an analytic continuation. Also, it is unique: its difference from another function suiting (6.1) must be constant (restrict to coordinate neighborhoods of points of the path and apply Chap. 2 [9.8a]). Again, there is a related definition.

6.2.2. *Algebraic functions on X.* An analytic function $\hat{f} : X \to \mathbb{P}^1_z$ satisfying $\hat{f}(x) = f(x)$ for all $x \in U_{x_0}$ is an analytic continuation or *extension* of f to X.

DEFINITION 6.8. Denote by $\mathcal{E}(X, x_0)$ all functions meromorphic in a neighborhood of x_0 that analytically continue along every path in X based at x_0.

Further, suppose there is compact Riemann surface $\bar{X}$ with $X = \bar{X} \setminus \boldsymbol{x}$ where $\boldsymbol{x}$ is a finite set of points on $\bar{X}$. Chap. 4 shows, if such a $\bar{X}$ exists, it is unique up to analytic isomorphism. If $\boldsymbol{x}$ consists of r points, call such an X an *r-punctured Riemann surface*. Dropping reference to r, call it just a punctured Riemann surface. This tacitly assumes r is a finite number.

DEFINITION 6.9. Suppose X is a punctured Riemann surface. Then, $\mathcal{E}(X,x_0)^{\mathrm{alg}}$ consists of the $f \in \mathcal{E}(X,x_0)$ for which both the following sets are finite.

(6.2a) All analytic continuations, $\mathcal{A}_f(X) = \{f_\gamma\}_{\gamma\in\Pi_1(X,x_0)}$ of f in X.
(6.2b) For $x' \in \boldsymbol{x}$, the limit endpoint values of f_γ along all $\gamma \in \Pi_1(X,x_0,x')$.

PROPOSITION 6.10. *Let D be a disk on X, and suppose $f : D \to \mathbb{P}^1_z$ is analytic. There is a partition $a = t_0 < t_0^* < t_1 < t_1^* < \cdots < t_{n-1}^* < t_n = b$ of $[a,b]$, coordinate neighborhoods (U_i,φ_i), a disk D_i centered about $\gamma(t_i)$ in U_i and $f_i \in \mathcal{H}(D_i)$, $i = 1,\dots,n-1$, with these properties.*

(6.3a) $D_i \cap D_{i+1} \neq \emptyset$ *and* $f_i(z) = f_{i+1}(z)$ *for* $z \in D_i \cap D_{i+1}$.
(6.3b) $\gamma(t) \in D_i$ *for* $t \in [t_i,t_i^*]$, $\gamma(t) \in D_{i+1}$ *for* $t \in [t_i^*,t_{i+1}]$, $i = 0,\dots,n-1$.
(6.3c) $f_0(z) = f(z)$ *for* $z \in U_{z_0}$.

Further, let γ^* be the path along the consecutive line segments $\gamma(t_i)$ to $\gamma(t_i^*)$, then $\gamma(t_i^*)$ to $\gamma(t_{i+1})$, $i = 0,\dots,n-1$. Then, $f_{\gamma^*} = f_\gamma$.

PROOF. The proof reduces to that of Chap. 2 Lem. 4.3 by using the definition of function and coordinate charts on a complex manifold. □

PROPOSITION 6.11 (The general monodromy theorem). *Let $\gamma_1,\gamma_2 : [a,b] \to X$ be two paths with $\gamma_1(a) = \gamma_2(a) = x_0$ and $\gamma_1(b) = \gamma_2(b) = x_1$. Suppose γ_1 and γ_2 are homotopic on X. Let U_{x_0} be a neighborhood of x_0 and $f : U_{x_0} \to \mathbb{P}^1_z$ Then, $f_{\gamma_1} = f_{\gamma_2}$ (*[**Ahl79**, p. 295] *and* [**Con78**, p. 219]*).*

PROOF. Let $F : [a,b] \times [0,1] \to X$ be a homotopy between γ_1 and γ_2 fixing points $x_a = x_0, x_b = x_1 \in X$. A continuous function on a compact space is absolutely continuous. From absolute continuity of F there are partitions

$$a = s_0 < s_1 < \cdots < s_n = b \text{ of } [a,b] \text{ and } 0 = t_0 < t_1 < \cdots < t_m = 1 \text{ of } [0,1]$$

so that $F : [s_i,s_{i+1}] \times [t_j,t_{j+1}] \to X$ has range in a coordinate chart $U_{i,j}$ on X and $\varphi_{i,j} : U_{i,j} \to \mathbb{C}$ has range in a disk.

Suppose h is meromorphic in a neighborhood of $F(s_i,t_j)$ and extensible on the range of F on $[s_i,s_{i+1}] \times [t_j,t_{j+1}]$. Denote the product of the paths

$$s \mapsto F(s,t_j) = F_{ij,1},\ s \in [s_i,s_{i+1}] \text{ and } t \mapsto F(s_{i+1},t) = F_{i+1j,2},\ t \in [t_j,t_{j+1}]$$

by μ^+_{ij}. Similarly, let μ^-_{ij} be the product of paths $t \mapsto F(s_i,t) = F_{ij,2}$, $t \in [t_j,t_{j+1}]$ and $s \mapsto F(s,t_{j+1}) = F_{ij+1,1}$, $s \in [s_i,s_{i+1}]$. From Chap. 2 Lem. 4.6, $h_{\mu^+_{ij}} = h_{\mu^-_{ij}}$.

Write the path γ_1 as the product of the paths $F_{i0,1}$, $i = 0,\dots,m$. Similarly, γ_2 is the product of the paths $F_{in,1}$, $i = 0,\dots,m$. We give a sequence of paths (with the same endpoints) that starts with γ_1, and ends with γ_2. The terms of the sequence differ from path-to-path in the chain by a product of paths of form $(\mu^+_{ij})^{-1}\mu^-_{ij}$ or of form $\gamma\gamma^{-1}$. This shows $f_{\gamma_1} = f_{\gamma_2}$. Simply replace $F_{i0,1}$ by

$$F_{i0,1}F_{i+10,2}F^{-1}_{i+10,2}(\mu^+_{i0})^{-1}\mu^-_{i0}$$

for each $i = 1,\dots,m$. These substitutions lead from γ_1 to the path that is the product of $F_{i1,1}$, $i = 0,\dots,m$. Continue inductively to the path γ_2, which is the product of $F_{i1,n}$, $i = 0,\dots,m$. □

Chap. 2 §4.4 defines the product of two paths $\gamma_i : [a_i,b_i] \to X$, $i = 1,2$, for which the end point of γ_1 is the initial point of γ_2. Many treatments on fundamental groups (like [Ma; Chap. 2]) restrict the domain interval for a path to $[0,1]$. The treatment here aids computation of the *Artin braid group* (Chap. 4 [11.8] and

Chap. 5). It has other virtues: If $\gamma_i : [a_i, b_i] \to X$, $i = 1, 2, 3$, are three paths with $\gamma_i(b_i) = \gamma_{i+1}(a_{i+1})$, $i = 1, 2$, then $\gamma_1(\gamma_2\gamma_3)$ and $(\gamma_1\gamma_2)\gamma_3$ are identical rather than just equivalent as in [**Ma67**, p. 59]. Thus, forming products is trivially *associative*.

6.3. Path equivalence classes form a group. We say $\gamma : [a, b] \to X$, a closed path with initial (and end) point $x_0 \in X$, is *based* at x_0. The set of paths based at x_0 is closed under taking products. Denote the (homotopy) equivalence class of γ by $[\gamma]$. Note: $[\gamma_1^*\gamma_2^*]$ is independent of the choice of $\gamma_i^* \in [\gamma_i]$, $i = 1, 2$. The function $\gamma : [a, b] \to X$ by $\gamma(t) = x_0$ is called a constant path; denote $[\gamma]$ by ϵ_{x_0}. The set of equivalence classes of paths in X based at x_0 is the *fundamental group* of X based at x_0.

THEOREM 6.12. *Equivalence classes of paths into X based at x_0 form a group, denoted $\pi_1(X, x_0)$, under the multiplication given by $[\gamma_1][\gamma_2] \stackrel{\text{def}}{=} [\gamma_1\gamma_2]$. The identity element is ϵ_{x_0}. The inverse of $[\gamma]$ is the class $[\gamma^{-1}]$ (*Chap. 2 *§4.4).*

PROOF. Consider $\gamma : [a, b] \to X$ and γ^{-1} as above. Let $s' = a+s(b-a)$ and consider the function $F : [a, 2b-a] \times [0, 1] \to X$ defined by

$$F(t, s) = \begin{cases} \gamma(t) & \text{for } t \in [a, s'] \\ \gamma(s') & \text{for } t \in [s', 2b-s'] \\ \gamma(2b-t) & \text{for } t \in [2b-s', 2b-a]. \end{cases} \tag{6.4}$$

So, F is a homotopy between $\gamma\gamma^{-1}$ and the constant path from $[a, 2b-a]$ into $\{x_0\}$.

From [9.12b], for $\gamma_0 : [a_0, b_0] \to \{x_0\}$, the paths $\gamma_0\gamma$ and $\gamma\gamma_0$ are equivalent to γ. Thus, $[\gamma][\gamma^{-1}] = \epsilon_{x_0}$, $[\gamma]\epsilon_{x_0} = [\gamma] = \epsilon_{x_0}[\gamma]$. This shows $\pi_1(X, x_0)$ is a group. □

The fundamental group *does* depend on the base point x_0, though its isomorphism class does not. Indeed, for $x_0, x_1 \in X$, let $\alpha : [a, b] \to X$ be a path with initial point x_0 and end point x_1. Define $\psi(x_0, x_1) : \pi_1(X, x_1) \to \pi_1(X, x_0)$ by

$$\psi(x_0, x_1)([\gamma]) = [\alpha\gamma\alpha^{-1}] \text{ for each } [\gamma] \in \pi_1(X, x_1).$$

Check that $\psi(x_0, x_1)$ is a homomorphism of groups inverse to the homomorphism $\psi(x_1, x_0)$: $[\gamma] \in \pi_1(X, x_0) \mapsto [\alpha^{-1}\gamma\alpha] \in \pi_1(X, x_0)$. Note: The isomorphism $\pi(x_0, x_1)$ depends on the choice of α if $\pi_1(X, x_0)$ is not an abelian group.

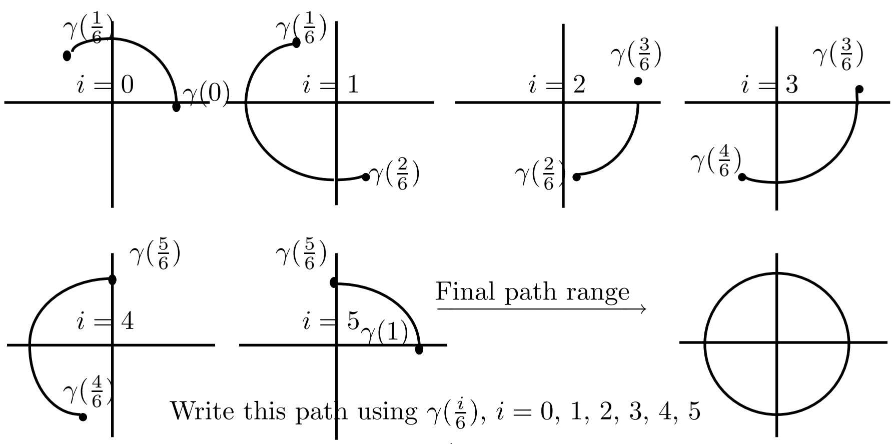
COROLLARY 6.13. *For $x_0, x_1 \in X$, $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic.*

Still, we eventually come to fundamental groups of members of a *family* of topological spaces (Chap. 5), where all members have the same fundamental group. Our most profound (the *braid* and *Hurwitz monodromy*) groups appear to account for different identifications among these fundamental groups.

6.4. Fundamental group of a circle. For any differentiable manifold X, there is a natural map from the fundamental group $\pi_1(X, x_0)$ computed with piecewise differentiable paths to the fundamental group computed with continuous paths, $\pi_1(X, x_0)^{\text{cont}}$. This induces an isomorphism (though we don't exploit this seriously) from Rem. 6.3. This point shows in a comparison of the two fundamental groups when $X = S^1$, a *circle* which we take to be the unit circle in $\mathbb{C}_z$. We give two proofs that it is isomorphic to $\mathbb{Z}$. The first explicitly uses simplicial paths. The other uses the universal covering space (Lem. 8.4).

Consider the path $\gamma^*_{|_{[a,b]}} : [a, b] \to S^1$ by $t \mapsto \cos(2\pi t) + i \sin(2\pi t)$, $t \in [a, b]$. For $n \geq 0$ an integer, denote $\gamma^*_{|_{[0,n]}}$ by γ^*_n, and let S^1 be the image of γ^*_1. Denote the inverse of $\gamma^*_{|_{[0,1]}}$ by $(\gamma^*)^{-1}_{|_{[0,1]}}$. Since $(\gamma_1^*)^n = \gamma_n^*$ it is consistent to define γ_{-n}^* to be $(\gamma_1^{*-1})^n$. For $n = 0$ let γ_0^* be the constant path mapping to 1.

FIGURE 5. Homotopically speaking, a path going nowhere. Traversal for $t \in [\frac{i}{6}, \frac{i+1}{6}]$, $i = 0, 1, 2, 3, 4, 5$



THEOREM 6.14. *The group $\pi_1(S^1, 1)$ is infinite cyclic with generator $[\gamma_1]$.*

PROOF. From Rem. 6.3 any nonconstant path $\gamma : [a, b] \to S^1$ is equivalent (Def. 6.4) to a product of paths with nonzero derivative. Each such is then image equivalent to $(\gamma^*)^{\epsilon}_{|_{[r,s]}}$ for some $r < s$ and $\epsilon \in \{\pm 1\}$. So, we can write the path as $\prod_{i=1}^{\ell} (\gamma^*)^{\epsilon_i}_{|_{[r_i,s_i]}}$ with $s_i = r_{i+1}$. Suppose ϵ_i and ϵ_{i+1} have opposite sign. Further subdivide one of paths corresponding to i or to $i{+}1$ to assume $[r_i, s_i]$ and $[r_{i+1}, s_{i+1}]$ have the same length. From (6.4),

$$(\gamma^*)^{\epsilon_i}_{|_{[r_i,s_i]}} (\gamma^*)^{\epsilon_{i+1}}_{|_{[r_{i+1},s_{i+1}]}}$$

is equivalent to the constant path with image $(\gamma^*)^{\epsilon_i}(r_i)$ [9.12a]. Thus the whole path is equivalent to a path with a smaller ℓ. An induction on the integer $\sum_{i=1}^{\ell} |\epsilon_{i+1} - \epsilon_i|$ shows γ is equivalent to γ_n^* for some integer n.

The proof is complete if γ_n^* is inequivalent to γ_m^* for $m \neq n$. Decompose $\gamma : [a, b] \to S^1$ into its real and imaginary parts: $\gamma = \gamma_1 + i\,\gamma_2$ where $\gamma_i : [a, b] \to \mathbb{R}$, $i = 1, 2$. Define $\deg(\gamma)$ through the formula

$$\begin{aligned} 2\pi i \,\deg(\gamma) &= \int_a^b (\gamma_1(t), \gamma_2(t)) \cdot (\tfrac{d\gamma_1}{dt}(t), \tfrac{d\gamma_2}{dt}(t))\, dt \\ &\quad + i \int_a^b (-\gamma_2(t), \gamma_1(t)) \cdot (\tfrac{d\gamma_1}{dt}(t), \tfrac{d\gamma_2}{dt}(t))\, dt \end{aligned}$$

(as in Chap. 2 Lem. 2.3). By direct computation $\deg(\gamma_n^*) = n$.

If γ is homotopic to γ_n^*, then Chap. 2 Lem. 2.3 shows $\deg(\gamma) = n$. As $\deg(\gamma)$ depends only on $[\gamma]$ [9.12d], $[\gamma_n^*]$ is distinct from $[\gamma_m^*]$ for $n \neq m$. □

Chap. 4 computes fundamental groups of many spaces from Thm. 6.14.

Let $\gamma : [a, b] \to X_1$ be a (simplicial) path. Consider $f \circ \gamma : [a, b] \to X_2$, and for $x_1 \in X_1$, denote $f(x_1)$ by x_2. For $[\gamma] \in \pi_1(X_1, x_1)$, $[f \circ \gamma] \in \pi_1(X_2, f(x_1))$ is independent of the choice of γ representing $[\gamma]$. To a product of paths $\gamma_1\gamma_2$ in X_1, apply the formula $f \circ (\gamma_1\gamma_2) = (f \circ \gamma_1)(f \circ \gamma_2)$. This shows $[f \circ \gamma_1][f \circ \gamma_2] = [f \circ (\gamma_1\gamma_2)]$.

LEMMA 6.15. *Conclude: f induces a homomorphism of groups*

$$f_*: \pi_1(X_1, x_1) \to \pi_1(X_2, x_2).$$

If f is one-one and onto then f_* is an isomorphism of groups.

EXAMPLE 6.16. Let $X_1 = X_2 = S^1$ and consider $\cos(2\pi t) + i\,\sin(2\pi t) = z(t)$. For a fixed positive integer n define a function f by the formula $f(z(t)) = z(nt) = \cos(2\pi nt) + i\,\sin(2\pi nt)$. Thus $f_*: \pi_1(S^1, 1) \to \pi_1(S^1, 1)$. Also, for γ_1^*, the generating path for $\pi_1(S^1, 1)$, $f \circ \gamma_1^*(t) = f(z(t))$. Therefore $f \circ \gamma_1^*$ is image equivalent to γ_n^*. Identify $\pi_1(S^1, 1)$ with $\mathbb{Z}$, the group of integers, by identifying the integer 1 with $[\gamma_1^*]$. Then, $f_*: \pi_1(S^1, 1) \to \pi_1(S^1, 1)$ sends the integer m to $f_*(m) = nm$. The image of f_* is the subgroup of $\pi_1(S^1, 1) = \mathbb{Z}$ that n generates.

6.5. Fundamental group of a product. Let (X, x_0) and (Y, y_0) be two differentiable manifolds with a base point. The projections onto each factor, $\mathrm{pr}_X: X \times Y \to X$ and $\mathrm{pr}_Y: X \times Y \to Y$, induce homomorphisms

$\mathrm{pr}_{X*}: \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0)$ and $\mathrm{pr}_{Y*}: \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(Y, y_0)$.

So, there is a homomorphism

$$(\mathrm{pr}_{X*}, \mathrm{pr}_{Y*}): \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \times \pi_1(Y, y_0). \tag{6.5}$$

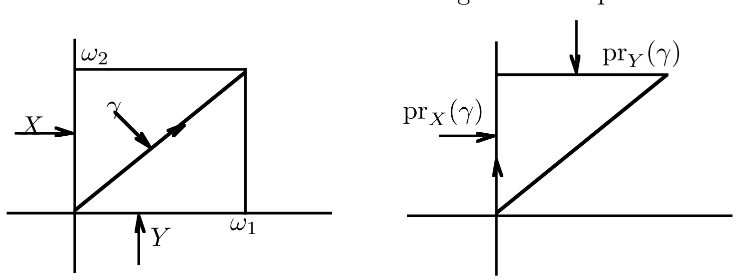
The right side is the product group with factors $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$.

THEOREM 6.17. *$\pi_1(X \times Y, (x_0, y_0))$ and $\pi_1(X, x_0) \times \pi_1(Y, y_0)$ are isomorphic.*

PROOF. Let f^X (resp. f^Y) map $X \to X \times Y$ by $f^X(x) = (x, y_0)$ (resp. map $Y \to X \times Y$ by $f^Y(y) = (x_0, y)$). For $\gamma: [a, b] \to X \times Y$ consider the paths $(f^X \circ \mathrm{pr}_X \circ \gamma) = \psi^X: [a, b] \to X \times Y$ and $(f^Y \circ \mathrm{pr}_Y \circ \gamma) = \psi^Y: [a, b] \to X \times Y$.

We show the map taking $([\gamma_1], [\gamma_2]) \in \pi_1(X, x_0) \times \pi_1(Y, y_0)$ to $f_*^X([\gamma_1]) f_*^Y([\gamma_2])$ in $\pi_1(X \times Y, (x_0, y_0))$ is inverse to $(\mathrm{pr}_{X*}, \mathrm{pr}_{Y*})$. This only requires showing γ is equivalent to $\psi^X \psi^Y$. Fig. 6.5 illustrates this when $X = Y = S^1$ and $X \times Y$ is the complex torus of Fig. 3 with $\omega_1 = 1$ and $\omega_2 = i$ [9.5b].

FIGURE 6. The diagonal recomposes itself



Write $\gamma(t) = (\gamma^X(t), \gamma^Y(t))$ for $t \in [a, b]$ and assume $[a, b] = [0, 1]$. Then γ is image equivalent to the path $(\gamma^X(\frac{t}{2}), \gamma^Y(\frac{t}{2}))$ for $t \in [0, 2]$. Also, ψ^X is the path $t \mapsto (\gamma^X(t), y_0)$ for $t \in [0, 1]$ and $(x_0, \gamma^Y(t-1))$ for $t \in [1, 2]$. Here is a homotopy between these paths running over $s \in [0, 1]$:

$$\gamma_s(t) = \begin{cases} (\gamma^X(\frac{t}{2-s}), y_0) & \text{for } t \in [0, s] \\ \gamma^X(\frac{t}{2-s}), \gamma^Y(\frac{t-s}{2-s})) & \text{for } t \in [s, 2-s] \\ (x_0, \gamma^Y(\frac{t-s}{2-s})) & \text{for } t \in [2-s, 2]. \end{cases}$$

□

EXAMPLE 6.18 (Continuation of §3.2.2). Here $X^i = \mathbb{C}/L(\omega_1^i, \omega_2^i)$ is

$$\{t_1\omega_1^i + t_2\omega_2^i \mid 0 \le t_i < 1,\ i = 1, 2\}$$

where $\omega_1^i/\omega_2^i \in \mathbb{C} \setminus \mathbb{R}$, $i = 1, 2$. For the lattice $\{m_1\omega_1^i + m_2\omega_2^i \mid m_1, m_2 \in \mathbb{Z}\}$ use the letter L_i, $i = 1, 2$. For $z \in \mathbb{C}$, there is a unique $\omega \in L_i$ with $z - \omega \in X^i$. Then $z - \omega$ *represents* the coset $z \bmod L_i \stackrel{\text{def}}{=} \{z + u \mid u \in L_i\}$ (as in §7.1). Let $\pi^i : \mathbb{C} \to \mathbb{C}/L_i$ be the map that takes z to $z \bmod L_i$. Then π^i is an analytic map. It becomes a homomorphism of groups if we make X^i into a group using this addition formula:

$$z_1 \bmod L_i + z_2 \bmod L_i \stackrel{\text{def}}{=} z_1{+}z_2 \bmod L_i[9.9d].$$

Suppose $L_1 \subseteq L_2$. Then, for $z \in \mathbb{C}$, the set

$$(\pi^1)^{-1}(z \bmod L_1) = \{z + \omega \mid \omega \in L_1\}$$

is in $(\pi^2)^{-1}(z \bmod L_2)$. So, the map f taking $z \bmod L_1$ to $z \bmod L_2$ depends only on $z \bmod L_1$, not on z. Identify $\pi_1(X^i, 0)$ with L_i (as in [9.9g]). The induced map f_* is the inclusion L_1 into L_2. For each $x_2 \in X_2$ the cardinality of the set $f^{-1}(x_2)$ is the order of the quotient group L_2/L_1 [9.7d].

Note: These concepts work equally well for finite unions of manifolds.

7. Permutation representations and covers

Two types of group theory arise in analyzing algebraic functions from Riemann's viewpoint. One is the presentation of fundamental groups, as free groups on generators with relations. Elementary examples of that do appear in many topology books (here too, starting with Chap. 4 §1.1). The second type is less common: Analyzing homomorphisms of fundamental groups into other groups. Motivating problems and sufficient group theory show how *finite* and *profinite* group theory apply to the study of moduli of Riemann surfaces. The group theory starts with permutation representations and their associated group representations.

7.1. Permutation representations. Denote by $\{\boldsymbol{x}\} = \{x_1, \ldots, x_n\}$ any set of n distinct elements. Let S_n be the collection of *permutations* of $\{\boldsymbol{x}\}$, and regard S_n as a group in the usual way. Multiplication of permutations corresponds to functional composition of maps on $\{\boldsymbol{x}\}$. Reminder: As the introduction states, we typically act with S_n on the *right* of elements from $\boldsymbol{x}$, though sometimes the presence of a second action forces us to act on the left.

7.1.1. *Permutation notation and actions.* Denote the identity element of S_n by 1. Here is an inefficient, though clear way to express the effect of $g \in S_n$:

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ (1)g & (2)g & \cdots & (n)g \end{pmatrix}$$

where $k = (j)g$ is the integer subscript of the image of x_j under g.

EXAMPLE 7.1. Suppose $n = 16$, and the display of g is

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 16 & 12 & 9 & 8 & 1 & 3 & 2 & 5 & 6 & 10 & 11 & 7 & 4 & 13 & 14 & 15 \end{pmatrix}.$$

The notation indicates g maps x_9 to x_6. *Disjoint cycle notation* for g represents it as a product of disjoint cycles of integers. It requires fewer symbols than the complete permutation notation. Also, it shortens computations in S_n by parsing the group action into memorable pieces. The disjoint cycle representation for g:

$$(1\,16\,15\,14\,13\,4\,8\,5)(2\,12\,7)(9\,6\,3).$$

The order of the disjoint cycles is unimportant; $(i)g$ goes to the right of i. That is, $(1)g = 16$ is right of 1, and the cycle closes at 5 because $(5)g$ is 1, back to the beginning. Exclude cycles of length 1 ($(10)g = 10$ gives a cycle (10)) for efficiency. An element of S_n is a *k-cycle*, $k > 1$ if it has one and only one cycle — of length k — of length bigger than 1.

For another unique, less orthodox way to write permutations see [9.17a].

Let G be any group. A degree n *permutation representation* of G is a homomorphism $T : G \to S_n$. Such a T is the same as giving an *action* of G on the set $S = \{x_1, \dots, x_n\}$.

With G a group and S a set, a *right action* is a function: $A = A_R : S \times G \to S$: $A(s, g) \mapsto (s)g$ with two action properties:

(7.1a) $(s)g_1g_2 = ((s)g_1)g_2$ for $s \in S$, $g_1, g_2 \in G$. Using A we would write this

$$A(A(s, g_1), g_2) = A(s, g_1g_2).$$

(7.1b) $(s)1_G = s$ for $s \in S$ (the identity in G leaves $s \in S$ fixed).

A *left action* is from a function $A_L : G \times S \to S$ with the action composite

$$A_L(g_1, A_L(g_2, s)) = A_L(g_1g_2, s).$$

An *orbit* of an action is the range of the set $s \times G$, under A, for some $s \in S$. The *kernel* of the action $\ker(A)$ consists of those $g \in G$ that act like the identity on S. The most important example is where G acts on the right cosets of a subgroup H of G. The set $Hg = \{hg\}_{h \in H}$ is a *right coset* of H in G. Two right cosets Hg and Hg' are either equal or have no elements in common. Assume there are exactly n distinct right cosets of H in G: $H, Hg_2, \dots, Hg_n$. Call n the *index* $(G : H)$ of H in G. Finding good representatives for cosets is an art (try [9.17c]).

The archetype of a right action: $A : (Hg', g) \mapsto Hg'g$, or $g \in G$ maps a right coset Hg' to $(Hg')g = Hg'g$. For any subgroup H there is both a set of right cosets of H and a set of left cosets of H. Only if H is normal in G are all right cosets also left cosets. The map $(g, g'H) \mapsto gg'H$ is a left action on left cosets. There are further actions of groups in [9.16]. We emphasize a right action because this is the natural action of fundamental groups acting on points as in Lem. 7.13.

Definition 7.2. Suppose G is a group with a normal subgroup H and another subgroup W. Assume $\langle H, W\rangle = G$ and $H \cap W = \{1\}$. We say G is the semi-direct product of H and W, written $H \times^s W$.

If $G = H \times^s W$, then elements of G act as automorphisms of H by conjugation. This is an action A: For $g \in G$, $A(g) : h \in H \mapsto g^{-1}hg \stackrel{\text{def}}{=} h^g$. This is a right action. The following lemma, in a left or right action form is in almost all graduate texts in algebra.

Lemma 7.3. *Each element of $H \times^s W$ has a unique expression as wh, $h \in H$, and $w \in W$. Suppose $A : W \to \mathrm{Aut}(H)$ is a homomorphism giving a right action of W on H. Then, there is a group G given as a semi-direct product of H and W. Multiplication in this group satisfies the formula $w_1h_1w_2h_2 = w_1w_2(h_1)A(w_2)h_2$.*

Remark 7.4 (Affine action). There is a memorable notation for multiplication by imitating matrix multiplication of lower triangular 2×2 matrices. Associate $w_i h_i$ with $\left(\begin{smallmatrix} w_i & 0 \\ h_i & 1 \end{smallmatrix}\right)$, $i = 1, 2$. Then, the multiplication in $H \times^s W$ imitates an expected matrix calculation:

$$\left(\begin{smallmatrix} w_1 & 0 \\ h_1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} w_2 & 0 \\ h_2 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} w_1 w_2 & 0 \\ (h_1)A(w_2)h_2 & 1 \end{smallmatrix}\right).$$

Further, $H \times^s W$ acts as permutations of H. For its matrix form, replace $h' \in H$ by the *vector* $(h', 1)$: $(h', 1)\left(\begin{smallmatrix} w & 0 \\ h & 1 \end{smallmatrix}\right) = ((h')A(w)h, 1)$ or $h' \mapsto (h')A(w)h$. The left action version with upper triangular matrices has a little glitch in it, unless H is an abelian group. That, however, comes up often in important examples (see [9.18]).

7.1.2. *Transitive and intransitive representations.* We discuss concepts that use coset representations. Lem. 7.7 shows how to go from the definition of action to the language of homomorphisms. When using groups acting on manifolds we often translate from actions into representations.

Definition 7.5. The *right coset* representation $T_H : G \to S_n$, defined by the subgroup $H \le G$, comes from the formula

(7.2) for $g \in G$, $i \in \{1, 2, \ldots, n\}$, $(i)T_H(g) = j$ with Hg_j the right coset equal to $Hg_i g$.

Denote the subgroup of elements $g \in G$ for which $T(g)$ fixes the integer j by $G(T, j) = G(j)$. For T a permutation representation, $\ker(T)$ is $\{g \in G \mid T(g) = 1_G\}$, the kernel of the action of G. Call T *faithful* if $\ker(T)$ consists only of 1_G. Also, T is transitive (G under T has one orbit) if for each $i \in \{1, 2, \ldots, n\}$, there is $g_i \in G$ with $(1)T(g_i) = i$. Then, $G(1)g_i$ is the set of $g \in G$ taking 1 to i. By definition, $\ker(T)$ is $\bigcap_{i=1}^n G(i)$. Assume T is transitive and $(1)T(g_i) = i$, $i = 1, \ldots, n$. Then, $g_i^{-1}G(1)g_i$, the *conjugate* of each element of $G(1)$ by g_i, equals $G(i)$. So, $G(1) \ldots, G(n)$ is a complete list of conjugates of $G(1)$ in S_n.

Definition 7.6. Let T_i be a degree n permutation representation of G, $i = 1, 2$. Suppose there is $h \in S_n$ with $h^{-1}T_1(g)h = T_2(g)$ for each $g \in G$. Then T_1 is *permutation equivalent* to T_2: T_1 and T_2 are equivalent as permutation representations.

Lemma 7.7. *In notation above, G acts on (right) cosets of $H \le G$, permuting them, and $T_H : G \to S_n$ is a homomorphism. The kernel is those $g \in G$ that fix each coset. This is the same as the elements of $\cap_{g \in G} g^{-1}Hg$. Reordering cosets of H in G changes the representation T_H only up to permutation equivalence.*

Suppose A_S (resp. $A_{S'}$) is an action of G on S (resp. S') with S and S' disjoint sets. Then, there is an action of G on $S \times S'$, the *direct product* action: $A \times A' : (S \times S') \times G \to S \times S'$ by $g \in G : (s, s') \in S \times S' \mapsto ((s)g, (s')g)$. There is also an action of G on $S \cup S'$, the *direct sum* action: $A \oplus A' : (S \dot\cup S') \times G \to S \dot\cup S'$ by $g \in G : s \in S \dot\cup S' \mapsto (s)g$ given by A if $s \in S$, and by A' if $s \in S'$. For $T : G \to S_n$ an arbitrary permutation representation, partition $\{1, \ldots, n\}$ into a disjoint union $X_1 \cup X_2 \cup \cdots X_t$ of the G orbits. Suppose $n_i = |X_i|$, $i = 1, \ldots, n$.

Theorem 7.8. *Let $T_H : G \to S_n$ be the right coset representation associated to the subgroup H of G. Then T_H is a transitive representation with* $\ker(T_H)$ *equal to $\bigcap_{g \in G} g^{-1}Hg$. Conversely, if $T : G \to S_n$ is a transitive representation of G, then T is permutation equivalent to T_H with $H = G(1)$. Generally, in the notation* *above for T, $T = \oplus_{i=1}^t T_i : G \to \oplus_{i=1}^t S_{n_i}$ presents T as the direct sum of right coset representations corresponding to subgroups of G.*

PROOF. For each $i \in \{1, 2, \dots, n\}$, formula (7.2) shows $(1)T_H(\sigma_i) = i$. So T_H is transitive. The subgroup $\ker(T_H)$ consists of the $g' \in G$ such that $Hg_ig' = Hg_i$, $i = 1, \dots, n$: $g' \in g_i^{-1}Hg_i$, $i = 1, \dots, n$. Each element in G has the form hg_i for some $h \in H$ and $i \in \{1, 2, \dots, n\}$. So, $g' \in \ker(T_H)$ if and only if $g' \in \bigcap_{g \in G} g^{-1}Hg$.

Let $T : G \to S_n$ be an arbitrary transitive permutation representation. Choose $g_1, \dots, g_n$ so that $(1)T(g_i) = i$, $i = 1, \dots, n$. Thus, the cosets $G(1)g_1, \dots, G(1)g_n$ are distinct. Conclude that (7.2), with $G(1)$ replacing H, gives $T_{G(1)}$. As

$$\{g \in G \mid (i)T(g) = j\} = g_j^{-1}G(1)g_i,$$

$(i)T(g) = j$ exactly if $(i)T_{G(1)}(g) = j$. This means $T_{G(1)}$ and T are the same permutation representation. We made choices in selecting the g_j s. So, independent of choices, the representations are permutation equivalent.

Now suppose the representation is not transitive. Since the orbits are all distinct, there is a natural map from the representation to the direct sum representation on the collection of orbits. □

7.1.3. *Primitive representations and equivariant maps.* A subgroup $H \le G$ is *normal* if $g^{-1}Hg = H$ for each $g \in G$. Only then is the set of pairwise products $HgHg'$ of two cosets a single coset, equal to Hgg'. So, the cosets have a natural group multiplication. Denote this set by G/H: Each element $\bar{g} = g \mod H \in G/H$ denotes the coset Hg. For H any subgroup of G, the *normalizer* of H in G is $N_G(H) = \{g \in G \mid g^{-1}Hg = H\}$. Similarly, define the *centralizer* of H in G:

$$\text{Cen}_G(H) = \{g \in G \mid g^{-1}hg = h \text{ for each } h \in H\} \text{ [9.15]}.$$

DEFINITION 7.9. Consider a transitive permutation representation $T : G \to S_n$ of G. Call T *primitive* if there are no groups properly between $G(1)$ and G. Let $G(1)$ be the subgroup of G that fixes 1. If T is transitive, then it is T is *doubly transitive* if for each $j \in \{2, \dots, n\}$ there is a $g \in G(1)$ with $(2)T(g) = j$: $G(1)$ is transitive on $\{2, \dots, n\}$.

When the notation shows G is in S_n, we drop the T notation for permutation representations. The *transitivity formula* for a chain of subgroups $K \le H \le G$ says that $(G : K) = (G : H)(H : K)$.

LEMMA 7.10. *Doubly transitive permutation representations are primitive.*

PROOF. Suppose $G \le S_n$ is doubly transitive. Let H be a subgroup of G properly containing $G(1)$. Choose $h \in H \setminus G(1)$. Then $(1)h = j \in \{2, \dots, n\}$. For any $j' \in \{2, \dots, n\}$, use double transitivity to produce g' with $(1)g' = 1$ and $(j)g' = j'$: $hg' \in H$ takes 1 to j'. So, the number of cosets of $G(1)$ in H is the same as the number of cosets of $G(1)$ in G. Apply the transitivity formula to the chain $G(1) < H \le G$ to conclude the index of H in G is 1 and T is primitive. □

Assume group G acts on two sets: It has an action A_S (resp. $A_{S'}$) on S (resp. S') with S and S' related by a function $f : S \to S'$. We say f commutes with (is *equivariant* for) these actions if $f((s,g)A_S) = (f(s), g)A_{S'}$ for $s \in S, g \in G$.

EXAMPLE 7.11 (Compatible permutation representations). For G a group and M a normal subgroup, let $u_M : G \to G/M$ be the natural homomorphism with kernel H. Suppose H_1 is a subgroup of G and H_2 is a subgroup of G/M for which

$f_M(H_1) \le H_2$. Then u_M induces a map $f_M : \{H_1 g \mid g \in G\} \to \{H_2 g \mid g \in G\}$. This map commutes with G acting on the cosets of H_1 and on the cosets of H_2.

7.1.4. *Representations from permutation representations.* [9.18] gives many examples of primitive groups that are not doubly transitive. For $g \in G$, some authors abuse notation to write $T(g) = (s_1) \cdots (s_t)$ where $s_1, \cdots, s_t$ are the integer lengths of the disjoint cycles of $T(g)$ (we usually omit cycles of length one) to indicate a cycle type (conjugacy class) in S_n. Denote the count of length one cycles in $T(g)$ by $\mathrm{t}(T(g))$, the *trace* of $T(g)$. For example, the permutation example of §7.1.1 has trace 2 and its cube has trace 5. We remind why $T(g)$ it is a trace.

Regard the formal symbols $\{x_1, \dots, x_n\}$ as basis vectors for a vector space V over a field F. Then each permutation $g \in S_n$ extends linearly to act on V. That is, applying $g \in G$ to $v = \sum_{i=1}^n a_i x_i \in V$ gives $\sum_{i=1}^n a_i x_{(i)g}$. Write the result of g on x_i to be $\sum_{j=1}^n a_{i,j} x_i$ with coefficients denoting what would appear in the ith position of a matrix M_g acting on the right of (row) vectors. When F has characteristic 0, the matrix M_g has trace $\sum_{i=1}^n a_{i,i}$, the count of the number of x_i s that g fixes. In each row and column the matrix M_g has exactly one non-zero entry and that is a 1. So, M_g is an element of the orthogonal group O_n: M_g times its transpose is the identity matrix. The determinant function is multiplicative on $n \times n$ matrices. Conclude that M_g has determinant $\mathrm{Det}(M_g) \stackrel{\mathrm{def}}{=} \mathrm{Det}(g)$ equal to ± 1. When the field F has characteristic p, the count of the integers fixed by g is the trace mod p. We may revert, when acting with matrices to a traditional left-hand action.

The result is that a degree n permutation representation T of a group G produces a homomorphism $\rho_T : G \to \mathrm{GL}_n(F)$. If T is a faithful permutation representation, then ρ_T is a faithful group representation: Its kernel is trivial. Any homomorphism $\rho : G \to \mathrm{GL}_n(F)$ is called a *representation* of G over the field F. With $V = F^n$, we often write V_T to indicate we mean V with the action through T. Then, for any representation, extend this notation to use V_ρ. In fact, group theory doesn't restrict to just finite dimensional representations, though we will. Most situations regard permutation representations as the same if they are equivalent. If $M \in \mathrm{GL}_n(F)$, then the two permutation representations $g \mapsto \rho(g)$ and $g \mapsto M^{-1}\rho(g)M$ are *(representation) equivalent*. Though two permutation representations may be inequivalent, their corresponding representations might be equivalent (§8.6.2 and [9.20]).

The group representation attached to the sum of permutation representations is the action on the direct sum of the vector spaces. When F has characteristic 0, every permutation representation of degree exceeding 1 is the direct sum of the identity representation and another representation. These are the only summands if and only if the permutation representation is doubly transitive [9.19d]. Further, the group representation of the direct product of two permutation representations is their *tensor product*; the trace is the product of the constituent traces [9.19a]. The *group ring* of G over F has the notation $F[G]$. The *product* of $\sum_{g\in G} a_g g$ and $\sum_{g\in G} b_g g$ (with $a_g, b_g \in F$) is given by convolution: $\sum_{g\in G} c_g g$ with $c_g = \sum_{h\in G} a_h b_{h^{-1}g}$, $g \in G$. A representation ρ then produces a homomorphism of associative rings: $\sum_{g\in G} a_g g \mapsto \sum_{g\in G} a_g \rho(g) \in \mathbb{M}_{\deg(\rho)}(F)$. Call an idempotent I in this ring G *invariant* if it commutes with multiplication by elements of G. That means the range of I is a G invariant space: I is a G invariant projection [9.19h].

7.2. Covering spaces. Let X and Y be differentiable (resp. analytic) manifolds. Assume $f : Y \to X$ is a differentiable (resp. analytic) map. We will often use that if f is one-one, and onto in a neighborhood of a point, then it has a differentiable (resp. analytic) *inverse* (Lem. 4.2). Suppose $\varphi : X \to X'$ is any map between spaces, and x_0 maps to x_0' under φ. As in Lem. 6.15, this induces a homomorphism on fundamental groups $\varphi_* : \pi_1(X, x_0) \to \pi_1(X', x_0')$ by mapping a closed path $\gamma : [a, b] \to X$ to $[\varphi \circ \gamma] \in \pi_1(X', x_0')$. This makes sense because composing with φ preserves homotopy classes of paths into X. Though obvious, it doesn't trivialize computing the image of $\pi_1(X, x_0)$ under φ_*.

DEFINITION 7.12 (Covering space). The pair (Y, f) (or just Y if there is no confusion) is a *covering space* (or cover) of X if each point $x \in X$ has a connected neighborhood (Chap. 2 §2.2.2) U_x with this property: for each connected component V of $f^{-1}(U_x)$, restricting f to V is a one-one and onto map $V \to U_x$.

7.2.1. *Degree of a cover.* Assume X is connected, and $f : Y \to X$ is a cover. Then, the cardinality of the fibers $|f^{-1}(x)|$, $x \in X$, being locally constant, must actually be constant. This is the *degree* $\deg(f)$ of f. We say (Y, f) is *finite*, or that f is a finite cover if $\deg(f) < \infty$.

Two covers $f_i : Y_i \to X$, $i = 1, 2$ are *equivalent* (as covers of X) if there is a one-one and onto continuous map $\psi : Y_1 \to Y_2$ with $f_2 \circ \psi = f_1$ [9.21]. Note: For any covering space (Y, f) of X, U an open subset of X, and V a union of connected components of $f^{-1}(U)$, the restriction of f to V gives a cover $(V, f_{|_V})$ of U.

A framework for considering equivalence classes of finite covers of a manifold X is the goal remaining to this subsection. This immediately reduces to considering connected finite covers (Y, f); we assume Y is a connected space. The classification hinges on producing an equivalence class, $T(Y, f)$, of permutation representations (§7.1) from an equivalence classes of covers (Y, f). We do that now.

Note: Covers in this section are what topologists call covers. In *algebraic geometry* the word cover includes complex analytic maps of manifolds having some fibers that *ramify* (their cardinality is smaller than the degree). The phrase then includes, for example, any nonconstant analytic map $f : Y \to \mathbb{P}^1_z$, with Y a compact Riemann surface and $\deg(f) \geq 2$. As the fundamental group of $\mathbb{P}^1_z$ is trivial, such an f must ramify (Chap. 4 Thm. 1.8). By the end of Chap. 4, a cover will include any surjective analytic map between compact complex manifolds with finite (point sets in their) fibers. Reference back to this chapter will speak of the unramified covers corresponding to subgroups of fundamental groups as in Thm. 7.16.

7.2.2. *Covers and permutation representations.* Let $f : Y \to X$ be a cover with $\gamma : [a, b] \to X$ a path having initial point x_0 and end point x_1.

LEMMA 7.13 (Action of path lifting). *For $y' \in Y$ with $f(y') = x_0$, there is a unique path $\tilde{\gamma} : [a, b] \to Y$ with $f \circ \tilde{\gamma} = \gamma$: the lift of γ with initial point y'.*

So, γ produces a unique map $\gamma_* : f^{-1}(x_0) \to f^{-1}(x_1)$ depending only on the image of γ in $\pi_1(X, x_0, x_1)$. In particular, consider paths $\gamma_i : [a_i, b_i] \to X$, $i = 1, 2$, with $\gamma_1(b_1) = \gamma_2(a_1)$ and $\gamma_1(a_1), \gamma_2(b_1), \gamma_2(b_2)$ respectively x_0, x_1, x_2. Then, there is a transitivity formula:

$$(\gamma_1 \cdot \gamma_2)_* = (\gamma_1)_* \circ (\gamma_2)_* : f^{-1}(x_0) \to f^{-1}(x_2). \tag{7.3}$$

PROOF. Each $\gamma(t)$ has a neighborhood U_t with f one-one on the connected components of $f^{-1}(U_t)$. The argument of Chap. 2 §3.3.2 works here as it did there, by assuming you have extended the path lifting $\tilde{\gamma}$ to an interval $[a, t']$ with $t' < b$.

Let $[r, s]$ be a closed nontrivial interval for which $t' \in [r, s]$ and there is neighborhood $U_{t'}$ of $\widetilde{(t')}$ containing $\gamma([r, s])$ with $U' \subset f^{-1}(U_{t'})$ a connected component on which f is one-one and $\gamma^*(t') \in U'$. For each $t \in [r, s]$ define $\tilde{\gamma}(t)$ to be the unique point of U' lying over $\gamma(t)$. Finish exactly as in Chap. 2 §3.3.2.

Now considering (7.3) Since the map γ_* is clearly continuous and varies continuously in a homotopy family, as a map on a finite set, it is a homotopy class invariant. So, γ_* depends only on the image of γ in $\pi_1(X, x_0, x_1)$. The path $\widetilde{\gamma_1 \cdot \gamma_2}$ starting at y' is the same as the path $\tilde{\gamma}_1 \cdot \tilde{\gamma}_2$ where $\tilde{\gamma}_2$ is the unique path starting at the end point of $\tilde{\gamma}_1$. The formula (7.3) just says the endpoint of both of these paths are the same. □

Label the points of $f^{-1}(x_0)$ as $\boldsymbol{y} = \{y_1, \ldots, y_n\}$. Consider a path $\gamma : [a, b] \to X$ based at x_0. Then, the end point of the lift of γ with initial point y_j, $j = 1, \ldots, n$ associates to γ and $\boldsymbol{y}$ a unique labeling of $f^{-1}(\gamma(b))$. A closed path γ gives an element of S_n, $T_{\boldsymbol{y}}(\gamma)$, as follows:

(7.4) $(i)T_{\boldsymbol{y}}(\gamma) = j$ with y_j the end point of the lift of γ with initial point y_i.

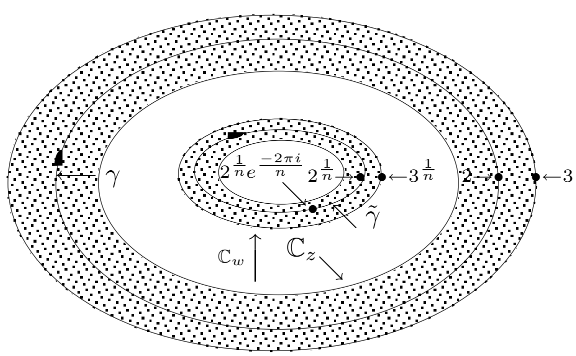
For $\gamma_1, \gamma_2 \in \Pi_1(X, x_0)$ (closed paths based at x_0) (7.3) gives

$$T_{\boldsymbol{y}}(\gamma_1\gamma_2) = T_{\boldsymbol{y}}(\gamma_1)T_{\boldsymbol{y}}(\gamma_2).$$

The right side consists of elements multiplied in S_n. So, $T_{\boldsymbol{y}}$ defines a permutation representation of $\pi_1(X, x_0)$ whose equivalence class we denote by $T(Y, f)$.

In Fig. 7, for example, $w \mapsto w^n = z$ gives the map $f : \mathbb{C}^*_w \to \mathbb{C}^*_z$ ($\mathbb{C}^* = \mathbb{C} \setminus \{0\}$). A lift of γ (a clockwise circle, compatible with our choices in Chap. 4) is $\tilde{\gamma}$ going $\frac{1}{n}$ of the way around a clockwise circle. The associated permutation is an n-cycle of S_n representing that $\tilde{\gamma}$ goes from the lift $y' = 2^{1/n}$ of $\gamma(0) = 2$ to $y'' = 2^{\frac{1}{n}}e^{\frac{-2\pi i}{n}}$, the point on $\tilde{\gamma}$ lying $\frac{1}{n}$ of the way around from y'. §7.2.3 discusses a traditional picture representing the nth power map as if it were the projection on a real coordinate.

Figure 7. An n-cycle of path liftings



7.2.3. *Impossible pictures.* We discuss the problem of representing covers by pictures in $\mathbb{R}^3$. Consider the ramified cover $f : U_{w:0,\infty} \to U_{z:0,\infty}$ by $w \mapsto w^n$ in Fig. 7. Points of $U_{w:0,\infty}$ over $z \in U_{z:0,\infty}$ correspond on the graph of f to $\mathbb{C} \times \mathbb{C}$ points on the line with constant second coordinate z. You can't draw pictures in $\mathbb{C} \times \mathbb{C} = \mathbb{R}^4$. So first year complex variables texts try to represent $U_{w:0,\infty}$ and $U_{z:0,\infty}$ as subsets of $\mathbb{R}^3$.

Let (x_1, x_2, x_3) be coordinates for $\mathbb{R}^3$, and let $x_3 = 0$ represent $U_{z:0,\infty}$ sitting in $\mathbb{R}^3 \setminus \{(0,0,0)\}$. Pictures try to represent an annulus around the origin in $U_{w:0,\infty}$ as a set M in $\mathbb{R}^3$ over an annulus D_0 in $U_{z:0,\infty}$. Then, points of M over $(x_1, x_2, 0) \in D_0$ are on the line in $\mathbb{R}^3$ whose points have first coordinates x_1 and x_2. That is, f appears as a coordinate projection. There is, however, no topological subspace M of $\mathbb{R}^3$ that can work! If there were, then a cylinder perpendicular to the plane $x_3 = 0$, with $(0,0,0)$ on its axis, would intersect M in a simple closed path winding n times around the cylinder. Represent such a path by $\gamma : [0,1] \to \mathbb{R}^3$ where $t \in [0,1]$ maps to

$$\gamma(t) = (\cos(2\pi nt), \sin(2\pi nt), x_3(2\pi nt)) \text{ and } x_3(2\pi n) = x_3(0).$$

Conclude: $w(t) = x_3(2\pi nt) - x_3(2\pi nt + 2\pi)$ is 0 for some value of t between 0 and $(n-1)/n$. So, the path isn't simple. The author has never seen such a picture attempt in the literature for any noncyclic cover, much less for more demanding nonsolvable groups. Still, we discuss this more in Chap. 4 §2.4 which also uses symbolic representations that assume we understand cyclic covers from their description in Chap. 2.

7.3. Pointed covers and a Galois correspondence. Let $f : Y \to X$ be a cover. Call the triple (Y, f, y') a *pointed cover* if $y' \in Y$. Then, we regard $f(x') = x_0$ as the base point for X, and (Y, f, y') is a pointed cover of (X, x_0).

DEFINITION 7.14. Suppose (Y, f_i, y_i'), $i = 1, 2$, are two pointed and connected covers of X. We say they are *compatibly pointed* (or compatible) if whenever we have covers $h : Z \to X$ and $h_j : Y_j \to Z$, with $h \circ h_j = f_j$, $j = 1, 2$, then $h_1(y_1) = h_2(y_2)$.

If it is clear a cover is pointed, we may refer just to the covering maps f_1 and f_2 to say these are compatible. Extension Lem. 8.1 shows the difference between a pointed cover on one hand, and a cover without a point on the other. Group theoretically this interprets as the difference between giving a subgroup of a group and giving a conjugacy class of subgroups.

7.3.1. *Fiber products of covers.* The basic theorems of Galois theory, including the construction of the Galois closure of a cover (§8.3), that translates geometrically using fiber products.

LEMMA 7.15. *Given connected covers $f_j : Y_j \to X$, $j = 1, 2$, of X, any connected component of $Y_1 \times_X Y_2$ is minimal among among connected covers (Y, f) of X factoring through each f_j. If the covers are compatibly pointed with $y_j' \in Y_j$, $j = 1, 2$, then a unique pointed component of $Y_1 \times_X Y_2$, $(Y, (f_1, f_2), (y_1', y_2'))$ is compatible with both (Y_i, f_i, y_i'), $i = 1, 2$.*

PROOF. Let Y be a connected component of $Y_1 \times_X Y_2$. Denote projection of Y on Y_j by pr_j. Consider any $(y_1, y_2) \in Y_1 \times_X Y_2$ lying over $x \in X$. Choose a neighborhood U_x of x for which there is a neighborhood $U_{y_j} \subset Y_j$ on which f_j maps one-one to U_x. Then, restricting (f_1, f_2) to $U_{y_1} \times_{U_x} U_{y_2}$ gives a one-one map that shows Y is a cover of X.

Now assume the covers are compatibly pointed. Let $x_0 \in X$ be $f_1(y_1') = f_2(y_2')$. Then, a unique component of $Y_1 \times_X Y_2$ contains (y_1', y_2'). □

Thm. 7.16 produces covers of any path-connected, locally path-connected space. For, however, our main applications where X is a (complex) manifold, it shows any cover of X is a (complex) manifold with a natural coordinate chart. It also says one cover of a space X dominates all others. This is the *universal* covering space $\tilde{X}$ corresponding to $H = \{1\} \le \pi_1(X, x_0)$.

THEOREM 7.16 (Unramified Galois correspondence). *Let (Y, f, y') be a pointed cover of (X, x_0). This canonically corresponds to a subgroup $H_{Y,f,y'} \le \pi_1(X, x_0)$ which we identify with $\pi_1(Y, y')$. The index $(\pi_1(X, x_0) : \pi(Y, y'))$ is $n = \deg(f)$. Any ordering $\boldsymbol{y} = \{y_1, \dots, y_n\}$ on the fiber $f^{-1}(x_0)$ with $y_1 = y'$ corresponds to a transitive permutation representation $T_{Y,f,\boldsymbol{y}}$ in which the stabilizer of 1 is $H_{Y,f,y'}$. If $y'' \in f^{-1}(x_0)$, then $H_{Y,f,y'}$ and $H_{Y,f,y''}$ are conjugate subgroups of $\pi_1(X, x_0)$ and we identify y'' with a coset of H in $\pi_1(X, x_0)$.*

Conversely, each subgroup $H \le \pi_1(X, X_0)$ of index n (possibly ∞) produces a canonical pointed (connected) degree n cover (Y_H, f_H, y'_H) of X. We regard y'_H as the H coset of the identity in $\pi_1(X, X_0)$. The fundamental group of Y_H maps one-one onto H under $(f_H)_*$.

Suppose H_1 and H_2 are two subgroups of $\pi_1(X, x_0)$. Then, the unique connected component of $Y_{H_1} \times_X Y_{H_2}$ containing (y'_{H_1}, y'_{H_2}) corresponds to the subgroup $H_1 \cap H_2$. The maximal pointed cover of X through which both f_1 and f_2 factor is $(Y_{\langle H_1, H_2\rangle}, f_{\langle H_1, H_2\rangle}, y'_{\langle H_1, H_2\rangle})$.

§7.3.2 consists of a proof of Thm. 7.16 and §8.1 has corollaries appropriate for covers that aren't pointed.

7.3.2. *Proof of Thm. 7.16.* Start with (Y, f, y'). Apply (7.4) to a closed path $\gamma : [a, b] \to X$ based at x_0. Use a specific ordering of $f^{-1}(x_0)$ with $y_1 = y'$. The lift of γ to a path with initial point y_1 is a closed path in Y based at y_1 if and only if $(1)T_{\boldsymbol{y}} = 1$. So we identify $\pi_1(Y, y_1)$ with $H(f, y_1)$, the subgroup of $\pi_1(X, x_0)$ stabilizing 1 under the map f_*.

Now consider how a subgroup H of $\pi_1(X, x_0)$ of index n canonically produces a degree n pointed cover of X. First: H produces an equivalence class of permutation representations of $\pi_1(X, x_0)$ of degree n (Thm. 7.8), with the coset of the identity corresponding to the integer 1 in the permutation representation.

Define Y_∞: As a set it is the collection of all equivalence classes of paths in X — not necessarily closed — with initial point x_0. For $\gamma \in Y_\infty$ let $f_\infty([\gamma])$ be the endpoint of γ. Define Y_H to be Y_∞ modulo the relation that equivalences

$$[\gamma_1] \text{ and } [\gamma_2] \text{ if } f_\infty([\gamma_1]) = f_\infty([\gamma_2]) \text{ and } [\gamma_1\gamma_2^{-1}] \in H.$$

Let $f_H : Y_H \to X$ be the map induced by f_∞ on the set Y_H. Now use that X is a connected manifold. For each $x \in X$ choose a path γ with initial point x_0 and endpoint x. A ball neighborhood U_x of x has this property: For $\gamma_1, \gamma_2 : [a', b'] \to U_x$, two paths with the same initial and endpoints, $\gamma_1\gamma_2^{-1}$ is equivalent to the constant path in U_x.

For each such pair (γ, U_x) consider the subset of Y_H represented by paths $\gamma\gamma_1$ with γ_1 a path in U_x with initial point x. Denote this subset by V_{γ,U_x}. We declare the topology on Y_H to have as a basis of open sets these V_{γ,U_x}s running over all pairs (x, U_x). For $y \in Y_H$ with $f_H(y) = x$, $f_H^{-1}(U_x)$ has n connected components, V_{γ_i,U_x}, $i = 1, \dots, n$, where $[\gamma_1\gamma_i^{-1}]$ runs over distinct coset representatives of H in $\pi_1(X, x_0)$. With this topology (Y_H, f_H) satisfies Def. 7.12. It also has an atlas of open sets inherited from X. If we show Y_H is Hausdorff, then (Y_H, f_H) is a cover of X. As usual, since X is Hausdorff, we have only to find disjoint open sets around two points over the same point of X. We have done exactly that above.

To complete classifying pointed covers of X, we show the following. Given (Y, f, y') a connected cover and $H(f, y')$ the corresponding subgroup of $\pi_1(X, x_0)$, and $(Y_{H(f,y')}, f_{H(f,y')}, y'_{H(f,y')})$ the cover of X associated to $H(f, y')$, then

(7.5) (Y, f, y') is equivalent to $(Y_{H(f,y')}, f_{H(f,y')})$.

For $y \in Y$ let $\gamma^* : [a, b] \to Y$ be a path from y' to y, and let $\psi(y) = f_H(\gamma^*)$. Follow the defined maps to see $\psi : Y \to Y_{H(f,\boldsymbol{y})}$ is a one-one map giving (7.5).

Suppose (Y_H, f_H, y_H) is the canonical cover defined by $H \le \pi_1(X, x_0)$. Let (Y_H, f_H, y'') by the same cover, those with a different point, $y'' \in f_H^{-1}(x_0)$. Any $\gamma \in \pi_1(Y, y_H, y'')$ defines a coset $H[\gamma]$ of H in $\pi_1(X, x_0)$. Conversely, the elements of $\pi_1(X, x_0)$ that stabilize $H[\gamma]$ are exactly the elements of the conjugate subgroup $[\gamma^{-1}]H[\gamma]$. That shows that using different points attached to a fixed cover correspond to subgroups conjugate to H.

Now suppose H_1 and H_2 are two subgroups of $\pi_1(X, x_0)$. We must show properties attached to the equivalence of two categories: Pointed covers of (X, x_0) and subgroups of $\pi_1(X, x_0)$. The notion of fiber product is a categorical construction. So, the association between $H_1 \cap H_2$ and $(Y_{\langle H_1,H_2\rangle}, f_{\langle H_1,H_2\rangle}, y'_{\langle H_1,H_2\rangle})$ is that they are the fiber products of the two givens in their respective categories. Def. 1.3 notes the fiber product for subsets of a set is just their intersection. As the intersection of two subgroups is a subgroup, the fiber product from subgroups of a group is just their intersection. For saying fiber product is categorical, see [9.3a]. Similarly, the correspondence between $\langle H_1, H_2\rangle$ and $(Y_{\langle H_1,H_2\rangle}, f_{\langle H_1,H_2\rangle}, y'_{\langle H_1,H_2\rangle})$ is that these are the *pushouts* of the two givens in their respective categories [9.3c].

8. Group theory and covering spaces

We won't be able to make explicit computations with covers until Chap. 4. Still, the topics of this section come from practical experience with covers. Following a discussion of algebraic functions (§8.2) and a geometric approach to the Galois closure of a cover (§8.3), we consider the decomposing covers (§8.4) and the relation between covers and locally constant bundles (§8.5). A problem from this on computing components of covers shows the power of an elementary piece from finite group representations (§8.6)

8.1. Corollaries of Thm. 7.16. Suppose (Y_i, f_i, y_i'), $i = 1, 2$, are any two pointed covers of (X, x_0). By an isomorphism $g : (Y_1, f_1, y_1') \to (Y_2, f_2, y_2')$ between them, we mean an isomorphism between Y_1 and Y_2 with these properties:

(8.1a) $g(y_1') = y_2'$ (g preserves basepoints); and
(8.1b) $f_2 \circ g = f_1$ (g commutes with projections).

The crucial point is that if two pointed covers are isomorphic, this isomorphism is unique. Suppose, however, we don't assume g preserves basepoints?

LEMMA 8.1 (Extension Lemma). *Consider a pair of covers (Y_i, f_i), $i = 1, 2$, without their basepoints, and any isomorphism g between them. Then, g maps the fiber $f_1^{-1}(x_0)$ one-one to $f_2^{-1}(x_0)$, and what g does to any one element of $f_1^{-1}(x_0)$ determines g. Further, isomorphisms between (Y_1, f_1) and (Y_2, f_2) correspond one-one with automorphisms $\mathrm{Aut}(Y_i, f_i)$ of (Y_i, f_i) (for either $i = 1$ or 2).*

Any automorphism of a cover (Y, f) of X lifts to an automorphism of the universal cover $(\tilde X, \tilde f)$ of X. If X is a complex manifold, then $\mathrm{Aut}(Y, f)$ is a group of complex analytic isomorphisms.

PROOF. Assume g that maps $y_1' \in f_1^{-1}(x_0)$ to $y_2' \in f_1^{-1}(x_0)$. Then, g is an isomorphism between (Y_1, f_1, y_1') and (Y_2, f_2, y_2'), and so it is unique. Let $A_{1,2}$ be the set of isomorphisms between (Y_1, f_1) and $Y_2, f_2)$. Then, we have an action of $\mathrm{Aut}(Y_1, f_1)$ (resp. $\mathrm{Aut}(Y_2, f_2)$) on the right (resp. left) of $A_{1,2}$:

$$\begin{aligned} A_1 : A_{1,2} \times \mathrm{Aut}(Y_1, f_1) \to A_{1,2} \text{ by } \quad (g, \alpha) \mapsto g \circ \alpha; \text{and} \\ A_2 : \mathrm{Aut}(Y_2, f_2) \times A_{1,2} \to A_{1,2} \text{ by } \quad (\beta, g) \mapsto \beta \circ g. \end{aligned}$$

For $g', g \in A_{1,2}$, $g^{-1}g' = \alpha$ is in $\mathrm{Aut}(Y_1, f_1)$. This shows $g \circ \alpha = g'$, and A_1 is transitive on $A_{1,2}$ (as in §7.1). Similarly, A_2 is transitive on $A_{1,2}$.

Now consider an automorphism α of (Y, f). Again, let (Y, f, y') with y' over x_0 be a corresponding pointed cover. Then, (Y, f, y') and $(Y, f, \alpha(y'))$ are pointed covers of (X, x_0). So, Thm. 7.16 shows they correspond to conjugate subgroups H and H_α: $H_\alpha = [\gamma^{-1}]H[\gamma]$ for some $[\gamma] \in \pi_1(X, x_0)$. A natural analytic isomorphism between (Y_H, f_H, y_H') and $(Y_{H_\alpha}, f_{H_\alpha}, y_{H_\alpha}')$ comes by mapping the homotopy class of $[\gamma']$ defining a point of Y_{H_α} (in §7.3.2) to $[\gamma][\gamma']$. The new base point (the coset of $[\gamma]$) has stabilizer $[\gamma^{-1}]H[\gamma]$. This automorphism lifts to the universal covering space, because premultiplying by $[\gamma]$ also defines it there. □

DEFINITION 8.2. Let $T_{\boldsymbol{y}} : \pi_1(X, x_0) \to S_n$ be the representation of (7.4) associated to (Y, f). The image of $\pi_1(X, x_0)$ is called the *(geometric) monodromy group*, $G(Y, f)$, of the cover. It is isomorphic to $\pi_1(X, x_0)/\bigcap_{i=1}^n \pi_1(Y, y_i)$ (Thm. 7.8).

Covers (Y, f) of a manifold (X, x_0) have two extremes. For most, $\mathrm{Aut}(Y, f)$ consists only of the identity element: We say (Y, f) has no automorphisms. The other extreme is in this definition.

DEFINITION 8.3. If $\mathrm{Aut}(Y, f)$ is transitive on the the fiber $f^{-1}(x_0)$, we say (Y, f) is Galois.

The Galois situation is our main tool, though what constantly arises in practice is the situation with no automorphisms. §8.3 has the details for distinguishing these and all the cases in between. An example of the Galois situation is the universal cover of (X, x_0) where the automorphism group is isomorphic to the whole fundamental group of (X, x_0). The fiber $f^{-1}(x_0)$ in this case corresponds to the elements of $\pi_1(X, x_0)$, and by translation these give a permutation of the points. Automorphisms also give a permutation of $f^{-1}(x_0)$. Still, from Lem. 8.8, only when $\pi_1(X, x_0)$ is abelian can we expect to canonically identify these two groups of permutations. The next lemma revisits Chap. 2 Prop. 3.2. As previously, use the notation $\tilde{f} : \tilde{X} \to X$ for the universal cover of X with paths starting at x_0 representing its points.

LEMMA 8.4. *In the notation above, let $[\gamma] \in \pi_1(X, x_0)$ and let $[\gamma']$ represent a homotopy class of paths on X with $\gamma' : [a, b] \to X$, $\gamma'(a) = x_0$ and $\gamma'(b) = x$. Then, multiplication by $[\gamma]^{-1}$ on the left of γ' induces an automorphism of $\tilde{X}$ giving an action $A_L : \pi_1(X, x_0) \times \tilde{X} \to \tilde{X}$. Regard the fiber $\tilde{f}^{-1}(x_0)$ as elements of $\pi_1(X, x_0)$. Then, the usual right action of $\pi_1(X, x_0)$ gives the group structure identifying $\pi_1(X, x_0)$ with the monodromy group of $\tilde{f}$.*

The exponential map $\exp : \mathbb{R} \to S^1$ by $\theta \mapsto e^{2\pi i\theta}$ presents $\mathbb{R}$ as the universal cover of S^1 with $\mathbb{Z}$ as its fundamental group. The path γ_n^* corresponds to $n \in \mathbb{Z}$ and the automorphisms of $(\mathbb{R}, \exp)$ identify with $\mathbb{Z}$ acting by translation. Similarly, the fundamental group of a complex torus $\mathbb{C}^n/L$ identifies with the lattice L.

PROOF. The universal covering space is unique up to homeomorphisms commuting with the map to X. One way to identify the fundamental group of a space X is to find any space $\tilde{X}$ with trivial fundamental group and a covering map $\tilde{f}:\tilde{X}\to X$. Given $x_0\in X$, any other cover of X that has trivial fundamental group must be isomorphic to $(\tilde{X},\tilde{f})$, and this isomorphism is unique up to composition on the left with an element of $(\tilde{X},\tilde{f})$. Since $\mathbb{R}$ and $\mathbb{C}^n$ are contractible, they have trivial fundamental group (Lem. 6.6). The map $\theta\in\mathbb{R}\mapsto e^{2\pi i\theta}$ is a covering map with the elements of $\mathbb{R}$ over 1 given by the integers. The permutation of the fiber over 1 given by the path γ_n^* is translation by n. The argument is similar for a complex torus. □

The next corollary tells when a map between spaces extends to a map between covers of the spaces.

COROLLARY 8.5. *Suppose $\varphi:X\to X'$ is a differentiable map between complex manifolds mapping a point $x_0\in X$ to $x_0'\in X'$. Let $\varphi_{H'}:Y'_{H'}\to X'$ be the cover defined by a subgroup $H'\le\pi_1(X',x_0')$. Then, there is a continuous (and so automatically differentiable) map $\psi:X\to Y'_{H'}$ with $\varphi_{H'}\circ\psi=\varphi$ if and only if the induced map $\varphi_*:\pi_1(X,x_0)\to\pi_1(X',x_0')$ has image in a conjugate of H'.*

PROOF. Suppose the induced map φ_* has image in a conjugate $m^{-1}H'm$ of H'. Let γ^* be a representative path in X' for which $[\gamma^*]=m$. Then, let $\gamma:[a,b]\to X$ start at x_0 and end at x. Define $\psi_{m,H'}:X\to Y'_{H'}$ by $\psi(x)$ is the class $m\cdot[\varphi\circ\gamma]\in Y'_{H'}$: the product of m and the image under ψ of γ. To show the map doesn't depend on γ, we consider another closed path γ' from x_0 to x. We are done if the closed path $(\gamma^*)^{-1}\cdot\psi(\gamma\cdot(\gamma')^{-1})\cdot\gamma^*$ in X' defines a closed path in $Y'_{H'}$. Since, however, $\gamma\cdot(\gamma')^{-1}$ is a closed path in X, its image under
phi_* is some $\rho\in m^{-1}H'm$ by hypothesis and the image of $(\gamma^*)^{-1}\cdot\psi(\gamma\cdot(\gamma')^{-1})\cdot\gamma^*$ is therefore $m\rho m^{-1}\in H'$. From the definition of $Y'_{H'}$ this exactly says the image path is closed.

Conversely, suppose there is such a $\psi:X\to Y'_{H'}$. Then, closed paths in X have image under ψ in X' that lift to closed paths in $Y'_{H'}$. So, the image group $\psi_*(\pi_1(X,x_0))=H^*$ is a subgroup of $\pi_1(X',x_0')$ whose corresponding cover Y'_{H^*} factors through $\psi_{H'}:Y'_{H^*}\to X'$. □

Suppose X is a connected complex manifold (like $U_{\boldsymbol{z}}=\mathbb{P}^1_z\setminus\{\boldsymbol{z}\}$). Define analytic continuation along a path from Def. 6.7. Consider the *extensible* functions $\mathcal{E}(X,x_0)$: complex analytic functions defined in a neighborhood of x_0 that have an analytic continuation along every path in X (as in Chap. 2 Def. 4.5). Let $\varphi:Y\to X$ be a cover with $y_0\in Y$ lying over x_0. Let $\gamma:[a,b]\to X$ be a path starting at x_0 with $\gamma^\dagger:[a,b]\to Y$ its unique path lift starting at y_0 (Lem. 7.13).

PROPOSITION 8.6. *There is an isomorphism (of rings) between $\mathcal{E}(Y,y_0)$ and $\mathcal{E}(X,x_0)$. In particular, for $(\tilde{X},\tilde{x}_0)$ the universal cover of (X,x_0), holomorphic functions on $\tilde{X}$ form a ring isomorphic to $\mathcal{E}(X,x_0)$. If φ is a finite cover of punctured Riemann surfaces, this induces an analytic isomorphism between $\mathcal{E}(Y,y_0)^{\rm alg}$ and $\mathcal{E}(X,x_0)^{\rm alg}$. These results hold with extensible meromorphic replacing extensible holomorphic functions.*

PROOF. Since φ is a cover, there is a disk neighborhood U_{x_0} of x_0 and a component U_{y_0} of $\varphi^{-1}(U_{x_0})$ with $y_0\in U_{y_0}$ on which φ maps one-one. So, restriction of a function $f\in\mathcal{E}(X,x_0)$ to U_{x_0} transports by φ^{-1} to a function $f\in U_{y_0}$. There is no harm in using the same notation to extend f along $\gamma^\dagger : [a,b] \to Y$ starting at y_0. Let γ be $\varphi \circ \gamma^\dagger$, and let $f^* : [a,b] \to \mathbb{P}^1_z$ be the continuous function defining the analytic continuation along γ. Define the analytic continuation of f along $\gamma^\dagger$ to be the same, f^*. This shows f is extensible in Y. Clearly, if f is algebraic (on X) it will also be algebraic on Y. □

8.2. The problem of identifying algebraic functions explicitly. Suppose $\tilde{\varphi} : \tilde{X} \to X$ is the universal covering space of a complex manifold X and $\tilde{x}$ lies over $x_0 \in X$. Then, similar to formation of complex torii and other quotient manifolds, it is natural to regard points of X as the orbits of the action of $\pi_1(X, x_0)$ on $\tilde{X}$. Riemann's approach was to identify the universal covering space of a Riemann surface as a simply connnected domain on the Riemann sphere. Consider the case of Prop. 8.6 when $Y = \tilde{X}$ and $X = U_{\boldsymbol{z}}$, with $|\boldsymbol{z}| \geq 3$. Riemann's Uniformization Theorem says $\tilde{X}$ is analytically isomorphic to a disk Δ in such a way that the map extends continuously to the boundaries (Chap. 4 Def. 7.9 for an elementary proof, or [**Spr57**, Thm. 9.6] for the more general case). So, $\mathcal{E}(U_{\boldsymbol{z}}, z_0)$ is ring isomorphic to convergent functions in a disk. We find it convenient to replace a disk by the analytically isomorphic upper half plane $\mathbb{H}$. This is the same exact space independent of $(z_0, \boldsymbol{z})$. What changes, however, with $\boldsymbol{z}$ is the identification of algebraic functions $F_{\boldsymbol{z}}$. Suppose $\varphi_{\boldsymbol{z}} : \mathbb{H} \to U_{\boldsymbol{z}}$ is this uniformization.

Elements of $\mathrm{PGL}_2(\mathbb{R})$ with positive determinant (Chap. 2 [9.14d]; this identifies with $\mathrm{PSL}_2(\mathbb{R})$) represent the action of complex analytic isomorphisms of $\mathbb{H}$. As $\boldsymbol{z}$ varies, a different subgroup $\Gamma_{\boldsymbol{z}}$ (though abstractly isomorphic as a group) of $\mathrm{PSL}_2(\mathbb{R})$ defines $U_{\boldsymbol{z}}$ as a quotient of $\mathbb{H}$.

Prop. 8.6 identifies extensible (meromorphic) algebraic functions on $U_{\boldsymbol{z}}$ with certain meromorphic functions $\mathcal{F}_{\boldsymbol{z}}$ on $\mathbb{H}$. Though, which ones? Given g^* meromorphic on $\mathbb{H}$, composing it with an analytic isomorphism of $\mathbb{H}$ produces a new meromorphic function on $\mathbb{H}$. We call the compositions of g^* with elements of $\Gamma_{\boldsymbol{z}}$ *transforms* by $\Gamma_{\boldsymbol{z}}$.

PROPOSITION 8.7. *Suppose f, meromorphic on $\mathbb{H}$, has only finitely many transforms under the action of $\Gamma_{\boldsymbol{z}}$ and a unique limit value as it approaches any point in $\mathbb{R} \cup \{\infty\}$. Then, f defines an algebraic element of $\mathcal{E}(U_{\boldsymbol{z}}, z_0)$ and conversely.*

OUTLINE. Let $\tilde{x} \in \mathbb{H}$ lie over $z_0 \in U_{\boldsymbol{z}}$. From Prop. 8.6, any meromorphic extensible function g on $U_{\boldsymbol{z}}$ identifies with a meromorphic function g^* on $\mathbb{H}$. Further, the analytic continuation of g around $[\gamma] \in \pi_1(U_{\boldsymbol{z}}, z_0)$ produces g^*_γ, the result of composing g^* with the analytic isomorphism of $\mathbb{H}$ associated to γ. If g is algebraic, then it has only finitely many analytic continuations, so the different transforms g^*_γ, running over $\gamma \in \pi_1(U_{\boldsymbol{z}}, z_0)$ are finite in number. Conversely, if the number of transforms of a meromorphic function g^* on $\mathbb{H}$ are finite in number, then the identification of g^* with $g \in \mathcal{E}(U_{\boldsymbol{z}}, z_0)$ gives a function with only finitely many analytic continuations. □

8.3. Galois theory and covering spaces. Use notation from Lem. 8.1: (Y, f) is a cover of X.

8.3.1. *Identifying automorphisms of a cover.* Having $\mathrm{Aut}(Y, f)$ act on a fiber $\{y_1, \ldots, y_n\} = f^{-1}(x_0)$ induces a homomorphism $\Lambda_{\boldsymbol{y}} : \mathrm{Aut}(Y, f) \to S_n$.

It is a mistake to confuse the Galois (geometric monodromy) group of a cover with its automorphism group, even if the cover is Galois. The next lemma efficiently differentiates $\mathrm{Aut}(Y,f)$ from $G(Y,f)$. It shows that having chosen a *right* action for $G(Y,f)$ forces using a left action of $\mathrm{Aut}(Y,f)$ on the set $\{1,\dots,n\}$.

LEMMA 8.8. *Let (Y,f) be a connected cover of X. The homomorphism $\Lambda_{\boldsymbol{y}}$ injects $\mathrm{Aut}(Y,f)$ onto the centralizer $\mathrm{Cen}_{S_n}(G(Y,f))$ of $G(Y,f)$ in S_n. This is isomorphic to $N_{\pi_1(X,x_0)}(\pi_1(Y,y_1))/\pi_1(Y,y_1)$ (§7.1) and $|\mathrm{Aut}(Y,f)|\le n$ with equality if and only if $\pi_1(Y,y_1)$ is normal in $\pi_1(X,x_0)$.*

PROOF. For $y\in Y$ let $\gamma^*:[a,b]\to Y$ be a path with initial point y_i and endpoint y. Consider $\psi\in\mathrm{Aut}(Y,f)$. Then $\psi\circ\gamma^*:[a,b]\to Y$ is the (unique) lift of $f\circ\gamma^*$ with initial point $\psi(y_i)$. So, if $i=1$ and $\psi(y_1)=y_1$, then $\psi\circ\gamma^*=\gamma^*$. Thus $\psi(y)=y$ for each $y\in Y$, and $\Lambda_{\boldsymbol{y}}$ is injective. This alone shows $|\mathrm{Aut}(Y,f)|\le n$.

In the above, assume $\gamma=f\circ\gamma^*$ is a closed path. If the endpoint of γ^* is y_j, then the endpoint of $\psi\circ\gamma^*$ is $\psi(y_j)$. Thus

$$(i)\Lambda_{\boldsymbol{y}}(\psi)^{-1}\circ T_{\boldsymbol{y}}(\gamma)\circ\Lambda_{\boldsymbol{y}}(\psi)=(i)T_{\boldsymbol{y}}(\gamma).$$

Equivalently, $\Lambda_{\boldsymbol{y}}(\psi)\in\mathrm{Cen}_{S_n}(G(Y,f))$. Conversely, for $\alpha\in\mathrm{Cen}_{S_n}(G(Y,f))$ define α to be a permutation of the points $\{y_1,\dots,y_n\}$ from its action on $\{1,\dots,n\}$. Still, use an action on the left: If $(i)\alpha=j$, write $\alpha(y_i)=y_j$). Our goal is to create an automorphism—also called α—on Y that extends this action on the fiber over x_0.

Take $i=1$ and γ^* as in the first paragraph above. Define ψ_{α,γ^*}:

(8.2) $\psi_{\alpha,\gamma^*}(y)$ is the endpoint of the lift of $f\circ\gamma^*$ with initial point $\alpha(y_1)$.

If we show $\psi_{\alpha,\gamma^*}(y)$ is independent of γ^* having endpoint y, then ψ_{α,γ^*} defines an element $\psi_\alpha\in\mathrm{Aut}(Y,f)$. For this purpose let γ^1 (resp., γ^2) be a path in Y with initial (resp., end) point y and end (resp., initial) point y_1. If $\psi_{\alpha,\gamma^*}(y)\ne\psi_{\alpha,\gamma^1}(y)$, then $\psi_{\alpha,\gamma^*\gamma^2}(y_1)\ne\psi_{\alpha,\gamma^1\gamma^2}(y_1)$. Therefore, $\psi_{\alpha,\gamma^*}(y)$ is independent of γ^* if and only if $\psi_{\alpha,\gamma^*}(y_1)$ is independent of γ^* for $\gamma^*\in\pi_1(Y,y_1)$. That is, we must show $\alpha(y_1)$ is the endpoint of the lift of $f\circ\gamma$ with initial point $\alpha(y_1)$ for each $\gamma\in\pi_1(Y,y_1)$.

With $\alpha(y_1)=y_j$, this is equivalent to $((1)\alpha)T(Y,f)(f\circ\gamma)=j$. (The right action of α on 1 is intentional—α did come from S_n.) For γ a closed path on Y with initial point y_1, $(1)T(Y,f)(f\circ\gamma)=1$ is automatic. Apply α to the right side of this and use that α commutes with $T(Y,f)(f\circ\gamma)$ to conclude from [9.15b]. Recall: $G(1)$ is the subgroup of $G(Y,f)$ leaving 1 fixed. Thm. 7.16 identifies $N_{\pi_1(X,x_0)}(\pi_1(Y,y_1))/\pi_1(Y,y_1)$ with $N_{G(Y,f)}(G(1))/G(1)$. □

8.3.2. *Fiber products and Galois closure.* We say a connected cover (Y,f) of X is a *Galois cover* (or is Galois) if $|\mathrm{Aut}(Y,f)|$ equals $n=\deg(f)$. By Lem. 8.8 this holds if and only if $\pi_1(Y,y_1)$ is a normal subgroup of $\pi_1(X,x_0)$). Each cover (Y,f) produces a Galois cover $(\hat{Y},\hat{f})$ of X called the Galois closure of (Y,f). If $H\le\pi_1(X,x_0)$ corresponds to Y, then $\cap g^{-1}Hg$ corresponds to $(\hat{Y},\hat{f})$. We use fiber products to give an alternate construction of it (Def. 1.3). It correctly displays the automorphism group action. We again warn: Don't confuse it with the geometric monodromy group, though they are isomorphic for a Galois cover.

Denote the fiber product of $Y\to X$ taken $n=\deg(f)$ times by

$$Y^n_X\overset{\text{def}}{=}Y\times_X\times\cdots\times_X Y.$$

Points of Y^n_X are n-tuples $(y_1',\dots,y_n')\in Y^n$ for which $f(y_i)=f(y_j)$ for all i and j. The *fat diagonal*, $\Delta_{Y,f,n}$, is the subset of n-tuples of Y^n_X with at least two equal coordinate entries. Remove it to form $Y^n_X\setminus\Delta_{Y,f,n}=U_{Y,f,n}$. We use a copy of S_n acting on the *left* of $\{1,\dots,n\}$ to give an action of automorphisms on this set:

(8.3) for $\sigma \in S_n$ and $\boldsymbol{y}' = (y'_1, \dots, y'_n) \in U_{Y,f,n}$, α_σ maps $\boldsymbol{y}'$ to

$$(y'_{\sigma(1)}, \dots, y'_{\sigma(n)}) = \alpha_\sigma(\boldsymbol{y}').$$

Restrict the natural map of Y^n_X to $X$to $U_{Y,f,n}$ to present $U_{Y,f,n}$ as a degree $n!$ cover of X with automorphism group containing S_n. The action of S_n is transitive on points mapping to x_0. Yet, $U_{Y,f,n}$ may not be connected. (We don't consider it a Galois cover of X.) Decompose $U_{Y,f,n}$ into connected components $\hat{Y}_1, \dots, \hat{Y}_t$. Let $\hat{f}_i$ be the restriction to $\hat{Y}_i$ of the projection map $U_{Y,f,n} \to X$, $i = 1, \dots, t$. A computation shows $\deg(\hat{f}_i) = |G(Y, f)|$ [9.22].

THEOREM 8.9. *The covers $(\hat{Y}_i, \hat{f}_i)$ are equivalent as covers of X, $i = 1, \dots, t$. Characterize members $(\hat{Y}, \hat{f})$ of this equivalence class from these properties.*

(8.4a) *$(\hat{Y}, \hat{f})$ is a Galois cover of X, with its group a transitive subgroup of S_n.*
(8.4b) *There is a commutative diagram of covers of X:*

$$\begin{array}{ccc} \hat{Y} & \xrightarrow{\hat{f}} & X \\ {\scriptstyle f_Y}\downarrow & \nearrow{\scriptstyle f} & \\ Y & & \end{array}$$

(8.4c) *For any Galois cover $\hat{g} : \hat{Z} \to X$ factoring through Y by $g_Y : \hat{Z} \to Y$, there is commutative diagram of covers of X:*

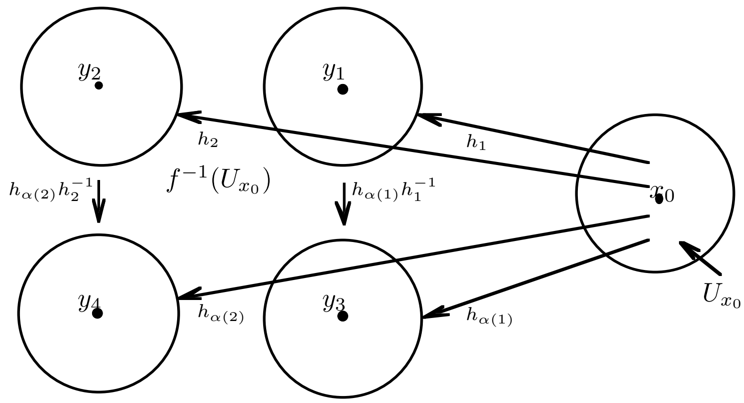
$$\begin{array}{ccccc} \hat{Z} & \xrightarrow{\hat{g}_Y} & \hat{Y} & \xrightarrow{\hat{f}} & X \\ & \searrow{\scriptstyle g_Y} & {\scriptstyle f_Y}\downarrow & \nearrow{\scriptstyle f} & \\ & & Y & & \end{array}$$

PROOF. Choose $y_1 \in Y$ lying over $x_0 \in X$. Thm. 7.3.2 identifies the subgroup of $\pi_1(X, x_0)$ corresponding to (Y, f) with $\pi_1(Y, y_1)$. It also identifies its conjugates (in $\pi_1(X, x_0)$) $\pi_1(Y, y_i)$ with y_i running over $f^{-1}(x_0)$. A Galois cover corresponds to a normal subgroup of $\pi_1(X, x_0)$. So, the *smallest* Galois cover mapping through (Y, f) corresponds to the *largest* normal subgroup, $H = \bigcap_{i=1}^n \pi_1(Y, y_i)$, of $\pi_1(X, x_0)$ contained in $\pi_1(Y, y_1)$. So, there is a cover with property (8.4c).

Let $\mathrm{pr}_1 : Y^n_X \to Y$ be projection onto the first factor, and let $f_{Y,i}$ be the restriction of pr_1 to $\hat{Y}_i$. Then, with $(\hat{Y}, \hat{f})$ (resp., f_Y) replaced by $(\hat{Y}_i, \hat{f}_i)$ (resp., $f_{Y,i}$) properties (8.4a) and (8.4b) hold, $i = 1, \dots, t$. This shows the map $h : \hat{Y}_i \to Y$ has degree 1: $(\hat{Y}_i, \hat{f}_i)$ and $(\hat{Y}, \hat{f})$ are equivalent covers of X. The proof is complete. □

Fig. 8 shows four discs on a degree 4 cover of $U_{\boldsymbol{z}}$ lying over a disk U_{z_0} around the base point. Assume the cover has monodromy group S_4. (Like that from a general degree 4 polynomial $f \in \mathbb{C}[w]$.) We visibly can see the action of any element $\alpha \in S_4$ on the four points of $f^{-1}(z_0)$ extend to the four disjoint disks over U_{z_0}. Yet, there is no continuous extension of any nonidentity α to $f^{-1}(U_{z_0})$. Lem. 8.8 says such extending αs must centralize the monodromy. We stipulated, however, this is S_4, a group with trivial center.

FIGURE 8. $\alpha = (1\,2)(3\,4) \in S_4$ tries, but fails, to be an automorphism of Y: The four discs on the left constitute $f^{-1}(U_{x_0})$



8.3.3. *Galois closure orbits.* Chap. 2 [9.5] has Galois exercises based on using fields. We now explain how these have analogs where we replace field extensions of a given field by covers of a given space. One tricky point: Composite of two fields makes sense only if there is given a priori a field L containing them both. As with the comments from §4.2.3 on local holomorphic functions, the next lemma shows fiber product of covers is dual to tensor product of fields. This analogy will come through even more when we deal with the field of meromorphic functions on a cover in Chap. 4 Prop. 2.11.

LEMMA 8.10. *Let K_i, $i = 1, 2$, be two finite extensions of a field K (having 0 characteristic). The ring $K_1 \otimes_K K_2$ is the direct sum of field extension of K. These summands are, up to isomorphism of extensions of K, in one-one correspondence with all compositions of K_1 and K_2.*

PROOF. Since the characteristic is 0 (only need separable extensions), the primitive element theorem says $K_2 = K(\alpha)$ for some $\alpha \in K_2$. Up to isomorphism of extensions, K_2/K is $K[x]/(f_2(x))$ with f_2 the irreducible polynomial for α over K. Factor f_2 as $\prod_{i=1}^{u} g_i(x)$ over K_1, with the g_i s monic and distinct. (Again use characteristic 0, or just that irreducible polynomials have no repeated roots.) Now apply Lem. 4.8 to write $K_1 \otimes K_2 = K_1[x]/(f_2(x))$ as $\oplus_{i=1}^{u} K_1[x]/(g_i(x))$. Since each of the g_i s is irreducible over K_1, each of the summands is a field. So each summand is a field generated by extensions of K isomorphic to K_1 and K_2.

Conversely, suppose L is a field containing K_1 and generated by K_1 and $K' = K(\alpha')/K$ with α' the image of α in an isomorphism of K_2/K with it. Then, L is isomorphic to one of the summands of $K_1 \otimes K_2$. This concludes the proof. $\square$

Suppose L_i/K (resp. $f_i : Y_i \to X$) is a field extension (resp. connected cover) of finite degree n_i, with G_i its Galois closure group and $\hat{L}_i/K$ (resp. $\hat{f}_i : \hat{Y}_i \to X$ its Galois closure field (resp. cover), $i = 1, 2$. As in Chap. 2 [9.6a], consider the fiber product H_f of G_1 and G_2 over the Galois group of the well-defined field extension $\hat{L}_1 \cap \hat{L}_2$. Then, $G(\hat{L}_1 \cdot \hat{L}_2/K)$ is H_f. The restriction of elements of H_f to $\hat{L}_i$ produces a permutation representation T_i, $i = 1, 2$. Now consider the direct product representation T_f of H_f induced from T_1 and T_2 (§7.1.2). The next lemma, in this analogy, shows different composites of field extensions correspond to the different components of the fiber product of the covers over X. The proof shows also that inequivalent composite extensions $L_1 \cdot L_2$ correspond one-one to orbits of T_f (compare with Chap. 2 [9.6c]).

LEMMA 8.11. *Let $g : Y \to X$ be the maximal cover through which $\hat{f}_i$, $i = 1, 2$, both factor. Then, g is a Galois cover. If M is its group, this induces homomorphisms $f_{i*} : G_i \to M$. Denote the fiber product of these group homomorphisms by H_c. Then, any connected component $\hat{Y}_{1,2}$ of $\hat{Y}_1 \times_X \hat{Y}_2$ (as a cover of X) is the minimal Galois cover of X factoring through $\hat{f}_i$, $i = 1, 2$. The group of this cover is H_c, a subgroup of $S_{n_1} \times S_{n_2}$ (acting on pairs (i, j), $1 \le i \le n_1, 1 \le j \le n_2$). Orbits of T_c correspond one-one to the components of $Y_1 \times_X Y_2$.*

PROOF. Let $\mathcal{C}_{\mathrm{Gal}}$ be the category of Galois covers of X up to isomorphism commuting with the map to X. Similarly, let $\mathcal{C}_{\mathrm{Nor}}$ be the category of normal subgroups of $\pi_1(X, x_0)$. The first part of the lemma is an equivalencing of fiber products in each of these categories (as at the end of the proof of Thm. 7.16). The fiber product for two normal subgroups of $\pi_1(X, x_0)$ is their intersection, which identifies the quotient as H_c in this case. Since the fiber product $\hat{Y}_1 \times_X \hat{Y}_2$ may not be connected, and therefore not Galois, this cannot be the fiber product in the category of Galois covers of X. A connected component, however, of it defines an equivalence class of connected and Galois covers. It is this that is the fiber product in the category $\mathcal{C}_{\mathrm{Gal}}$.

Now consider the statement on orbits of T_c. Since H_c factors through G_i, with its representation T_i, $i = 1, 2$, it makes sense to form the direct (tensor) product T_c of T_1 and T_2. Direct summands in the category of permutation representations correspond to components of covers in the category of covers of X. Since permutation representations correspond to equivalence classes of covers, to show the statement on orbits we have only to show that the direct product permutation representation T_c corresponds to the fiber product $Y_1 \times_X Y_2$. This is the equivalence of direct product in their respective categories. □

8.4. Imprimitive covers and wreath products. Suppose $f : Y \to X$ is a (connected) cover, and f factors through another cover $f_1 : Y_1 \to X$. That gives a series of covers $Y \xrightarrow{f_2} Y_1 \xrightarrow{f_1} X$. We say $f_1 \circ f_2$ is a decomposition of f if $\deg(f_i) > 1$, $i = 1, 2$. If there is no such decomposition of f, we say it is *indecomposable* or *primitive*. Equivalence two decompositions if the their corresponding covers $f_1 : Y_1 \to X$ are equivalent to give equivalence classes of decompositions. As $G(Y, f) \le S_n$, denote the subgroup stabilizing 1 by $G(Y, f)(1)$.

LEMMA 8.12. *The monodromy group $G(Y, f)$ is a primitive subgroup of S_n if and only if f is primitive (Def. 7.9). Equivalence classes of decompositions of f correspond one-one with subgroups properly between $G(Y, f)$ and $G(Y, f)(1)$.*

PROOF. Choose a basepoint $y_1 \in Y$ to apply Thm. 7.16. Groups between $G(Y, f)$ and $H_1 = \{g \in G(Y, f) \mid (1)g = 1\}$ correspond one-one to decompositions of f. In particular, f is primitive if and only if there no decomposition of f. □

Suppose G and H are groups, with $G_1 \le G$ and $H_1 \le H$. Let $T_{G_1} : G \to S_n$ and $T_{H_1} : H \to S_m$ be corresponding coset representations. Use T_{G_1} to have G act on H^n, the product of n copies of H:

(8.5) $g \in G$ acts by $(h_1, \ldots, h_n) \mapsto (h_{(1)T_{G_1}(g)}, \ldots, h_{(n)T_{G_1}(g)})$.

This gives a natural permutation representation $T_{H\wr G} : H\wr G \stackrel{\text{def}}{=} H^n \times^s G \to S_{nm}$ acting on a set $L = \{1_1, \dots, 1_m, 2_1, \dots, 2_m, \dots, n_1, \dots, n_m\}$ by this formula:

$$(i_j)T_{H\wr G}(h_1, \dots, h_n, g) = (i)T_G(g)_{(j)T_H(h_i)}.$$

Call $T_{H\wr G}$ the *wreath product* representation of T_G and T_H. Then, $H \wr G$ is the wreath product of G and H, though this assumes we know the corresponding permutations representations. Now consider how the wreath occurs in covering theory.

DEFINITION 8.13. Suppose $\psi : \hat{G} \to G$ is a cover of groups. Let T_{G_1} (resp. $T_{\hat{G}_1}$) be a faithful permutation representation of G (resp. $\hat{G}$). Call $T_{\hat{G}_1}$ an *extension* of T_{G_1} if ψ maps some conjugate of $\hat{G}_1$ maps surjectively to G_1: $T_{\hat{G}_1}$ extends T_{G_1}.

LEMMA 8.14. *Suppose $f : Y \to X$ is a (connected) cover, and f factors as a series of covers $Y \xrightarrow{f_2} Y_1 \xrightarrow{f_1} X$. Let G_{f_i} be the group of the Galois closure of f_i, with T_{f_i} the corresponding permutation representations, $i = 1, 2$. Use similar notation for f. Then, T_f extends T_{f_1}, G_f is a transitive subgroup of $G_{f_2} \wr G_{f_1}$ and $G_{f_1}(1)$ maps surjectively to the group G_{f_2}. Further, $G_f = G_{f_2} \wr G_{f_1}$ if and only if the kernel of $G_f \to G_{f_1}$ is isomorphic to $G_{f_2}^{deg(f_1)}$.*

PROOF. Choose a base point $y_0 \in Y$ and so image base points in Y_1 and X. Apply Thm. 7.16 to identify G_f (resp. G_{f_2}, G_{f_1}) with permutation representations of $\pi_1(X, f(y_0))$ (resp. $\pi_1(Y_1, f_2(y_0)), \pi_1(X, f(y_0))$) from the cosets of $\pi_1(Y, y_0)$ (resp. $\pi_1(Y, y_0), \pi_1(Y_1, f_2(y_0))$). The permutation representation of G_f comes from the image $G_f(1)$ of $\pi_1(Y, y_0)$ in G_f. Similarly, the permutation representation of G_{f_1} comes from the image of $G_{f_1}(1)$ of $\pi_1(Y_1, f_2(y_0))$ in G_{f_1}. As $\pi_1(Y_1, f_2(y_0))$ contains $\pi_1(Y, y_0)$, T_f extends T_{f_1}. All coset permutation representations are transitive.

With $x_0 = f(y_0)$, let $W = y_1, \dots, y_{\deg(f_1)}$ be the points of Y_1 lying over x_0. Similarly, let $W_i = \{y_{i,j_i}\}_{j_i=1,\dots,\deg(f_2)}$ be the points of Y lying over y_i. Intersecting the conjugates of $G_f(1)$ gives $K = \ker(G_f \to G_{f_1})$. So, K acts as permutations on each W_i, $i = 1, \dots, \deg(f_1)$. Restrict $G_f(1)$ to W_1 for the group G_{f_2} in the representation T_{f_2}. Similarly, identifying all sets W_i, embeds K as a subgroup of $G_{f_2}^{\deg(f_1)}$. This identifies G_f with a subgroup of the wreath product. Since the order of G_f is $|G_{f_1}||K|$, the index of G_f in $G_{f_2} \wr G_{f_1}$ equals $(G_{f_2}^{\deg(f_1)} : K)$. This gives the last statement of the lemma. □

8.5. Representations and groupoids. Rather than define *groupoid* generally, we present a classical case for later use. The idea is that of Deligne and Grothendieck. Deligne has a notion of (fundamental group) *realizations*. We think of these as ways a space declares its presence through types of analytic continuation. This helps us to explain the *profinite* fundamental group of a complex manifold (Chap. 4 §8.2). Mastering the *Hurwitz monodromy group* in Chap. 5 simplifies if we understand how a fundamental group depends on a base point. That leads to generalizing what will serve as a base point. Tangential base points (Chap. 2 §8.4) are an example. We get much mileage from a particularly significant parameter space, the classical j-line (Chap. 4 §7.8). This follows [**De89**, §10] which used the related λ-line.

8.5.1. *A law of composition.* Suppose $\mathcal{C}_X$ is the category of unramified covers of an complex manifold X. For $\varphi : Y \to X$ an unramified cover and $\psi : X' \to X$ any map of complex manifolds, there is a natural contravariant map $\psi^* : \mathcal{C}_X \to \mathcal{C}_{X'}$ through fiber products: $\psi^*(\varphi) =: X' \times_X Y \to X'$.

LEMMA 8.15. *The map ψ^* preserves fiber products. For $\varphi_1, \varphi_2 \in \mathcal{E}_X$:*

$$\psi^*(\varphi_1 \times_X \varphi_2) = \psi^*(\varphi_1) \times_{X'} \psi^*(\varphi_2).$$

PROOF. Seeing this set theoretically makes it clear the cover structures are compatible. First: Identify $(Y_1 \times_X Y_2) \times_X X'$ with $(Y_1 \times_X X') \times_{X'} (Y_2 \times_X X')$ by mapping (y_1, y_2, x') all lying over a given $x \in X$ to $((y_1, x'), (y_2, x'))$. Then, both maps send this element to x'. □

Let $\hat{\varphi} : \hat{Y} \to X$ be the Galois closure of this cover. Suppose this has group G. Then G acts faithfully and transitively on the fibers of $\hat{\varphi}$. On $\hat{Y} \times \hat{Y} \to X \times X$ let G act diagonally: $(\hat{y}_1, \hat{y}_2)g \stackrel{\text{def}}{=} ((\hat{y}_1)g, (\hat{y}_2)g)$.

Denote $\hat{Y} \times \hat{Y}/G$, the orbits of the action of G, by $\mathcal{G}$. Let $\mathcal{G}_{i,j}$ be the pullback of $\mathcal{G}$ to $X \times X \times X$ induced from the projection of $X \times X \times X$ on its (i, j) factors. For example, $\mathcal{G}_{1,2}$ consists of triples $(\hat{y}_1, \hat{y}_2, x_3)$ with $\hat{y}_i \in \hat{Y}$, $i = 1, 2$, and $x_3 \in X$.

This gives a *composition law* $\mathcal{G}_{1,2} \times \mathcal{G}_{2,3} \to \mathcal{G}_{1,3}$ respecting fibers over $X \times X \times X$. Here is what that means. For $(x_1, x_2, x_3) \in X \times X \times X$, let $(\hat{y}_1, \hat{y}_2, \hat{x}_3)$ (resp. $(x_1, \hat{y}'_2, \hat{y}'_3)$) represent a point of $\mathcal{G}_{x_1,x_2}$ the fiber of $\mathcal{G}_{1,2}$ (resp. $\mathcal{G}_{2,3}$) over (x_1, x_2) (resp. (x_2, x_3)). The composition law $\mathcal{G}_{x_1,x_2} \times \mathcal{G}_{x_2,x_3} \to \mathcal{G}_{x_1,x_3}$ uses the following formula. There is a unique $g \in G$ taking $\hat{y}'_2$ to $\hat{y}_2$. Define the product of $(\hat{y}_1, \hat{y}_2, x_3)$ and $(x_1, \hat{y}'_2, \hat{y}'_3)$ to be $(\hat{y}_1, x_2, (\hat{y}'_3)g)$.

We say $\mathcal{G} = \hat{Y} \times \hat{Y}/G \to X \times X$ is a *groupoid*. Most significant is that it induces a groupoid in $\mathcal{F}_{X'}$ by pullback, for each $\psi : X' \to X$.

8.5.2. *Fundamental groupoid.* There is a *fundamental groupoid* that dominates all (classical) groupoids over X. We define this directly, as it will appear in Chap. 5.

Consider this data: $x_1, x_2 \in X$, and D_i a simply connected (path-connected) neighborhood of x_i on X, $i = 1, 2$. Suppose $x'_i \in D_i$, $i = 1, 2$. To read the next lemma correctly, emphasize the word *canonical*.

LEMMA 8.16. *There is a canonical isomorphism (dependent on (D_1, D_2)):*

$$\psi_{D_1,D_2} : \pi_1(X, x_1, x_2) \to \pi_1(X, x'_1, x'_2).$$

PROOF. For γ_i any path from x_i to x'_i in D_i, $i = 1, 2$, map $\gamma \in \pi_1(X, x_1, x_2)$ to $[\gamma_1^{-1} \cdot \gamma \cdot \gamma_2] = [\gamma_1^{-1}][\gamma][\gamma_2] \in \pi_1(X, x'_1, x'_2)$. Under the hypotheses, $[\gamma_i]$ depends only on x_i, x'_i, D_i and not the particular choice of path. That shows the lemma. We will, however, confront repeatedly the dependence of ψ_{D_1,D_2} on (D_1, D_2). □

DEFINITION 8.17. The fundamental groupoid $\mathcal{P}_X$ of X consists of the disjoint union $\dot\cup_{x_1,x_2 \in X} \pi_1(X, x_1, x_2)$. The composition law for $\pi_1(X, x_1, x_2) \times \pi_1(X, x_2, x_3)$ is the usual path multipication: $[\gamma_{1,2}] \in \pi_1(X, x_1, x_2)$ times $[\gamma_{2,3}] \in \pi_1(X, x_2, x_3)$ is $[\gamma_{1,2}][\gamma_{2,3}] \in \pi_1(X, x_1, x_3)$.

Restriction of $\mathcal{P}_X$ to the diagonal of $X \times X$ is the *local system* of fundamental groups $\dot\cup_{x_1 \in X} \pi_1(X, x_1)$. For $x \in X$, restrict $\mathcal{P}_X$ to $X \times \{x\} \subset X \times X$ to get the universal cover of (X, x). Now we trace through an action of a groupoid on various locally constant sets.

8.5.3. *Action of a groupoid.* We recognized already that the category $\mathcal{C}_X$ consists of *locally constant finite sets* on X. That means, given $f : Y \to X$ an unramified cover, the topology on Y comes from an open cover $\mathcal{U}$ of X so that $f_U : Y_U \to U$ makes of Y_U a finite collection of disjoint copies of U. Generalizing the notion of covers allows defining related locally constant structures. We concentrate here on $\mathbb{V}_X$, the category of *locally constant —or flat —vector bundles* on X. Suppose V is a vector space over $\mathbb{C}$ (say, $\mathbb{C}^n$). Then, there is a natural fiber preserving addition and scalar multiplication with the expected properties on $V \times U$. An object $\mathcal{V} \in \mathbb{V}_X$ consists of an analytic map $L : \mathcal{V} \to X$ of manifolds with an open cover $\mathcal{U}$ having the following properties.

(8.6a) For $U \in \mathcal{U}$, there is an analytic isomorphism $\psi_U : \mathcal{V}_{U_i} \to V \times U_{\gamma(t_i)}$ so that $L_U : \mathcal{V}_U \to U$ and $\mathrm{pr}_U \circ \psi_U : \mathcal{V}_U \to U$ are the same.

(8.6b) Local constancy: For $U, U' \in \mathcal{U}$, with $U \cap U'$, an element of $\mathrm{GL}_n(\mathbb{C})$ gives $\psi_U^{-1} \circ \psi_{U'}$ restricted to $V \times (U \cap U')$ along each fiber.

(8.6c) A fiber preserving complex analytic addition and multiplication by $\mathbb{C}$ on $\mathcal{V}$ restricts over each $U \in \mathcal{U}$ to that structure on $V \times U$.

Note the right action in (8.6b). We say $\mathcal{V}$ is a *rank* n (locally constant, or *flat*) bundle. Two flat bundles $\mathcal{V}_1$ and $\mathcal{V}_2$ are *bundle isomorphic* if there is a compatible open cover $\mathcal{U}$ for both and a fiber preserving analytic isomorphism $\psi : \mathcal{V}_1 \to \mathcal{V}_2$. Suppose ψ intertwines (8.6) for $\mathcal{V}_2$ relative to $\mathcal{U}$ to that for $\mathcal{V}_1$ so that for each $U \in \mathcal{U}$, an element $g_U \in \mathrm{GL}_n(\mathbb{C})$ gives $\psi_{1,U}^{-1} \circ \psi \circ \psi_{2,U}$. Then, ψ is a *flat isomorphism*. Warning: Some bundle isomorphisms have no corresponding flat isomorphism.

Example 8.18 (Flat bundle from a cover). Let $f : Y \to X$ be a degree n cover (element of $\mathcal{C}_X$). For each $x \in X$, denote the space spanned over $\mathbb{C}$ by the points of $f^{-1}(x)$ by V_x. We explain why $\mathcal{V}_f \stackrel{\text{def}}{=} \dot\cup_{x \in X} V_x$ is a locally constant vector bundle on X by taking L_f to be the natural projection. Suppose $U \le X$ is open, $x' \in U$ and f_U identifies Y_U with $\dot\cup_{y' \in f^{-1}(x')} U_{y'}$ where $U_y \le Y$ maps one-one onto U. This means we have n sections to the map f_U. We also call these $y_1', \dots, y_n'$. So, for each $x \in U$, $\{y_i'(x)\}_{i=1}^n$ is a basis for V_x. Then, we have a natural analytic manifold topology on $\mathcal{V}_f$ by identifying $\mathcal{V}_{f,U}$ with $\mathbb{C}^n \times U$ by mapping the standard basis of $\mathbb{C}^n$ to $y_1'(x), \dots, y_n'(x)$ running over $x \in U$.

Suppose $\mathcal{P}$ is a groupoid on X and $\mathcal{V} \in \mathbb{V}$. Regard $\mathcal{P}$ as a locally constant bundle of sets over $X \times X$. Consider the fiber products $\mathrm{pr}_i^*(\mathcal{V}) \stackrel{\text{def}}{=} \mathcal{V} \times_X (X \times X)$, using $\mathrm{pr}_i : X \times X \to X$, projection on the ith factor, $i = 1, 2$. We say $\mathcal{P}$ acts on $\mathcal{V}$ if there is a fiber preserving analytic map

$$A_X : \mathrm{pr}_1^*(\mathcal{V}) \times_{X \times X} \mathcal{P} \to \mathrm{pr}_2^*(\mathcal{V}). \tag{8.7}$$

Regard each term $\mathcal{P}$, $\mathrm{pr}_1^*(\mathcal{V})$ and $\mathrm{pr}_2^*(\mathcal{V})$ as a locally constant bundle over $X \times X$.

Denote the vector space $\mathbb{C}^n$ (with its canonical basis understood) by V, so that there is an action of $\mathrm{GL}_n(\mathbb{C})$ on the right of V. (To adjust to a left action on $\mathrm{GL}_n(\mathbb{C})$, see Ex. [9.16f].) For $x_0 \in X$, and n a positive integer, consider pairs $(\mathcal{V}, m_{x_0})$ with $\mathcal{V}$ a flat bundle of rank n, and m_{x_0} a fixed vector space isomorphism of $\mathcal{V}_{x_0}$ with V, by $\mathbb{V}_{x_0,n}$. Compose m_{x_0} with any element of $\mathrm{GL}_n(\mathbb{C})$ gives a natural action of $\mathrm{GL}_n(\mathbb{C})$ on the pairs $(\mathcal{V}, m_{x_0})$.

Proposition 8.19. *The fundamental groupoid $\mathcal{P}_X$ acts on every $\mathcal{V} \in \mathbb{V}_X$. Each $(\mathcal{V}, m_{x_0}) \in \mathbb{V}_{x_0,n}$ produces $\alpha_{\mathcal{V},m_{x_0}} \in \mathrm{Hom}(\pi_1(X, x_0), \mathrm{GL}_n(\mathbb{C}))$ and the map $(\mathcal{V}, m_{x_0}) \mapsto \alpha_{\mathcal{V},m_{x_0}}$ is one-one and onto. Flat rank n bundles up to flat isomorphism correspond to elements of* $\mathrm{Hom}(\pi_1(X, x_0), \mathrm{GL}_n(\mathbb{C}))/G$.

Proof. Consider $(x, x') \in X \times X$ and $[\gamma] \in \pi_1(X, x, x')$. We give an action: $A_X((v, x, x'), [\gamma]) = (v', x, x')$ with $v \in \mathcal{V}_x$ and $v' \in \mathcal{V}_{x'}$. The construction, for a given path γ, is exactly as in the proof of Lem. 7.13. We set appropriate notation.

If $\gamma : [a,b] \to X$, then there is a partition $t_0 = a < t_1 < \cdots < t_n = b$ and contractible open subsets $U_{\gamma(t_i)}$, $i = 0,\dots,n$, with $U_{\gamma(t_i)} \cap U_{\gamma(t_{i+1})}$ contractible, $i = 0,\dots,n-1$, so the following holds.

(8.8a) $\psi_{U_i} : \mathcal{V}_{U_i} \to V \times U_{\gamma(t_i)}$ is one of the maps given by (8.6).

(8.8b) $\gamma_{[t_{i-1},t_{i+1}]} \le U_{\gamma(t_i)}$, $i = 0,\dots,n$, with the provisos $t_{-1} = a$ and $t_{n+1} = b$.

Since the path γ has the information about the endpoints in it, we may simplify notation by rewriting our expression for A_X as $A_X(v,[\gamma]) = v'$ with v (resp.v') in the beginning (resp. end) point of γ. Inductively define $A_X(v,\gamma_{[t_0,t_{k+1}]}) = v_{k+1}$:

$$A_X(A_X(v,[\gamma_{[t_0,t_k]}]),[\gamma_{[t_k,t_{k+1}]}]) = A_X(v_k,[\gamma_{[t_k,t_{k+1}]}]) = (v_k)(\psi_{U_k})^{-1} \circ \psi_{U_{k+1}}.$$

That defines the action for a particular path. We need to know the result doesn't depend on the partition, nor on the homotopy class of γ. Starting from the definition of the action on γ with a partition, apply the General Monodromy Theorem 6.11 proof. (Our contractibility assumptions on the U_i s allow us to use this proof.) Line-for-line this shows A_X depends only on the homotopy class $[\gamma]$ and not on γ.

Define $\alpha_{\mathcal{V}}$ as $\prod_{k=0}^{n-1}(\psi_{U_k})^{-1} \circ \psi_{U_{k+1}}$. We use that the constituent elements are in $\mathrm{GL}_n(\mathbb{C})$ (locally constant as a function of $x \in X$), and that the result is independent of the homotopy class of the path to see it is a homomorphism. Now consider when two flat bundles are flat isomorphic.

Notice that the collection of isomorphisms $\psi_U : \mathcal{V}_U \to V \times U$ gives a cocycle condition: For U, U', U'' intersecting nontrivially,

$$(\psi_U^{-1} \circ \psi_{U'}) \circ (\psi_{U'}^{-1} \circ \psi_{U''}) = \psi_U^{-1} \circ \psi_{U''}.$$

Apply Lem. 2.2 to see that $\mathcal{V}$ identifies with the disjoint union of $\cup_{U\in\mathcal{U}} V \times U$ modulo the equivalence of points on $V \times U$ with $V \times U'$ on the overlap of $U \cap U'$ by $\psi_U^{-1} \circ \psi_{U'}$. Using this, a flat isomorphism between $\mathcal{V}_1$ and $\mathcal{V}_2$ interprets as the existence of $g_U \in \mathrm{GL}_n(\mathbb{C})$ for which

$$g_U^{-1} \circ \psi_{1,U}^{-1} \circ \psi_{1,U'} \circ g_{U'} = \psi_{2,U}^{-1} \circ \psi_{2,U'}.$$

In running around any path given by a sequence of U_i s, the conclusion is that $\alpha_{\mathcal{V}_1}$ differs from $\alpha_{\mathcal{V}_2}$ on this path by conjugation by g_{U_0}. That effect is determined by its effect on m_{x_0}. This concludes the proof of the theorem. □

8.6. Complete reducibility and covers with equivalent flat bundles. Flat bundles appear in a few well-known papers long ago. [**Gun67**, p. 97], from which the author first heard of these subjects many years ago, cites [**We38**] and [**At57**]. Riemann knew of the distinction between holomorphic vector bundles and flat bundles through his investigation general ordinary differential equations versus differential equations with ordinary singular points. This topic appears in Chap. 4. An advanced reader will note we have yet to define general holomorphic bundles.

8.6.1. *Decomposing the representations of a cover.* A cover $f : Y \to X$ has a flat bundle on X associated with it (Ex. 8.18). Let $\rho_X \in \mathrm{Hom}(\pi_1(X,x_0),\mathrm{GL}_n(\mathbb{C}))$ be the associated homomorphism. We explore the natural map $\mathcal{E}_X \to \mathcal{V}_X$, especially noting it is not injective. [**Sch70**] and [**Fri73**] are sources for practical problems in which this becomes significant. In particular, Chap. 4 [11.12] uses Riemann's Existence Theorem on the groups of [9.20] to produce primitive, inequivalent covers whose fibers products are reducible. This is a chance to introduce the significant topic of *complete reducibility* for fundamental groups representations.

DEFINITION 8.20. Let G be a group and F a field. Suppose $\rho : G \to GL_n(F)$ is a representation of G. Then, ρ has an *invariant subspace* $V \le F^n$ if $\rho(g)$ maps V into V for each $g \in G$. A representation is *irreducible* if it has no invariant subspace. Two invariant subspaces V and W (for φ) are *complements* if V and W span F^n, and $V \cap W = \{0\}$. Call ρ completely reducible if every ρ invariant subspace V has a complement.

Recall: With R a ring, $r \in R$ is an *idempotent* if $r^2 = r$. Idempotents in $\mathbb{M}_n(F)$ are the matrices of projection onto subspaces of F^n.

LEMMA 8.21. *Suppose V is a ρ invariant subspace. If F has characteristic 0, then V has a complement.*

PROOF. Let $P : F^n \to V$ be any projection onto V: Choose a basis $v_1, \dots, v_k$ of V, extend to a basis $v_1, \dots, v_n$ of V, and define P by $\sum_{i=1}^n a_i v_i \mapsto \sum_{i=1}^k a_i v_i$. Then, $P^2 = P$ and P is an *idempotent*. So, too is $I_n - P$=P', and it defines a complementary space by projection. If P commutes with the action of G, then $I_n - P$ would also be a G invariant subspace. To get this, *average over* G: Replace P with $P_G = \frac{1}{|G|}\sum_{g\in G} \rho(g)^{-1}P\rho(g)$. Since each term $\rho(g)^{-1}P\rho(g)$ acts like the identity on V, for $v \in V$, $(v)P_G = \frac{1}{|G|}\sum_{g\in G}(v)\rho(g)^{-1}P\rho(g) = v$. □

[9.19] applies the complete reducibility of finite group representations when F has zero characteristic. Complete reducibility does in general if either G is infinite or F has positive characteristic [9.17]. If a representation ρ is completely reducible, then we may write F^n as $\oplus_{i=1}^k V_i$, a direct sum of invariant and irreducible subspaces for the action of G. Another notation for this is $\rho = (\rho_1, \dots, \rho_k)$ with ρ_i restriction of ρ to the space V_i: ρ is the direct sum of the actions of the ρ_i, $i = 1, \dots, k$.

The notation $\mathbf{1}_G$ is for the one-dimensional representation of G where the action of G leaves each vector fixed. Given any representation ρ there is natural *conjugate representation* $\bar{\rho}$: $g \mapsto \bar{\rho}(g)$ by applying $^-$ to each entry of $\rho(g)$.

8.6.2. *Components of fiber products.* Suppose $f_i : Y_i \to X$ is a connected cover of degree n_i, with $\rho_{f_i} \in \mathrm{Hom}(\pi_1(X, x_0), \mathrm{GL}_{n_i}(\mathbb{C}))$ the corresponding element from Prop. 8.19, $i = 1, 2$. Then, ρ_{f_1} and ρ_{f_2} induce the tensor product representation $\rho_{f_1} \otimes \rho_{f_2} \in \mathrm{Hom}(\pi_1(X, x_0), \mathrm{GL}_{n_1 n_2}(\mathbb{C}))$. Let G_i be the group of a Galois closure $\hat{Y}_i \to X$ of f_i, $i = 1, 2$. Lem. 8.11 shows each of these representations factors through a faithful representation of $G = G_1 \times_H G_2$ for some group H that is a quotient of both G_1 and G_2. Here G is the group of the minimal Galois cover of X factoring through f_1 and f_2. Use the notation $\rho_{f_1} \otimes \rho_{f_2}$ for this representation, too. Since G is a finite group, each representation is completely reducible. Any representation of G_i induces a representation of G through the canonical projection of G onto G_i. Write $\rho_{f_i} = \oplus_{j=1}^{k_i} V_{i,j}$, $i = 1, 2$, indicating the irreducible representations of G coming from those of G_i, $i = 1, 2$.

PROPOSITION 8.22. *The number of connected components of the fiber product $Y_1 \times_X Y_2$ is the same as the number of times the identity appears in $\rho_{f_1} \otimes \rho_{f_2}$. In turn, this is the same as the number of distinct pairs (j, j') where $V_{1,j}$ is equivalent to the conjugate of $V_{2,j'}$.*

If $G = G_1 = G_2$, and $\rho_1 = \rho_2$, $Y_1 \times_X Y_2$ has at least two connected components. In this case it has precisely two if and only if the permutation representation associated with f_1 (or with f_2) is doubly transitive.

Proof. Apply Lem. 8.11 to conclude there are as many connected components in $Y_1 \times_X Y_2$ as the number of orbits in the direct product applied to G of the permutation representations attached to f_1 and f_2. This counts the appearances of the identity in the corresponding representation which in turn counts the number of appearances of the identity in $\rho_{f_1} \otimes \rho_{f_2}$. Use the representation theory reminders in [9.19b] to see this also counts the number of pairs (j, j') listed in the statement of the proposition. This completes the first part of the proof.

Suppose $\rho_T = \oplus_{j=1}^k V_{T,j}$ is the decomposition of $\rho_1 \otimes \rho_2$ given in the statement into irreducible representations (over $\mathbb{C}$). A permutation representation is the same as its conjugate. So, for each $V_{T,j}$, its conjugate also appears in the summands of ρ_T. If $\rho_1 = \rho_2$ and $G = G_1 = G_2$, besides the identity in both ρ_1 and ρ_2, there must exist at least one other pair indexed by (j, j') of conjugate representations. From [9.19d], $k = 2$ if and only if the permutation representation is doubly transitive. If, however, $k \geq 3$, there will be at least three pairs (j, j') indicating corresponding pairs of conjugate represenations. This concludes the proof. □

9. Exercises

We apply group theory exercises here to geometric applications in Chap. 4. [**FH91**] contains a hurried encyclopedic account of classical representations. Yet, it doesn't cover our later needs. [**Ben91**] (very concise) and older relaxed texts like [**Ha63**] work for Riemann surface applications requiring deeper group theory. We have exercises that prepare some characteristic p representations. These appear in *Modular Towers* (Chap. 5). Representation theory changes as much as Riemann surface theory. As [**Lam98**, p. 369] notes, it is about 100 years old. Even such topics as *higher characters* from its beginnings — unlike linear characters these do determine the group — have still an uncertain place in the theory.

9.1. Constructing manifolds. Call a topological space a *pre-manifold* if it has coordinate charts, but is not necessarily Hausdorff. We characterize Hausdorff.

(9.1a) Show the space of Ex. 2.4 is not Hausdorff.

(9.1b) Prove Lemma 2.5 using the argument before it.

(9.1c) Let $\{(X_{\alpha_i}, \varphi_{\alpha_i})\}_{\alpha_i \in I_i}$ (resp., $\{(Z_\alpha, \varphi_\alpha)\}_{\alpha \in I}$) be topological data for X_i (resp., Z), $i = 1, 2$. Let $f_i : X_i \to Z$, $i = 1, 2$ be continuous. Show

$$\{(X_{\alpha_i} \times X_{\alpha_j}) \cap (X_1 \times_Z X_2), (\varphi_{\alpha_i}, \varphi_{\alpha_j})\}_{(\alpha_i, \alpha_j) \in I_1 \times I_2}$$

gives topologizing data on $X_1 \times_Z X_2$ with continuous projections pr_i : $W \stackrel{\text{def}}{=} X_1 \times_Z X_2 \to X_i$, $i = 1, 2$. Further, W is Hausdorff if X_1, X_2 and Z are. Use this to prove Lemma 4.3.

(9.1d) Let $f : X \to Y$ be continuous, with X and Y pre-manifolds. Let γ : $[0, 1] \to Y$ be a path. If a continuous $\gamma_1 : [0, 1) \to X$ lies over $\gamma_{[0,1)}$ ($f \circ \gamma_1(t) = \gamma(t)$ for $t \in [0, 1)$). Show: For all pairs (γ, γ_1), there is at most one extension of γ_1 to a path $\gamma_1^* : [0, 1] \to Y$ if and only if the diagonal in $X \times_Y X$ is closed. Call an f satisfying this *separated*.

(9.1e) With f in d) separated, consider extending γ_1 to $\gamma_1^* : [0, 1] \to Y$. Show: Such γ_1^* exists (for each γ_1) if and only if f is a proper map (§2.2).

Consider some manifolds (differentiable) from vector calculus.

(9.2a) If X_i is n_i-dimensional, $i = 1, 2$, show $X_1 \times X_2$ is n_1+n_2-dimensional.

(9.2b) The n-sphere is $S^n = \{\boldsymbol{x} \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = 1\}$. Here is some data for defining a manifold structure on S^n:

$$U^+ = \{(x_1, \dots, x_{n+1}) \in S^n \mid x_{n+1} > 0\}$$

and $R_{\boldsymbol{x}}$ is any rotation of the sphere that that takes $\boldsymbol{x}$ to $(0, \dots, 0, 1)$. Let $U_{\boldsymbol{x}}$ be the image of U^+ under $R_{\boldsymbol{x}}^{-1}$, and define $\varphi_{\boldsymbol{x}}$ to be $\text{pr} \circ R_{\boldsymbol{x}}$ where $\text{pr}(\boldsymbol{x}) = (x_1, \dots, x_n)$. Show the $(U_{\boldsymbol{x}}, \varphi_{\boldsymbol{x}})$'s are a differentiable atlas on S^n.

(9.2c) Consider $f \in \mathbb{R}[x_1, \dots, x_n]$ and the set $X_f = \{\boldsymbol{x} \in \mathbb{R}^n \mid f(\boldsymbol{x}) = 0\}$. Let $X_f^0 = \{\boldsymbol{x} \in X_f \mid \nabla(f)(\boldsymbol{x}) \neq 0\}$ (Lemma 3.2). State a differentiable version of the *implicit function theorem* [**Rud76**, p. 224] from Chap. 2 §6.2.

(9.2d) Assume $n = 3$ in c) and two open sets U_1 and U_2 with these properties: $\frac{\partial f}{\partial x_1}$ is nonzero in U_1 and $\frac{\partial f}{\partial x_3}$ is nonzero in U_2. Apply c) to conclude there is a differentiable transition function $\varphi_2 \circ \varphi_1^{-1}$ for the pair (U_1, U_2).

(9.2e) If X_f^0 is nonempty, show it is a differentiable $n-1$ dimensional manifold.

(9.2f) State a complex analog of c) for $f \in \mathbb{C}[z_1, \dots, z_n]$ using complex partials. How does this show the complex version of X_f^0 is an $n-1$ dimensianal analytic manifold?

(9.2g) Apply the fundamental theorem of algebra [**Ahl79**, p. 122] to show the manifold in f) cannot be compact.

Fiber products and pushouts are categorical constructions. Chap. 4 [11.10] continues this exploration.

(9.3a) The fiber product of two maps $f_i : Y_i \to X$, $i = 1, 2$, satisfies the following universal property: If $f : Y \to X$ factors through each of the f_i s, then f factors through (f_1, f_2). Further, (f_1, f_2) is universal for this property.

(9.3b) The pushout for $f_i : Y_i \to X$, $i = 1, 2$, satisfies a reverse diagram to the fiber product. It is the maximal object through which both f_i, $i = 1, 2$, factor. For subsets of a set, the pushout would be the union. Show the pushout of pointed covers is exactly as given in Thm. 7.16.

(9.3c) For subgroups of a group, the union is not a group. Show the subgroup generated by the two groups is the pushout.

9.2. Complex structure and torii. Going from $\mathbb{R}$ to $\mathbb{C}$ is partly a linear algebra constraint. Use the identifications $\{L_n\}_{n=1}^\infty$ of $\mathbb{R}^{2n}$ and $\mathbb{C}^n$ in §3.1.2. Consider replacing $\{L_n\}_{n=1}^\infty$ by the sequence $\{L'_n\}_{n=1}^\infty$ of linear (invertible) maps (from $\mathbb{R}^{2n} \to \mathbb{C}^n$). Denote $(x_1, y_1, \dots, x_n, y_n) \mapsto (-y_1, x_1, \dots, -y_n, x_n)$ by J_n.

(9.4a) Show with L'_n in place of L_n, though the functions labeled analytic in any neighborhood of an analytic manifold X will change, the set of n-dimensional analytic manifolds remains the same.

(9.4b) Show, for analytic manifolds X and Y (possibly of different dimensions), the set of analytic maps X to Y using $\{L_n\}_{n=1}^\infty$ map naturally to the corresponding set using $\{L'_n\}_{n=1}^\infty$.

(9.4c) Show $\{L'_n\}_{n=1}^\infty$ gives the same analytic functions on each analytic manifold as $\{L_n\}_{n=1}^\infty$ if and only if $L'_n = B_n \circ L_n$ with $B_n \in \text{GL}_n(\mathbb{C})$ for all n. Further, this is equivalent to $L'_n \circ J_n = i \cdot L'_n$ for all n. Hint: Check on $\mathbb{C}$ linear combinations of $z_1, \dots, z_n$ in $\mathbb{C}^n$ using L'_n. Also: Invertible $\mathbb{R}$ linear maps $\mathbb{C}^n \to \mathbb{C}^n$ are in $\text{GL}_n(\mathbb{C})$ if and only if they commute with i.

(9.4d) Consider the case $L = L_n : \mathbb{R}^{2n} \to \mathbb{C}^n$ by

$$(x_1, y_1, \dots, x_n, y_n) \mapsto (x_1 - iy_1, \dots, x_n - iy_n) = (\bar{z}_1, \dots, \bar{z}_n)$$

for examples where using $\{L'_n\}_{n=1}^{\infty}$ changes a given analytic structure. Hint: See Chap. 4 §**??**.

Consider the topology of the torus of Fig. 3.

(9.5a) Show the complex torus $\mathbb{C}/L(\omega_1,\omega_2)$ of § 3.2.2 is compact.

(9.5b) Suppose $R > 3r$ with $r, R \in \mathbb{R}$. The *torus*, $T_{r,R;\boldsymbol{x}_0,\boldsymbol{v}} =$, with radii (r, R) centered at $\boldsymbol{x}_0 = (0,0,0) \in \mathbb{R}^3$ and perpendicular to $\boldsymbol{v} = (0,0,1)$ has this underlying set of points:

$$\{\boldsymbol{x}_0 + R(\cos(\theta),\sin(\theta),0) + r(\cos(\theta)\cos(\beta),\sin(\theta)\cos(\beta),\sin(\beta))\}_{\theta,\beta\in[0,2\pi]}.$$

Show $T_{r,R;\boldsymbol{x}_0,\boldsymbol{v}}$ is differentiably isomorphic to $\mathbb{C}/L(\omega_1,\omega_2)$.

(9.5c) Consider the two torii in Fig. 2: Assume one is $T = T_{r,R;\boldsymbol{x}_0,\boldsymbol{v}}$, the other $T' = T_{r,R;\boldsymbol{x}'_0,\boldsymbol{v}'}$ for vectors $\boldsymbol{x}'_0, \boldsymbol{v}' \in \mathbb{R}^3$ and $T \cap T' = \emptyset$. Call T and T' *unknotted* if for any $C > 0$ there is a continuous function

$$F : [0,1] \times \mathbb{R}^3 \setminus T \to \mathbb{R}^3 \setminus T$$

with $F(0,y) = y$ for $y \in \mathbb{R}^3 \setminus T$ and $|F(1,y)| > C$ for $y \in T'$. Otherwise they are knotted. Show there are two knotted torii in $\mathbb{R}^3$.

(9.5d) Regard $\mathbb{R}^3$ as in in $\mathbb{R}^4$: It is the set of $\boldsymbol{x} \in \mathbb{R}^4$ with $x_4 = 0$. Extend the definitions above to show any pair of torii in $\mathbb{R}^3$ is unknotted in $\mathbb{R}^4$.

We start discussing the nature of the lattice attached to a complex torus.

(9.6a) Let $\mathbb{C}/L(\omega_1,\omega_2) = X$ be a complex torus with lattice $L(\omega_1,\omega_2) = L$ as in Ex. 6.18. For $z_1, z_2 \in \mathbb{C}$ define $m(z_1 \bmod L, z_2 \bmod L)$ to be $z_1 + z_2 \bmod L$. Define the inverse of $z \bmod L$ to be $-z \bmod L$. Show X is a differentiable group with multiplication m.

(9.6b) For $t \in \mathbb{R}$, let $z(t) = \cos(2\pi t) + \sqrt{-1}\,\sin(2\pi t)$. Use $f : X \to S^1 \times S^1$ by $t_1\omega_1 + t_2\omega_2 \mapsto (z_1(t), z_2(t))$ to conclude that $\pi_1(X, 0 \bmod L)$ identifies with L as a group isomorphic to $\mathbb{Z}^2$, pairs of integers.

(9.6c) Suppose $x_1, x_2 \in S^1$ generate an infinite group $\langle x_1, x_2\rangle$. Consider the collection $T_N = \{x_1^j x_2^{j'}\}_{-N\le j,j'\le N}$ for large N to conclude 1 is a limit point for $\langle x_1, x_2\rangle$. Conclude: $w_1, w_2 \in \mathbb{C}$, $\mathbb{C}/L(w_1,w_2)$ satisfies the conditions of Lem. 2.3 only if w_1, w_2 lie on different lines through the origin.

Consider comparing two lattices of complex torii. With $L_i = L(\omega_{1,i},\omega_{2,i})$, $i = 1, 2$, continue Ex. 6.18. Assume $\lambda_i = \frac{\omega_{1,i}}{\omega_{2,i}} \in \mathbb{C} \setminus \mathbb{R}$, $i = 1, 2$.

(9.7a) Assume $\lambda_2 = \frac{a\lambda_1+b}{c\lambda_1+d}$ for some $a,b,c,d \in \mathbb{Z}$ for $ad - bc = 1$. Show $\mathbb{C}/L_1 = X_1$ and $\mathbb{C}/L_2 = X_2$ are analytically isomorphic. Hint: Map $t_1\omega_{1,1} + t_2\omega_{2,1}$ to $t_1(a\omega_{1,1} + b\omega_{2,1})\alpha + t_2(c\omega_{1,1} + d\omega_{2,1})\alpha$ with $\alpha \in \mathbb{C}$ satisfying

$$(a\omega_{1,1} + b\omega_{2,1})\alpha = \omega_{1,2} \text{ and } (c\omega_{1,1} + d\omega_{2,1})\alpha = \omega_{2,2}.$$

(9.7b) Why assume $ad - bc = 1$ in a)? Why must we have a,b,c,d in $\mathbb{Z}$, rather than just $a,b,c,d \in \mathbb{R}$?

(9.7c) Suppose $L_1 \subset L_2$. Consider $f : X_1 \to X_2$ given in Ex. 6.18. Show there exist $\omega_1, \omega_2 \in L_2$ and $n_1, n_2 \in \mathbb{Z}$ with these properties: $L(\omega_1,\omega_2) = L_1$; and the complex numbers

$$z(k_1,k_2) = (\tfrac{k_1}{n_1})\omega_1 + (\tfrac{k_2}{n_2})\omega_2,\ 0 \le k_i \le n_i,\ i = 1,2,$$

give the n_1n_2 distinct elements $z \bmod L_1$ mapping to $0 \bmod L_2$. Hint: Apply the Elementary Divisor Theorem Chap. 2 [9.15] to get a basis $\{\boldsymbol{u}_i\}_{i=1}^2$ of L_2 and integers $n_1, \ldots, n_2$ so that $\{n_i\boldsymbol{u}_i\}_{i=1}^2$ generates L_1.

(9.7d) Conclude for $x \in X_1$ that $x + z(k_1, k_2) \bmod L_1$ are the distinct elements of X_1 mapping $f(x)$ under f.

Now we describe holomorphic differentials on a complex torus.

(9.8a) Let L be a lattice in $\mathbb{C}_z$. Define ω_α on one of the local coordinate charts $\varphi_\alpha(U_\alpha) \subset \mathbb{C}_z$ for $\mathbb{C}/L$ to be the differential dz (As in Ex. 6.18). Show this defines a global differential form ω_L on $\mathbb{C}/L$, and the divisor of this form is 0. Hint: Use that the transition functions, on connected subsets of $\varphi_\alpha(U_\alpha \cap U_\beta)$ have the form $z \mapsto z + \beta$.

(9.8b) Accept without proof that any meromorphic function has divisor of degree 0. Conclude: Holomorphic differentials on $\mathbb{C}/L$ have degree 0 divisor; so they are constant multiples of ω_L.

(9.8c) A g dimensional complex torus has the form $A = \mathbb{C}^g/L$ where L is a $\mathbb{Z}$ module having dimension $2g$ and such that $\mathbb{R}L = \mathbb{C}^g$ (a *lattice*). Imitate b) to show holomorphic differentials on A form a dimension g vector space.

[9.8c] considers complex torii. Since $\mathbb{C}^g$ is contractible, $\pi_1(A, \mathbf{0})$ identifies with L. We now see all differentiable groups have an abelian fundamental group.

(9.9a) Suppose that $\gamma_{0,i}$ and $\gamma_{1,i}$ are homotopic paths in a space X, $i = 1, 2$, and that the end point of $\gamma_{0,1}$ is equal to the initial point of $\gamma_{0,2}$. Show $\gamma_{0,1}\gamma_{0,2}$ is homotopic to $\gamma_{1,1}\gamma_{1,2}$.

(9.9b) Show the associative rule for multiplying paths.

(9.9c) Let ψ_1 and ψ_2 be two isomorphisms between $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ as in Corollary 1.19. Show $\psi_2^{-1} \circ \psi_1$ is an *inner automorphism* of $\pi_1(X, x_0)$. That is, it is given by conjugation by an element of $\pi_1(X, x_0)$.

(9.9d) A group G is *differentiable* G if it is a differentiable manifold, and its multiplication and inverse are both differentiable maps. Similarly, there is the notion of *analytic group*. Show a complex torus $\mathbb{C}^g/L$ (L a lattice) is an analytic group.

(9.9e) Suppose M is a subvariety of $\mathrm{GL}_n(\mathbb{C})$ (defined by a finite number of equations in the n^2 coordinates of the entries), closed under multiplication and inverse. Show M is an analytic group.

(9.9f) For G a differentiable group consider $f_1 : G \to (G, 1)$ (resp. $f_2 : G \to (1, G)$) by $g \mapsto (g, 1)$ (resp., $g \mapsto (1, g)$). Show for $[\gamma_1], [\gamma_2] \in \pi_1(G, 1)$:

$$m_*((f_1)_*[\gamma_1])(f_2)_*[\gamma_2]) = [\gamma_1][\gamma_2].$$

(9.9g) Continuing b), show $\pi_1(G, 1)$ is an abelian group. Conclude: A differentiable manifold X with a nonabelian fundamental group (as often in Chap. 4) has no differentiable group structure.

9.3. $\mathbb{P}^n$ compactification. Use the notation of §4.3.

(9.10a) Consider $h \in \mathbb{C}(w)$, $h = h_1/h_2$, with $(h_1, h_2) = 1$. Let $m = h_2(w)z - h_1(z)$ as in Ex. 4.7. Show the $\mathbb{P}^1_z \times \mathbb{P}^1_w$ compactification of $\{(z, w) \mid m(z, w) = 0,\ z \notin \boldsymbol{z}\}$ is a manifold.

(9.10b) Consider $\mathbb{P}^n = \mathbb{C}^{n+1} \setminus \{0\}/\mathbb{C}^*$. Induct on n to show $\mathbb{P}^n = U_0 \dot\cup \mathbb{P}^{n+1}$. Inductively define a topology: neighborhoods of $\boldsymbol{x} \in \mathbb{P}^{n-1}$ are the image in $\mathbb{P}^n$ of neighborhoods of $(0, v_1, \ldots, v_n) \in \mathbb{C}^{n+1}$.

(9.10c) Prove directly in $\mathbb{P}^n$: Any infinite sequence has a limit point. Hint: Any infinite sequence has an infinite subsequence in U_i for some i.

Fiber products help construct new manifolds from old. Use notation of §4.2.3.

(9.11a) Generalize the $\mathbb{P}^2$ compactification of $h(w) - g(z)$ from Ex. 4.3.3.

(9.11b) Conclude the proof of Prop. 4.9 by noting $\mathcal{L}^h_{z'}[(z-z')^{1/e_1}, (z-z')^{1/e_2}]$ is a proper subring of $\mathcal{P}^h_{z',[e_1,e_2]}$, though its quotient field equals $\mathcal{P}_{z',[e_1,e_2]}$.

(9.11c) Finish the hyperelliptic case of Ex. 4.3.3: $\mathbb{P}^1 \times \mathbb{P}^1$-compactification gives a manifold while no $\mathbb{P}^2$-compactification ever does.

(9.11d) Apply b) to $f : X \to \mathbb{P}^1_z$ of degree at least 3. Then, $V = X \times_{\mathbb{P}^1_z} X$ contains the diagonal Δ and it consists of the union of this and another compact set V'. Show V' has a manifold structure from its embedding in $X \times X$ if and only if there is only one ramified point over each branch point of f and that ramification order is 2. That is, f is a *simple-branched* cover.

(9.11e) Show global meromorphic functions on $\mathbb{P}^n$ are ratios of (same degree) homogeneous polynomials in the coordinates of $\mathbb{P}^n$. Show there is no analytic map $\psi : \mathbb{P}^2 \to \mathbb{P}^1$. Hint: A ratio of same degree polynomials has a singularity at common zeros.

(9.11f) Assume $\bar{X} \subset \mathrm{pr}^2_{z,w,u}$ is a compact manifold, and $(z_0, w_0, u_0) \in \bar{X}$ is the intersection of L_1 and L_2 in Prop. 4.13. Show there is no other value $z_0' \neq z_0$ so $L_1 - z_0' L_2$ is tangent to $\bar{X}$. Hint: Otherwise, $u' = (L_1 - z_0' L_2)/z$ and $w' = (L_1 - z_1' L_2)/z$ give local coordinates for $\bar{X}$ in a neighborhood of $(0,0) \in\in \mathbb{C}_{u'} \times \mathbb{C}_{w'}$ though both functions ramify at $(0,0)$.

9.4. Paths and vector fields. Let X be a manifold.

(9.12a) Show each (simplicial) path $\gamma : [a,b] \to X$ is image equivalent to $\gamma_1 : [0,1] \to X$. Show each nonconstant path is image equivalent to a path constant on no interval.

(9.12b) Assume X is contractible (Def. 5.8). Suppose $\gamma : [a,b] \to X$ is a path with initial point x_0 and endpoint x_1. Form the function $G : [a,b] \times [0,1] \to X$ by $G(t,s) = f(\gamma(t), s)$. Use this to show *all* paths in X with initial point x_0 and endpoint x_1 are homotopic.

(9.12c) Let $\gamma : [a,b] \to \mathbb{R}^n$ be a simplicial path. Let $\boldsymbol{f} = (f_1, \dots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$ be defined and continuous on the image of $[a,b]$. Consider

$$\sum_{i=1}^{n} \int_a^b f_i(\gamma(t)) \frac{d\gamma_i}{dt}\, dt \stackrel{\text{def}}{=} \int_\gamma \boldsymbol{f} \cdot d\boldsymbol{x},$$

the *line integral of* $\boldsymbol{f}$ *along* γ.

(9.12d) If γ_1 and γ are image equivalent paths in $\mathbb{R}^n$, show line integrals along them are equal (use change of variables formula from Chap. 2 Lem. 2.3).

(9.12e) Let $F : [a,b] \times [0,1] \to \mathbb{R}^n$ be a homotopy between paths γ_0 and γ_1 (write $F(t,s) = \gamma_s(t)$) in $\mathbb{R}^n$. Assume $\boldsymbol{f}$ is continuous on the image of F. Show the line integral of $\boldsymbol{f}$ along γ_s is a continous function of s.

For a differentiable path $\gamma : [0,1] \to U$ with U open in $\mathbb{R}^n$, there may not exist a vector field T_U having γ as an integral curve, though *locally* this is so.

(9.13a) If T_U exists explain why $\gamma(t_1) = \gamma(t_2)$ implies $\frac{d\gamma}{dt}(t_1) = \frac{d\gamma}{dt}(t_2)$.

(9.13b) Let V be a neighborhood of the line segment $t \to (t, 0, \dots, 0) \in \mathbb{R}^n_{\boldsymbol{t}}$, $t \in [0,1]$. Assume there is a one-one differentiable $\Gamma : V \to U$ with $\Gamma(t, 0, \dots, 0) = \gamma(t)$. Show $\frac{\partial \gamma}{\partial t_1}(\boldsymbol{t})$ (applying $\frac{\partial}{\partial t_1}$ to all coordinates of Γ) produces a vector field on $\Gamma(V)$ with γ an integral curve.

(9.13c) Assume $\frac{d\gamma}{dt}$ is never 0. Consider $H_t = \{\boldsymbol{w} \in \mathbb{R}^n \mid \boldsymbol{w} \cdot \frac{d\gamma}{dt} = 0\}$. Find differentiable one-one $F : \mathbb{R}^n \to \mathbb{R}^n$ with $F(\boldsymbol{t}) = \gamma(t_1) + \boldsymbol{w}(t_1, t_2, \dots, t_n)$

with $\boldsymbol{w}(t_1, t_2, \ldots, t_n) \in H_{t_1}$ linear in $(t_2, \ldots, t_n)$ (t_1 fixed). Hint: Apply the chain rule.

(9.13d) How does F give Γ in b)?

Returning to (5.3) we relate

$$(f_{\alpha,1}(\boldsymbol{y}_\alpha), \ldots, f_{\alpha,n})(\boldsymbol{y}_\alpha) \text{ to } (f_{\beta,i}, \ldots, f_{\beta,n})(\psi_{\beta,\alpha}(\boldsymbol{y}_\alpha))$$

.

(9.14a) Apply both sides of (5.3) to the coordinate function $y_{\beta,j}$ to get

$$f_{\beta,j}(\psi_{\beta,\alpha}(\boldsymbol{y}_\alpha)) = \sum_{i=1}^{n} f_{\alpha,i} \frac{\partial \psi_{\beta,\alpha,j}}{\partial y_{\alpha,i}}(\boldsymbol{y}_\alpha)$$

where $\psi_{\beta,\alpha,j}$ is the jth coordinate of $\psi_{\beta,\alpha}$. That is, the f_β s are the result of applying the Jacobian matrix of $\psi_{\beta,\alpha}(\boldsymbol{y}_\alpha)$ to the f_α s.

(9.14b) Consider the case $\psi = \psi_{(x,y),(r,\theta)} : \mathbb{R}^2_{r,\theta} \to \mathbb{R}^2_{x,y}$ by $(r, \theta) \mapsto (x, y)$. Express $\frac{\partial}{\partial x}$ as $f_r \frac{\partial}{\partial r} + f_\theta \frac{\partial}{\partial \theta}$ by applying both to $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Do the same for $\frac{\partial}{\partial y}$, expressing it as $f'_r \frac{\partial}{\partial r} + f'_\theta \frac{\partial}{\partial \theta}$. Applying $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ to $f(x, y)$ and evaluating at $(r\cos(\theta), r\sin(\theta))$ is the same as applying $J(\psi_{(x,y),(r,\theta)})(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta})$ to $f(r\cos(\theta), r\sin(\theta))$.

(9.14c) Generalize b) to say (as in (5.4))

$$J(\psi_{\boldsymbol{y}_\beta,\boldsymbol{y}_\alpha})^{-1}\Big(\frac{\partial}{\partial y_{\alpha,1}}, \ldots, \frac{\partial}{\partial y_{\alpha,n}}\Big) = \Big(\frac{\partial}{\partial y_{\beta,1}}, \ldots, \frac{\partial}{\partial y_{\beta,n}}\Big).$$

9.5. Permutation groups. Suppose $G \le S_n$ is transitive. Def. 7.9 defines primitive subgroup of S_n.

(9.15a) For $g \in N_G(G(1))$, multiplication of g on the *left* of the distinct right cosets $G(1)\sigma_1, \ldots, G(1)\sigma_n$ of $G(1)$ permutes these cosets. Conclude: This induces a homomorphism $\psi : N_G(G(1))/G(1) \to \mathrm{Cen}_{S_n}(G)$.

(9.15b) Show ψ is an isomorphism because both groups have order equal

$$|\{i \in \{1, 2, \ldots, n\} \mid \sigma(i) = i \text{ for each } \sigma \in G(1)\}|.$$

(9.15c) Show $N_G(G(1))/G(1)$ (or $\mathrm{Cen}_{S_n}(G)$) is trivial if G is primitive and $G(1)$ is nontrivial.

(9.15d) Show a nontrivial normal subgroup of a primitive group is transitive.

(9.15e) Show a primitive subgroup of S_n containing a 2-cycle is S_n. Conclude any transitive group generated by 2-cycles is S_n. Hint: Consider the normal subgroup generated by the conjugates of the 2-cycle.

Let G be a centerless group, $\mathrm{Aut}(G)$ its automorphisms and $T : G \to S_n$ faithful transitive permutation representation.

(9.16a) Explain this from [**Isa94**, p. 43]: In general neither $(gH)A(g') \stackrel{\text{def}}{=} gHg'$ nor $(gH)A(g') \stackrel{\text{def}}{=} (gg')H$ define an action on left cosets of H in G.

(9.16b) Let S be the collection of conjugates of a subgroup H of the group G, with the action by conjugation by elements of G: $S = g^{-1}Hg_{g \in G}$ and the right action of $g' \in G \mapsto (g')^{-1}g^{-1}Hgg'$. What is the coset representation associated with this transitive action, and when is it faithful?

(9.16c) Show (conjugation by) G is normal in $\mathrm{Aut}(G)$. The *outer automorphism* group $\mathrm{Out}(G)$ of G is the quotient $\mathrm{Aut}(G)/G$. Show the natural map $\psi_T : N_{S_n}(G) \to \mathrm{Out}(G)$ has kernel $\mathrm{Cen}_{S_n}(G)$ (§7.1.3; compare with [9.15c]).

(9.16d) Denote the image of ψ_T in $\mathrm{Out}(G)$ by $\mathrm{Out}_T(G)$. Show $\mathrm{Out}_T(G) = \mathrm{Out}(G)$ if and only if $G(T,1)$ (§7.1.2) has exactly n images under $\mathrm{Aut}(G)$. Hint: Associate to $\alpha \in \mathrm{Aut}(G)$ an element of S_n defined up to $\mathrm{Cen}_{S_n}(G)$ if it maps among the conjugates of $G(T,1)$. Show [9.20b] gives examples where T is doubly transitive and $\mathrm{Out}(G) \neq \mathrm{Out}_T(G)$.

(9.16e) Case: $G = A_n$ (resp. $G = S_n$), $n \geq 4$, in its standard representation T. Show $\mathrm{Out}(S_n) = \{1\}$ (resp. $\mathrm{Out}_T(A_n) = \mathrm{Out}(A_n) = \mathbb{Z}/2$) if and only if S_n (resp. A_n) has exactly n transitive subgroups of index n under $\mathrm{Aut}(G)$. Hint: Intransitive subgroups have small orders. (See [9.17b].)

(9.16f) Set notation in the proof of Prop. 8.19 to change to a left action of GL_n.

We will need the following facts later.

(9.17a) For each i, $2 \leq i \leq n$, consider $L_i = \{1_n, (1\,i), (2\,i), ..., (i-1\,i)\} \subset S_n$ (1_n indicates the identity). Show each $x \in S_n$ has a unique product representation as $x = x_1 x_2 \dots x_n$ with $x_i \in L_i$. (This gives a technique to generate random elements of S_n with uniform distribution.) Hint: For $g \in S_n$ if $(n)g = i$, let $h = g(i\,n)$ and induct on n.

(9.17b) [**Isa94**, p. 79-80] bases $\mathrm{Out}(S_n) = \{1\}$, if $n \neq 6$, on two observations:
- If $\alpha \in \mathrm{Aut}(S_n)$ permutes transpositions, then conjugating by some $g \in S_n$ gives α. Hint: Elements of $(L_i)\alpha$ in a) then have a unique integer of common support.
- If $n \neq 6$, among elements of order 2, the conjugacy class of transpositions has a unique cardinality.

(9.17c) Let $T_H : G \to S_n$ be a permutation representation. Show all cosets of H have the form Hg^i, $i = 0, \dots, n-1$, if and only if g is an n-cycle in T_H.

(9.17d) Suppose F has characteristic p which also divides the order of finite group G. Show a faithful permutation representation of G cannot be completely reducible. Hint: Reduce to $G = \langle g \rangle$ with g having order p.

(9.17e) Suppose G is a free group on $r \geq 2$ generators. Find representations $\varphi : G \to \mathrm{GL}_r(\mathbb{C})$ that are not completely reducible. Hint: Map G into an upper-triangular, not diagonal, matrix group.

9.6. Affine groups as permutation representations. Let $H \leq GL_k(F)$ with $F = \mathbb{F}_q$. Regard $G = \{\left(\begin{smallmatrix} A & v \\ 0 & 1 \end{smallmatrix}\right) \mid A \in H,\ v \in V = \mathbb{F}_q^k\}$ as a group $V \times^s H$ as in Rem. 7.4. Note: If a nonabelian group replaced $\mathbb{F}_q^k$, then $A(v') + v$ should more naturally be written $v + A(v')$.

(9.18a) Suppose $\{0\} < V_1 < V$ is an H invariant space. Then, $V_1 \times^s H$ is a subgroup of G properly containing H. Show conversely, a group properly between H and G has the form $V_1 \times^s H$ with H invariant V_1.

(9.18b) Embed V in G by $v \mapsto \left(\begin{smallmatrix} 1 & v \\ 0 & 1 \end{smallmatrix}\right)$. Have G act on V by $\left(\begin{smallmatrix} A' & v' \\ 0 & 1 \end{smallmatrix}\right)$ maps $v \mapsto A(v) + v' = v^*$: equivalent to $\left(\begin{smallmatrix} A' & v' \\ 0 & 1 \end{smallmatrix}\right)$ multiplies $\binom{v}{1}$ to $\binom{v^*}{1}$. Show this gives a faithful transitive permutation representation of G.

(9.18c) From a) the representation of b) is primitive if and only if H acts irreducibly. Suppose $H = \langle A \rangle$ has a single matrix generator, which we use to makes V into an $F[z]$ module by having $f(z) \in F[z]$ map $v \in V$ to $f(A)(v)$. The *elementary divisor theorem* (Chap. 2 §9.15) says $V \equiv \oplus_{i=1}^t F[z]/(f_i)$

(as an $F[z]$ module). Example: If $v = (a, b) \in F^2$, and $A = \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}$, $f(z) = z^2 + z + 1$, then $f(z)(v) = \begin{pmatrix} 6 & 12 \\ 4 & 10 \end{pmatrix}(v) = (6a + 12b, 4a + 10b)$. We can uniquely choose the f_is monic so that $f_1|f_2|\cdots|f_t$. Show G is primitive if and only if $t = 1$ and f_1 is an irreducible polynomial.

(9.18d) The multiplicative group $\mathbb{F}_{p^n}^*$ is cyclic. Let α be a generator, and A the matrix of α acting on $\mathbb{F}_p^n$ by regarding it as $\mathbb{F}_{p^n}$. In equation form: $v \in \mathbb{F}_{p^n} \mapsto \alpha v \in \mathbb{F}_{p^n}$. Show $V \times^s \langle A \rangle$ is doubly transitive on F.

(9.18e) From Def. 7.9, b) is doubly transitive if and only if H is transitive on $V \setminus \{0\}$. When $H = \langle A \rangle$, show G is doubly transitive if and only if, for some isomorphism of $\mathbb{F}_{p^n}$ and $(\mathbb{F}_p)^n$, A acts like multiplication by $\alpha \in \mathbb{F}_{p^n}^*$.

9.7. Group representations. In this exercise consider representations over any field containing $\mathbb{Q}$.

(9.19a) Show that the direct product of two permutation representations as a group representation is the tensor product of the two group representations. Therefore the trace is the product of the traces.

(9.19b) Finish showing the number of orbits is the same as the number of appearances of the identity.

(9.19c) Let $T_i : G \to S_{n_i}$, $i = 1, 2$, be permutation representations for which $\mathrm{t}(T_1(g)) = \mathrm{t}(T_2(g))$ for each $g \in G$ (as in §7.1). Show $n_1 = n_2$ and $T_1(g)$ and $T_2(g)$ are conjugate in S_{n_1} for each $g \in G$. Hint: Induct on the length of the highest disjoint cycles and compare $\mathrm{t}(T_1(g))$ and $\mathrm{t}(T_1(g^r))$ for some prime r dividing the order of g.

(9.19d) Show $\frac{1}{|G|}\sum_{g\in G}\mathrm{t}(T(g))$ counts the orbits of a permutation representation T. Hint: Put the additive operator t on the outside of the sum by regarding $T(g)$ as a permutation matrix. Each orbit I gives a 1-dimensional invariant subspace spanned by $\sum_{i\in I} x_i$ (as in §7.1.4).

(9.19e) Show the collection of $L_{\mathrm{C}} = \sum_{u\in C} u$ with C a conjugacy class of G, span the G invariant idempotents of $\mathbb{C}[G]$. For ρ any representation, $\frac{1}{|G|}\sum_{g\in G}\mathrm{t}(\rho(g))$ counts appearances of $\mathbf{1}_G$ in ρ. Hint: $\frac{1}{|G|}\sum_{g\in G}\rho(g)$ is an idempotent, and its trace equals the dimension of its range.

(9.19f) *Orthogonality Relations*: Let ρ_V and ρ_W be representations of G on respective spaces V and W. Show $\mathrm{t}(\rho_{V^*\otimes W}(g)) = \mathrm{t}(\rho_V(g))\mathrm{t}(\rho_W(g))$ gives

$$\sum_{g\in G}\mathrm{t}(\bar\rho_V(g))\mathrm{t}(\rho_W(g)) = \dim_{\mathbb{C}}\mathrm{Hom}_{\mathbb{C}[G]}(V, W).$$

Further, this dimension gives the appearances of $\mathbf{1}_G$ in $\mathrm{Hom}_{\mathbb{C}}(V, W)$. Hint: For ρ' irreducible, $\mathbf{1}_G$ appears exactly once in $\mathrm{Hom}_{\mathbb{C}}(V_{\rho'}, V_{\rho'})$.

(9.19g) Show V_T is $\mathbf{1} \oplus V'$ with V' irreducible if and only if T is doubly transitive. Hint: Apply d) to count appearances of $\mathbf{1}_G$ in $V_T \otimes V_T$; use that $\mathbf{1}_G$ appears in $\rho_1 \otimes \rho_2$ with ρ_1, ρ_2 irreducible only if $\rho_2 = \bar\rho_1$.

(9.19h) Suppose the representation $T : G \to S_n$ is doubly transitive. Show G does not contain a subgroup H of degree $m < n$ intransitive in T. Hint: Count appearances of $\mathbf{1}_G$ in $V_T \otimes V_{T_H}$ using d).

Denote the finite field of $q = p^r$ for p a prime by $\mathbb{F}_q$. Let $G = \mathrm{GL}_k(F)$ be the $k \times k$ invertible matrices with coefficients in the field $F = \mathbb{F}_q$. Write $\mathbb{P}^{k-1}(F)$ for lines through the origin in $\mathbb{F}_q^k$: $\{\alpha\boldsymbol{v} \mid \alpha \in F\}$ for some $\boldsymbol{v} \in \mathbb{F}_q^k \setminus \{\boldsymbol{0}\}$. Then, G has a permutation action $T_{k,F}$ on $\mathbb{P}^{k-1}(F)$ induced from its action on $\mathbb{F}_q^k$. Let $\psi : \mathbb{F}_q^k \to \mathbb{F}_q$ be a nonconstant linear map (*linear functional*). Denote linear functionals up to multiplication by elements of $\mathbb{F}_q \setminus \{0\}$ by $\hat{\mathbb{P}}^{k-1}(F)$, with a permutation action $\hat{T}_{k,F}$: For $\psi \in \mathbb{P}^{k-1}(F)$ and $A \in \mathrm{GL}_k(F)$, $\psi^A(\boldsymbol{v}) \stackrel{\text{def}}{=} \psi((\boldsymbol{v})A^{-1})$ for $\boldsymbol{v} \in \mathbb{F}_q^k$.

(9.20a) Show $T_{k,F}$ is doubly transitive of degree $n(q) = \frac{(q^k-1)}{(q-1)}$ for $k > 1$.

(9.20b) Show $\hat{T}_{k,F}$ also has degree $n(q)$ and is doubly transitive, though $T_{k,F}$ and $\hat{T}_{k,F}$ are not permutation equivalent. Hint: Show the stabilizer in G of a hyperplane in $\mathbb{P}^{k-1}(F)$ fixes no point.

(9.20c) Show $\mathrm{t}(\hat{T}_{k,F}(g)) = \mathrm{t}(T_{k,F}(g))$, so $\hat{T}_{k,F}$ and $T_{k,F}$ are equivalent as representations [9.19c]. Hint: $\hat{T}_{k,F}(g)$ is induced from the transpose of g, and a matrix and its transpose are conjugate.

(9.20d) As in [9.18d], identify $\mathbb{F}_q^k$ with $\mathbb{F}_{q^k}$ as vector spaces over $\mathbb{F}_q$ to find $\alpha \in \mathbb{F}_{q^k}$ producing $A \in \mathrm{GL}_k(F)$ with $T_{k,F}(A)$ and $\hat{T}_{k,F}(A)$ both $n(q)$-cycles.

(9.20e) Assume: T_1, T_2 are inequivalent degree n doubly transitive representations of a group G; they are equivalent as group representations; and $T_1(g) = T_2(g) = (1\,2\,\ldots\,n)$ for some $g \in G$. Let D be the orbit of 1 under $G(T_1, 1)$ in the representation T_2. Use double transitivity to show D is a *difference set*: $\{d_i - d_j \mid d_i \ne d_j \in D\}$ contains each nonzero integer mod n with the same multiplicity t [**Fri73**]. Further, $t \cdot (n-1) = |D| \cdot (|D| - 1)$. Example: For $k = 3$, $q = 2$, $n = 7$ in b), $D = \{1, 2, 4\}$ and $t = 1$.

9.8. Easy Galois covers.

(9.21a) Suppose X and Y_i are differentiable manifolds, and that $f_i : Y_i \to X$ are covering maps, $i = 1, 2$. Assume $\psi : Y_1 \to Y_2$ is any *continuous* map with $f_2 \circ \psi = f_1$. Show ψ is a map of differentiable manifolds. Also: ψ is analytic if X is a complex manifold.

(9.21b) Let $f : Y \to X$ be a finite cover of degree n. Use that X is connected to show $|f^{-1}(x)|$ is n for each $x \in X$.

(9.21c) Consider $X_1 = \{x + \sqrt{-1}\,y \in S^1 \mid y > 0\}$ and $X_2 = S^1$. Show, for $n > 0$, the map of Ex. 6.16 restricted to X_1 is not a covering map.

(9.21d) Follow the notation of Ex. 6.18 and of [9.7]. Let L and L_i, with $L_i \subseteq L$, $i = 1, 2$, be lattices. Show that if $f_i : X_i = \mathbb{C}/L_i \to \mathbb{C}/L$ by $z \bmod L_i \mapsto z \bmod L$, then the covers (X_i, f_i) are equivalent if and only if $L_1 = L_2$.

(9.21e) Let $X_i = X$, $i = 1, \ldots, n$, and let Y be the disjoint union of the X_i's. What is the automorphism group of the cover $Y \to X$ obtained by mapping each point of Y to its corresponding point in X?

(9.21f) Let $f : Y \to X$ be a cover and consider a subgroup G of $\mathrm{Aut}(Y, f)$ of order equal to $\deg(f)$. Assume that, for some point $x_0 \in X$, G acts transitively on the set $f^{-1}(x_0)$. Show f restricted to any connected component of Y gives a Galois cover of X.

(9.21g) Let $X = Y = \mathbb{C} \setminus \{0\}$. Show $f : Y \to X$ by $z \mapsto z^n$ is a Galois cover. Hint: Consider $\psi_k : z \mapsto e^{2\pi\sqrt{-1}\,k} z$, $0 \le k \le n-1$.

(9.21h) Let X_i, $i = 1, 2$, be as in [9.7c] with $L_1 \subset L_2$. Show $f : X_1 \to X_2$ in Ex. 6.18 is a Galois cover. Hint: Consider $\psi_{k_1,k_2} : z \bmod L_1 \mapsto z + z(k_1, k_2) \bmod L_1$.

(9.21i) Consider a) with $f \in \mathbb{C}[y]$ and $f(y) = y^n + c_{n-2}y^{n-2} + \cdots + c_1 y$. Assume the greatest common divisor of the set $\{n$ and i with $c_i \neq 0\}$ is 1. Show $\mathrm{Aut}(Y, \mathrm{pr})$ is trivial. Hint: Apply Liouville's Theorem [**Ahl79**, p. 122] to see elements of $\mathrm{Aut}(Y, f)$ have the form $y \mapsto ay + b$ for some $a, b \in \mathbb{C}$.

9.9. Imprimitive and Frattini covers. This discussion on imprimitivity continues in Chap. 4 [11.13]

(9.22a) Let $\pi_1(X, x_0)$ be the fundamental group of a connected differentiable manifold X. Let $H\sigma_1, \ldots, H\sigma_n$ be the distinct cosets of a subgroup $H \leq \pi_1(X, x_0)$ of index n corresponding to the cover (Y, f) (with fiber $\{y_1, \ldots, y_n\}$ over x_0). Consider the points of $U_{Y,f,n}$ (§8.3.2) over x_0 that connect by a path to $(y_1, \ldots, y_n)$. Show these correspond to distinct n-tuples of cosets: $\{(H\sigma_1\sigma, H\sigma_2\sigma, \ldots, H\sigma_n\sigma) \mid \sigma \in \pi_1(X, x_0)\}$. Why is this the same as $|G|$? Conclude $\deg(\hat{f}_i) = |G(Y, f)|$ (as prior to Thm. 8.9).

(9.22b) Show components of $Y \times_X Y$ of degree 1 over Y correspond to elements of $\mathrm{Aut}(Y, f)$ (Lem. 8.8). If $f : Y \to X$ has automorphisms, and f is not a cyclic Galois cover of prime degree, show $G(Y, f)$ is imprimitive. How does [9.21i] give explicit imprimitive covers with no automorphisms?

(9.22c) Show (Y, f) decomposes if and only if $Y \times_X Y \to X$ properly factors through a fiber product of form $Y' \times_X Y'$. If so, show $Y' \times_X Y' \setminus \Delta$ is a nontrivial component of $Y \times_X Y$.

Let $K \subset \hat{L} \subset \hat{M}$ be a chain of fields with $\hat{M}/K$ (resp. $\hat{L}/K$) Galois with group G^* (resp. G). This is a *Frattini chain* if the only subfield $K \leq T \leq \hat{M}$ with $T \cap \hat{L} = K$, is $T = K$. Denote restriction of elements of G^* to $\hat{L}$ by $\mathrm{rest} : G^* \to G$.

(9.23a) Suppose $T = \hat{M}^H$ is the fixed field of a subgroup H of G^*. Show $T \cap \hat{L} = K$ is equivalent to $\mathrm{rest} : H \twoheadrightarrow G$. Hint: Use that $T \cap \hat{L} = K$ allows extending any automophism of $\hat{L}$ to $T \cdot \hat{L}$ to be the identity on T.

(9.23b) Show a) is equivalent to this group statement: If $H \leq G^*$ and $\mathrm{rest}(H) = G$, then $H = G^*$ (the map $\mathrm{rest} : G^* \to G$ is a *Frattini cover*). Hint: $\mathrm{rest}(H) = G$ is equivalent to $\hat{M}^H \cap \hat{L} = K$.

(9.23c) Suppose $\hat{X} \to \hat{Y} \to Z$ is a sequence of covers with $\psi_X : \hat{X} \to Z$ Galois with group G^* and $\psi_Y : \hat{Y} \to Z$ Galois with group G. Let $\psi : G^* \to G$ be the natural map and assume ψ is a Frattini cover. Show the equivalence with this. For any sequence $\hat{X} \to W \to Z$ of covers with $W \neq Z$, there is a proper cover of Z that $W \to Z$ and $\hat{Y} \to Z$ factor through.

9.10. Laplacian. The *Laplace operator* $\nabla^2 = \frac{\partial}{\partial x}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\frac{\partial}{\partial y}$ on $\mathbb{R}^2_{x,y}$ acts on $C^\infty(\mathbb{R}^2)$. It generalize to a Riemann surface X (see Chap. 4 §11.11 for $\wedge$ product). Locally in $z = x + iy$, write a differential 1-form (not necessarily holomorphic) on an open set $U \subset \mathbb{C}$ as $\omega = p(x, y)\, dx + q(x, y)\, dy$. Consider $*\omega = -q dx + p dy$. Write $w = u + iv$ for the real and imaginary components of the variable for $\mathbb{C}_w$.

(9.24a) With $z = f(w)$, suppose $f : V \subset \mathbb{C}_w \to U \subset \mathbb{C}_z$ is analytic, one-one and onto from V to U. Write $w = u(x, y) + iv(x, y)$ as the local inverse of f. Express ω as $\Omega_(u, v) = p(x(u, v), y(u, v))\, dx(u, v) + q(x(u, v)\, y(u, v))\, dy(u, v)$. Show $*\Omega_(u, v) = -Q(u, v)\, du + P(u, v)\, dv$ equals $*\omega$ expressed in u and v. Hint: Apply the Cauchy-Riemann equations: $\frac{\partial x}{\partial u} = \frac{\partial y}{\partial v}$ and $\frac{\partial x}{\partial v} = -\frac{\partial y}{\partial u}$.

(9.24b) Conclude from a): On any Riemann surface X, $*$ defines a linear map on differentiable 1-forms.

(9.24c) Show these further properties of $*$: Its square is multiplication by -1, $\omega_\wedge * \omega_= (p^2+q^2) dx \wedge dy$, and $*\omega = i\omega$ if ω is holomorphic. Conclude: ω is holomorphic if and only if $d\omega = 0$ and $*\omega = i\omega$.

(9.24d) Consider $*d = -\frac{\partial}{\partial y} dx + \frac{\partial}{\partial x} dy$ acting on differentiable functions. So, for f differentiable on X, $*df = *d(f)$ is well-defined, and it extends to 1-forms: $p(x,y)\, dx + q(x,y)\, dy \mapsto *d(p(x,y)) \wedge dx + *d(q(x,y)) \wedge dy$. Show $*d(\omega)$ is $-d*\omega$. Define $\nabla^2(f)$ by $d*df = \nabla^2(f) dx \wedge dy$. Argue why this defines a (complex) Laplacian ∇^2_X on a 1-dimensional complex manifold.

(9.24e) Suppose differentiable f on X has a corresponding $\lambda \in \mathbb{C}$ with $\nabla^2_X(f) = \lambda f\, dx \wedge dy$ (everywhere locally). Call λ an *eigenvalue* of ∇^2_X. If $f_i : Y_i \to X$ are inequivalent covers of X, with equivalent locally flat bundles (Defn8.6), over X, $i = 1, 2$, show their Laplacians have the same eigenvalues.

CHAPTER 4

RIEMANN'S EXISTENCE THEOREM

This chapter introduces the foundation of the book: The construction of all compact Riemann surfaces through Riemann's classification of the branched covers of the sphere (Thm. 2.6). Still, one cover at a time, won't give us much useful information. We need to know the nature of families of related covers. The Existence Theorem serves well, though it takes additional ideas to find a useful naming scheme for the families. This chapter's nontraditional treatment of modular curves motivates many general ideas in Chap. 5.

1. Presentations of fundamental groups of Riemann surfaces

Our command of Riemann's Existence Theorem requires combinatorial ability to list finite quotients of the fundamental group of $U_{\boldsymbol{z}}$. Thm. 1.8 tells us $\pi_1(U_{\boldsymbol{z}})$ is a free group on $r-1$ generators (with $r = |\boldsymbol{z}|$) and more. It is the basis for describing families of covers (Chap. 5) of $\mathbb{P}^1_z$. Our main computational tools for this are Hurwitz monodromy actions. These are on explicit sets running from types of *Nielsen classes* (§3.2) to special fundamental group generators of Riemann surfaces defined by Nielsen classes (§9.2).

1.1. Presentations and free products. Most fundamental groups appear as quotients of free groups. Further, we define the kernel of that quotient by listing specific *relation* elements in the kernel. We recognize the smallest normal subgroup containing these relations as the kernel. A *presentation*, however, doesn't list all relations from this normal subgroup condition. Presenting groups as quotients of free groups this way is convenient for forming their quotients. To see whether a group G is a quotient of some fundamental group, we need only check if specific generators of G satisfy a tiny list of relations. This suits how we form compact Riemann surfaces from unramified covers of $U_{\boldsymbol{z}}$. Still, this often leaves a tough problem. How to check if an expression from the free group is in that kernel.

For S a set, we first define the group $F(S)$ that S *freely generates*. The following construction is of a free group with relations. Generalizing this to groups generated freely by subgroups is a categorical rather than quotient construction.

For $s \in S$ and $n \in \mathbb{Z}$, use the symbol s^n to denote the pair (s, n). If $t \in S$ and $m \in \mathbb{Z}$ then $s^n = t^m$ if and only if $s = t, n = m$.

Elements of $F(S)$ are (finite) sequences $\boldsymbol{s}^{\boldsymbol{n}} = (s_1^{n_1}, \cdots, s_k^{n_k})$ satisfying

$$k \in \mathbb{N}; s_1, \cdots, s_k \in S;\ n_1, \cdots, n_k \in \mathbb{Z} \setminus \{0\};\ \text{and } s_i \neq s_{i+1},\ i = 1, \cdots, k-1.$$

Regard the sequence $\emptyset$ with no elements as an element of $F(S)$. Denote $(t_1^{m_1}, \cdots, t_\ell^{m_\ell}) \in F(S)$ by $\boldsymbol{t}^{\boldsymbol{m}}$. Define the product of $\boldsymbol{s}^{\boldsymbol{n}}$ and $\boldsymbol{t}^{\boldsymbol{m}}$ by cancellation to be the elimination of any consecutive terms of the form tt^{-1}. Formally, Find the smallest integer u with this property: $t_u^{-m_u} \neq s_{k-u+1}^{n_{k-u+1}}$; but $t_i^{-m_i} = s_{k-i+1}^{n_{k-i+1}}$, $i = 1, \cdots, u-1$. Then

$$\begin{aligned} \boldsymbol{s^n t^m} = \quad & (s_1^{n_1}, \cdots, s_{k-u}^{n_{k-u}}, \alpha, t_{u+1}^{m_{u+1}}, \cdots, t_\ell^{m_\ell}) \\ \text{where} \quad & \alpha = \begin{cases} (s_{k-u+1}^{n_{k-u+1}}, t_u^{m_u}) & \text{for } t_u \neq s_{k-u+1} \\ t_u^{n_{k-u+1}+m_u} & \text{for } t_u = s_{k-u+1}. \end{cases} \end{aligned} \tag{1.1}$$

With this multiplication $F(S)$ is a group with $\emptyset$ the identity. For example, an induction on the length of the sequence of the middle term, in a product of 3 terms, suffices to establish the associative law. The inverse of $\boldsymbol{s^n}$ is $(s_k^{-n_k}, \cdots, s_1^{-n_1})$.

For a group G and a subset S of G, denote by $\langle S \rangle$ the subgroup of G that S generates. The elements $\boldsymbol{s^n} \in F(S)$ for which $s_1^{n_1} \cdots s_k^{n_k}$ is the identity in G form a subset $\bar{R}(S)$ called the *relations satisfied by* S. It is a normal subgroup of $F(S)$.

Definition 1.1. Let S be a set of generators of a group G. A sequence $\{r_1, r_2, \dots\}$ of $F(S)$ is a *presentation* of G if $\bar{R}(S)$ is the smallest *normal* subgroup of $F(S)$ containing $\{r_1, r_2, \dots\}$. We say $\{r_1, r_2, \dots\}$ generates $\bar{R}(S)$. A presentation is *finite* if both S and $\{r_1, r_2, \dots\}$ are finite sets.

It is standard to denote $(s_1^{n_1}, \dots s_k^{n_k}) = \boldsymbol{s^n} \in F(S)$ by $s_1^{n_1} \cdots s_k^{n_k}$ when this symbol could not be confused with the product in another group.

Example 1.2. Let $G = \mathbb{Z}^2$, the additive group of integer pairs. Let $s_1 = (1, 0)$ and $s_2 = (0, 1)$. Take for S the set $\{s_1, s_2\}$. Then $\{s_1 s_2 s_1^{-1} s_2^{-1}\}$ is a presentation of G. Indeed, $\bar{R}(S) = [F(S), F(S)]$, the commutator subgroup of $F(S)$. [11.7c]

Example 1.3. Take for S the set $\{s_1, s_2, \dots, s_r\}$. From now on we denote $F(S)$ by F_r. There is a natural map from F_r to F_{r-1} that maps s_i to itself, $i = 1, \dots, r-1$, and s_r to $s_{r-1}^{-1} s_{r-2}^{-1} \cdots s_1^{-1}$. A nonidentity element of $\bar{R}(S)$ becomes 1 when you make the above substitution for s_r. Therefore such an element involves s_r, and $\{s_1 \cdots s_r\}$ gives a presentation of F_{r-1}.

The following treatment on *free products* of groups, from [**Wae48**], appears also in [**Ma67**, p. 97-100]. Let $G_1, \dots, G_t$ be groups. We define their free product G by its properties. There are homomorphisms $\alpha_i : G_i \to G$, $i = 1, \dots, t$, satisfying this condition: For any group H and homomorphisms $\beta_i : G_i \to H, i = 1, \cdots, t$, there exists a unique homomorphism $\beta : G \to H$ with

$$\beta \circ \alpha_i = \beta_i, \ i = 1, \dots, t. \tag{1.2}$$

Modern terminology might suggest the term *free sum* or pushout; it generalizes for arbitrary groups the direct sum of abelian groups[11.10a]. We now show a free product exists. From the definition it is unique up to isomorphism.

Define $T(\boldsymbol{G}) = T(G_1, \cdots, G_t)$ as those (finite) sequences $(x_1, \cdots, x_n)$ where each x_k is a nonidentity element of one of the groups G_i, and where consecutive terms of the sequence are in different groups. Each $g \in G_i$ acts faithfully on the right of $T(\boldsymbol{G})$ as a permutation $\alpha_i(g)$ given by the following formula. For $g \in G_i$ and $(x_1, \dots, x_n) \in T(\boldsymbol{G})$, $\alpha_i(g)$ maps $(x_1, \dots, x_n)$ to this element:

(1.3a) $(x_1, \dots, x_n g)$ if $x_n \in G_i$ and $x_n g \neq 1_{G_i}$;
(1.3b) $(x_1, \dots, x_{n-1})$ if $x_n \in G_i$ and $x_n = g^{-1}$;
(1.3c) $(x_1, \dots, x_n)$ if $x_n \notin G_i$ and $g = 1_{G_i}$;
(1.3d) $(x_1, \dots, x_n, g)$ if $x_n \notin G_i$ and $g \notin 1_{G_i}$; and
(1.3e) (g) if $(x_1, \dots, x_n) = \emptyset$.

Let $\mathrm{Per}(T(\boldsymbol{G}))$ be the group of (right action) permutations of $T(\boldsymbol{G})$. Then G is the subgroup of $\mathrm{Per}(T(\boldsymbol{G}))$ that the images of the G_i under the homomorphisms α_i, $i = 1, \ldots, t$, generate.

LEMMA 1.4. *The group G just defined is a free product of $G_1, \ldots, G_t$.*

PROOF. Express a given nonidentity element γ of G (in *reduced form*) as

$$\alpha_{i_1}(g_{i_1}) \cdots \alpha_{i_n}(g_{i_n})$$

where g_{i_k} is a nonidentity element of G_{i_k} and $i_k \neq i_{k+1}$, $i = 1, \ldots, n-1$. This expression is unique. Apply γ to $\emptyset$ (as in (1.3e)) to get $(g_{i_1}, \ldots, g_{i_n})$.

Suppose $\beta_i : G_i \to H$, $i = 1, \ldots, t$, is any collection of homomorphisms. Define $\beta : G \to H$ as follows: $\beta(\alpha_{i_1}(g_{i_1}) \cdots \alpha_{i_n}(g_{i_n}))$ is equal to $\beta_{i_1}(g_{i_1}) \cdots \beta_{i_n}(g_{i_n})$. Induction on the lengths of the reduced forms of two elements of G shows that β is a homomorphism. Clearly β is the unique homomorphism satisfying (1.2). □

1.2. Fundamental groups of unions of spaces. Let X be a connected union of finitely many differentiable manifolds. Suppose U and V are open subsets of X with $U \cup V = X$, and U, V and $U \cap V$ nonempty and *connected*. For topological spaces Y and Z with Y a subspace of Z and $y_0 \in Y$, denote the induced homomorphism $\pi_1(Y, y_0) \to \pi_1(Z, y_0)$ by $i(Y, Z)_*$.

THEOREM 1.5 (Seifert-van Kampen). *Let $x_0 \in U \cap V$. For H a group, let $\beta(U) : \pi_1(U, x_0) \to H$ and $\beta(V) : \pi_1(V, x_0) \to H$ be two homomorphisms for which*

$$\beta(U) \circ i(U \cap V, U)_* = \beta(V) \circ i(U \cap V, V)_*. \tag{1.4}$$

Then, there is a unique homomorphism $\beta(X) : \pi_1(X, x_0) \to H$ with

$$\beta(U) = \beta(X) \circ i(U, X)_* \text{ and } \beta(V) = \beta(X) \circ i(V, X)_*. \tag{1.5}$$

In using Thm. 1.5, don't forget $U \cap V$ must be connected. Neglecting this would lead to concluding the torus has trivial fundamental group (Fig. 1).

REMARK 1.6. Commutativity of this diagram characterizes $\pi_1(X, x_0)$:

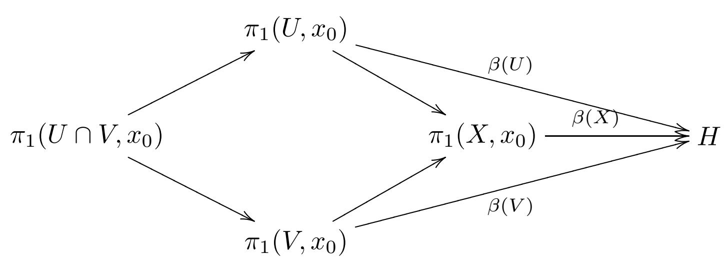
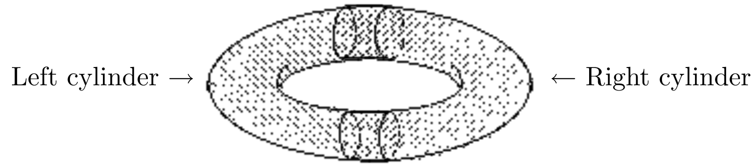


FIGURE 1. Two cylinders try to share the fundamental group of a torus, but they connect poorly.



1.3. Proof of Seifert-van Kampen, Thm. 1.5. This is a special case of [**Ma67**, p.114-22]. We give the proof in four subsections.

1.3.1. *$\pi_1(U, x_0)$ and $\pi_1(V, x_0)$ generate $\pi_1(X, x_0)$.* Let $\gamma : [a, b] \to X$ represent an element of $\pi_1(X, x_0)$. Find $t_0 = a < t_1 < \cdots < t_n = b$ so either U or V entirely contains the image of $\gamma_{|_{[t_i,t_{i+1}]}}$, $i = 1, \ldots, n$. Let U_i be either U or V, so U_i contains the image of $\gamma_{|_{[t_i,t_{i+1}]}}$. As $\gamma(t_i)$ lies in both U_{i-1} and U_i there is a path γ_i in $U_{i-1} \cap U_i$ joining x_0 to $\gamma(t_i)$, $i = 1, \ldots, n-1$. Then each of the following closed paths is in U_i for the corresponding value of i:

$$\gamma'_0 = \gamma_{|_{[t_0,t_1]}} \gamma_1^{-1},\ i = 0, \qquad \gamma'_i = \gamma_i \gamma_{|_{[t_i,t_{i+1}]}} \gamma_{i+1}^{-1},\ i = 1, \ldots, n-2,$$
$$\text{and } \gamma'_{n-1} = \gamma_{n-1}\gamma_{|_{[t_{n-1},t_n]}}.$$

The product $\gamma'_0 \cdots \gamma'_{n-1}$ is equivalent to γ. Write $[\gamma]$ as

$$i(U_0, X)_*([\gamma'_0])i(U_1, X)_*([\gamma'_1]) \cdots i(U_{n-1}, X)_*([\gamma'_{n-1}]), \tag{1.6}$$

a product of paths, each from $\pi_1(U, x_0)$ or $\pi_1(V, x_0)$.

1.3.2. *Condition for existence of β.* It is natural to define $\beta([\gamma])$ from (1.6):

$$\beta(U_0)([\gamma'_0])\beta(U_1)([\gamma'_1]) \cdots \beta(U_{n-1})([\gamma'_{n-1}]). \tag{1.7}$$

We show, if (1.6) is the identity, then so is (1.7); β is well-defined.

Let $F : [a, b] \times [0, 1] \to X$ be a homotopy between γ and the constant path:

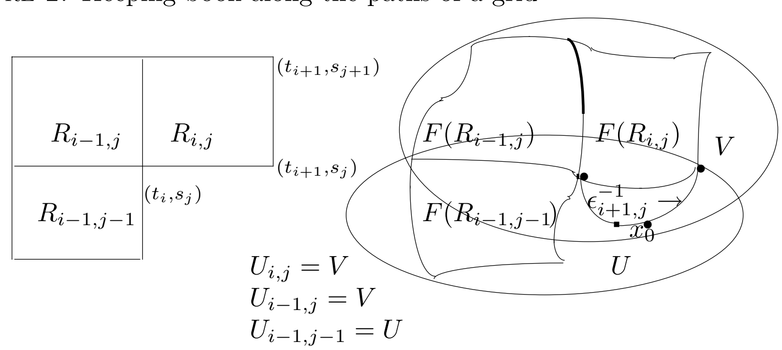
$$F(t, s) = \gamma_s(t),\ \gamma_0(t) = \gamma(t),\ \text{and } \gamma_1(t) = x_0.$$

Refine the subdivision $t_0 = a < t_1 < \cdots < t_n = b$ to find $s_0 = 0 < \cdots < s_m = 1$ so $U_{i,j}$, one of U or V, contains the image under F of each rectangle

$$R_{i,j} = \{(t, s) \mid s_j \le s \le s_{j+1},\ t_i \le t \le t_{i+1}\}.$$

Let $V_{i,j}$ be the intersection of $U_{i-1,j}, U_{i-1,j-1}$ and $U_{i,j}$. This refinement doesn't change the value of (1.7). Choose a path $\epsilon_{i,j} : [a, b] \to V_{i,j}$ with initial point x_0 and end point $\gamma_{s_j}(t_i) = F(t_i, s_j)$. When $F(t_i, s_j) = x_0$, choose $\epsilon_{i,j}$ to be the constant path, and choose $\epsilon_{i,0}$ to be γ_i (as in §1.3.1), $i = 1, \ldots, n-1$.

FIGURE 2. Keeping book along the paths of a grid



1.3.3. *Grid following paths.* Denote the path $t \in [t_i, t_{i+1}] \mapsto F(t, s_j)$ (resp., $s \in [s_j, s_{j+1}] \mapsto F(t_i, s)$) by $F_{|_{[t_i,t_{i+1}]\times s_j}}$ (resp., $F_{|_{t_i\times[s_j,s_{j+1}]}}$). Let $\gamma_{i,j}$ be the path $\epsilon_{i,j}(F_{|_{[t_i,t_{i+1}]\times s_j}})(\epsilon_{i+1,j})^{-1}$. Let $\delta_{i,j}$ be the path $\epsilon_{i,j}(F_{|_{t_i\times[s_j,s_{j+1}]}})(\epsilon_{i,j+1})^{-1}$. Define $g_{i,j}$ to be the image under $\beta(U_{i,j})$ of the homotopy class of $\gamma_{i,j}$ in $\pi_1(U_{i,j}, x_0)$, $i = 0, \ldots, n-1$; $j = 0, \ldots, m-1$. Note: (1.4) implies $g_{i,j}$ is also the image under $\beta(U_{i,j-1})$ of the class of $\gamma_{i,j}$ in $\pi_1(U_{i,j-1}, x_0)$, $i = 0, \ldots, n-1$; $j = 1, \ldots, m$. So, we consistently define $g_{i,m}$ to be $\beta(U_{i,m-1})$, the image of $\gamma_{i,m}$ in $\pi_1(U_{i,m-1}, x_0)$, $i = 0, \ldots, n-1$. Similarly, $\delta_{i,j}$ gives $h_{i,j} \in H$, $i = 0, \ldots, n$; $j = 0, \ldots, m-1$.

Since the boundary of $R_{i,j}$ (traversed clockwise) is homotopic to a constant path in $R_{i,j}$, its image under F is homotopic to a constant path in $U_{i,j}$. Therefore

$$(F_{|_{t_i\times[s_j,s_{j+1}]}})(F_{|_{[t_i,t_{i+1}]\times s_{j+1}}}) \text{ is homotopic to } (F_{|_{[t_i,t_{i+1}]\times s_j}})(F_{|_{t_{i+1}\times[s_j,s_{j+1}]}})$$

in $U_{i,j}$. Conclude:

$$\gamma_{i,j}\delta_{i+1,j} \text{ is homotopic to } \delta_{i,j}\gamma_{i,j+1} \text{ in } U_{i,j}. \tag{1.8}$$

Denote the identity in H by 1_H. An application of $\beta(U_{i,j})$ gives

(1.9a) $g_{i,j}h_{i+1,j} = h_{i,j}g_{i,j+1}$, $i = 0, \ldots, n-1$; $j = 0, \ldots, m-1$.
(1.9b) As a consequence of $F(t, 1) = F(a, s) = F(b, s) = x_0$:

$$g_{i,m} = 1_H,\ i = 0, \ldots, n-1;\ h_{0,j} = h_{n,j} = e_H,\ j = 0, \ldots, m-1.$$

Finally, (1.7) is the same as

$$g_{0,0}g_{1,0}\cdots g_{n-1,0}. \tag{1.10}$$

1.3.4. (1.9a) *and* (1.9b) *imply* (1.10) *is* 1_H. From (1.9b), $g_{0,0}\cdots g_{n-1,0}h_{n,0}$ equals (1.10). From (1.9a), this is $g_{0,0}\cdots h_{n-1,0}g_{n-1,1}$. Repeat using (1.9a) and (1.9b) to see (1.10) is $g_{0,1}g_{1,1}\cdots g_{n-1,1}$. Inductively: (1.10) is $g_{0,j}g_{1,j}\cdots g_{n-1,j}$ for each j. With $j = m$, (1.9b) shows this is 1_H.

Since $\pi_1(X, x_0)$ is a pushout of the homomorphisms $\pi_1(U \cap V, x_0) \to \pi_1(U, x_0)$ and $\pi_1(U \cap V, x_0) \to \pi_1(V, x_0)$, this uniquely defines $\pi_1(X, x_0)$ [11.10b] and concludes the proof.

1.4. Classical generators on an r-punctured sphere. Let Y be a subspace of a space X. Then Y is a *retract* of X if there is a continuous map $f : X \to Y$ such that $f(y) = y$ for $y \in Y$. The sequence of maps

$$Y \xrightarrow{i(Y,X)} X \xrightarrow{f} Y$$

induces the sequence of homomorphisms of groups

$$\pi_1(Y, y_0) \xrightarrow{i(Y,X)_*} \pi_1(X, y_0) \xrightarrow{f_*} \pi_1(Y, y_0)$$

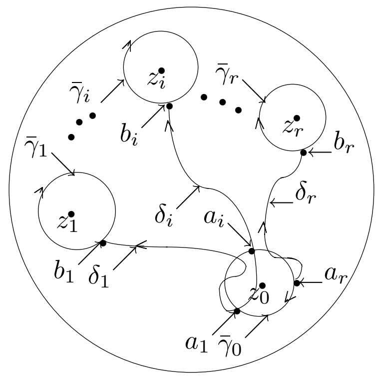
where $f_* \circ i(Y, X)_*$ is the identity. This splitting of the sequence of groups means $\pi_1(X, y_0)$ is the direct product of $\pi_1(Y, y_0)$ and the kernel of f_*.

Definition 1.7. A retract Y of X is a *deformation retract* of X if there exists a continuous map $F : X \times [0, 1] \to X$ for which $F(x, 0) = x$ and $F(x, 1) = f(x)$ for $x \in X$, and $F(y, s) = y$ for $y \in Y$, $s \in [0, 1]$.

For each $s \in [0, 1]$ the map F, restricted to $X \times s$, induces a continuous map $\pi_1(X, y_0) \to \pi_1(X, y_0)$. (Regard these fundamental groups as topological spaces with the discrete topology.) Such a map is clearly independent of s. For $s = 0$ this map is the identity, and for $s = 1$ the image of this map identifies with $\pi_1(Y, y_0)$. So, f_* identifies the fundamental groups $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$.

1.4.1. *Defining classical generators.* Chap. 2 §1.1 introduced the *r-punctured sphere*: $\mathbb{P}^1_z \setminus \{\boldsymbol{z}\} \stackrel{\text{def}}{=} U_{\boldsymbol{z}}$, r distinct points $z_1, \dots, z_r$ removed from $\mathbb{P}^1_z$.

FIGURE 3. Example classical generators based at z_0



Let z_0 be a point on $U_{\boldsymbol{z}}$. Let D_i be a disc with center z_i, $i = 1, \dots, r$. Assume these discs are disjoint and each excludes z_0. Let b_i be a point on the boundary of D_i. Regard this boundary, oriented clockwise, as a path $\bar\gamma_i$ with initial and end point b_i. Finally, let δ_i be a simple *simplicial* (Chap. 2 Def. 2.1) path with initial point z_0 and end point b_i. Assume, also, that δ_i meets none of $\bar\gamma_1, \dots, \bar\gamma_{i-1}, \bar\gamma_{i+1}, \dots, \bar\gamma_r$, and it meets $\bar\gamma_i$ only at its endpoint.

With D_0 a disc with center z_0 and disjoint from each of the discs $D_1, \dots, D_r$, consider the first point of intersection of δ_i and the boundary $\bar\gamma_0$ of D_0. Call this point a_i. Suppose $\delta_1, \dots, \delta_r$ satisfy two further conditions:

(1.11a) they are pairwise nonintersecting, excluding their initial point z_0; and
(1.11b) $a_1, \dots, a_r$ appear in order clockwise around $\bar\gamma_0$.

Since the paths are simplicial this last condition is independent of the choice of D_0, at least for D_0 sufficiently small.

With these conditions, the ordered collection of closed paths $\delta_i \bar\gamma_i \delta_i^{-1} = \gamma_i$, $i = 1, \dots, r$, in Fig. 3 are *classical generators* (for $\boldsymbol{z}$) based at z_0. We say γ_i is a *classical loop around* z_i. In our case this has a precise meaning.

1.4.2. *Main Theorem for classical generators of* $\pi_1(U_{\boldsymbol{z}}, z_0)$. Chap. 5 deforms classical generators compatible with deformations of the set $\{\boldsymbol{z}\} = \{z_1, \dots, z_r\}$. Such deformations produce very complicated sets of classical generators. Thus the generality of our next result.

THEOREM 1.8. *Let* $(\gamma_1, \dots, \gamma_r)$ *by any collection of classical generators for* $\boldsymbol{z} = (z_1, \dots, z_r)$ *based at* z_0 *on the* r*-punctured sphere* $\mathbb{P}^1_z \setminus \{\boldsymbol{z}\}$. *Then the homotopy classes* $[\gamma_1] = s_1, \dots, [\gamma_r] = s_r$ *generate* $\pi_1(\mathbb{P}^1_z \setminus \{\boldsymbol{z}\}, z_0)$ *with the one relation* $s_1 \cdots s_r$*: The Product-One condition. So,* $\pi_1(\mathbb{P}^1_z \setminus \{\boldsymbol{z}\}, z_0)$ *is isomorphic to* F_{r-1} *through the presentation* $\{s_1 \cdots s_r\}$ *(Ex. 1.3).*

If $[\gamma'_1] = s'_1, \dots, [\gamma'_r] = s'_r$ *is another collection of classical generators, then there is a* $\pi \in S_r$ *so that* s'_i *is conjugate to* $s_{(i)\pi}$, $i = 1, \dots, r$.

1.5. Proof of classical generators Thm. 1.8. For the statement on the presentation of $\pi_1(U_{\boldsymbol{z}}, z_0)$, induct on r. For $r = 0$, write $\mathbb{P}^1$ as the union of

$$\mathbb{P}^1 \setminus \{\infty\} = U_1 \text{ and } \mathbb{P}^1 \setminus \{0\} = U_2$$

as in Chap. 3 Ex. 3.2.1. Apply Thm. 1.5 (just §1.3.1). For $r \geq 1$ we show $\pi(U_{\boldsymbol{z}}, z_0)$:

(1.12a) $\gamma_1 \cdots \gamma_r$ is homotopic (on $U_{\boldsymbol{z}}$) to the identity.

(1.12b) $[\gamma_1], \dots, [\gamma_{r-1}]$ are free generators of the fundamental group.

These suffice to show the statement gives a correct presentation of $\pi_1(U_{\boldsymbol{z}}, z_0)$ if we show any relation among $s_1, \dots, s_r$ is in the group generated by products of conjugates of the product-one condition. Hints: Do an induction starting with a nontrivial relation containing no subproduct conjugate to the product one relation, and having a minimal number of appearances of s_r. No appearances of s_r is impossible from (1.12b); by conjugating shift the any one appearance of s_r to the far right. We divide the proof of (1.12) into 4 parts to separate the conceptual proof from a technical preliminary.

1.5.1. *Polygonal paths.* We show the set of paths $\gamma_1, \dots, \gamma_r$ is (simultaneously) homotopic to a set of simple polygonal paths based at z_0, intersecting only at z_0; and that $\gamma_1 \cdots \gamma_r$ is homotopic to a simple polygonal path based at z_0.

Choose D_0 so a_i is the only intersection of δ_i and $\bar{\gamma}_0$, $i = 1, \dots, r$. This is possible because $\delta_1, \dots, \delta_r$ are simplicial. For an integer $n > 2$, let $\bar{\gamma}_i^*$ be the regular n-gon inscribed in $\bar{\gamma}_i$ as a clockwise path from the vertex b_i. Chap. 2 Lem. 4.3 allows replacing each δ_i by a polygonal path homotopic to δ_i (with its endpoints fixed), so as to assume our classical generators are polygonal paths.

We explain the formation of the shaded region around the polygonal path δ_i in Fig. 4. The points b_i' and b_i'' are the vertices of $\bar{\gamma}_i^*$ next to b_i. Draw the lines through b_i' and b_i'' parallel to the last segment of δ_i, and let $d = d_n$ be the maximum of the distances between these lines and the last segment. Now continue drawing the lines at a distance d parallel to each segment of δ_i. For large n: the lines parallel to the last segment meet $\bar{\gamma}_0$ at points a_i' and a_i''; the paths δ_i^* and δ_i^{**} traced by these lines on either side of δ_i are simple and have segments corresponding one-one with the segments of δ_i. The shaded region (bounded by δ_i^*, δ_i^{**}, the two sides of $\bar{\gamma}_i^*$ next to b_i, and the line segments a_i to a_i' and a_i to a_i'') meets none of the corresponding shaded regions around δ_j for $j \neq i$. In addition, the path going from a_i to a_i', then along δ_i^*, and then from b_i' to b_i is homotopic (with a_i and b_i fixed) to δ_i through a homotopy of simple polygonal paths that stay within the shaded region and, until the end, do not meet δ_i.

Indeed, with a few choices of lines separating the *elbows* and *ends* of the shaded region from the intermediate stretches — this may require a larger value of n — we can make the homotopy canonical. To illustrate, consider the *elbow* of the last two segments of δ_i. The lines ℓ' and ℓ'' (perpendicular, respectively, to the last and second last segments of δ_i) that meet at P outline this elbow in Fig. 4. In this region the homotopy takes points along the projection from P. In general, the homotopy carries points of δ_i^* along the perpendicular to the corresponding segment of δ_i.

Let λ_i^* (resp., λ_i^{**}) be a path tracing the ray from z_0 to a_i' (resp., z_0 to a_i''). Finally, let γ_i^* be the part of $\bar{\gamma}_i^*$ with initial point b_i' and end point b_i''. Then,

$$\gamma_i' = \lambda_i^* \delta_i^* \gamma_i^* (\delta_i^{**})^{-1} (\lambda_i^{**})^{-1}, \ i = 1, \dots, r,$$

are simple, polygonal, pairwise nonintersecting (except at z_0) paths that are respectively homotopic to $\gamma_1, \dots, \gamma_r$ on $U_{\boldsymbol{z}}$.

Let $\bar{a}_i$ be the midpoint of the arc from a_i'' to a_{i+1}', $i = 1, \ldots, r-1$. Denote the path along the two straight line segments from a_i'' to $\bar{a}_i$, and then from $\bar{a}_i$ to a_{i+1}' by ϵ_i^*. Then the following simple polygonal path, γ', is homotopic on $U_{\boldsymbol{z}}$ to $\gamma_1' \cdots \gamma_r'$, and thus to $\gamma_1 \cdots \gamma_r$:

$$\lambda_1^* \delta_1^* \gamma_1^* (\delta_1^{**})^{-1} \epsilon_1^* \delta_2^* \gamma_2^* (\delta_2^{**})^{-1} \epsilon_2^* \cdots \epsilon_{r-1}^* \delta_r^* \gamma_r^* (\delta_r^{**})^{-1} (\lambda_r^{**})^{-1}. \tag{1.13}$$

This homotopy shows the interior of a polygonal sector of the disc (marked off clockwise) from a_1' to a_r'', together with the shaded regions of Fig. 4 and the interiors of $\bar{\gamma}_1^*, \ldots, \bar{\gamma}_r^*$ to be part of one connected component of $\mathbb{P}^1 \setminus \gamma'$.

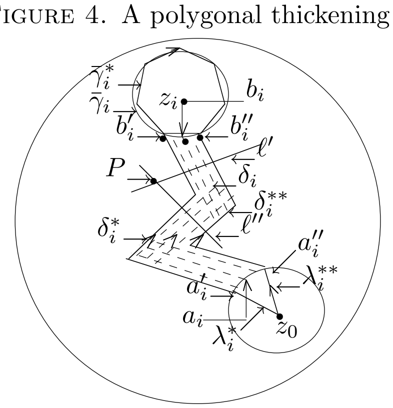
1.5.2. *Homotopy of $\gamma_1 \cdots \gamma_r$ to 1.* It suffices to show γ' is homotopic to the identity. The *Jordan curve theorem* says the complement of the simple closed path γ' on $\mathbb{P}^1$ consists of two components. For a polygonal path, however, this is fairly easy ([**He62**, p. 146] or [11.3a]). The *Schwartz-Christoffel transformation* ([**He66**, p.351-3] or §6.6) gives a one-one continuous map φ' from the closed upper hemisphere on $\mathbb{P}^1$ to $\mathbb{P}^1$, analytic on the open hemisphere, that maps the equator onto γ'. With no loss we assume φ' maps onto the component excluding z_r. From the last line of §1.5.1, none of $z_1, \ldots, z_r$ are in the image of φ'. Since the closed upper hemisphere is simply connected, so is the image of the one-one map φ' on $U_{\boldsymbol{z}}$. Thus, γ' is homotopic to the identity (see §6.6).

1.5.3. *Retraction of $U_{\boldsymbol{z}}$ onto $\gamma_1' \cup \cdots \cup \gamma_{r-1}'$.* To simplify our discussion, identity a simple path with its collection of image points. Notice this further use of the Jordan curve theorem (for polygonal paths). The path λ_{r-1}^{**} divides the interior W of γ' into two parts. The collection of points $\{z_1, \ldots, z_{r-1}\}$ is accessible from one side of λ_{r-1}^{**}, and z_r from the other. So, $\{z_1, \ldots z_{r-1}\}$ and $\{z_r\}$ lie in distinct components of $W \setminus \lambda_{r-1}^{**}$ [11.3b]. In the above replace γ' with following path:

$$\gamma'' = \lambda_1^* \delta_1^* \gamma_1^* (\delta_1^{**})^{-1} \epsilon_1^* \cdots \epsilon_{r-2}^* \delta_{r-1}^* \gamma_{r-1}^* (\delta_{r-1}^{**})^{-1} (\lambda_{r-1}^{**})^{-1}. \tag{1.14}$$

§1.5.2 shows there is a continuous one-one map φ'' from the upper hemisphere mapping the equator onto the path γ''; and mapping onto the component of $\mathbb{P}^1 \setminus \gamma''$ that includes z_r, but excludes $\{z_1, \ldots, z_{r-1}\}$.

FIGURE 4. A polygonal thickening of δ_i



The upper hemisphere minus $(\varphi'')^{-1}(z_r)$ clearly retracts to the equator. Therefore the closure of the component of $\mathbb{P}^1 \setminus \gamma''$ containing z_r, with z_r removed, retracts to γ''. Denote the closure of the other component by X''. Similarly, denote the closure of the component of $\mathbb{P}^1 \setminus \gamma_i'$ containing z_i by X_i, $i = 1, \dots, r-1$. Let Y_i be the quadrilateral with vertices a_i'', $\bar{a}_i$, a_{i+1}' and z_0, $i = 1, \dots, r-2$. Retract Y_i onto the union of the two sides defined by $\{a_i', z_0\}$ and $\{a_i'', z_0\}$. Since

$$X'' = X_1 \cup \cdots \cup X_{r-1} \cup Y_1 \cup \cdots \cup Y_{r-2},$$

this retracts X'' onto $X_1 \cup \cdots \cup X_{r-1}$. Apply the Schwartz-Christoffel transformation to retract $X_i \setminus \{z_i\}$ onto γ_i', $i = 1, \dots, r-1$. This retracts $U_{\boldsymbol{z}}$ onto $\gamma_1' \cup \cdots \cup \gamma_{r-1}'$.

1.5.4. $[\gamma_1], \dots, [\gamma_{r-1}]$ *generate* $\pi_1(\mathbb{P}^1 \setminus \{\boldsymbol{z}\}, z_0)$ *freely.* The retraction of §1.5.3 reduces this to showing $[\gamma_1'], \dots, [\gamma_{r-1}']$ generate $\pi_1(\lambda_1' \cup \cdots \cup \gamma_{r-1}', z_0)$ freely.

Let c_i be a vertex of γ_i^* different from b_i' or b_i'' (Fig. 4), $i = 1, \dots, r-1$. Take U to be $\gamma_1' \cup \cdots \cup \gamma_{r-1}' \setminus \{c_{r-1}\}$ and V to be $\gamma_1' \cup \cdots \cup \gamma_{r-1}' \setminus \{c_1, \dots, c_{r-2}\}$. Then $\gamma_1' \cup \cdots \cup \gamma_{r-2}'$ is a deformation retract of U; γ_{r-1}' is a deformation retract of V; and $\{z_0\}$ is a deformation retract of $U \cap V$. From Thm. 1.5, $\pi_1(\gamma_1' \cup \cdots \cup \gamma_{r-1}', z_0)$ is a free product of $\pi_1(U, z_0)$ and $\pi_1(V, z_0)$.

To complete the proof of the theorem, consider another r-tuple of classical generators: $[\gamma_1'] = s_1', \dots, [\gamma_r'] = s_r'$. Identify the point around which s_i' loops as the unique point $z' \in \boldsymbol{z}$ for which $s_i' \mapsto 1$ in the natural map $\pi_1(U_{\boldsymbol{z}}, z_0) \to \pi_1(U_{\boldsymbol{z}'}, z_0)$ where $\boldsymbol{z}' \dot{\cup} \{z'\} = \boldsymbol{z}$. So, there is a $\pi \in S_r$ for which s_i' loops around $z_{(i)\pi}$. An easy homotopy of both $\gamma_{(i)\pi}$ and γ_i' has these properties.

- It moves only points on these paths within the outermost of $\bar{\gamma}_{(i)\pi}$ and $\bar{\gamma}_i'$.
- The homotopies end so the respective bounding path to the discs about $z_{(i)\pi}$ are the same.

At time t in the homotopy of $\gamma_{(i)\pi}$ denote the resulting path by $\gamma_{(i)\pi,t}$. In Fig. 5: γ_i' remains constant in the homotopy; $\bar{\gamma}_{(i)\pi,1}$ is $\bar{\gamma}_i'$; and only the end portion of $\delta_{(i)\pi,t}$ varies in the homotopy. With $\gamma_{(i)\pi,1}$ replacing $\gamma_{(i)\pi} = \gamma_{(i)\pi,0}$ (and the other $r-1$ paths fixed), the equivalence classes in $\pi(U_{\boldsymbol{z}}, z_0)$ give the same elements $s_1, \dots, s_r$. With no loss, as in Fig. 5, assume $\gamma_{(i)\pi}$ and γ_i' are respectively $\delta_{(i)\pi}\bar{\gamma}_{(i)\pi}(\delta_{(i)\pi})^{-1}$ and $\delta_i'\bar{\gamma}_i'(\delta_i')^{-1}$. The homotopy class of $\delta_{(i)\pi,1}(\delta_i')^{-1}$ conjugates the former to the latter. That completes the proof of the theorem.

REMARK 1.9. Massey notes [**Ma67**, p. 125]:

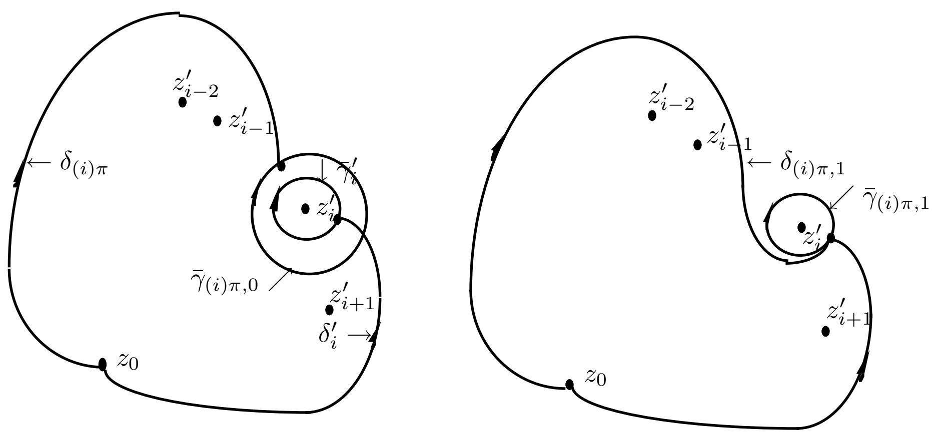
> To actually apply the Seifert-van Kampen Theorem, it is usually necessary to use the properties of deformation retracts.

2. Ramified covers from the Existence Theorem

Return to the notation of §2.1. Let $\psi : Y \to X$ be a nonconstant analytic map between two connected compact Riemann surfaces. The first part of the Existence Theorem is a combinatorial formula for contructing such ramified covers ψ.

2.1. Nonconstant maps of Riemann surfaces. Let $\psi : Y \to X$ be a nonconstant analytic map of *compact* connected Riemann surfaces. For any subset V of X denote $\psi^{-1}(V)$ by Y_V. If V is a point $x \in X$, simplify Y_V to be Y_x, the *fiber* over x. Recall the definition of unramified cover from Chap. 3 Def. 7.12.

FIGURE 5. Comparing two loops around $z_{(i)\pi}$



2.1.1. *The divisor of ramification.* We first attach a multiplicity to a point in a fiber. The outcome is that all fibers of ψ will have the same degree.

LEMMA 2.1. *The map ψ is open and so is surjective. Two analytic functions $\psi_i : Y \to \mathbb{P}^1_z$, $i = 1, 2$, with exactly the same zeros and poles (with multiplicity) on X differ by multiplication by a constant.*

For some integer n, $|Y_x| = n$ for all but finitely many $x \in X$. For $x \in X$, $|Y_x| \le n$. Let $D(\psi)$ be those x with $|Y_x| < n$. Then $Y_{X\setminus D(\psi)} \to X \setminus D(\psi)$ is an unramified cover.

Representing restriction of ψ around any point y_0 by an analytic function in a disk allows assigning a multiplicity e_{y_0} to y_0 in $Y_{\psi(y_0)}$. This gives a degree of the fiber Y_x by $\deg(Y_x) \stackrel{\text{def}}{=} \sum_{y\in Y_x} e_y$ and all fibers of ψ have degree n.

If $X = \mathbb{P}^1_z$, then the divisor (ψ) of the meromorphic function ψ has degree 0. Any meromorphic function on Y comes from an analytic map where $X = \mathbb{P}^1_z$.

PROOF. If ψ maps open sets to open sets, then the range of ψ is open. Since X is compact, the range of ψ is also closed. As X is connected, that means the range is the only possible nontrivial open and closed set, X. The statement that ψ is open is local: We have only to show it maps small open sets to small open sets. [**Ahl79**, p. 131] (as it is used below) shows ψ is locally an open map. Apply this by considering two analytic functions $\psi_i : Y \to \mathbb{P}^1_z$, $i = 1, 2$, with the same divisor of zeros and poles on Y. Then, the ratio ψ_1/ψ_2 has no zeros, and no poles. It gives an analytic map to $\mathbb{P}^1_z$ missing ∞ for example. So, it must be constant.

Let f be a nonconstant analytic function on an open connected subset U on $\mathbf{C}$, and let $z_0 \in U$. There is a neighborhood V of z_0 on which f is one-one if and only if $\frac{df}{dz}(z_0) \ne 0$ [**Ahl79**, p. 131]. Suppose $\frac{df}{dz}(z_0) \ne 0$. Then there is a neighborhood U_{z_0} of z_0 for which $\frac{df}{dz}$ is not 0 and f restricted to U_{z_0} is one-one. Let $\{(U^Y_\alpha, \varphi^Y_\alpha)\}_{\alpha\in I}$ (resp., $\{(U^X_\beta, \varphi^X_\beta)\}_{\beta\in J}$) be an atlas for the manifold Y (resp. X).

Consider the set R of $y \in Y$ with

$$\frac{d}{dz}(\varphi^X_\beta \circ \psi \circ (\varphi^Y_\alpha)^{-1})(\varphi^Y_\alpha(y)) = 0 \tag{2.1}$$

for some $\alpha \in I$, $\beta \in J$ with $y \in U_\alpha^Y \cap \psi^{-1}(U_\beta^X)$. The condition is independent of the choice of α and β (as in Chap. 3 Lem. 5.2). If R is infinite, then R has a limit point y_0. We show this leads to a contradiction.

There exists $\alpha \in I$ and $\beta \in I$ with $y_0 \in U_\alpha^Y$ and $\psi(y_0) \in U_\beta^X$. The zeros of $\frac{d}{dz}(\varphi_\beta^X \circ \psi \circ (\varphi_\alpha^Y)^{-1})$ have limit point $\varphi_\alpha^Y(y_0)$. So $\varphi_\beta^X \circ \psi \circ (\varphi_\alpha^Y)^{-1}$ is constant in a neighborhood of $\varphi_\alpha^Y(y_0)$ [**Ahl79**, p. 127], and ψ is constant in a neighborhood of y_0. The points of Y with a neighborhood on which ψ is constant is an open set contained in R. Any accumulation point of it is therefore an accumulation point of R. The above argument shows this set is closed. Since Y is connected, the existence of y_0 shows ψ is constant on all of Y, contrary to assumption. So R is finite.

Each $y \in Y \setminus R$ has a connected neighborhood U_y of y to which the restriction of ψ is a one-one function. Let $x \in X \setminus \psi(R)$. For each $y \in R$, let U_y be a neighborhood of y with $x \notin \psi(U_y)$. As ψ is one-one on U_y, U_y contains at most one point of Y_x. The cover $\{U_y\}_{y\in Y}$ of the compact space Y contains a finite subcover. Therefore Y_x is finite. Now consider neighborhoods of points of Y_x.

Let V_x be a connected neighborhood of x contained in $\psi(U_y)$ for each $y \in Y_x$. Then the connected components of Y_{V_x} are $\{U_y \cap Y_{V_x}\}_{y\in Y_x}$, and the restriction of ψ to each of these is one-one. From Chap. 3 Def. 7.12, ψ restricted to $Y_{X\setminus\psi(R)}$ is a cover, and the fibers have constant cardinality (Chap. 3 [9.21b]).

Now consider a fiber Y_x with $x \in D(\psi)$. Expression (2.1) generalizes. Any point $y \in Y_x$ gives a well-defined integer e_y: The minimal $e \geq 1$ with

$$\frac{d^e}{dz^e}(\varphi_\beta^X \circ \psi \circ (\varphi_\alpha^Y)^{-1})(\varphi_\alpha^Y(y)) \neq 0.$$

This is the *ramification index* of ψ at y (Chap. 2 Def. 7.6). Suppose $|Y_x| = t$. [**Ahl79**, p. 131] shows $f = \varphi_\beta^X \circ \psi \circ (\varphi_\alpha^Y)^{-1}$ is e to 1 in a neighborhood of $\varphi_\alpha^Y(y)$ with y removed. So, in some small punctured neighborhood $V_x^0 = V_x \setminus \{x\}$ of x, the punctured neighborhoods $U_1^0, \dots, U_t^0$ above V_x^0 have this property: $\psi_{U_i^0} : U_i^0 \to V^0$ is everywhere e_i to 1. Since the degree of each fiber over $x \in V_{x_0}$ is n, conclude $\sum_{y\in Y_x} e_y = n$. This is the formula stated in the lemma.

Now assume $X = \mathbb{P}_z^1$. So, Chap. 4 §5.3.1 assigns to ψ a well-defined divisor: $Y_0 - Y_\infty$. Its degree is $\deg(Y_0) - \deg(Y_\infty) = n - n = 0$. Finally, let f be any global meromorphic function on Y. Then, locally f is a ratio of two holomorphic functions on a disk. At each point of the disk this defines a map to $\mathbb{P}_z^1$ which is analytic, even at the zeros of the denominator (Chap. 2 §4.6). So, f is an analytic map to $\mathbb{P}_z^1$. □

We often refer to a cover $\psi : Y \to X$ by the pair (Y, ψ). With the hypotheses of Lem. 2.1, call (Y, ψ) a *ramified cover of X of degree n*: $\deg(\psi) = n$. Then $D(\psi)$ consists of the *branch points* of ψ.

DEFINITION 2.2. Let $\psi : Y \to X$ be an analytic map of 1-dimensional complex manifolds (not necessarily compact or connected). If $(\psi)^{-1}(K)$ is compact for each compact subset K of X and $|(\psi)^{-1}(x)| = n$ for all but a discrete subset of points $x \in X$, then (Y, ψ) is a *finite* ramified cover of degree n. Denote the set $\{x \mid |Y_x| \neq n\}$ by $D(\psi)$.

2.1.2. *s-equivalence of covers.* Let $\psi^i : Y^i \to X$, $i = 1, 2$, be two finite ramified covers of X. Then (Y^1, ψ^1) and (Y^2, ψ^2) are *s(trong)-equivalent* (as ramified covers of X) if there is a one-one and onto continuous map $\psi : Y^1 \to Y^2$ for which $\psi^2 \circ \psi = \psi^1$. Colloquially: There is an isomorphism that commutes with the projection maps to the base. In §3.2.2 this corresponds to the notion of *absolute* s-equivalence; there is no extra condition on the s-equivalence of these covers.

Then, ψ is automatically an analytic isomorphism [11.2]. Clearly $D(\psi^1) = D(\psi^2)$. Using the phrase s-equivalence differentiates this from other equivalences of covers that appear later. The compactification process for covers of complex manifolds in higher dimensions does not necessarily produce a manifold, as it does in dimension 1 (Thm. 2.6). Still, the notion of s-equivalence makes sense and we extend its use to many situations.

Let D be a finite subset of the connected 1-dimensional compact complex manifold X. Cor. 2.9 classifies s-equivalence classes of finite ramified covers $\psi : Y \to X$ with $D(\psi) \subseteq D$. Restricting ψ to $Y_{X\setminus D(\psi)}$ gives an unramified cover. Therefore explicitly completing such a classification requires explicitly presenting the fundamental group $\pi_1(X \setminus D, x_0)$ for $x_0 \in X \setminus D$.

2.2. Constructing ramified covers. Now take X to be the Riemann sphere, $\mathbb{P}^1 = \mathbb{P}^1_z$. Versions of these results work in the general case [11.11].

2.2.1. *Product-One Condition.* Label points of $D(\psi)$ as $\{\boldsymbol{z}\} = \{z_1, \dots, z_r\}$. Let $z_0 \in \mathbb{P}^1 \setminus D(\psi) = U_{\boldsymbol{z}}$. Let $(\gamma_1, \dots, \gamma_r) = \boldsymbol{\gamma}$ be *classical generators* for $\pi_1(U_{\boldsymbol{z}}, z_0)$. A labeling $\boldsymbol{y} = (y_1, \dots, y_n)$ of the points of Y lying over z_0 determines a transitive permutation representation $T(\boldsymbol{y})$ of $\pi_1(U_{\boldsymbol{z}}, z_0)$ of degree n. This is as in Chap. 3 Thm. 7.16, except we now have additional information. Denote $T(\boldsymbol{y})([\gamma_i])$ by $g_i \in S_n$, $i = 1, \dots, r$, and let $G(\boldsymbol{g})$ be the subgroup of S_n the g_i s generate.

LEMMA 2.3. *With the hypotheses above, $g_1 \cdots g_r = 1$. Conversely, given elements $g_i \in S_n$, $i = 1, \dots, r$ satisfying $g_1 \cdots g_r = 1$, there exists a unique homomorphism $\psi_* : \pi_1(U_{\boldsymbol{z}}, z_0) \to S_n$ mapping γ_i to g_i, $i = 1, \dots, r$. This canonically produces a(n unramified) cover $\psi : Y^0 \to U_{\boldsymbol{z}}$ whose components correspond one-one to the orbits of $G(\boldsymbol{g})$ on $\{1, \dots, n\}$.*

PROOF. Thm. 1.8 says $\pi_1(U_{\boldsymbol{z}}, z_0)$ is a free group on $\boldsymbol{\gamma}$ modulo the product one relation $[\gamma_1 \cdots \gamma_r] = 1$ in the fundamental group. This implies the quotient relation

$$[\gamma_1 \cdots \gamma_r] = [\gamma_1] \cdots [\gamma_r] = g_1 \cdots g_r = 1.$$

Conversely, the product-one relation on the g_i s implies there is a homomorphism having the desired properties. The corresponding permutation representations on the orbits of $G(\boldsymbol{g})$ correspond to connected covers of $U_{\boldsymbol{z}}$. □

DEFINITION 2.4. We call the r-tuple $\boldsymbol{g} = (g_1, \dots, g_r)$ in Lem. 2.3 a *branch cycle description of the cover* $\psi : Y \to \mathbb{P}^1$ with respect to $\boldsymbol{\gamma}$.

The group $G(\boldsymbol{g})$ is the *monodromy group* of the ramified cover (Y, ψ) (with respect to $\boldsymbol{y}$). Refer to an r-tuple $\boldsymbol{g}' \in S_n^r$ for which there is β in S_n with $\beta^{-1} g_i \beta = g_i'$, $i = 1, \dots, r$, as *absolutely equivalent* to $\boldsymbol{g}$.

2.2.2. *Compactification of unramified Riemann surface covers.* The first part of Riemann's Existence Theorem, the part so technically useful, is that there is a unique compactification of any finite cover $\psi^0 : Y^0 \to U_{\boldsymbol{z}}$ to a cover $\psi : Y \to \mathbb{P}^1_z$ of compact Riemann surfaces. We now show this.

Let D_i be the disc about z_i in Fig. 3, $i = 1, \dots, r$. Consider $Y_{D_i} \to D_i$, the restriction of ψ over D_i. Then, $\bar{\gamma}_i$ generates $\pi_1(D_i \setminus \{z_i\}, b_i)$ which maps naturally to $\pi_1(\mathbb{P}^1 \setminus D(\psi), b_i)$. Identify $\pi_1(\mathbb{P}^1 \setminus D(\psi), b_i)$ with $\pi_1(\mathbb{P}^1 \setminus D(\psi), z_0)$ using the path δ_i (of Fig. 3). Apply unique pathlifting along δ_i (Chap. 3 Lem. 7.13). So, the labeling on $\boldsymbol{y}$ uniquely labels points of Y over b_i.

With this, the permutation from $\bar{\gamma}_i$ on the fiber over b_i is g_i. Write $Y_{D_i\setminus\{z_i\}}$ as a disjoint union of connected components $\cup_{j=1}^{t_i} M_{i,j}$. Up to s-equivalence as a cover of $D_i \setminus \{z_i\}$, each $M_{i,j}$ corresponds to an orbit of $\pi_1(D_i \setminus \{z_i\}, b_i)$ on $\{1, 2, \dots, n\}$. Disjoint cycles in the decomposition of the generator g_i determine the orbits (Chap. 2 Prop. 7.4). The degree of $M_{i,j}$ as a cover is the length of its corresponding cycle, $i = 1, \dots, t$. Thus, g_i determines the covers $M_{i,1}, \dots, M_{i,t_i}$ (and their degrees).

Suppose $z_i = 0$ and D_0 is a disc about the origin in $\mathbb{C}$. Then, for each integer $e > 0$, the s-equivalence class of the connected cover of degree e is represented by

$$M' = \{(w, z) \in \mathbb{C} \times \mathbb{C} \mid w^e = z\}_{D_0\setminus\{0\}} \xrightarrow{\text{proj. on } z} D_0 \setminus \{0\}.$$

For each $z \in D_0 \setminus \{0\}$, let D_z be a disc about z contained in $D_0 \setminus \{0\}$. The components of M'_{D_z}, with their projections to $D_0 - \{0\}$, give an atlas on M'.

LEMMA 2.5. *The space $M' \cup \{(0, 0)\} = M$ has a complex manifold structure (extending that of M') that makes it a ramified cover of D_0 with exactly one point over* 0*. Indeed, M is analytically isomorphic to a disc.*

PROOF. The mapping $(w, z) \mapsto w$ gives a homeomorphism of $M' \cup \{(0, 0)\}$ to the subset of $\mathbb{C}$ that lies over D_0 via the map $w \mapsto w^e$. This subset is a disc around the origin, so it is complex analytically isomorphic to D_0. With this identification of M with D_0, add it to the atlas to conclude the manifold property. Compactness of the inverse image of a compact subset of D_0 follows easily (Def. 2.2). □

2.2.3. *From unramified to ramified covers.* Now for **Riemann's Existence Theorem**: Equivalence classes of ramified covers $\psi : Y \to X$ with $D(\psi)$ contained in a given set D' correspond exactly to classes of permutation representations of $\pi_1(X \setminus D', z_0)$ (Chap. 3 §7.2.2). Our next two results give formal restatements.

THEOREM 2.6. *Let $\boldsymbol{z} = \{z_1, \dots, z_r\}$ be a collection of r distinct points of $\mathbb{P}^1_z$. There is a one-one correspondence between connected unramified covers of $U_{\boldsymbol{z}}$ and connected covers of $\mathbb{P}^1$ ramified over a subset of $\boldsymbol{z}$.*

PROOF. From the opening remarks of this subsection we must show that if $\psi' : Y' \to \mathbb{P}^1 \setminus D'$ is an unramified cover, then there exists a unique ramified cover $\psi : Y \to \mathbb{P}^1$ such that $Y_{\mathbb{P}^1\setminus D'}$ is equivalent to (Y', ψ').

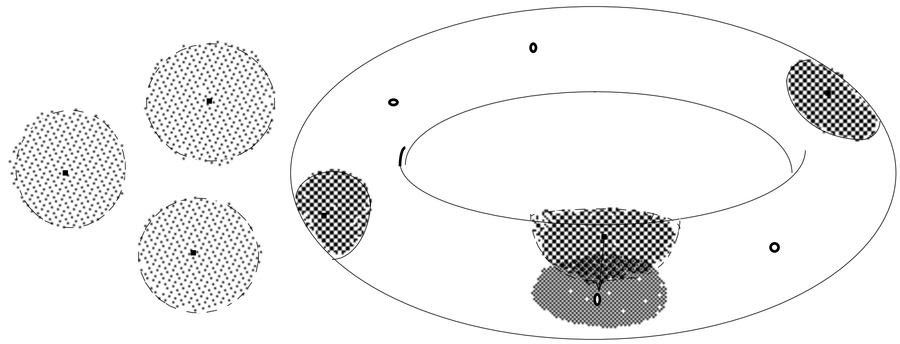
Use the notation prior to Lem. 2.5. For each $i = 1, \dots, r$, it shows how to add just one point $m_{i,j}$ to each component $M_{i,j}$, $j = 1, \dots, t_i$, of $Y'_{D_i\setminus\{z_i\}}$ to obtain a disjoint union $\cup_{j=1}^{t_i} \bar{M}_{i,j} = Y_i$ of manifolds with these properties.

(2.2a) There is a ramified covering map $\psi_i : Y_i \to D_i$.
(2.2b) $\psi_i^{-1}(D_i \setminus \{z_i\})$ is equivalent to $Y'_{D_i\setminus\{z_i\}}$.
(2.2c) $\bar{M}_{i,j}$ is analytically isomorphic to a disc.

The identification of $\bar{M}_{i,j}$ with a disc in (2.2c), $j = 1, \dots, t_i$; $i = 1, \dots, r$, added to an atlas for Y' gives an atlas for $Y = Y' \bigcup_{i,j}\{m_{i,j}\}$. Extend ψ' to $\psi : Y \to \mathbb{P}^1$ by mapping $m_{i,j}$ to z_i, $j = 1, \dots, t_i$; $i = 1, \dots, r$. Then Y_{D_i} is equivalent to Y_i, $i = 1, \dots, r$. Now we show Y is a compact manifold.

Since Y has an atlas, it is a manifold if it is Hausdorff. But $\mathbb{P}^1$ is Hausdorff. Thus if $y_1, y_2 \in Y$ with $\psi(y_1) \neq \psi(y_2)$, then we get $\psi^{-1}(U_1)$ and $\psi^{-1}(U_2)$, disjoint open sets, respectively, containing y_1 and y_2, by taking U_1 and U_2 to be disjoint open sets of $\mathbb{P}^1$, respectively, containing $\psi(y_1)$ and $\psi(y_2)$. Also, Y' is a manifold. Thus we only need consider $y_1, y_2 \in Y$ distinct points with $\psi(y_1) = \psi(y_2) = z_i$ for

FIGURE 6. Virtual neighborhoods awaiting a disc call—see Fig. 7



some $i = 1, \dots, r$. Therefore $y_1 = m_{i,\ell}$ and $y_2 = m_{i,k}$ for some $\ell \neq k$ between 1 and t_i. In particular, $\bar{M}_{i,\ell}$ and $\bar{M}_{i,k}$ are disjoint open sets, respectively, containing y_1 and y_2. The Hausdorff property follows.

For $z \in \mathbb{P}^1$ let D_z be a disc neighborhood of z. If $D_z \setminus \{z\}$ contains no points of D', then each component of Y_{D_z} contains a point of $\psi^{-1}(z)$. Thus the open sets Y_{D_z} form a neighborhood base for Y_z. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be an open cover of Y. The fiber Y_z is contained in a finite union U_z of the sets U_α, so $Y_{D_z} \subset U_z$ for some choice of D_z. Since $\mathbb{P}^1$ is compact, $\mathbb{P}^1 = \cup_{i=1}^t D_{z_i}$ and $Y = \cup_{i=1}^t U_{z_i}$. Thus $\mathcal{U}$ has a finite subcover, and Y is compact.

The theorem is complete if we show $\psi : Y \to \mathbb{P}^1_z$ is unique. Let $\psi_1 : Y^1 \to \mathbb{P}^1$ be a ramified cover with $Y^1_{\mathbb{P}^1 \setminus D'}$ equivalent to (Y', ψ'), and therefore to $Y_{\mathbb{P}^1 \setminus D'}$. Thus there is an analytic isomorphism $\varphi : Y^1_{\mathbb{P}^1 \setminus D'} \to Y_{\mathbb{P}^1 \setminus D'}$. If φ extends to Y^1 then Lem. 2.1 shows Y^1 and Y are analytically isomorphic. Let $y \in (\psi_1)^{-1}(z_i)$ for some $i = 1, \dots, r$. Let U be a connected open neighborhood of y contained in some coordinate neighborhood with $\psi_1(U)$ contained in D_i. Since U is connected, φ maps $U \setminus \psi_1^{-1}(z_i)$ into $\bar{M}_{i,j}$ for some j. Riemann's removable singularities theorem extends φ to y uniquely [11.2b]. □

Conspicuous among covers of $U_{\boldsymbol{z}}$ that now compactify to a manifold are those from an algebraic function $f(z) \in \mathcal{E}(U_{\boldsymbol{z}}, z_0)$, labeled as X^0_f in Chap. 3 Prop. 3.12.

DEFINITION 2.7. Call the manifold compactification X_f of X^0_f (or more sloppily, of f) from Thm. 2.6 its *rs-compactification*. This theorem says any manifold compactification of X^0_f will have a unique complex extending structure. Still, this notation differentiates X_f from a different compactification that might not have a manifold structure (as in Chap. 3 §4.2).

2.3. Combinatorial RET, algebraic and abelian covers. Let $\varphi : X \to \mathbb{P}^1_z$ be an analytic map of compact Riemann surfaces with $\boldsymbol{z}$ the branch points of φ. For $z' \in \mathbb{P}^1_z$ consider $D_{\varphi,z'} = D_{z'}$, the divisor of $\varphi - z'$ on X (Chap. 3 §5.3.1). For $z' = \infty$, interpret $D_{\varphi,\infty}$, the *polar* divisor, as counting (with multiplicity) points on X over ∞.

2.3.1. *An atlas from a compact cover.* For $z' \notin \boldsymbol{z} \cup \{\infty\}$, and $D_{z'} = \sum_{j=1}^n x_j$, choose a neighborhood $U_{z'}$ of z' and U_{x_i} so φ is invertible on U_{x_i}. As in Chap. 3 Prop. 3.12, use (U_{x_i}, φ) as a coordinate chart around x_i as $\varphi \overset{\text{def}}{=} w_{x_i} : U_{x_i} \to U_{z'} \subset$

$U_{\boldsymbol{z}} \setminus \{\infty\} \subset \mathbb{C}_z$. We extend this around ramified points (when $z' \in \boldsymbol{z}$) and the possibility $z' = \infty$, where e_i is the ramification index of x_i in the fiber $X_{z'}$ (§2.1), and $\{x_1, \dots, x_t\} = X_{z'}$. First, assume $z' \neq \infty$. As in applying (2.2), for some coordinate neighborhood (U_{x_i}, ψ_{x_i}) of x_i, (with $\varphi_{x_i}(x_i) = 0$) there is a branch of e_ith root of the function $\varphi \circ \psi_{x_i}^{-1} : \mathbb{C} \to \mathbb{C}$. So, there is a well defined function — designate it $w_{x_i} = \varphi^{1/e_i}$ — one-one in a neighborhood of x_i with w_{x_i} giving a coordinate chart about x_i. (Again select $U_{z'}$ to avoid ∞ and any other points of $\boldsymbol{z}$.) If $z' = \infty$, use $w_{x_i} = 1/\varphi^{1/e_i}$ instead.

Definition 2.8. Call $\{(U_x, w_x)\}_{x \in X}$ the *atlas for X from φ*. In basing a construction on this atlas, we must guarantee the result does not depend on the choice of branches of e_ith roots; we have made no canonical choice for these here.

2.3.2. *Algebraic and abelian covers of $\mathbb{P}^1_z$.* Combined with Nielsen classes (§3.2), Cor. 2.9 is the statement we use most often in describing types of covers.

Corollary 2.9. *Let $\boldsymbol{z} = \{z_1, \dots, z_r\}$ as in Thm. 2.6. Each set of classical generators $(\gamma_1, \dots, \gamma_r) = \boldsymbol{\gamma}$ for $\boldsymbol{z}$ based at $z_0 \in \mathbb{P}^1_z \setminus D'$ determines a one-one correspondence between equivalence classes of the following sets:*

(2.3a) *connected covers $\psi : Y \to \mathbb{P}^1_z$ with $D(\psi) \subseteq D'$ and $deg(\psi) = n$; and*
(2.3b) *r-tuples $\boldsymbol{g} = (g_1, \dots, g_r) \in S_n^r$ with $G(\boldsymbol{g})$ transitive, and $g_1 \cdots g_r = 1$.*

For a representative $\psi : Y \to \mathbb{P}^1_z$ of (2.3a) *and a labeling $\boldsymbol{y} = (y_1, \dots, y_n)$ of $\psi^{-1}(z_0)$, the correspondence produces a unique representative $\boldsymbol{g}$ of the class of* (2.3b)*; and the disjoint cycles of g_i identify with points of $\psi^{-1}(z_i)$, $i = 1, \dots, r$.*

Proof. From Thm. 2.6, elements of (2.3a) correspond to equivalence classes of unramified covers of $U_{\boldsymbol{z}}$. Excluding the last line, the corollary follows from the discussion prior to Def. 2.4. Given a representative $\psi : Y \to \mathbb{P}^1$ of a class of (2.3a), and a labeling $\boldsymbol{y}$ of $\psi^{-1}(z_0)$, the discussion following Def. 2.4 shows connected components of $Y_{D_i \setminus \{z_i\}}$ correspond uniquely to the disjoint cycles of g_i, $i = 1, \dots r$, in the correspondence of (2.3). Then, (2.2) gives a correspondence of the points of $\psi^{-1}(z_i)$ with the components of $Y_{D_i \setminus \{z_i\}}$, $i = 1, \dots, r$. This gives the corollary. □

Chap. 2 Thm. 8.8 describes all abelian algebraic functions of z. We compare that precise description with Cor. 2.9. An abelian cover $\varphi : X \to \mathbb{P}^1_z$ is one that is the compactification of a cover of $U_{\boldsymbol{z}}$ with abelian monodromy group. The same terminology is useful in describing *nilpotent* or *solvable* covers of any Riemann surface (or of any manifold if there is an appropriate construction of the compactification).

Definition 2.10 (Algebraic cover of $\mathbb{P}^1_z$). Call a cover of compact Riemann surfaces $\varphi : X \to \mathbb{P}^1_z$ *algebraic* if there is a second analytic map $f : X \to \mathbb{P}^1_w$ so that for some $z' \in U_{\boldsymbol{z}}$, f separates points in the fiber $X_{z'}$: $f(x') \neq f(x'')$ for distinct points $x', x'' \in X_{z'}$. Then, $\mathbb{C}(z, f) \stackrel{\text{def}}{=} \mathbb{C}(X)$ is the *field of functions* of X.

If $\varphi' : X' \to \mathbb{P}^1_z$ is s-equivalent to φ (§2.1.2), then φ is algebraic if and only if φ' is.

Proposition 2.11 (Algebraists' RET). *Every algebraic cover $\varphi : X \to \mathbb{P}^1_z$ is s-equivalent to to an rs-compactification (Def. 2.7) X_f of an algebraic function (*Chap. 3 *Prop. 3.12). The lattice of fields between $\mathbb{C}(z, f(z))$ and $\mathbb{C}(z)$ is dual to the lattice of covers $\varphi_Y : Y \to \mathbb{P}^1_z$ through which φ factors.*

Suppose $\hat{L}$ is the Galois closure of $\mathbb{C}(z, f(z)) = L$ over $\mathbb{C}(z)$, with branch points $\boldsymbol{z} = \{z_1, \dots, z_r\}$. Then a set of classical generators, $\gamma_1, \dots, \gamma_r$, for $\pi_1(U_{\boldsymbol{z}}, z_0)$ defines a set of embeddings $\psi_i : \hat{L} \to \mathcal{P}_{z_i, e_i}$ with e_i the ramification index of f over z_i.

Consider the restrictions $g_{z_i,\psi_i} \in G_f$ *of the canonical generator of* $G(\mathcal{P}_{z_i,e_i}/\mathcal{L}_{z_i})$ *to* $\hat{L}$, $i = 1,\dots,r$ *(*Chap. 2 *Lem. 7.9). Then* $(g_{z_1,\psi_1},\dots,g_{z_r,\psi_r}) = \boldsymbol{g}$ *generates* $G\hat{L}/\mathbb{C}(z)$ *and satifies the product-one condition.*

Any abelian cover of $\mathbb{P}^1_z$ *is the rs-compactification of an explicit algebraic function* f *from branches of log. So, each abelian cover of* $\mathbb{P}^1_z$ *is an algebraic cover.*

Proof. Consider the function $f : X \to \mathbb{P}^1_w$. As in Rem. 2.14, this produces an analytic structure on X. The phrase, *f is a meromorphic function on X*, means f and φ give same analytic structure on X.

As usual form $U_{\boldsymbol{z}} \subset \mathbb{P}^1_z$. Let V be an open set in $\varphi^{-1}(U_{\boldsymbol{z}})$ on which φ maps one-one to a disk D in $\mathbb{P}^1_z$. Use the notation φ_V^{-1} for the inverse map. Then, $f_D = f \circ \varphi_V^{-1} : D \to \mathbb{P}^1_w$ is meromorphic. Now we show the analytic continuations of f_D along paths in $U_{\boldsymbol{z}}$ satisfy Chap. 2 (1.1), properties. Chap. 2 Prop. 6.4 guarantee f_D is an algebraic function of z.

Let $z_0 \in D$, $x_1 \in V$ over z_0 and let $\gamma^* : [a,b] \to X$ be the unique lift to $\varphi^{-1}(U_{\boldsymbol{z}})$ starting at x_1. Consider analytic continuation of f_D along $\gamma \in \Pi_1(U_{\boldsymbol{z}}, z_0)$: $f_{D,\gamma}(t)$ is the function defined by $f \circ \gamma^*(t)$. This gives an analytic continuation according to Chap. 2 Def. 4.1. Further, analytic continuation gives only finitely many possible functions, the functions defined by f at the finite set of points above z_0. Similarly, test what happens as we approach the points $z' \in \boldsymbol{z}$. We evaluate f points with a limit on X. So the values remain bounded around a point of the range.

Now consider $\hat{L}$, the Galois closure of $\mathbb{C}(z, f(z)) = L$ over $\mathbb{C}(z)$, with branch points $\boldsymbol{z} = \{z_1,\dots,z_r\}$. First note that each element among the r classical generators $\gamma_1,\dots,\gamma_r$ defines an embedding of $\hat{L}$ in the corresponding $\mathcal{P}_{z_i,e_i}$. Write $\gamma_i = \delta_i\bar{\gamma}_i\delta_i^{-1}$ (as in Fig. 3), then δ_i gives an analytic contuation of f and all its conjugates to a disk neighborhood about z_i. Then, Chap. 2 Lem. 7.9, gives the desired embedding ψ_i. Generation and product-one conditions follow because they hold for the classical generators.

Finally consider when the cover φ has abelian monodromy. Chap. 2 (8.8) gives a branch cycle description with values in an abelian group. This was the hypothesis for producing an abelian function through branches of log. So, Chap. 2 Thm. 8.8 says branches of log display this unique cover (up to s-equivalence). □

2.3.3. *New covers from subfields of algebraic function fields.* Def. 3.5 explains normal fiber products of compact Riemann surface covers. This shows Prop. 2.11 directly gives many covers with nonabelian monodromy group as algebraic. §6 explains why any of the competing definitions of algebraic apply to algebraic covers.

Many uses of Riemann's Existence Theorem (including for the Inverse Galois Problem) require knowing covers are algebraic and more. Given f attesting to an algebraic cover, there is a unique $h(w) = w^n + \sum_{j=0}^{n-1} u_j(z)w^j \in \mathbb{C}(z)[w]$ (monic and irreducible in w) relating f to z in Prop. 2.11. We eventually need the minimal field (of *definition*) containing all coefficients (in z) of those u_j s, $j = 0,\dots,n-1$. We usually want the minimal such field as f varies. It is inefficient (sometimes hopeless), outside special cases, to compute f or h to find this out. There should be a good reason for doing such calculations. For example, theory might show there is a good choice of f, yet give reasons for looking more deeply at the algebraic relation. Our examples will show when theory is not yet sufficient to tell everything we want. Then, computing h may give us new clues about theory.

The best situation is that among these fields, as f varies, there is one that is minimal in that any nontrivial isomorphism of that field gives a new cover. This is the situation when the *field of moduli* is a field of definition (§6.2); §8.6 gives the first step in investigating this possibility and variant questions. *This is a question that tacitly assumes there is such an* f: One reason why Thm. 2.13 is so important.

Given that $\varphi : X \to \mathbb{P}^1_z$ is algebraic, we know that nonconsant elements of $\mathbb{C}(X)$ give all ways that X covers the Riemann sphere.

COROLLARY 2.12. *Each field L properly between $\mathbb{C}$ and $\mathbb{C}(X)$ corresponds to a cover $\psi : X \to Y$ with Y algebraic and the embedding $f \in \mathbb{C}(Y) \mapsto f \circ \psi \in \mathbb{C}(X)$ identifies $\mathbb{C}(Y)$ with L. Conversely, a cover ψ corresponds to subfield L.*

PROOF. For $w \in L$ nonconstant, $x \in X \mapsto w(x)$ gives a cover $\varphi_w : X \to \mathbb{P}^1_w$. Apply Thm. 2.11 to L between $\mathbb{C}(X)$ and $\mathbb{C}(w)$. Prop. 6.3 shows the converse. □

Though we do not complete showing all covers of compact Riemann surfaces are algebraic until Chap. 5 §**??**, we record that here. Examples in the remainder of this chapter emphasize aspects of applying Riemann's Existence Theorem. Several concentrate on showing the historical attention given to finding functions displaying covers as algebraic.

THEOREM 2.13. *Each cover $\varphi : X \to \mathbb{P}^1_z$ of compact Riemann surfaces is not only algebraic, it is $\mathbb{P}^1$-algebraic.*

REMARK 2.14 (Warning on constructing f in Def. 2.10). Suppose X is any compact Riemann surface and $f : X \to \mathbb{P}^1_w$ is any differentiable map with but finitely many points at which *df* is 0. There are many such maps. Thm. 2.6 says f induces a complex structure on X. Chances are, however, that complex structure will differ from that we started with. That is why it is difficult to construct an f that demonstrates a cover of $\mathbb{P}^1_z$ is algebraic.

2.4. Cuts and impossible pictures. Chap. 3 §7.2.3 discusses problems with traditional renderings of covers. Even the case when the degree n is 2, as in § 7.1. Assuming Y has a presentation as a sphere with g handles in $\mathbb{R}^3$, presenting the map ψ by a picture in $\mathbb{R}^3$ can be confusing. Still, something akin to Fig. 7 appears in many books; for example, [**Con78**, p. 243].

It includes all the usual elements, especially the *cuts*. We understand from [**Ne81**] that Gauss suggested cuts to Riemann (see §10.2). We don't rely on these cuts. Still, it will be valuable to see what they represent and how we can use symbols from them to draw pictures of the covers. The short and general description in §2.4.3 suffices for an alternate description of the manifold. The slower treatment in §2.4.1 establishes that the idea behind cuts is that covers are a locally constant structure.

2.4.1. *The simplest possible cuts.* The left of Fig. 7 represents a disc snipped and separated along a radius on the nonpositive real axis $\mathbb{R}^{\le 0} \stackrel{\text{def}}{=} \{x \le 0\}\dot\cup\{\infty\}$ from $-\infty$ to 0. Our perspective is taken from looking along the front edge. So the cut side that is on top has label T and the edge along the bottom has label B. The mathematical reality, however, is that (unlike the figure) we shouldn't separate the two sides of the cut (on either disk) by lifting one above the other. Rather, we intend just to remove the cut $\mathbb{R}^{\le 0}$, including the points (0 and ∞) at the ends of the cut. To continue the explanation, call the result of this $U_{z,l}$ and the corresponding figure on the right $U_{z,r}$. At each point z' of either of these two figures, the ring of functions we call analytic in a disc about z' (entirely within $U_{z,l}$ or $U_{z,r}$) identifies with the ring of analytic functions on $\mathbb{P}^1_z$ about that same disc regarded as on $\mathbb{P}^1_z$. Now let S be an open strip on $\mathbb{P}^1_z$ along the cut. Remove from S the negative real axis $\mathbb{R}^{<0} \stackrel{\text{def}}{=} \{x < 0\}\dot\cup\{\infty\}$ to leave two open substrips on each side of S.

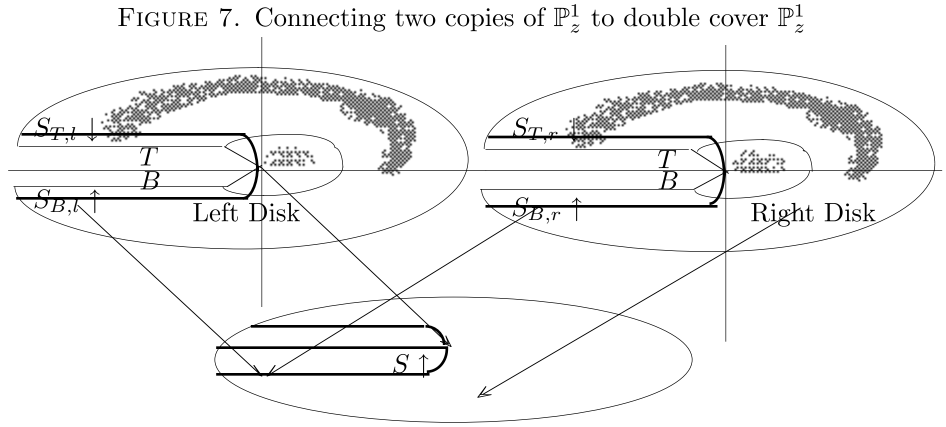
We want to consider the copies $S_{T,l}$ and $S_{B,l}$ (resp. $S_{T,r}$ and $S_{B,r}$) on $U_{z,l}$ (resp. $U_{z,r}$). These appear in Fig. 7. We also need two copies of S, S_l and S_r. Identify the analytic functions on each with those of S, exactly as you would expect from S being on $\mathbb{P}^1_z$.

The complex space X^0 we construct to cover $U_{0,\infty}$ consists of four pieces: $U_{z,l}$, $U_{z,r}$ and S_l and S_r. The map from all four pieces to $\mathbb{P}^1_z$ is exactly from the identification of each with a subset of $\mathbb{P}^1_z$. The only item left unsaid is the identification of points of $U_{z,l}$, $U_{z,r}$ and S_l and S_r between each of these four pieces. We don't want to identify these with points of $\mathbb{P}^1_z$ for this purpose. That would just give (two copies of) the manifold $U_{0,\infty}$ back. Here is the right final identification.

(2.4a) Points of $S_{T,l}$ identify with points of the corresponding strip on S_l, but $S_{B,l}$ identifies with the corresponding strip on S_r.
(2.4b) Identify points of $S_{T,r}$ with the points of the corresponding strip on S_r, but identify $S_{B,r}$ with the corresponding strip on S_l.
(2.4c) Make no further identifications.

Consider the path $\bar\gamma : [0,1] \to U_{0,\infty}$ by $t \in [0,1] \mapsto e^{-2\pi i t}$ and let $\bar\gamma_1$ be its lift to X starting at $1 \in U_{z,l}$. We follow it to what happens as it gets to the different pieces. As t increases to $\frac{1}{2}$, within the points of $S_{B,l}$, switch to points we identify with them on S_r. Now cross $\mathbb{R}^{<0}$ on S_r, to get to points that identify with points on $S_{T,r}$. Finally, complete $\bar\gamma_1$ around to 1 on $U_{z,r}$. Total result: Traversing the unique lift of $\bar\gamma$ (a clockwise path) starting at $1 \in U_{z,l}$ ends at $1 \in U_{z,R}$. Exercise: Do the same with the lift of $\bar\gamma$ starting at $1 \in U_{z,R}$ to see it ends at $1 \in U_{z,l}$.

Figure 7. Connecting two copies of $\mathbb{P}^1_z$ to double cover $\mathbb{P}^1_z$



2.4.2. *Any cycle, and any one cut.* Instead of using the labels l(eft) and r(ight) in §2.4.1, we might have used x_1 and x_2 by associating everything on the left with the point $1 \in U_{z,l}$ renamed to x_1, and everything on the right with the point $1 \in U_{z,r}$ renamed to x_2. There was no loss in our picture of changing left to right to regard it as going from bottom to top. We generalize this in two steps. First: consider any integer n, two distinct points z_1' and z_2' on $\mathbb{P}^1_z$ and any simple, piecewise simplicial path $\delta_{z_1',z_2'} : [a,b] \to \mathbb{P}^1_z$ starting at z_1' and ending at z_2'.

Let $U_{\delta_{z_1',z_2'},j}$ be a copy of $\mathbb{P}^1_z$ minus the range of $\delta_{z_1',z_2'}$, $j = 1,\dots,n$. Think of these copies listed from left to right, according to their numbering ($U_{\delta_{z_1',z_2'},j}$ on the far right). Let $S_{\delta_{z_1',z_2'}}$ be a thin open strip playing the same role toward $\delta_{z_1',z_2'}$ as did S toward $\mathbb{R}^{\le 0}$ (starting from ∞ and going toward 0 along the negative real axis). Then, consider copies of $S_{\delta_{z_1',z_2'}}$, $S_{\delta_{z_1',z_2'},j}$, $j = 1,\dots,n$, with their corresponding substrips $S_{\delta_{z_1',z_2'},j,T}$ (on the left of $\delta_{z_1',z_2'}$) and $S_{\delta_{z_1',z_2'},j,B}$ (on the right of $\delta_{z_1',z_2'}$). Let $\bar\gamma : [0,1] \to U_{z_1',z_2'}$ by $t \mapsto z_2' + r_0 e^{-2\pi i t + t_0}$ so $\bar\gamma$ meets $\delta_{z_1',z_2'}$ precisely once and not when $t = 0$ ($\bar\gamma(0) = z_0 \notin \delta_{z_1',z_2'}$). Label the point on $U_{\delta_{z_1',z_2'},j}$ above z_0 as x_j, $j = 1,\dots,n$.

Lemma 2.15. *Let $g \in S_n$. Then, there is a canonical equivalence on the union of the open sets $U_{\delta_{z_1',z_2'},j}$ and $S_{\delta_{z_1',z_2'},j}$, $j = 1,\dots,n$, so the following holds.*

(2.5a) *The resulting equivalence classes form a complex manifold X^0 giving an unramified cover $\varphi^0 : X^0 \to U_{z_1',z_2'}$.*

(2.5b) *The unique lift of $\bar\gamma$ starting at x_j ends at $(j)g$, $j = 1,\dots,n$.*

So, $(g, \delta_{z_1',z_2'})$ produces a canonical ramified cover $\varphi : X \to \mathbb{P}^1_z$ of compact Riemann surfaces, ramified only over z_1' and z_2', the completion of φ^0 from Thm. 2.6.

Proof. We do the case $g = (1\,2\,\cdots\,n)$ and leave the adjustments for the general case as an exercise. Most of even this case imitates the case $n = 2$. To simplify notation, drop extra reference to the path $\delta_{z_1',z_2'}$. The map of the union of the S_j s and U_j s to $U_{z_1',z_2'}$ is by identifying the points (and the local complex functions) on these sets with those on $\mathbb{P}^1_z$. The only item left unsaid is the identification of points of the S_j s with corresponding points of the $S_{T,j}$ s and $S_{B,j}$ s.

(2.6a) Identify points of $S_{T,j}$ with the points of the corresponding strip on S_j, but identify $S_{B,j}$ with the corresponding strip on S_{j+1}.

(2.6b) Make no further identifications, except for $j = n$, we take $j + 1$ to be 1.

Do the rest of the lemma as [11.17a] requests. □

Remark 2.16 (Locally constant structures). Chap. 3 Ex. 8.18 uses that degree n unramified covers are equivalent to locally constant bundles on $\{1,\dots,n\}$. Such structures, over $U_{\boldsymbol z}$ for example, are equivalent to looking at elements of $\mathrm{Hom}(\pi_1(U_{\boldsymbol z}), S_n)$. In Lem. 2.15, the sets S_j and U_j are simply connected. So above these sets, the cover consists of n connected copies of each of these sets. Using cuts is equivalent to explicitly laying out this locally constant structure.

2.4.3. *Any r rooted cuts.* Look again at the case of one cut. We may turn this into two rooted cuts by selecting any point z_0 along the cut. For simplicity assume for now it is not one of the endpoints of the cut. Now follow the procedure below.

Fig. 3 has the notation for the construction of classical generators. We show how the paths $\delta_1,\dots,\delta_r$ correspond one-one with r rooted cuts by the following simple device. Extend δ_i to a path $\bar\delta_i$ by adding the ray from b_i to z_i, $i = 1,\dots,r$. Thm. 1.8 says the *rooted bush* formed by the union of $\bar\delta_1,\dots,\bar\delta_r$ has simply connected complement, an essential property for having a collection of cuts on $U_{\boldsymbol z}$.

Any sequence of covers $Y \xrightarrow{\varphi_u} \mathbb{P}^1_u \xrightarrow{\varphi_{u,z}} \mathbb{P}^1_z$ gives three covers for which we would like an algorithm to precisely relate branch cycle descriptions. Especially, we have applications that allow computing classical generators for $\mathbb{P}^1_u$ automatically from classical generators for $\mathbb{P}^1_z$. This would allow constructing a branch cycle description for φ_u immediately from such a description for $\varphi_z = \varphi_{u,z} \circ \varphi_u$ (§6.3).

Suppose $\varphi : X \to \mathbb{P}^1_z$ is a ramified cover with branch cycles $\boldsymbol{g}$ from the classical generators that give $\bar{\delta}_1, \dots, \bar{\delta}_r$. This assumes a labeling $x_1, \dots, x_r$ of X_{z_0}. Then, we form a cover $\varphi_c : X_c \to \mathbb{P}^1_z$ from the cut construction canonically identifies with φ. Here are the ingredients.

(2.7a) Label copies of $\mathbb{P}^1_z$ as $\mathbb{P}^1_{z,j} = \mathbb{P}^1_j$, $j = 1, \dots, r$. On each remove the points labeled $z_0, z_1, \dots, z_r$ and call the result $\mathbb{P}_j$.

(2.7b) Use each element g_i and the cut $\bar{\delta}_i$ from z_0 to z_i to attach the $\mathbb{P}_j$ s along the lift of the ith cut. When done, compactify what we get.

We use the word *triangle* on a Riemann surface to mean a(clockwise oriented) boundary of a topological disk with the boundary divided into three oriented simplicial segments (edges) by three points called its vertices (Fig. 8). Call the triangle with its *interior* (which makes sense as the region to the right of the boundary) a (simplicial) *simplex*. The proof of Prop. 2.18 consists of describing these attachments and forming from them a natural triangulation of the result.

DEFINITION 2.17. A *triangulation* of a compact Riemann surface X is a cover of it by simplices satisfying these conditions. The simplex sides meet other simplices in their sides (in opposite orientation), and no two simplices have overlapping interiors. Let n_v (resp. n_e, n_s) be the number of vertices (resp. edges, simplices). The *Euler characteristic* of the triangulation is the alternating sum $n_v - n_e + n_s$.

Form a pre-manifold $\mathbb{P}^{\pm}_j$ (not Hausdorff) from $\mathbb{P}_j$ by replacing each point z along any one of the $\bar{\delta}_i$ s (minus its endpoints) by two points: z^+ and z^-. We put a new topology on a quotient relation on the union of $\{\mathbb{P}^{\pm}_j\}_{j=1}^n$. This uses an expected neighborhood basis at all points, except the pairs labeled z^+ and z^-: Disks not meeting any of the cuts $\bar{\delta}_1, \dots, \bar{\delta}_r$. The right neighborhood basis around z^+ and z^- on a cut use the following. Write $D_{j,z}$, a disk around z (on $\bar{\delta}_i$), as a union of $D^+_{j,z}$ and $D^-_{j,z}$: $D^+_{j,z}$ (resp. $D^-_{j,z}$) is all points on and to the left (resp. right) of $\bar{\delta}_i$.

PROPOSITION 2.18. *Compactifying X^0_c gives a cover $\varphi_x : X_c \to \mathbb{P}^1_z$ unramified over z_0. A map giving the equivalence to φ takes x_j to the point identified with z_0 on $\mathbb{P}^1_j$. Let t_i be the number of disjoint cycles in g_i, $i = 1, \dots, r$. The cuts from $\bar{\delta}_1, \dots, \bar{\delta}_r$ produce a triangulation of X_c with $n_s = 2nr$ simplices, $3nr$ sides and $2n + \sum_{i=1}^r t_i$ vertices. So the Euler characteristic of X_c is $2n + \sum_{i=1}^r t_i - nr$.*

PRECISE CUT PASTING. Form X^0_c as an equivalence relation on $\cup_{j=1}^n \mathbb{P}^{\pm}_j$. Suppose g_i maps k to l and z lies on $\bar{\delta}_i$. Then, identify $z^- \in \mathbb{P}^{\pm}_k$ with $z^+ \in \mathbb{P}^{\pm}_l$. In the resulting set, take a neighborhood of z^- to be $D^+_{l,z} \cup D^-_{k,z}$ identified along the part of $\bar{\delta}_i$ running through z.

Interpret the path $\delta_i \bar{\gamma}_i \delta_i^{-1} = \gamma_i$ in Fig. 3 as follows.

(2.8a) The lift of δ_i starting at z_0 on $\mathbb{P}^1_k$ rides along the right edge of the g_i-cut on $\mathbb{P}^1_k$ until it gets to $\bar{\gamma}_i$.

(2.8b) The initial point of $\bar{\gamma}_i$ is on the $^-$-edge of the g_i-cut on $\mathbb{P}^1_k$; it ends at the $^-$-edge of the g_i-cut on $\mathbb{P}^1_l$.

(2.8c) The lift of δ_i^{-1} starting at z_j on $\mathbb{P}^1_l$ rides along the $^-$ edge of g_i-cut on $\mathbb{P}^1_l$ until it gets to z_0.

So traversing the lift of γ_i from $z_0 \in \mathbb{P}^1_k$ will end at $z_0 \in \mathbb{P}^1_l$. Consider the small clockwise circle about z_0 denoted $\bar{\gamma}_0$ in Fig. 3. Our construction shows that traversing a lift of $\bar{\gamma}_0$ has the same effect on the points over z' in the range of $\bar{\gamma}_0$ as the product $\Pi(\boldsymbol{g}) = 1$ has on the integers $\{1, \dots, n\}$. It leaves them fixed. So, a deleted neighborhood of z_0 has above it n disjoint copies of that neighborhood on X^0_c. According to Lem. 2.5, the compactification does not ramify over z_0.

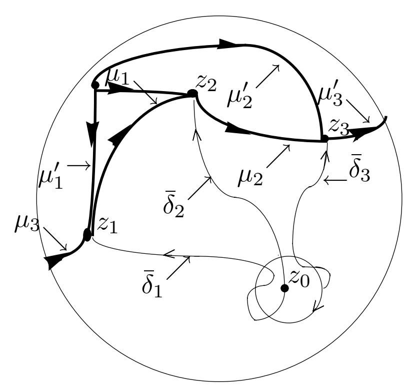
Triangulate $\mathbb{P}^1_z$ using the cuts $\bar{\delta}_1, \dots, \bar{\delta}_r$ and the proof of Thm. 1.8. Especially recall §1.5.2 showing the outside of the product of the classical generator paths bounds a disk. From this, draw paths μ_i from z_i to z_{i+1}, $i = 1, \dots, r-1$, and μ_r from z_r to z_1 with the following properties. The closed path $\mu_1 \cdot \mu_2 \cdots \mu_r$ bounds a closed (topological) disk $\bar{\Delta}_\infty$ that meets the $\bar{\delta}_i$ s only at the endpoint z_i s. From any point z_∞, interior to $\bar{\Delta}_\infty$, draw paths $\mu'_1, \dots, \mu'_r$, intersecting only at their beginning point, entirely in $\bar{\Delta}_\infty$ from z_∞ to the respective z_i s.

Triangulate $\mathbb{P}^1_j$ by listing the three ordered edges of the triangles:

$$(\bar{\delta}_i, \mu_i, \bar{\delta}^{-1}_{i+1}),\ i = 1, \dots, r-1, \quad (\bar{\delta}_r, \mu_r, \bar{\delta}^{-1}_1),$$
$$((\mu'_i)^{-1}, \mu'_{i+1}, \mu^{-1}_i),\ i = 1, \dots, r-1, \quad (\mu'_r)^{-1}, \mu'_1, \mu^{-1}_r). \tag{2.9}$$

Now, triangulate X_c using the following simple principle. Each of the $2r$ triangles in (2.9) bounds a simplex with exactly two endpoints in $\boldsymbol{z}$. Let S be one of these. Remove the two points from $\boldsymbol{z}$ in the boundary; call this S^0. It is simply-connected, and $\varphi_c : X_c \to \mathbb{P}^1_z$ is unramified over it. So, S^0 has n connected components $S^0_1, \dots, S^0_n$ over it. With each take the closure in X_c (adding back points of X_c over $\boldsymbol{z}$). These simplices give the triangulation of X_c. Just count to get the statement of the proposition. □

FIGURE 8. Cuts for a triangulation of X_c when $r = 3$



REMARK 2.19. The expression for the Euler characteristic in Prop. 2.18 is $2 - 2g_{\boldsymbol{g}} = \chi_X$, appearing in Prop. 3.10. This shows, all triangulations of a compact Riemann surface X from presenting φ using cuts, have the same Euler characteristic. We leave the following observations to the many topology books that treat Euler characteristic in detail and generality. We will do exercises in that direction to illustrate how it works.

(2.10a) χ_X is an invariant of the homeomorphism class of the compact of X (whether from cuts or not).
(2.10b) If the Euler characteristic is 2 then X is topologically a sphere: genus 0.
(2.10c) If the Euler characteristic is 0 then X is topologically a torus (as in Chap. 3 Fig. 2): genus 1.

To conclude these results from a triangulation of X in either case requires only laying out on the sphere (resp. torus) an *equivalent triangulation* [11.6].

REMARK 2.20 (Using a branch point as a base point). The beginning literature on Riemann surfaces has figures with cuts. Often the cuts don't have an obvious base point z_0 attached to them. That early literature is usually about the nature of integrals of meromorphic differentials around closed paths. So the fundamental group action is through the first homology group $H_1(U_{\boldsymbol{z}})$. As in Lem. 7.1, analytic continuations of the primitive give a complicated analytic continuation action (of course, not through a finite group). Since this is about integration, [11.16b] explores how to use a branch point as a base point for the cuts.

2.5. Residues and traces. Cauchy's Residue Theorem (Chap. 2 §5.4.4) implies the sum of the residues of any meromorphic differential ω on $\mathbb{P}^1_z$ is 0. We prove the same holds on any compact Riemann surface X. Then we give Abel's famous necessary condition for a divisor on X to be the divisor of a meromophic function $\varphi : X \to \mathbb{P}^1_z$. That it is also sufficient is the cornerstone of the theory of Riemann surfaces (§7.6 for surfaces of genus 0 and 1, and Chap. 5 §**??** in general).

2.5.1. *Sum of the residues is 0.* Let $\omega \in \mathcal{M}^1(X)$ be a meromorphic differential on the compact Riemann surface X. Chap. 2 §4.3 has the definition of the residue of a meromorphic differential at $z_0 \in \mathbb{C}_z$. Since X is compact, ω has but finitely many poles (as in the argument for Lem. 2.1). So, it has only finitely many points at which there is a nonzero residue. There are two approaches to showing the sum of the residues of ω is 0. We use here Green's Theorem, to have available the exterior calculus for later. Another approach, reducing the sum of the residues to exactly Cauchy's Theorem in the plane comes from uniformization [11.11].

2.5.2. *Orientation and Green's Theorem.* When we say a path bounds a closed disk $\bar{D}'$ in X we mean here that the oriented path has the disc on its left. Suppose X is 2-dimensional differentiable manifold with atlas $\{U_\alpha, \varphi_\alpha\}_{\alpha\in I}$. Use (x_α, y_α) for the variables of the range of $\varphi_\alpha : U_\alpha \to \mathbb{R}^2$. For $\alpha, \beta \in I$, use $F_{\beta,\alpha} = \varphi_\beta \circ \varphi_\alpha^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ for the transition function on $\varphi_\alpha(U_\alpha \cap U_\beta)$. A differential 2-form on X consists of giving $f_\alpha(x_\alpha, y_\alpha)\, dx_\alpha \wedge dy_\alpha$ for each $\alpha \in I$, satisfying these two conditions.

(2.11a) $f_\alpha(x_\alpha, y_\alpha) : \mathbb{R}^2 \to \mathbb{C}$ is differentiable on U_α.
(2.11b) $f_\beta(F_{\beta,\alpha}(x_\alpha, y_\alpha)) = \mathrm{Det}(J(F_{\beta,\alpha}(x_\alpha, y_\beta)))f_\alpha(x_\alpha, y_\alpha)$ where $J(F)$ denotes the Jacobian matrix as in Chap. 3 Lem. 3.2.

[11.4] reminds that 2-forms appear to form integrals over 2-dimensional subsets of X. The change of variables $(x_\alpha, y_\alpha) \mapsto (y_\alpha, x_\alpha)$ would change the sign of this 2-form. In the case $f_\alpha(x_\alpha, y_\alpha)$ is invariant under this transformation, the new contribution to integrating over U_α would subtract from, not add to, the integral. Fortunately, that is not an allowable transformation of coordinates on a 1-dimensional complex manifold. The (x_α, y_α) coordinates come from the complex coordinates $x_\alpha + iy_\alpha$. Any analytic change of $x_\alpha + iy_\alpha$ leaves the sign of the determinant positive [11.4a].

DEFINITION 2.21 (Orientation). An *orientation* on a differentiable dimension 2 manifold is a choice of subatlas for which the determinant of the coordinate transformation Jacobian is always positive.

PROPOSITION 2.22 (Green's Theorem). *Suppose ω is meromorphic 1-form in a domain $D \subset X$. The residue of ω has a well-defined meaning at each $x' \in D$. Denote the set of $x' \in D$ at which ω has a nonzero residue by $R_\omega(D)$. Let γ be a disjoint union $\gamma_1, \dots, \gamma_t$ of simple closed paths on D, where each γ_i is the counterclockwise boundary of a closed topological disk $\bar{D}_i \subset D$. Assume ω has no poles on γ and all its residues are in $\cup_{i+1}^t \bar{D}_i$. Then, $\frac{1}{2\pi i}\int_\gamma \omega = \sum_{x' \in R_\omega(D)} \mathrm{Res}_{x'}(\omega)$. In particular, if $D = X$, then $\frac{1}{2\pi i}\int_\gamma \omega = 0$.*

More generally, let ω' be any differentiable differental 1-form on the domain $D \setminus \cup_{i+1}^t \bar{D}_i$ as above. Then, there is a differential 2-form $d\omega'$ on D so that

$$\int_\gamma \omega = \int_{D \setminus \cup_{i=1}^t \bar{D}_i} d\omega. \tag{2.12}$$

PROOF. We show the last paragraph first. Use the notation from above for a 2-dimensional differentiable manifold. Then, on $\varphi_\alpha(U_\alpha)$ from the coordinate chart $(U_\alpha, \varphi_\alpha)$, express ω' as $f_\alpha(x_\alpha, y_\alpha)\, dx_\alpha + g_\alpha(x_\alpha, y_\alpha)\, dy_\alpha$. The production of the differential 2-form from ω' comes from the exterior derivative:

$$\begin{aligned} d(\omega') = & \quad df_\alpha \wedge dx_\alpha + dg_\alpha \wedge dy_\alpha \\ = & \quad \tfrac{\partial f_\alpha}{\partial y_\alpha}\, dy_\alpha \wedge dx_\alpha + \tfrac{\partial g_\alpha}{\partial x_\alpha}\, dx_\alpha \wedge dy_\alpha = (\tfrac{\partial g_\alpha}{\partial x_\alpha} - \tfrac{\partial f_\alpha}{\partial y_\alpha})\, dx_\alpha \wedge dy_\alpha. \end{aligned} \tag{2.13}$$

We must establish this is a 2-form: (2.11b) holds [11.23b]. Then, the integration on the right of (2.12) is independent of the coordinate chart. We already know that is true of the integration on the left from Chap. 2 Lem. 2.3. Then, the conclusion is a consequence of Green's Theorem from vector calculus in the plane. While [**Rud76**, p. 272] has a complete treatment, as our paths are semi-simplicial, the case of bounding by rectangles suffices.

To apply the result to the first paragraph requires only noting that if we are on a 1-dimensional complex manifold, then locally an analytic differential has the form $f_\alpha(z_\alpha)\, dz_\alpha$. In that case the Cauchy-Riemann equations immediately imply $d(f_\alpha(z_\alpha)\, dz_\alpha) = 0$ [11.4b]. □

REMARK 2.23. Apply Thm. 2.25 to the translate of φ by a constant, $\varphi - c$, $c \in \mathbb{C}$. Conclude $\deg(D_{z'})$ is constant running over all $z' \in \mathbb{P}^1_z$, a case of Lem. 2.1.

2.5.3. *Traces of differentials and functions.* Let $\varphi : X \to \mathbb{P}^1_z$ be an analytic map of compact Riemann surfaces. Use notation from the coordinate chart from φ (Def. 2.8). Denote meromorphic differentials on X by $\Gamma(X, \mathcal{M}^1)$. Suppose $\omega \in \Gamma(X, \mathcal{M}^1)$, and $\boldsymbol{z}$ is the branch point set of φ. For $z' \notin \boldsymbol{z}$, consider $D_{z'} = \sum_{j=1}^n x_j$. Since z' is not a branch point, there is a neighborhood $U_{z'}$ of z' and U_{x_i} so φ is invertible on U_{x_i}. To keep our neighborhoods straight, denote the inverse of φ on U_{x_i} by φ_i^{-1}. For each $i \in \{1, \dots, n\}$, $\varphi_i^{-1} : U_{z'} \to U_{x_i}$ is a section for φ. Denote the local variable on U_{x_i} by w_i. On $\varphi_i(U_{x_i})$ write ω as

$$h_i(w_i \circ \varphi_i^{-1}(z))\, d(w_i \circ \varphi_i^{-1}(z)).$$

Define $\mathrm{t}(\omega)$ on $U_{z'}$ as a differential in z by $\sum_{i=1}^n h_i(w_i \circ \varphi_i^{-1}(z))\, d(w_i \circ \varphi_i^{-1}(z))$.

We extend this around ramified points (when $z' \in \boldsymbol{z}$) where e_i is the ramification index of x_i in the fiber $X_{z'}$ and $\{x_1, \dots, x_t\} = X_{z'}$. Let $\zeta_{e_i} = e^{2\pi i/e_i}$, exactly as in Lem. 2.5. To simplify notation designate φ as z. The only extension of $\mathrm{t}(\omega)$ that gives the same values over a deleted neighborhood of $U_{z'}$ requires the expression $\sum_{i=1}^{t}\sum_{j=0}^{e_i-1} h_i(\zeta_{e_i}^j z^{1/e_i})d(\zeta_{e_i}^j z^{1/e_i})$ for $\mathrm{t}(\omega)$. Write $dz = e_i z^{\frac{e_i-1}{e_i}} dw_i$ to reexpress the contribution around x_i as

$$\sum_{j=0}^{e_i-1} \frac{h_i(\zeta_{e_i}^j z^{1/e_i})}{e_i z^{\frac{e_i-1}{e_i}}}\, dz. \tag{2.14}$$

So, (2.14) is a Laurent series in z^{1/e_i} times dz, symmetric in $\{\zeta_{e_i}^j z^{1/e_i}\}_{j=0}^{e_i-1}$, the conjugates of z^{1/e_i} over $\mathbb{C}\{\{z\}\}$. Conclude: Each term in $\mathrm{t}(\omega)$, the *trace* of ω is a Laurent series in z (times dz), and $\mathrm{t}(\omega)$ is a differential on $\mathbb{P}^1_z$.

REMARK 2.24. There is a similar definition of trace for meromorphic functions (elements of $\mathbb{C}(X)$) on X. Further, the following extensions are also easy: We may replace $\mathbb{P}^1_z$ by any Riemann surface Y (not necessarily compact) and $\varphi : X \to Y$ is a ramified cover. Recall: Regard meromorphic differentials (resp. functions) on Y as meromorphic differentials (resp. functions) on X by pullback (Chap. 3 §5.3.3).

THEOREM 2.25. *Given a ramified cover $\varphi : X \to Y$ of Riemann surfaces, the trace $\mathrm{t} = \mathrm{t}_{X/Y}$ from meromorphic differentials on X to those on Y is a $\mathbb{C}$-linear. It maps homolomorphic differentials to holomorphic differentials. In particular, if $Y = \mathbb{P}^1_z$, then the range of t on holomorphic differentials is 0.*

If $\omega \in \Gamma(Y, \mathcal{M}^1)$, then $\mathrm{t}(\varphi^*(\omega)) = \deg(X/Y)\omega$.

PROOF. Consider the statements on holomorphicity. If ω is holomorphic, each h_i above is holomorphic. From (2.14), $\mathrm{t}(\omega)$ has a pole of order no more than $\frac{e_i-1}{e_i}$ at z'. The order, however, of the pole must be an integer. That means it has no pole at z' and ω is holomorphic. As there are no holomorphic differentials on the sphere (Chap. 3 Ex. 5.17), $\mathrm{t}(\omega)$ vanishes.

More generally, if ω is any differential, then its trace has the same sum of residues as does ω. This comes back to the case the differential is locally dx_i/x_i with its trace locally reexpressed as $\sum_{j=0}^{e_i-1} h(\zeta_{e_i}^j z^{1/e_i})/e_i z^{\frac{e_i-1}{e_i}} dz$ with $h_i = 1/x_i$. The final equation is a consequence of the definitions and Rem. 2.24. □

2.6. Abel's necessary condition. With X a compact Riemann surface, let $\Gamma(X, \Omega)$ be the vector space of global holomorphic differentials on X. We don't know its dimension yet, though Lem. 6.14 shows it is $g_X = g_{\boldsymbol{g}}$ (as in Thm. 3.10). Suppose $D^0 = \sum_{i=1}^n x_i^0$ and $D^\infty = \sum_{i=1}^n x_i^\infty$ are two degree n divisors on X. We allow some points repeated with multiplicity.

Consider those n-tuples of paths $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_n)$ for which there is $\sigma \in S_n$, with γ_i having beginning point x_i^0 and end point $x_{(i)\sigma}^\infty$, $i = 1, \dots, n$. Denote these by $\Pi_1(X, D^0, D^\infty)$. If σ is the appropriate permutation, define the endpoint evaluation map by $E_{D^0,D^\infty}(\boldsymbol{\gamma}) = \sigma$. When the support of D^∞ consists of distinct points, this defines π uniquely, otherwise it is a coset of the subgroup of permutations stabilizing the ordered set $(x_1^\infty, \dots, x_n^\infty)$.

2.6.1. *Integrating a basis of holomorphic differentials.* Abel's necessary condition tests for existence of a meromorphic degree n function on X whose divisor of zeros (resp. poles) is D^0 (resp. D^∞). It is tacit that D^0 and D^∞ have no common support and are both positive divisors. Lem. 2.1 says the divisor of zeros and poles determine a function up to multiplication by a constant.

The test on integrals is made efficient by using a basis $\mathcal{B} \stackrel{\text{def}}{=} (\omega_1, \dots, \omega_u)$ for $\Gamma(X, \Omega)$. We integrate the entries of $\mathcal{B}$ along elements of $\Pi_1(X, D^0, D^\infty)$. Such integrals are equivalent to evaluating analytic continuations of a branch of a primitive (Chap. 2 §4.3). So, the monodromy theorem says results will only depend on homotopy classes of such paths (with their endpoints fixed; Chap. 2 Thm. 8.3). Denote these $\pi_1(X, D^0, D^\infty)$. For these definitions we may allow common support to D^0 and D^∞. When, however, $D^0 = D^\infty$, write $\pi_1(X, D^0)$ for the homotopy classes of n-tuples of closed paths. In this case, the paths in an ordered n-tuples of paths may each have a different end point than beginning point. The case $D^0 = nx_0$ is allowed, to indicate an n-tuple of closed paths.

From Thm. 2.6, each meromorphic function on X gives an analytic map $\varphi : X \to \mathbb{P}^1_z$. This gives a map from $\boldsymbol{\gamma} \in \pi_1(X, D^0, D^\infty)$ to the integral of $\mathcal{B}$ over $\boldsymbol{\gamma}$:

$$\text{Int}_{D^0,D^\infty} = \text{Int}_{X,D^0,D^\infty}(\boldsymbol{\gamma}) \stackrel{\text{def}}{=} \int_{\boldsymbol{\gamma}} \mathcal{B} = (\sum_{j=1}^{n} \int_{\gamma_j} \varphi_1, \dots, \sum_{j=1}^{n} \int_{\gamma_j} \varphi_u).$$

THEOREM 2.26. *The range of* Int_{X,mx_0}, *for* $x_0 \in X$, *is an abelian subgroup* L_X *of* $\mathbb{C}^u$, *independent of either* z_0 *or* $m \geq 1$. *A change of basis for* $\Gamma(X, \Omega)$ *changes* L_X *by the action (on the left) of some element of* $\text{GL}_n(\mathbb{C})$.

Suppose there is a nonconstant analytic map $\varphi : X \to \mathbb{P}^1_z$ *with* $D^0 = \varphi^{-1}(0)$ *and* $D^\infty = \varphi^{-1}(\infty)$. *Then,* $\ker(\text{Int}_{D^0,D^\infty}) \neq \emptyset$ *and the range of* $\text{Int}_{D^0,D^\infty}$ *is* L_X.

Let $\boldsymbol{z}$ *be the branch points of* φ, *and suppose* $0 \notin \boldsymbol{z}$ *(resp.* $\infty \notin \boldsymbol{z}$*). Then,* $\pi_1(U_{\boldsymbol{z}}, 0)$ *(resp.* $\pi_1(U_{\boldsymbol{z}}, \infty)$*) has a faithful left (resp. right) action on* $\ker(\text{Int}_{D^0,D^\infty})$. *Therefore,* $\{E_{D^0,D^\infty}(\boldsymbol{\gamma})\}_{\boldsymbol{\gamma} \in \ker(\text{Int}_{D^0,D^\infty})}$ *contains the monodromy group* G_φ *of* φ. *This holds even if* $0 \in \boldsymbol{z}$ *(resp.* $\infty \in \boldsymbol{z}$*) using a tangential base point at* 0 *(resp.* ∞*).*

2.6.2. *Proof of Thm. 2.26 and integrations along* $\gamma \in \pi_1(U_{\boldsymbol{z}}, z_0)$. Consider $\boldsymbol{\gamma}, \boldsymbol{\gamma}' \in \pi_1(X, mx_0)$. Then, the component wise product $\boldsymbol{\gamma} \cdot \boldsymbol{\gamma}' = (\gamma_1 \cdot \gamma_1', \dots, \gamma_1 \cdot \gamma_1')$ is in $\pi_1(X, mx_0)$. Apply Int to these to see the range is independent of m and is an abelian group. Given another basis $\mathcal{B}'$, there exists $A \in \text{GL}_n(\mathbb{C})$ so that $A(\mathcal{B}) = \mathcal{B}'$. Therefore $A(\int_\gamma(\mathcal{B})) = \int_\gamma A(\mathcal{B})$ has range in $A(L_X)$.

Now suppose φ exists. Start with the case D^0 and D^∞ have n distinct points in their support. Let $\gamma \in \pi_1(U_{\boldsymbol{z}}, 0, \infty)$, and define $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_n)$ so γ_i is the unique lift of γ starting at x_i^0. Write $\gamma : [0,1] \to U_{\boldsymbol{z}}$ to define $(\gamma_1(t), \dots, \gamma_n(t))$ for $t \in [0,1]$, an ordering of $\varphi^{-1}(t)$.

Apply Thm. 2.25 to $\text{Int}_{D^0,D^\infty}(\boldsymbol{\gamma})$ by designating the trace from φ by t_φ. Then, $\text{Int}_{D^0,D^\infty}(\boldsymbol{\gamma})$ is just $(\int_\gamma \text{t}_\varphi(\omega_1), \dots, \text{t}_\varphi(\omega_u))$. Since each of the integrand entries is 0, this shows that any element of $\pi_1(U_{\boldsymbol{z}}, 0, \infty)$ defines an element of $\ker(\text{Int}_{D^0,D^\infty})$.

If either D^0 or D^∞ has support with multiplicity, connect 0 and ∞ by paths $\gamma_{z'}$ and $\gamma_{z''}$ to respective points z' and z'' that lie (excluding endpoints) entirely in $U_{\boldsymbol{z}}$. Let γ denote a path in $U_{\boldsymbol{z}}$ connecting z' and z''. There is still an n-tuple of lifts of the path $\gamma_0 \cdot \gamma \cdot \gamma_\infty^{-1} : (0,1) \to X_0$ (avoiding endpoints). Now form the paths to replace $t \mapsto (\gamma_1(t), \dots, \gamma_n(t))$ by taking the closure of these lifted paths in X. The integral is 0, again from Thm. 2.25.

Similarly, we may compose on the right of $\text{Int}_{D^0,D^\infty}$ by $\pi_1(U_{\boldsymbol{z}}, \infty)$ so long as 0 and ∞ are not in $\boldsymbol{z}$. Suppose, however, $0 \in \boldsymbol{z}$ (the case for ∞ is analogous). Then, $\pi_1(U_{\boldsymbol{z}}, 0)$ doesn't make sense.

Chap. 2 §8.4 has the notion of a tangential base point. We need a convenient (nonempty) simply connected open set $D_{\boldsymbol{v}}$ tangent to 0 in $U_{\boldsymbol{z}}$. The choice there was a disk with 0 on the boundary, defined by a tangent vector $\boldsymbol{v}$ to 0. Let $\lambda : [0,1] \to \bar{D}_{\boldsymbol{v}}$ be a path with these properties: $\lambda(0) = 0$, restriction to $(0,1]$ has range in $D_{\boldsymbol{v}}$ and $\lambda(1) = z' \in D_{\boldsymbol{v}}$. Consider paths (minus beginning and endpoint) given by $\lambda_{(0,1]} \cdots \gamma \lambda^{-1}_{(0,1]}$, γ representing an element of $\pi_1(U_{\boldsymbol{z}}, D_{\boldsymbol{v}}, z')$. This has n distinct lifts to X. Their closures have their beginning and end points in D^0.

Up to homotopy, these paths don't depend on z' or λ (though it does on $D_{\boldsymbol{v}}$). So, up to homotopy, composition of these paths defines a group $\pi_1(U_{\boldsymbol{z}}, D_{\boldsymbol{v}})$ with an action on the left of $\ker(\mathrm{Int}_{D^0,D^\infty})$. The isomorphism class of the group is the same no matter the choice of $D_{\boldsymbol{v}}$. There is, however, no canonical isomorphism between the groups if you change $D_{\boldsymbol{v}}$ to another tangential disk [11.16a].

3. Nielsen classes and Hurwitz monodromy

This section introduces combinatorial group theory that helps display the myriad covers from Cor. 2.9. §4 uses this to illustrate Riemann's Existence Theorem. We suggest the reader go between the two sections on a first reading; we put many concepts together in this section. That includes interpretation of the genus of a compact surface, and the related fiber product and Galois closure of compact covers topics. Braid and Hurwitz monodromy representations are critical to this book. [**Ar25**], [**Ar47**], [**Bi75**], [**Boh47**], [**Ch47**], [**KMS66**], [**Ma34**], [**Mar45**] hint at early literature on the Braid group. None, however, of these sources apply these to the families of Riemann surface covers. Further, the Hurwitz monodromy group is a modest player in them though some of their combinatorics, especially [**Boh47**] and [**KMS66**], appears in our picture.

3.1. Artin Braids and Hurwitz monodromy. Let F_r be the free group on the elements of $S = \{s_1, \dots, s_r\}$. Since F_r is a free group, any r words $w_1, \dots, w_r$ in S determine a homomorphism of F_r into itself by mapping the ordered r-tuple $(s_1, \dots, s_r) = \boldsymbol{s}$ respectively to $(w_1, \dots, w_r)$. So, given $\boldsymbol{s}$, any other r-tuple, $(s'_1, \dots, s'_r)$, of generators of F_r determines an element of the automorphism group $\mathrm{Aut}(F_r)$ of F_r. Denote the set of (ordered) r-tuples of generators of F_r by $\mathcal{G}_{F_r}$.

3.1.1. *Automorphisms of $\pi_1(U_{\boldsymbol{z}}, z_0)$ permuting classical generators.* Certain automorphisms of $\pi_1(U_{\boldsymbol{z}}, z_0)$ play a big role from here on. Chap. 5 describes the geometry that produces them. Here they are a combinatorial tool.

Let Q_i be the permutation of $\mathcal{G}_{F_r}$ that sends entries of $(s_1, \dots, s_r) = \boldsymbol{s}$ (in order) to the new r-tuple of generators

$$(3.1) \qquad (s_1, \dots, s_{i-1}, s_i s_{i+1} s_i^{-1}, s_i, s_{i+2}, \dots, s_r),\ i = 1, \dots, r-1.$$

The *Artin braid group* (of degree r), is the subgroup of permutations of $\mathcal{G}_{F_r}$ that $Q_1, \dots, Q_{r-1}$ generate. We denote it B_r.

LEMMA 3.1. *Any $\boldsymbol{s} \in \mathcal{G}_{F_r}$ gives a faithful map $\psi_{\boldsymbol{s}} : B_r \to Aut(F_r)$: $Q \in B_r$ maps to the automorphism that takes $\boldsymbol{s}$ to $(\boldsymbol{s})Q$. Suppose $Q \in B_r$ and $\alpha \in \mathrm{Aut}(F_r)$. Then, Q acts on α by this formula: $(\boldsymbol{s})\alpha^Q \stackrel{\text{def}}{=} (\boldsymbol{s})Q^{-1}\alpha Q$. The action of B_r on inner automorphisms of F_r is trivial. Also, $\psi_{\boldsymbol{s}}$ is a 1-cocycle on the group B_r: $(QQ')\psi_{\boldsymbol{s}} = ((Q')\psi_{\boldsymbol{s}})(Q)\psi_{\boldsymbol{s}}^{Q'}$.*

PROOF. The effect of $Q \in B_r$ on any one $\boldsymbol{s} \in \mathcal{G}_{F_r}$ determines it. So, $\psi_{\boldsymbol{s}}$ is faithful. Notice that conjugation by w commutes with the action of Q_i on $\boldsymbol{s}$. As these are generators of B_r, this implies $\alpha^Q = \alpha$ for $Q \in B_r$ if α is conjugation by w. Check how both sides of the cocycle condition act on $\boldsymbol{s}$:

$$(\boldsymbol{s})QQ' = (\boldsymbol{s})(QQ')\psi_{\boldsymbol{s}} = ((\boldsymbol{s})Q)Q' = (((\boldsymbol{s})Q'Q'^{-1})(Q)\psi_{\boldsymbol{s}})Q' = ((\boldsymbol{s})(Q')\psi_{\boldsymbol{s}})(Q)\psi_{\boldsymbol{s}}^{Q'}.$$

This concludes the lemma. □

3.1.2. *Hurwitz monodromy quotient of the braids.* The word *cocycle* in Lem. 3.1 has a more complicated meaning than in Chap. 3 §5.4.1 where it was a condition on transition functions. This is a group cocycle, for a group acting on a nonabelian group (rather than on a module). Our emphasis is that $\psi_{\boldsymbol{s}}$ is a cocycle, not a homomorphism. The *Hurwitz monodromy group* (of degree r) is the quotient of B_r by the normal subgroup generated by

$$Q(r) = Q_1 Q_2 \cdots Q_{r-1} Q_{r-1} \cdots Q_2 Q_1. \tag{3.2}$$

Denote this quotient group by H_r.

Observations from the following proposition will appear in examples of §4. It simplifies reading Chap. 5 to be already acquainted with these. Let $\bar{R}$ be the normal subgroup of F_r that $s_1 \cdots s_r = u_{\boldsymbol{s}}$ generates (Ex. 1.3). Denote $F_r/\bar{R}$ by G_r.

PROPOSITION 3.2. *The following properties hold for B_r (acting on $\mathcal{G}_r$).*

(3.3a) *Each $Q \in B_r$ maps $s_1 \cdots s_r$ to itself and s_i to a conjugate of s_j for some j (dependent on i). This induces a homomorphism $\Psi_{r,*} : B_r \to S_r$ (the Noether representation) mapping Q_i to $(i\,i{+}1) \in S_r$, $i = 1, \dots, r$.*
(3.3b) *The Q_i s have these relations: $Q_iQ_j = Q_jQ_i$, $1 \le i \le j \le r-1$; $j \ne i-1$ or $i+1$, and $Q_iQ_{i+1}Q_i = Q_{i+1}Q_iQ_{i+1}$, $i = 1, \dots, r-2$.*
(3.3c) *Elements of* $\ker(B_r \to H_r)$ *induce inner automorphisms of G_r.*

PROOF. Each formula is a simple computation on the effect of sides of the equation on elements of S. For example, since S is a set of generators, to see (3.3a) note that the result of applying any Q_i to S is another generating set. Then, induct on the length of a word in the Q_i s to conclude the result from the application of Q_i which maps s_i to a conjugate of s_{i+1} and s_{i+1} to s_i.

The first relation of (3.3b) is obvious, for Q_i and Q_j with i and j separated, move indices with no common support. The other formula follows from a renaming of the indices and showing that $Q_1Q_2Q_1 = Q_2Q_1Q_2$ in its application to (s_1, s_2, s_3):

$$\begin{aligned}
(s_1, s_2, s_3)Q_1Q_2Q_1 = \quad & (s_1s_2s_1^{-1}, s_1s_3s_1^{-1}, s_1)Q_1 = (s_1s_2s_3s_2^{-1}s_1^{-1}, s_1s_2s_1^{-1}, s_1) \\
(s_1, s_2, s_3)Q_2Q_1Q_2 = \quad & (s_1s_2s_3s_2^{-1}s_1^{-1}, s_1, s_2)Q_2 = (s_1s_2s_3s_2^{-1}s_1^{-1}, s_1s_2s_1^{-1}, s_1).
\end{aligned}$$

Finally, consider an extension of this computation.

$$\begin{aligned}
(s_1, \dots, s_r)Q(r) = \quad & (s_1s_2s_1^{-1}, s_1s_3s_1^{-1}, \dots, s_1s_rs_1^{-1}, s_1)Q_{r-1} \cdots Q_1 \\
= \quad & (u_{\boldsymbol{s}}s_1u_{\boldsymbol{s}}^{-1}, s_1s_2s_1^{-1}, s_1s_3s_1^{-1}, \dots, s_1s_rs_1^{-1}).
\end{aligned}$$

As $u_{\boldsymbol{s}}$ has image the identity in the group G_r, $Q(r)$ induces conjugation by s_1^{-1} in G_r. If $Q \in B_r$ maps $(s_1, \dots, s_r)$ to $(s_1', \dots, s_r')$, then $QQ(r)Q^{-1}$ gives this chain of mappings: $\boldsymbol{s} \mapsto \boldsymbol{s}' \mapsto (s_1'\boldsymbol{s}'(s_1')^{-1})Q^{-1} = s_1'\boldsymbol{s}(s_1')^{-1}$. Everything in $\ker(B_r \to H_r)$ is a product of powers of elements of form $QQ(r)Q^{-1}$. So, this shows (3.3c). □

3.2. s-equivalences on Nielsen classes. The original definition of Nielsen class is from [**Fri77**]. Special cases appearing in [**Fri73**], and many illustrating examples related to elliptic curves in [**Fri78**]. They loom large in the books of Matzat-Malle and Voelklein. The former calls them *generating s-systems* [**MM95**, p. 26] (our r is their s) and the latter uses the name *ramification type* [**Vö96**, p. 37] for the most closely related definition.

An old literature on simple branched covers influenced classical geometers ([**Cl1872**], [**Hu1891**]). This continued through papers of Lefschetz, Segre and Zariski. Simple branched covers apply to the moduli space of genus g curves, knot types and Lefschetz pencils (of surfaces). Our interest came through complex multiplication and modular curves. We found every finite group produces a modular curve-like setup (Chap. 5 §**??**). S(trong)-equivalences and r(educed)-equivalences classes on elements of Nielsen classes give geometric meanings to some valuable group properties. These showed the Inverse Galois Problem fit very generally with many classical problems. A reader will require time to acclimate to these.

3.2.1. *Setup for Nielsen classes.* Consider any cover of compact connected Riemann surfaces $\varphi : X \to \mathbb{P}^1_z$ with r branch points $\boldsymbol{z}$. Denote the degree of the cover by n. Thm. 2.6 shows one way to picture how that cover arises. Choose an ordered r-tuple of classical generators $\boldsymbol{s}$ for $\pi_1(U_{\boldsymbol{z}}, z_0)$. Then φ and an ordering of the points of X over z_0 determines the image of the entries of $\boldsymbol{s}$ in the monodromy group G of the cover: Each s_i in $\boldsymbol{s}$ maps to some $g_i \in G$.

Conversely, given $\boldsymbol{s}$ and $\boldsymbol{g} = (g_1, \dots, g_r)$ generators of G satisfying the product-one condition $g_1 \cdots g_r = 1$, interpreting $\boldsymbol{s}$ as cuts (§2.4.3) attached according to the branch cycle description $\boldsymbol{g}$ produces φ (Def. 2.4).

As $\boldsymbol{s}$ runs over all classical generators, Thm. 1.8 gives this data attached to φ:

(3.4a) an associated group $G = G(\boldsymbol{g})$;
(3.4b) a permutation representation $T : G \to S_n$; and
(3.4c) conjugacy classes $\mathbf{C} = (\mathrm{C}_1, \dots, \mathrm{C}_r)$ of G into which entries of $\boldsymbol{g}$ fall in some order (denoted $\boldsymbol{g} \in \mathbf{C}$).

Further, running over all possible classical generators $\boldsymbol{s}$, the collection of images of $\boldsymbol{s}$ (branch cycle descriptions $\boldsymbol{g}$) that correspond to φ all fall in this set:

$$\mathrm{Ni}(G, \mathbf{C}, T) = \{(g_1, \dots, g_r) \mid \prod_{i=1}^{r} g_i = 1,\ G(\boldsymbol{g}) = G \le S_n,\ \boldsymbol{g} \in \mathbf{C}\}. \tag{3.5}$$

We often use $\Pi(\boldsymbol{g})$ in place of $\prod_{i=1}^{r} g_i$. Then, (3.5) is the Nielsen class of (r-tuples in G) corresponding to $(G, \mathbf{C}, T)$. Elements in this set are *Nielsen class representatives*.

3.2.2. *The s(trong)-equivalences on* $\mathrm{Ni}(G, \mathbf{C}, T)$. Consider the subgroup of S_n that normalizes G and permutes entries of $\mathbf{C}$. Denote this $N_{S_n}(G, \mathbf{C}) = N_T(G, \mathbf{C})$. For convenience we list some equivalences on a Nielsen class that will appear later. For N any group between G and $N_T(G, \mathbf{C})$, let $n \in N$ act on $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C}, T)$ by

$$\boldsymbol{g} \mapsto n\boldsymbol{g}n^{-1} \stackrel{\mathrm{def}}{=} (ng_1n^{-1}, \dots, ng_rn^{-1}).$$

Denote the orbits for this action by $\mathrm{Ni}(G, \mathbf{C}, T)/N$.

We reserve a special notation, for two cases:

(3.6a) $\mathrm{Ni}(G, \mathbf{C}, T)^{\mathrm{abs}}$ when $N = N_T(G, \mathbf{C})$ for *absolute s-equivalence classes* (of Nielsen class representatives); and

(3.6b) $\mathrm{Ni}(G,\mathbf{C},T)^{\mathrm{in}}$ when $N=G$ and T is the regular representation (acting on cosets of the identity subgroup), for *inner s-equivalence classes*.

In applyings Prop. 3.2, for an element $Q\in B_r$, when possible use the notation q for its image in H_r. For all s-equivalences, Prop. 3.2 gives an action of H_r that preserves these equivalence classes. Here is how the generator $q_i\in H_r$ acts on $\boldsymbol{g}\in \mathrm{Ni}(G,\mathbf{C},T)/N$, corresponding to (3.1):

$$(3.7)\qquad (\boldsymbol{g})q_i \stackrel{\text{def}}{=} (g_1,\dots,g_{i-1},g_ig_{i+1}g_i^{-1},g_i,g_{i+2},\dots,g_r),\ i=1,\dots,r-1.$$

As in Chap. 3 §7.1.2 denote the elements $g\in G$ with $(1)T(g)=1$ by $G(T,1)$.

Suppose G is abelian. Then, the action of $q\in H_r$ permutes the entries of $\boldsymbol{g}$ according to $\Psi_{r,*}(q)\in S_r$. This holds for inner classes. We give some standard situations that generalize this using the commutator notation $(g_1,g_2)\stackrel{\text{def}}{=} g_1g_2g_1^{-1}g_2^{-1}$. Let $G^*=(G,G)=\langle (g_1,g_2)\mid g_1,g_2\in G\rangle$ be the commutator subgroup of G.

For G any finite group and $H\triangleleft G$, suppose T is a transitive permutation representation of G and $T^{G/H}$ is the induced representation of G/H from the cosets of $G(T,1)/(G(T,1)\cap H\equiv G(T,1)\cdot H/H$. The next lemma follows from the definitions. We will see this situation come up often. We do not assume $\mathbf{C}$ is a set of conjugacy classes whose elements lie outside H. So it is possible some entries of $\mathbf{C}$ will become trivial mod H.

LEMMA 3.3. *Mapping* $\boldsymbol{g}\in \mathrm{Ni}(G,\mathbf{C},T_G)$ *to the r-tuple with entries reduced modulo H produces a natural map* $\psi_{G,\mathbf{C},T_G;H}:\mathrm{Ni}(G,\mathbf{C},T_G)\to \mathrm{Ni}(G/H,\mathbf{C}/H,T_{G/H})$. *This commutes with the action of B_r: $\psi_{G,\mathbf{C},T_G;H}$ is B_r equivariant (*Chap. 3 §*7.1.3).*

Any N between G and $N_{S_n}(G,\mathbf{C})$ that also normalizes H produces an H_r equivariant map $\mathrm{Ni}(G,\mathbf{C},T_G)/N\to \mathrm{Ni}(G/H,\mathbf{C},T_{G/H})/(N/H)$.

3.3. Normal fiber products and Galois closure. We inspect the fiber product of two compact Riemann surfaces $\varphi_i: X_i\to \mathbb{P}^1_z$ by comparing two natural choices. According to Prop. 4.9 the naive fiber product $X_1\times_{\mathbb{P}^1_z} X_2$ will produce an analytic manifold at a point (x_1',x_2') lying over $z'\in\mathbb{P}^1_z$ if and only if at least one of the corresponding pairs of ramification orders $e_{x_i'/z'}$ is 1, $i=1,2$. It also showed there really should be be $d=(e_{x_1'/z'},e_{x_2'/z'})$ distinct points (with ramification orders $[e_1',e_2']$ over z') in this fiber product corresponding to the pair (x_1',x_2'). Riemann's Existence Theorem combinatorially gives that by forming a fiber product in the category of compact Riemann surfaces (Prop. 3.4).

3.3.1. *Fiber products of compact Riemann surfaces.* For a given compact Riemann surface Y let $\mathcal{C}^{\mathrm{c}}_X$ be the category of finite covers $\varphi: X\to Y$ of compact Riemann surfaces where a map between two $\varphi_i: X_i\to Y$, $i=1,2$, is a map of Riemann surfaces $\psi: X_1\to X_2$ that commutes with the maps to Y: $\varphi_2\circ\psi=\varphi_1$. Let $\boldsymbol{y}$ be the union of the branch points for φ_1 and φ_2, and denote $Y\setminus\{\boldsymbol{y}\}$ by $U_{\boldsymbol{y}}$. It is often useful to indicate lengths of disjoint cycles of an element $g\in S_n$ by symbols like $(s_{i,1})\cdots(s_{i,t_i})$ (Chap. 3 §7.1.4).

Let $\varphi_i^0: X_i^0\to U_{\boldsymbol{y}}$ be the restriction of φ_i over $U_{\boldsymbol{y}}$. Compatible with Def. 1.3, form the unramified fiber product map $\varphi_1^0\times_{U_{\boldsymbol{y}}}\varphi_2^0: X_1^0\times_{U_{\boldsymbol{y}}} X_2^0\to U_{\boldsymbol{y}}$. This may have several components, even if each of the X_i^0 are connected (see and Chap. 3 §8.6.1 and§ 5.1). Thm. 7.16 uses an ordering of points above some base point y_0. With this it corresponds to components of the fiber product a pair of subgroups H_1 and H_2 of $\pi_1(U_{\boldsymbol{y}},y_0)$. The component of the fiber product corresponds to the subgroup $H_1\cap H_2$. The maximal pointed cover of $U_{\boldsymbol{y}}$ through which both pr_1 and

pr_2 factor comes from the subgroup $\langle H_1, H_2\rangle = H$ generated by H_1 and H_2. Then, the monodromy group of the fiber product component defined by (H_1, H_2) is the fiber product $G_{H_1} \times_{G_H} G_{H_2}$.

PROPOSITION 3.4. *Let $\varphi_1 \times^c \varphi_2 : X_1 \times^c_Y X_2 \to Y$ be the extension of $\varphi_1^0 \times_{U_{\boldsymbol{y}}} \varphi_2^0$ to the unique manifold completion of $X_1^0 \times_{U_{\boldsymbol{y}}} X_2^0$ given by Cor. 2.9. This satisfies the categorical fiber product in the category $\mathcal{C}^c_Y$.*

Suppose $Y = \mathbb{P}^1_z$ (write $\boldsymbol{z}$ for $\boldsymbol{y}$), and ${}_1\boldsymbol{g}$ and ${}_2\boldsymbol{g}$ are respective branch cycles relative to a classical set of generating homotopy classes for $\pi_(Y_{\boldsymbol{z}}, z_0)$ and orderings of the points X_{i,z_0} of X_i above z_0, $i = 1, 2$. Branch cycles for $\varphi_1 \times^c \varphi_2$ are then

$$(({}_1g_1, {}_2g_1), \dots, ({}_1g_r, {}_2g_r)) \in G_{H_1} \times_H G_{H_2}$$

given by their action on the orbit of points on the component over z_0.

Let $z_i \in \boldsymbol{z}$ and let x_1' (resp. x_2') be a point of X_i above z_i. Assume x_k' corresponds to the orbit of ${}_kg_i$ labeled by its disjoint cycle ${}_kg_i'$ (of length ${}_ks_i'$) in the disjoint cycle decomposition of ${}_kg_i$, $k = 1, 2$. Then, points of $\varphi_1 \times^c \varphi_2 : X_1 \times^c_{\mathbb{P}^1_z} X_2$ over both x_1' and x_2' correspond one-one with orbits of $({}_1g_i', {}_2g_i')$ on pairs of letters in the respective orbits of the cycles ${}_1g_i'$ and ${}_2g_i'$.

PROOF. Since $\varphi_1 \times^c \varphi_2$ is a map of compact Riemann surfaces, it is in the right category. To show it is a fiber product consider what happens if we have maps of compact Riemann surfaces $\varphi : W \to Y$, and $\psi_i : W \to X_i$, $i = 1, 2$, so that $\varphi_i \circ \psi_i = \varphi$, $i = 1, 2$. We only need show there is a unique map $\alpha : W \to X_1 \times^c_Y X_2$ that suits the other maps. Restrict all the existing maps and Riemann surface covers over $U_{\boldsymbol{y}}$, and use ^{0} superscripts to indicate that. Our previous understanding of fiber product produces the corresponding $\alpha^0 : W^0 \to (X_1 \times^c_Y X_2)^0$. Now apply the unique completion property of Cor. 2.9 to get α which then automatically has all desired properties.

Almost everything else is a restatement of previous propositions, though we comment further on the last paragraph of the statement. By relabeling the points in the fibers of X_i over z_0, assume with no loss that ${}_1g_i'$ acts as $(a_1 \dots a_{e_1})$ and ${}_2g_i'$ acts as $(b_1 \dots b_{e_1})$. The final statement says that $({}_1g_i', {}_2g_i')$ has $d = (e_1, e_2)$ orbits of length $[e_1, e_2]$ on the pairs $\{(a_u, b_v)\}_{1\le u\le e_1, 1\le v\le e_2}$ [11.12a]. □

DEFINITION 3.5. In Prop. 3.4, $\varphi_1 \times^c \varphi_2 : X_1 \times^c_Y X_2 \to Y$ is the *normal* fiber product of φ_1 and φ_2.

REMARK 3.6 (Use of the word normal). In many problems the fiber product appears as an auxiliary construction. Whether the naive or normal is a better choice depends on circumstances. Usually, however, the normal is best. In our category $\mathcal{C}^c_Y$ it would appear we are stuck with considering only manifolds. For higher dimensional manifolds this result does not work, because it is possible that two manifold (ramified) covers $\varphi_i : X_i \to \mathbb{P}^n$, with $n \ge 2$, $i = 1, 2$, have no manifold fiber product. That is, there is no manifold completion of the fiber product with these properties:

(3.8a) It is the expected fiber product restricted over the unramified locus.
(3.8b) It is a finite cover of $\mathbb{P}^n$.

The correct extension of the Prop. 3.4 uses normal analytic sets (§8.5).

3.3.2. *Geometry of the Galois closure.* Consider a cover $f : Y \to X$ of degree $n = \deg(f)$ with an attached permutation representation $T_f = T : G \to S_n$. When f is an unramified cover, Chap. 3 §8.3.2 constructs the Galois closure of this cover. We want to do the same when the cover ramifies. While the construction goes through using either the naive or normal fiber product (§3.3), we emphasize the latter. So, from this point, when we say fiber product of two covers, we are referring to the normal fiber product.

When f is unramified, we took the fiber product $Y_f^n \stackrel{\text{def}}{=} Y_X^n$ of φ, n times. Now take the normal fiber product, so the resulting set is a manifold. Then, Y_X^n has components where each point has at least two of the coordinates identical. These form the *fat diagonal*. Remove components of this fat diagonal to give Y^*, which (exactly as in Chap. 3 Thm. 8.9) has as many components as $(S_n : G)$. List one of these components as $\hat{Y}$. Points in $\hat{Y}$ over the branch points no longer have the form of an n-tuple of points in Y. The stabilizer in S_n of $\hat{Y}$ is a conjugate of G. Normalize by choosing $\hat{Y}$ so the stabilizer is actually G.

LEMMA 3.7. *Then, $\hat{\varphi} : \hat{Y} \to X$ is Galois with group G.*

If $X = \mathbb{P}^1_z$ and the cover was in the Nielsen class $\mathrm{Ni}(G, \mathbf{C}, T)$, with $T : G \to S_n$ a faithful permutation representation, the cover $\hat{\varphi}$ has the same conjugacy classes $\mathbf{C}$, but the representation is the regular representation. The Galois cover $\hat{Y} \to Y$ has group $G(1) = G(T, 1)$ where T acts on $G(1)$ cosets. The next lemma (from [**Fri77**, Lem. 2.1]) is just the compactified version of Chap. 3 Lem. 8.8.

LEMMA 3.8. *The centralizer of G in $N_{S_n}(G, \mathbf{C})$ induces the automorphisms of X that commute with φ_T.*

Consider any permutation representation $T' : G \to S_{n'}$. This provides $\varphi_{T'} : X_{T'} \to \mathbb{P}^1_z$; $X_{T'}$ is the quotient $\hat{X}/G(T', 1)$ (with $G(T', 1)$ as in §3.2.1).

From Thm. 2.6 the next observations follow from the analogous statements for unramified covers in Chap. 3 §8.3. A cover (Y, ψ) is *Galois* if the order of $\mathrm{Aut}(Y, \psi)$ is n, as big as it can be. The construction above gives a unique minimal Galois cover $\hat{Y} \xrightarrow{\hat{\psi}} X$ fitting in a commutative diagram, *the Galois closure diagram*

$$\begin{array}{ccc} \hat{Y} & \xrightarrow{\hat{\psi}_Y} & Y \\ & \hat{\psi} \searrow & \big\downarrow \psi \\ & & X \end{array} \tag{3.9}$$

Suppose $X = \mathbb{P}^1_z$, and $\boldsymbol{g}$ is a branch cycle description of the cover with respect to canonical generators of $\pi_1(U_{\boldsymbol{z}}, z_0)$. The group $\mathrm{Aut}(\hat{Y}, \hat{\psi})$, isomorphic to $G(\boldsymbol{g})$, canonically identifies with elements of $S_{\hat{n}}$ that centralize the image of $G(\boldsymbol{g})$ in its right regular representation where $\hat{n} = \deg(\hat{\psi})$.

For any subgroup H of $\mathrm{Aut}(\hat{Y}, \hat{\psi})$ let $\bar{H}$ be the subgroup of $\pi_1(\mathbb{P}^1 \setminus D(\psi), z_0)$ that maps onto H. From $\bar{H}$ we obtain a cover $\psi_H : Y_H \to \mathbb{P}^1$ (Chap. 3 Thm. 8.9) that fits in a commutative diagram

$$\begin{array}{ccc} \hat{Y} & \xrightarrow{\hat{\psi}_H} & Y_H \\ & \hat{\psi} \searrow & \big\downarrow \psi_H \\ & & \mathbb{P}^1_z \end{array} \tag{3.10}$$

where $Y \to Y_H$ is Galois with group isomorphic to H. This is a version of the classical *Galois correspondence*.

COROLLARY 3.9. *Let T_H be the coset representation of the group $G(\boldsymbol{g})$ corresponding to a subgroup H. Then $T_H(\boldsymbol{g}) = (T_H(g_1), \dots, T_H(g_r))$ is a description of the branch cycles for the cover $\psi_H : Y_H \to \mathbb{P}^1$.*

PROOF. Let $T_{\bar{H}}$ be the coset representation of $\pi_1(\mathbb{P}^1 \setminus D(\psi), z_0)$ corresponding to the subgroup $\bar{H}$, and let $\hat{H}$ be the kernel of the map from $\pi_1(\mathbb{P}^1 \setminus D(\psi), z_0)$ given by $[\gamma_i] \to \sigma_i$, $i = 1, \dots, r$, as in Cor. 2.9. Recall that $\hat{H}$ is the maximal normal subgroup of $\pi_1(\mathbb{P}^1 \setminus D(\psi), z_0)$ contained in $\bar{H}$, and the quotient $\bar{H}/\hat{H}$ is isomorphic to H. Then $(T_{\bar{H}}([\gamma_1]), \dots, T_{\bar{H}}([\gamma_r])) = T_H(\boldsymbol{g})$. Since the left side consists of a branch cycle description for (Y_H, ψ_H), this concludes the corollary. □

3.4. Riemann-Hurwitz and the genus of a cover of $\mathbb{P}^1_z$. Let $\boldsymbol{g}$ correspond to $\psi : Y \to {\mathbb{P}_z}^1$ as in Cor. 2.9. Indicate lengths of disjoint cycles of g_i by the symbol $(s_{i,1}) \cdots (s_{i,t_i})$ (Chap. 3 §7.1.4). Points of Y corresponding to cycles of length greater than 1 are *ramified points of* ψ. The *index* of g_i, $\text{ind}(g_i)$, is the integer $\sum_{j=1}^{t_i}(s_{i,j} - 1) = n - t_i$.

3.4.1. *The appearance of $g_{\boldsymbol{g}}$.* Consider the quantity $g_{\boldsymbol{g}}$ defined by the *Riemann-Hurwitz formula*:

$$2(n + g_{\boldsymbol{g}} - 1) = \sum_{z_i \in D(\psi)} \text{ind}(g_i). \tag{3.11}$$

Note!: The following lemma requires Y to be connected. Chap. 3 Ex. 5.12 defines the differential $d\psi$ of the function ψ.

PROPOSITION 3.10. *The expression $t_\psi = \sum_{z_i \in D(\psi)} \text{ind}(g_i) - 2n$ is even. So, $g_{\boldsymbol{g}}$ in (3.11) is an integer. Further, t_ψ is the degree of the divisor $(d\psi)$. Finally, $t_\psi = t_{\boldsymbol{g}}$ depends only on Y, and not on ψ or n, and $g_{\boldsymbol{g}} = (t_\psi + 2)/2$ is nonnegative.*

PROOF. The determinant of (the matrix for) g_i is $(-1)^{\text{ind}(g_i)}$ (Chap. 3 §7.1.4); check for each disjoint cycle. The product-one condition implies an even number of g_i s have determinant -1. So, for an even number of g_i s, $\text{ind}(g_i)$ is odd. In particular, $\sum_{i=1}^r \text{ind}(g_i)$ is even, and $g_{\boldsymbol{g}}$ is an integer.

Suppose $\{\varphi_\alpha, U_\alpha\}_{\alpha \in I}$ is the coordinate chart for Y from ψ (Def. 2.8). We may assume the local expression for ψ at $y \in Y$ is $\psi \circ \varphi_\alpha^{-1}(z_\alpha)$ with $\varphi_\alpha(y) = 0$. Then, the leading term is $a_u z_\alpha^u$ ($a_u \neq 0$) and the divisor of $d\psi$ at y is y^{u-1}. For y over z, if $z \in \mathbb{C}$, then $u = e_y$. If, however, $z = \infty$, then $u = -e_y$, and the divisor of $d\psi$ at y is $-e_y - 1$. The expression $-e_y - 1$ summed over $y \in Y_\infty$ is the same as the sum over $e_y - 1 - 2e_y$. Since $\sum_{y \in Y_\infty} e_y = n$ (Lem. 2.1), this gives the formula.

Now use that Y is connected so that $G(\boldsymbol{g})$ is transitive. We show

$$\sum_{i=1}^r \text{ind}(g_i) - 2(n-1)$$

is nonnegative. When all the σ's are 2-cycles the result follows if $r \geq 2(n-1)$. That is immediate from the first part of Lem. 3.11. To reduce to that case, write each g_i as $\prod_{u=1}^{\text{ind}(g_i)} h_{u,i}$ with each $h_{u,i}$ a 2-cycle. With $\boldsymbol{h}_i = (h_{1,i}, \dots, h_{\text{ind}(g_i),i})$, replace $\boldsymbol{g}$ by the juxtaposition of these $\boldsymbol{h}_i$ s: $\boldsymbol{h} = (\boldsymbol{h}_1, \dots, \boldsymbol{h}_r)$. Then, $\boldsymbol{h}$ satisfies the product-one condition and $\langle \boldsymbol{h} \rangle$ is transitive. (It is S_n: Chap. 3 [9.15e].) Further, $g_{\boldsymbol{h}} = g_{\boldsymbol{g}}$. So, the general formula for the genus of $\boldsymbol{g}$ follows from the case for 2-cycles.

We have only to show $g_{\boldsymbol{g}}$ does not depend on ψ. If ψ^*, however, is another function, then t_ψ and t_{ψ^*} are the respective degrees of the two differentials $d\psi$ and

$d\psi^*$ on the compact Riemann surface Y. The result follows from the statement in §5.3.1 that these degrees are equal. □

3.4.2. *Non-negativity of $g_{\boldsymbol{g}}$.* Let $\mathrm{Ni}(G, \mathbf{C}, T)$ be the Nielsen class for the group G and r of its conjugacy classes $\mathbf{C}$, with $T : G \to S_n$ faithful and transitive.

LEMMA 3.11 (2-cycle Braids). *For $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C}, T)$, $t_{\boldsymbol{g}} = \sum_{i=1}^{r} \mathrm{ind}(g_i) - 2n$ is independent of the choice of $\boldsymbol{g}$. When $\boldsymbol{g}$ consists of 2-cycles in S_n generating a transitive subgroup, $(t_{\boldsymbol{g}} + 2)/2 = g_{\boldsymbol{g}} \geq 0$.*

PROOF. The index of an element in S_n is independent of its conjugacy class. Since the conjugacy classes of entries of any $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C}, T)$ differs only by permutation from any other, the expression $t_{\boldsymbol{g}}$ is independent of the choise of $\boldsymbol{g}$.

Now apply transitivity of $G(\boldsymbol{g})$ and assume $\boldsymbol{g}$ has entries consisting of 2-cycles. There must be a series of $n-1$ entries of $\boldsymbol{g}$ so that, after the first, each consists of (i_1, i_2) with i_1 in the support of the previous 2-cycles, and i_2 is not. Apply an element $Q \in B_r$ to $\boldsymbol{g}$ to braid these so the $n-1$ entries just chosen come together as the first $n-1$ of the 2-cycles (for help, see [11.8]). Then, the product of the first $n-1$ 2-cycles is an n-cycle.

Now we use the product-one condition: $\prod_{i=1}^{n-1} g_i \prod_{i=n}^{r} g_i = 1$. Since $\prod_{i=1}^{n-1} g_i$ is an n-cycle, that implies $\prod_{i=n}^{r} g_i$ is also. Therefore $\langle g_i, i \geq n-1 \rangle$ is also transitive. Now apply the previous argument to $(g_n, \dots, g_r)$ to conclude there are at least $n-1$ of them, giving a total of at least $2(n-1)$. This concludes the proof. □

DEFINITION 3.12 (The genus). Prop. 3.10 defines the *genus* g_ψ of a compact Riemann surface Y presented as a cover $\psi : Y \to \mathbb{P}^1_z$.

Other books on Riemann surfaces give examples of computing g_Y from (3.11). Rarely, however, do they discuss a branch cycle description of ψ and such examples are usually abelian covers from branches of logs (Thm. 8.8 as in Prop. 2.11).

A topologist might say they have an *easier* proof of the Riemann-Hurwitz formula. That suggested proof is likely dependent on having a *triangulation* of Y. The formula then interprets as expressing the *Euler characteristic* of Y (see Rem. 2.19). There are many ways to prove this formula. No matter what the proof, interpreting the integer $g_{\boldsymbol{g}}$, the *genus* of Y, is the key point.

3.5. Hurwitz spaces; inner s-equivalence and conjugacy classes. For each s-equivalence we must consider sets of corresponding covers.

3.5.1. *Notation for Hurwitz spaces.* Suppose $\mathrm{Ni}(G, \mathbf{C})$ is a Nielsen class with r conjugacy classes. Then, any cover in the Nielsen class has an attached set $\boldsymbol{z}$ of r distinct branch points. Label the space of these unordered branch points as U_r. §4.2.1 identifies U_r with $(\mathbb{P}^1_z)^r \setminus \Delta_r / S_r$. For each s-equivalence, the classes of covers with a given $\boldsymbol{z}$ as branch point set is the same as the number of s-equivalence classes in the Nielsen class.

Label the collection of equivalence classes of covers in a given s-equivalence class by using the notation $\mathcal{H}$, denoting a *Hurwitz space*, usually with extra decoration to indicate the type of s-equivalence classes.

There are $|\mathrm{Ni}(G, \mathbf{C}, T)\mathrm{abs}|$ absolute s-equivalence classes of covers with branch points $\boldsymbol{z} \in U_r$ with the data $(G, \mathbf{C}, T)$ attached to them. Prop. 2.18 shows it requires a choice of classical generators (or cuts) canonically corresond these two sets. Denote the set of classes of $\mathrm{Ni}(G, \mathbf{C}, T)\mathrm{abs}$ covers by $\mathcal{H}(G, \mathbf{C}, T)^{\mathrm{abs}}$. A point $\boldsymbol{p} \in \mathcal{H}(G, \mathbf{C}, T)^{\mathrm{abs}}$ has as a representative a cover $\varphi_{\boldsymbol{p}} : X_{\boldsymbol{p}} \to \mathbb{P}^1_z$.

Inner s-equivalence of covers, corresponds exactly to (3.6b). The following pairs correspond to a point $\boldsymbol{p} \in \mathcal{H}_G \stackrel{\text{def}}{=} \mathcal{H}(G, \mathbf{C})^{\text{in}}$:

$$(\hat{\varphi} : \hat{X} \to \mathbb{P}^1_z,\ G(\hat{X}/\mathbb{P}^1_z) \xrightarrow{\alpha} G). \tag{3.12}$$

A given such pair is quivalent to $(\hat{\varphi}' : \hat{X}' \to \mathbb{P}^1_z,\ G(\hat{X}'/\mathbb{P}^1_z) \xrightarrow{\alpha'} G)$ if

(3.13) $\hat{\psi} : \hat{X} \to \hat{X}'$ with $\hat{\varphi}' \circ \hat{\psi} = \hat{\varphi}$ induces α'.

For example: Suppose $g \in G$ maps $\hat{X} \to \hat{X}$, changing α by conjugation by g. Then, composing α with g gives a cover inner equivalent to (3.12). On the other hand, composing α with an outer automorphism of G gives a new equivalence class.

Proposition 3.13. *Given a (faithful) permutation representation $T : G \to S_n$, there is a natural map $\Psi_{\text{in,abs}} : \mathcal{H}(G, \mathbf{C})^{\text{in}} \to \mathcal{H}(G, \mathbf{C}, T)^{\text{abs}}$ by*

$$(\hat{\varphi} : \hat{X} \to \mathbb{P}^1_z,\ G(\hat{X}/\mathbb{P}^1_z) \xrightarrow{\alpha} G) \mapsto \varphi : \hat{X}/\alpha^{-1}(G(T,1)) \to \mathbb{P}^1_z.$$

This map is $|N_{S_n}(G, \mathbf{C})/G|$ to 1 over every point of $\mathcal{H}(G, \mathbf{C}, T)^{\text{abs}}$.

Definition 3.14. An element $\boldsymbol{g} \in \text{Ni}(G, \mathbf{C})$ is a H-M (Harbater-Mumford) representative if r is even and $\boldsymbol{g} = (g_1, g_1^{-1}, \dots, g_{r/2}, g_{r/2}^{-1})$.

Example 3.15 (Comparing H_r inner and absolute orbits). There is a general problem that arises when applying prop. 3.13. Suppose $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ represent two distinct elements of $\mathcal{H}(G, \mathbf{C})^{\text{in}}$ that lie over the same element of $\mathcal{H}(G, \mathbf{C}, T)^{\text{abs}}$. When is there a $q \in H_r$ that takes $\boldsymbol{g}_1$ to $\boldsymbol{g}_2$?

With T_n the standard representation of S_n, each element of $\mathcal{H}(A_n, \mathbf{C}, T_n)^{\text{abs}}$ has exactly two from $\mathcal{H}(A_n, \mathbf{C})^{\text{in}}$ above it. Suppose $\boldsymbol{g}$ is an H-M rep. from $g_1, \dots, g_{r/2}$ where there is $\alpha \in S_n \setminus A_n$ such that $\alpha g_i \alpha^{-1} = G_i^{-1}$, $i = 1, \dots, r/2$. Then, $(\boldsymbol{g})q = \alpha \boldsymbol{g} \alpha^{-1}$ with $q = q_1 q_3 \cdots q_{r-1} \in H_r$. That solves relating the inner and absolute H_r orbits in this case.

3.5.2. *Conjugacy classes and multiplier groups.* Many times one conjugacy class will appear several times in $\mathbf{C}$. It is easy to label conjugacy classes in S_n. One tricky event is when several entries of $\mathbf{C}$ are distinct conjugacy classes in $G(\boldsymbol{g}) \le S_n$, but generate the same conjugacy class in S_n. We give easy examples here.

Suppose C is a conjugacy class in a group G consisting of elements having order m. Then, for $k \in (\mathbb{Z}/m)^*$ denote the kth powers of elements of C by C^k. For a collection $\mathbf{C}$ of conjugacy classes use the notation $\mathbf{C}^k$, $k \in \hat{\mathbb{Z}}^*$ (integers relatively prime to the order of elements in $\mathbf{C}$).

Definition 3.16. Call $\mathbf{C}$, conjugacy classes in G, a *rational union* if $\mathbf{C}^k = \mathbf{C}$ (both sides counted with multiplicity) for all $k \in \hat{\mathbb{Z}}^*$. There is always a natural *rationalization* $\mathbf{C}'$ of $\mathbf{C}$: The minimal rational collection of conjugacy classes containing $\mathbf{C}$.

Let T_n be the standard representation of S_n, $n \ge 3$. As in Chap. 3 §7.1.4, indicate conjugacy classes in S_n with a simple notation. Give C_i by its cycle type: $(s_{i,1}) \cdots (s_{i,t_i})$, $i = 1, \dots, r$. As $\sum_{j+1}^{t_i} s_{i,j} = n$, it is often (not always) convenient to order the $s_{i,j}$ by size: $s_{i,j} \le s_{i,j+1}$. Recall: This class is in A_n if and only if $n - t_i$ (its index, §3.4) is even.

For any homomorphism $\psi : H \to G$ (containment of H in G is the standard case) a conjugacy class C in H generates a conjugacy class C_G in G: For $g \in \mathrm{C}$, C_G is the collection of conjugates of g in G.

DEFINITION 3.17 (Multiplier group). Let C be a conjugacy class C in G whose elements have order m. The *multipier group* of C is $M_{\mathrm{C}} \stackrel{\text{def}}{=} \{k \in (\mathbb{Z}/m)^* \mid \mathrm{C}^k = \mathrm{C}\}$. The *multiplier field* K_{C} is the fixed field in $\mathbb{Q}(e^{2\pi i/k})$ of M_{C}.

3.5.3. *Multiplier groups and fields in A_n.* Each conjugacy class in S_n is rational. It is more complicated for A_n. The following results give valuable examples.

LEMMA 3.18. *For a conjugacy class* C *in A_n, there are two possibilities for $\mathrm{C}_{S_n} = (s_1)\cdots(s_t)$: $\mathrm{C}_G = \mathrm{C}$, or $\mathrm{C}_G = \mathrm{C}\dot{\cup}h\mathrm{C}h$ with $h = (1\,2)$. The former happens if and only if there is an even length cycle or a product of an odd number of disjoint 2-cycles that centralizes any $g \in \mathrm{C}$. The latter happens if and only if*

(3.14) *all the s_j s are odd, $j = 1, \ldots, t$, and distinct.*

PROOF. Suppose h is either an m-cycle with m even or it is product of m disjoint 2-cycles with m odd. Then $S_n = A_n \dot{\cup} hA_n$. If h centralizes $g \in \mathrm{C}$, then the orbit of hA_n on g is the same as that of A_n and $\mathrm{C}_{S_n} = \mathrm{C}$.

Conversely, by the class equation if C_{S_n} is larger than C, some nontrivial element of $S_n \setminus A_n$ centralizes g. Suppose m is the length of a disjoint cycle in g and there are t_m of these. Denote by g_m the product of all these disjoint m-cycles in g. Write g as the product of these g_m s running over all distinct integers m. Denote the centralizer of $(1\, m+1 \,\ldots\, (t_m-1)m+1)\ldots(m\, 2m \,\ldots\, t_m m)$ by C_m. Then, the centralizer of g is isomorphic to the direct product of the C_m s.

Now we check that the group C_m is the wreath product

$$\mathbb{Z}/m \wr S_{t_m} = (\mathbb{Z}/m)^{t_m} \times^s S_{t_m} \text{ (Chap. 3 §8.4)}$$

regarded as a subgroup of S_{mt_m}. The copy of $(\mathbb{Z}/m)^{t_m}$ identifies with products of powers of the disjoint cycles in g_m. A $\pi \in S_{t_m}$ maps $(i_1, \ldots, i_{t_m}) \in (\mathbb{Z}/n)^{t_m}$ to $(i_{(1)\pi}, \ldots, i_{(t_m)\pi})$. Example: $\pi = (1\,2)$ acts in S_{mt_m} as $(1\, m{+}1)(2\, m{+}2)\cdots(m\, 2m)$, a product of m disjoint 2-cycles. If m is even then C_m contains an m-cycle, that is not in A_{mt_m}. If m is odd, but larger than 1, a 2-cycle $\pi \in S_{t_m}$ acts as a product of m disjoint 2-cycles in A_{mt_m}. So, C_m is in A_{mt_m} if and only t_m is 1 and m is odd. That concludes the proof. □

Assume $g \in \mathrm{C}$ with $\mathrm{C}_{S_n} = (s_1)\cdots(s_t)$ satisfies (3.14), the only possible nonrational conjugacy classes in A_n. The next proposition checks which of those are rational when $\mathrm{C} = (n)$ (n is odd); [11.18b] outlines the general case [**Fri95b**, p. 332].

Recall: p^u *exactly divides* n (written $p^u||n$) if p^u divides n, but p^{u+1} does not. Also, use Euler's Theorem that if p is an odd prime, the invertible integers $(\mathbb{Z}/p^u)^*$ (of $\mathbb{Z}/p^u$) is a cyclic group.

PROPOSITION 3.19 (Irrational Cycles). *Consider the case $n > 4$ is odd and $g \in \mathrm{C}$ with $C_{S_n} = (n)$. Suppose n is not a square. Let J be those primes p that exactly divide n to an odd power $p^{u(p)}$. For any $p \in J$, let $k \in (\mathbb{Z}/n)^*$ have these properties: its image in $(\mathbb{Z}/p^{u(p)})^*$ generates this cyclic group; and its image in $(\mathbb{Z}/p'^{u'})^*$ is 1 for primes $p' \neq p$ that divide n. Then, g^k and g are not conjugate in A_n:* C *is not a rational conjugacy class.*

Denote $\sqrt{\prod_{p\in J}(-1)^{(p-1)/2}p}$ by α_n. For all odd n, $K_{\mathrm{C}} = \mathbb{Q}(\alpha_n)$.

Conversely, if n is an odd square, g^k is conjugate to g in A_n for all $k \in (\mathbb{Z}/n)^*$: C *is a rational conjugacy class.*

PROOF. Suppose n is not a square. With k (and $p \in J$) as in the statement, we show g^k and g aren't conjugate in A_n. With no loss, $g = (1 \,\ldots\, n)$. So g^k maps

$i \mapsto i{+}k \mod n$, $i = 1, \dots, n$. Multiplication by k gives a permutation τ_k of the integers modulo n. Then, $\tau_k^{-1} g \tau_k$ equals g^k:

$$(ki)\tau_k^{-1} g \tau_k = (i) g \tau_k = (i+1)\tau_k = ki + k.$$

We characterize those k with τ_k not in A_n. Apply the Chinese remainder theorem to write $(\mathbb{Z}/n)^* = \prod_{i=1}^t (\mathbb{Z}/p_i^{u_i})^*$ with $p_1, \dots, p_t$ distinct (odd) primes. So, it suffices to check if $\tau_k \in A_n$ for $k = \boldsymbol{k}_i = (1, \dots, 1, k_i, 1, \dots, 1)$; the only non-identity entry is k_i, a generator of the cyclic group $(\mathbb{Z}/p_i^{u_i})^*$, in the i-th position. Consider what happens with k equal $(k_1, 1, \dots, 1)$.

First, assume $t = 1$, $u_1 = u$ and $k_1 = k$. Consider the cycle structure of τ_k $\mathbb{Z}/p^u$. Multiplication by k on integers of $\mathbb{Z}/p^u$ exactly divisible by p^i, $i < u$, gives one orbit of length $p^{u-i} - p^{u-i-1}$. For each i between 0 and $u-1$, this cycle has even length—not in A_n. (The orbit for $i = u$ has length 1.) Thus, the permutation is a product of u elements not in A_n (and it fixes exactly one integer). The total permutation from multiplication by k is in A_n if and only if u is even.

For the general case, write $\mathbb{Z}/n$ as $\mathbb{Z}/p_1^{u_1} \times \mathbb{Z}/n'$. Multiplication by k is the identity on the second coordinate. Thus, it stabilizes each coset $\mathbb{Z}/p_1^{u_1} \times k'$ with $k' \in \mathbb{Z}/n'$. In particular, τ_k is the product of n' elements coming from the first case above. Thus, $\tau_k \in A_n$ if and only if u_1 is even. The converse comes by noting it suffices to check the elements $\boldsymbol{k}_i$ above.

Finally, we identify the field $\hat{\mathbb{Q}}_n$. Identify the kernel of $\mu : (\mathbb{Z}/n)^* \to \mathbb{Z}/2$ by $k \in (\mathbb{Z}/n)^*$ maps to $\tau_k \mod A_n$. In the above notation, $\boldsymbol{k}_i$ goes to 1 if and only if $i \in J$. The unique quadratic extension of $\mathbb{Q}$ inside $\mathbb{Q}(\zeta_{p_j})$ is $\mathbb{Q}\left(\sqrt{(-1)^{(p_j-1)/2} p_j}\right)$. Conclude by noting the kernel of μ is of index 2 in $(\mathbb{Z}/n)^*$ and it fixes α_n. □

Example 3.20. Suppose C_1, C_2 and C_3 are respectively the conjugacy classes of the 5-cycles in A_5 given by $g_1 = (1\,2\,3\,4\,5)$, $g_2 = (1\,3\,5\,2\,4)$ and g_1 again. Then, C_1, C_2, C_3 is not a rational union because the conjugacy class of g_1 appears with multiplicity 2, while its square appears only with multiplicity 1. The collection $\mathbf{C}' = (C_1, C_2, C_1, C_2)$ is its rationalization.

Example 3.21 (Rational conjugacy classes in A_9). The conjugacy classes of A_9 that don't remain the same in S_9 are those that become $(1)(3)(5)$ of (9) in S_9. In general, counting the partitions of n into distinct odd integers is a nontrivial combinatorial business (see [11.18d]). [**Al99**] says the number of partitions of n by odd distinct integers equals partitions of n with all parts $\neq 2$, at least 6 apart and at least seven apart if both parts are even. For $n = 25$ this count is

$$12 = |(25), \{(i, 25-i), 1 \le i \le 9, i \neq 2, (1, k, 25-k-1), 7 \le k \le 9\}|.$$

According to Prop. 3.19, there are two rational conjugacy classes A_9 that become (9) in S_9. From [11.18b] the two conjugacy classes C for which $\mathrm{C}_{S_n} = (1)(3)(5)$ are not rational and $M_{\mathrm{C}} = \mathbb{Q}(\sqrt{-3 \cdot 5})$.

4. Applications of the Existence Theorem

This section should surprise the reader at how simple group theory, starting with dihedral groups, reveals serious classical topics. We develop two skills.

- Creating notation for calculating collections of covers.
- Finding algebraic functions to give coordinates on such collections.

For any group G denote by $\mathrm{Aut}(G)$ the full set of automorphisms of G, and by $\mathrm{Inn}(G)$ the autmorphisms induced by conjugation by G. The first nonabelian group that comes up in Galois theory is the dihedral group. Prop. 2.11 shows all abelian covers are algebraic. Covers $\varphi : X \to \mathbb{P}^1_z$ with dihedral monodromy group, even when X has genus 0, are not obviously algebraic. Part of Abel's Theorem is equivalent to displaying functions that show this. There is more to such covers than one would expect from its group theory alone.

We start slowly with dihedral covers, because there is so much history in them, especially about coordinates. §4.1 is a case that function theoretically is almost trivial, though its applications require careful coordinates.

4.1. Dihedral — a ka Tchebychev — polynomials. Suppose a degree n cover $\varphi : X \to \mathbb{P}^1_z$ has genus 0 ($g_X = 0$) and branch cycles $\boldsymbol{g} = (g_1, \dots, g_r)$ (relative to some choice of classical generators) with at least one totally ramified place. That means some g_i, say g_r, is an n-cycle in $G(\boldsymbol{g}) \le S_n$. At first examples use the standard representation T_n of S_n restricted to $G(\boldsymbol{g})$. Apply Riemann-Hurwitz to conclude $\sum_{i=1}^{r-1} \mathrm{ind}(g_i) = n - 1$.

4.1.1. *Cyclic covers and Redei functions.* An element of S_n has index $n-1$ if and only if it is an n-cycle. We draw conclusions from this and the product-one condition, $\Pi(\boldsymbol{g}) = 1$. If there is another n-cycle among the branch cycles, then $r = 2$. By conjugating by an element of S_n we may take $g_1 = (1 \dots n)$ and $g_2 = g_1^{-1}$. There is unique absolute Nielsen class of genus 0 covers with at least two n-cycles: $\mathrm{Ni}(\mathbb{Z}/n, \mathbf{C}_{n,n}, T_n)^{\mathrm{abs}}$. Further, in that class there is exactly one absolute s-equivalence class representing the Nielsen class: $\mathbf{C}$ consists of C and C^{-1}, a conjugacy class in $\mathbb{Z}/n$ and its inverse. The case $n = 2$ is trivial.

For $n \ge 3$, there are $\varphi(n)/2$ inner Nielsen classes of such covers,

$$\mathrm{Ni}(\mathbb{Z}/n, (\mathrm{C}^j, \mathrm{C}^{-j}))^{\mathrm{in}}, \text{ with } (j,n) = 1,\ j \le n/2.$$

As C contains one element, there are two inner s-equivalence class representing each Nielsen class: One with $g \in \mathrm{C}$ (resp. $g^{-1} \in \mathrm{C}^{-1}$) the branch cycle for z_1 (resp. z_2); another with the branch cycles switched.

These abelian covers we can produce by hand. Cases like this where G has a nontrivial center present special problems, as we'll see later. Just consider $\mathbb{P}^1_w \to \mathbb{P}^1_z$ by $w \mapsto w^n$: 0 and ∞ map respectively to 0 and ∞. Put the branch points anywhere using $\alpha \in \mathrm{PGL}_2(\mathbb{C})$ (say $\alpha = \frac{z - z_1}{z - z_2}$) that maps z_1, z_2 to $0, \infty$. Then, $w \mapsto \alpha^{-1}((\alpha(w))^n)$ gives a representing cover $\varphi_{\mathrm{C},\mathrm{C}^{-1},\boldsymbol{z}} : \mathbb{P}^1_w \to \mathbb{P}^1_z$ in the absolute s-equivalence class with branch points $\boldsymbol{z} = \{z_1, z_2\}$. Further, it is z_i that maps to z_i, $i = 1, 2$, by $\varphi_{\mathrm{C},\mathrm{C}^{-1},\boldsymbol{z}}$. We've explicitly written a representative of very s-equivalence class of covers in the Nielsen class.

§4.2.1 discusses r-equivalence classes. In this equivalence, all the covers $\varphi_{\mathrm{C},\mathrm{C}^{-1},\boldsymbol{z}}$ are equivalent. There is just one element in any Nielsen class, for we can put the branch points where we want, and switch the branch points, too. Recall: $\mathbb{P}^1_z(\mathbb{F}_q)$ the values on the Riemann sphere in the finite field $\mathbb{F}_q$ (Chap. 2 [9.19]).

Example 4.1 (Redei functions). The problem solved by Redei functions is to consider the collection of covers $\varphi_{\mathrm{C},\mathrm{C}^{-1},\boldsymbol{z}}$ up to changing φ to $\alpha^{-1}\varphi \circ \alpha$ with $\alpha \in \mathrm{PGL}_2(\mathbb{Q})$. Assume $n \ge 3$ is odd. That is we trying to describe *rational* r-equivalence repesentatives in this Nielsen class. If φ has coefficients in $\mathbb{Q}$, then the set $\{z_1, z_2\}$ is a $\mathbb{Q}$-set (see Lem. 6.4). [**LN83**] discusses Redei functions in detail. They give the easiest examples of *exceptional* functions $f : \mathbb{P}^1_w \to \mathbb{P}^1_z$ that map one-one when restricted to $\mathbb{P}^1_w(\mathbb{F}_q)$, for infinitely many prime powers q. They are perfect for standard cryptography applications, as are Dickson polynomials and other dihedral cover examples.

The branch points $\{0,\infty\}$ and $\{z_1, z_2 \mid z_1 = \sqrt{m}, z_2 = -\sqrt{m}$, m a square-free integer represent the $\mathbb{Q}$ absolute r-equivalence classes [11.15a].

4.1.2. *Twisted Chebychev — a ka Dickson — polynomials.* Here are the conditions for absolute Nielsen classes of Chebychev covers $\varphi : X \to \mathbb{P}^1_z$:

(4.1a) X has genus $g_X = 0$ and $\deg(\varphi)$ is an odd prime p;
(4.1b) $G \le S_p$ is a subgroup of $\mathbb{Z}/p \times^s (\mathbb{Z}/p)^* \stackrel{\text{def}}{=} \mathbb{A}_p$ (acting on $\mathbb{Z}/p$);
(4.1c) $\mathbf{C}$ has an entry, say C_r, that is a p-cycle; and
(4.1d) φ is not a cyclic cover.

Tacitly the permutation representation throughout is the degree p representation T_p on $\mathbb{Z}/p$. Recall, we represent elements of $\mathbb{A}_p$ by 2×2 matrices $\left(\begin{smallmatrix} a & 0 \\ b & 1 \end{smallmatrix}\right)$ with multiplication from matrix multiplication (Chap. 3 Rem. 7.4). Using §4.1.1, we have just one p-cycle of conjugacy classes. Elements of order p are conjugate in $\mathbb{A}_p$ to $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. Denote this conjugacy class C_p. The other conjugacy classes in $\mathbb{A}_p$ correspond one-one with non-identity elements of $(\mathbb{Z}/p)^*$. Denote the corresponding conjugacy class to $a \in (\mathbb{Z}/p)^*$ by C_a. For A a subgroup of $(\mathbb{Z}/p)^*$, $\mathbb{Z}/p \times^s A$ is the corresponding subgroup of $\mathbb{A}_p$. Prop. 4.2 and Cor. 4.3 is from [**Fri70**].

PROPOSITION 4.2. *There is only one absolute Nielsen class satisfying* (4.1). *It is* $\mathrm{Ni}(G, \mathbf{C})^{\mathrm{abs}}$ *with* $\mathbf{C} = (\mathrm{C}_{-1}, \mathrm{C}_{-1}, \mathrm{C}_p) \stackrel{\text{def}}{=} \mathbf{C}_{(-1)^2 \cdot p}$ *and* $G = \mathbb{Z}/p \times^s \langle -1 \rangle$. *Further, there is one element in this Nielsen class. More generally, for any odd* $n > 0$, *there is a unique absolute representative in the absolute Nielsen class of* $\mathrm{Ni}(D_n, \mathbf{C}_{(-1)^2 \cdot n})^{\mathrm{abs}}$.

PROOF. With no loss in an absolute Nielsen class take branch cycles so that $\boldsymbol{g}$ has $g_r = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. The other g_is are in C_a, $a \in (\mathbb{Z}/p)^* \setminus \{1\}$, which acts as multiplication by a on $\mathbb{Z}/p$. If m_a is the order of a, then this action has $\frac{p-1}{m_a}$ orbits of length m_a, and one orbit of length 1. The index of such a g_i is thus $\frac{p-1}{m_a}(m_a - 1)$. Now apply Riemann-Hurwitz (3.11) using that $g_X = 0$: $p - 1 = \sum_{i=1}^{r-1} \frac{p-1}{m_{a_i}}(m_{a_i} - 1)$. The expression $\frac{m-1}{m}$ $(m \ge 2)$ is at least $\frac{1}{2}$, with equality if and only if $m = 2$ $(m_a = -1$. Since $r - 1 \ge 2$, the result is $r - 1 = 2$, and $g_i \in \mathrm{C}_{-1}$, $i = 1, 2$.

Now we see there is only one element in this Nielsen class. Fix g_3, and take $g_1 = \left(\begin{smallmatrix} -1 & 0 \\ b & 1 \end{smallmatrix}\right)$, with g_2 determined by the product-one relation: $g_1 g_2 g_3 = 1$. Normalize further by conjugating the 3-tuple by a power of g_3:

$$\left(\begin{smallmatrix} 1 & 0 \\ -k & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} -1 & 0 \\ b & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ k & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} -1 & 0 \\ b+2k & 1 \end{smallmatrix}\right).$$

So, by choosing k so $b + 2k = 0 \mod p$ gives a unique representative of $\mathrm{Ni}(\mathbb{Z}/p \times^s \langle -1 \rangle, \mathbf{C}_{(-1)^2 \cdot p})^{\mathrm{abs}}$.

□

Up to reduced equivalence, we may place the three branch points $\boldsymbol{z} = \{z_1, z_2, z_3\}$ by whatever three points we want. Given $\varphi : X \to \mathbb{P}^1_z$ (unique up to equivalence) in this equivalence class, we find a polynomial $T_p(w)$ with branch points $-2, +2, \infty$ reduced equivalent to it. Further, from the branch cycle description there is exactly one unramified point of X over each of -2 and $+2$ (use the corresponding between points over branch points and disjoint cycles of the branch cycles). So, by ordinary equivalence, put these at $-1, +1, \infty$ respectively. This is a less trivial case than previously for producing a function on the covers to show they are algebraic.

COROLLARY 4.3.

PROOF. There is one element in the absolute s-equivalence classes of polynomials with dihedral group cover. Suppose f is a monic degree n polynomial over $\bar{F}$ that gives a branched cover $\mathbb{P}^1_T \to \mathbb{P}^1_z$ with two finite branch points $z_1, z_2 \in \bar{F}$, both ramified of order 2. The following observations occur in [**Fri70**]. The geometric Galois group of the Galois closure is a dihedral group. If n is odd, then the Nielsen class of the cover is $\mathrm{Ni}(D_n, \mathbf{C}_{n\cdot 2^{\frac{n-1}{2}}\cdot 2^{\frac{n-1}{2}}})$. Further, since the normalizer of D_n in S_n has no center, any cover with branch points $\{\infty, z_1, z_2\}$ in this Nielsen class is determined up to a unique isomorphism. So, if the unordered branch points are defined over F, then the cover is represented by a unique polynomial over F. As $z_1 + z_2$ are defined over F, changing z to $z - (\frac{z_1+z_2}{2})$ normalizes further to assume the branch points sum to 0. Call these *normalized Chebychev polynomials*. From these observations the following are clear. For any $d \in F^*$, and odd positive integer n define the *Dickson Polynomial* $D_n(a, w)$ to be $a^{n/2}T_n(a^{-1/2}w)$. As a varies we get all the normalized Chebychev polynomials. Clearly two such polynomials are isomorphic over F if and only multiplication by some $b \in F$ maps the branch points of one to the other. $\square$

If, however, n is even, the conjugacy classes defining the Nielsen class are distinct and the branch points all are defined over F. A compensating fact is that $N_{S_n}(D_n)$ has a nontrivial centralizer $Z_{S_n}(D_n) = \langle (1\ \dots\ n)^{n/2} \rangle$ in S_n (multiplication by -1 leaves $1 + n/2$ invariant modulo n. [**Tu95**] [**Wel69**] [**LN73**] [**Mu80-02**]

4.2. $\mathrm{PGL}_2(\mathbb{C})$ **action, r-equivalence and hyperelliptic covers.** §3.5 explains the sets of covers in $\mathcal{H}(G, \mathbf{C}, T)^{\mathrm{abs}}$ and other s-equivlaence classes. The group $\mathrm{PGL}_2(\mathbb{C})$, as one-one analytic maps of $\mathbb{P}^1_z$ enters immediately to give from each s-equivalence class, a new equivalence (r(educed)-equivalence) from it.

4.2.1. *$\mathbb{P}^r$ as $(\mathbb{P}^1_z)^r/S_r$ and r-equivalence.* Identify the elements of $\mathbb{P}^r$ (Chap. 3 §4.3) as nonzero monic polynomials in a variable z of degree at most r. For example, if $(a_0, a_1, \dots, a_r)$ represents a point of $\mathbb{P}^r$, and $z_0 \neq 0$, by scaling it by $\frac{1}{z_0}$ assume with no loss $z_0 = 1$. Then, take the polynomial associated to this point as $z^r + \sum_{i=0}^{r-1} (-i)^{r-i} a_{r-i} z^i$. There is a natural permutation action of $\pi \in S_r$ on the entries of $(\mathbb{P}^1_z)^r$: $\pi : (z_1, \dots, z_r) \mapsto (z_{((1)\pi}, \dots, z_{(r)\pi})$. Denote the set of distinct r-tuples of elements of $(\mathbb{P}^1_z)^r$ by $U^r = (\mathbb{P}^1_z)^r \setminus \Delta_r$. Call Δ_r the *fat diagonal*: The locus were two or more equal entries.

PROPOSITION 4.4. *Represent the natural quotient map*

$$\Psi_r : (z_1, \dots, z_r) \in (\mathbb{P}^1_z)^r \mapsto \{\boldsymbol{z}\} \in (\mathbb{P}^1_z)^r/S_r$$

by sending $(z_1, \dots, z_r)$ to the polynomial $\prod_{i=1}^r (z - z_i)$ in z: If $z_i = \infty$, replace $(z - z_i)$ by 1. This canonically identifies Ψ_r with degree $n!$ analytic map of complex manifolds $(\mathbb{P}^1_z)^r \to \mathbb{P}^r$ [11.14a]. Identify unordered sets of r branch distinct points *as an affine subspace U_r of $\mathbb{P}^r$; the complement of the classical discriminant locus D_r identified with the image of Δ_r [11.14b].*

If $\varphi : X \to \mathbb{P}^1_z$ represents an s-equivalence class of covers in a given Nielsen class Ni, then the collection $\{\alpha \circ \varphi : X \to \mathbb{P}^1_z\}_{\alpha\in\mathrm{PGL}_2(\mathbb{C})}$ gives the set of covers r-equivalent to φ. To any cover $\varphi : X \to \mathbb{P}^1_z$, associate the unordered collection of its branch points $\boldsymbol{z} \in \mathbb{P}^r$. This branch point map produces a map we will never lose sight of in the rest of this book.

Suppose we have a group G, conjugacy classes $\mathbf{C}$ in G, a permutation representation $T : G \to S_n$ and $G \le N \le N_{S_n}(\mathbf{C})$. For covers in the set of r-equivalence classes use the notation $\mathcal{H}(G,\mathbf{C},T)/N^{\mathrm{rd}}$. We have a special notation $\mathcal{H}^{\mathrm{abs,rd}}$ and $\mathcal{H}^{\mathrm{in,rd}}$ for the associated reduced absolute and inner equivalence classes of covers.

The action of $\alpha \in \mathrm{PGL}_2(\mathbb{C})$ on $(\mathbb{P}^1_z)^r$ by $(z_1,\dots,z_r) \mapsto (\alpha(z_1),\dots,\alpha(z_r))$ maps U^r into itself.

Proposition 4.5. *The actions of* $\mathrm{PGL}_2(\mathbb{C})$ *and of* S_r *on* U^r *commute. This gives a complex analytic map* $\Psi^{\mathrm{rd}}_r : (\mathbb{P}^1_r)^r \to \mathrm{PGL}_2(\mathbb{C})\backslash(\mathbb{P}^1_r)^r/S_r \overset{\mathrm{def}}{=} J_r$ *factoring through the space* $\mathrm{PGL}_2(\mathbb{C})\backslash(\mathbb{P}^1_r)^r \overset{\mathrm{def}}{=} \Lambda_r$. *For all r the spaces Λ_r and J_r are normal affine varieties, though for $r \ge 5$, neither is a manifold.*

Then, Ψ^{rd}_r induces a natural map of any Hurwitz space $\mathcal{H}(G,\mathbf{C},T)/N^{\mathrm{rd}}$ to J_r.

Proof. [11.14c] □

Refer to the induced map $\Lambda_r \to J_r$ also as Ψ^{rd}_r when that causes no confusion.

Corollary 4.6. *The space J_4 (resp. Λ_4) naturally identifies with $\mathbb{P}^1_j \setminus \{\infty\}$ (resp. $\mathbb{P}^1_\lambda \setminus \{0,1\infty\}$) and $\Lambda_4 \to J_4$ compactifies to a Galois covering map with group S_3, ramified over $j = 0$, $j = 1$ and $j = \infty$ with branch cycles identified with* $((1\,3\,5)(2\,4\,6),(1\,2)(3\,4)(5\,6),$ *RETURN.*

4.2.2. *Hyperelliptic covers.* For $G \le S_n$ denote its intersection with A_n by ${}^+G$: Indicating the elements of positive sign in this representation. For any degree n cover $\varphi : X \to Y$, its monodromy group G_φ is a subgroup of S_n. Similarly, for any finite group G consider the collection of faithful transitive permutation representations (up to permutation equivalence, ${}^+P_G$ that give an embedding of G in an alternating group. In that case

Lemma 4.7. *If G has no normal subgroup of index 2, then ${}^+P_G$ consists of all faithful permutation representations of G. This holds if G is generated by elements of odd order, or if G is 2-perfect.*

Conversely, suppose $H \triangleleft G$ and has index 2. Then, RETURN

Proof. ${}^+_TG = G$ for each $T \in {}^+P_G$ if and only if consists of faithful permutation representations. □

Example 4.8 (H-M reps. and r-equivalence). Recall the definition of H-M reps. from Def. 3.14. In the Nielsen class $\mathrm{Ni}(G,\mathbf{C},T)^{\mathrm{abs}}$, consider an H-M rep. Suppose $r = 4$. Reduced equivalence when r=4 comes about because of the linear fractional transformations that flip any two pairs of branch points. In particular, the setup for $q_1q_3^{-1}$ action on an H-M rep. takes the representative $\boldsymbol{g} = (g_1, g_1^{-1}, g_2, g_2^{-1})$ to $(g_1^{-1}, g_1, g_2^{-1}, g_2)$. The action of this element fixes the absolute class $\boldsymbol{g}$ if there is some element $g \in N_{S_n}(\mathbf{C})$ that conjugates the first 4-tuple to the second. (If we were doing inner r-equivalence, g would be in G.)

In all cases you get the following conclusion about any cover $\varphi : X \to \mathbb{P}^1_z$ (in an H-M rep. orbit) if this happens. There is another map $\varphi' : X \to \mathbb{P}^1_z$ and $\alpha : \mathbb{P}^1_z \to \mathbb{P}^1_z$ having order 2, so $\alpha \circ \varphi' = \varphi$. Take $\mathbb{P}^1_w tobethequotientof \mathbb{P}^1_z$ by α: Giving a map $\mu : \mathbb{P}^1_z \to \mathbb{P}^1_w$ branched at two points. Example case: the cover is cyclic of odd prime degree p. The degree 2 cover, however, is not a Galois cover with group D_p.

4.3. Involution dihedral covers. Last week we produced modular curves from dihedral involution-covers. Today (Wed. Mar. 19, at 3PM) I'll generalize this to a much larger class of j-line covers that we may compare with modular curves. This retains observations from Riemann's generalization of Abel's Theorem.

We do this by reflecting on application valuable properties of ALL finite groups. The analog: Dihedral groups are to modular curves as general p-perfect groups are to generalizing j-line covers. Modular curves come in series related to a prime p. The analog says each finite group comes with a series related to any prime p dividing its order.

This approach to the regular version of the Inverse Galois problem has a valuable structure.

1. It includes the most famous theorems in diophantine geometry as special cases.

2. It exposes difficult modular representation problems beyond what group theorists classically study. (John Thompson noted some are analogs to such topics as the Golod-Shafarevich class field tower.)

3. Modular representation theory interprets simple properties of the Modular Tower levels.

Once we get by the initial definitions (like the mapping class group), the relation to group theory comes clear. We saw last time that modular curves are algebraic precisely because of the relation between j-invariants that comes from the dihedral involution realizations. So, parameter spaces for dihedral involutions covers are algebraic using specific coordinates from Abel's Theorem. The next step is developing analogs of this to Modular Towers through theta nulls. I'll conclude today with the first topic in that direction – half-canonical classes.

2. Finding all abelian covers of a compact surface is equivalent to finding all functions on the surfaces. How is it tacit in this description to know the surface is algebraic? Example: From the fundamental group, you know about abelian covers. Yet, ... The distinction between hyperelliptic and general: You know a hyperelliptic surface is algebraic.

5. Braid orbits

Take $r = 4$ and $G = S_5$. Let C_1 and C_3 be the conjugacy classes of 2-cycles in S_5, C_2 the conjugacy class of a 3-cycle and C_4 the conjugacy class of a 5-cycle. Consider the Nielsen class $\mathrm{Ni}(S_5, \mathbf{C})/S_5 = \mathrm{Ni}^+$:

$$\{\boldsymbol{g} = (g_1, \dots, g_4) \mid g_1 g_2 g_3 g_4 = 1, \langle \boldsymbol{g} \rangle = S_5 \text{ and } \boldsymbol{g} \in \mathbf{C}\}/S_5.$$

(5.1a) How many elements are in Ni^+?

(5.1b) Let $\psi : \pi_1(U_{\boldsymbol{z}}) \to S_5$ map a fixed set $\bar{g}_1, \dots, \bar{g}_4$ into some element of Ni^+. Why is the cover corresponding to such a homomorphism a genus 0 compact Riemann surface minus a finite set of points?

(5.1c) Represent S_5 on the 10 unordered distinct pairs of integers from $\{1,\dots,5\}$: $T : S_5 \to S_{10}$. Example: $(1\,2\,3\,4\,5)$ has two orbits on these 10 pairs. What are the lengths of the disjoint cycles of T applied to an element of the conjugacy class of a 3-cycle in S_5?

(5.1d) Compose ψ with T to get $T \circ \psi = \psi' : \pi_1(U_{\boldsymbol{z}})$. What is the genus of the curve at the top of the corresponding cover $X = X_\psi \to \mathbb{P}^1_z$?

(5.1e) Does the isomorphism class of X_ψ depend on ψ (assuming ψ is in the Nielsen class Ni^+)?

5.0.1. *Genus of the corresponding degree 10 covers.* Let $\boldsymbol{g}$ be a branch cycle description of the cover from Ni^+ in [11.20]. Compute the genus g of ${}_+\mathcal{T}_{\boldsymbol{p}}^{(2)}$ from Riemann-Hurwitz:

$$2(10 + g - 1) = \sum_{i=1}^{4} \mathrm{ind}(R(g_i)). \tag{5.2}$$

Suppose g_1 and g_3 are 2-cycles from S_5. Then, $R(g_i)$ has shape (2)(2)(2) in the representation R, $i = 1, 3$. Similarly, if g_2 is a 3-cycle, $R(g_3)$ has shape (3)(3)(3). Finally, $R(g_4)$ has shape (5)(5). Thus, the total contribution to the right side of (11.21) is $2 \cdot 3 + 6 + 2 \cdot 4 = 20$ and $g = 1$.

Next: Compute Ni^+ modulo conjugation by S_5. Choose S_5 representatives with g_4 equal $g_\infty = (1\,2\,3\,4\,5)^{-1}$. Divide Ni^+ into two sets T_1 and T_2: $\boldsymbol{g} \in T_1$ has g_1 and g_2 with no integers of common support, and $\boldsymbol{g} \in T_2$ has g_1 and g_2 with one integer of common support. Conjugate by a power of $\boldsymbol{g}_\infty$ to assure elements of T_1 have $g_1 = (1\,j)$ with $j = 2$ or 3. Similarly, elements in T_2 have 1 as common support of g_1 and g_2. From this, list $\mathrm{Ni}^{+,\mathrm{abs}}$.

5.0.2. *Covers with group A_5.* (3)(3)(3)(5): Suppose $g_3 = (1\,2\,3)$.

(5.3a) Ramification: g_1g_2 is (2)(2), assume missing integer is 1, so to get product a 5-cycle: may assume g_1g_2 is $(2\,5)(3\,4)$. Now everthing is fixed and need only count number of ways to write g_1g_2 is a product of two three cycles. Hint: Products of two 3-cycles giving $(2\,5)(3\,4)$: You get one element from $(4\,2\,5)(2\,3\,4)$. Now conjugate the pair $((4\,2\,5),(2\,3\,4))$ by the centralizer of $(2\,5)(3\,4)$, the group $\langle(2\,5)(3\,4),(2\,4)(3\,5)\rangle$.

(5.3b) If g_1g_2 is (3), then conjugate by $\langle g_3\rangle$ to assume common integer is 1, and $g_1g_2 = (1\,4\,5)$. Hint: Take $(g_1,g_2) = ((1\,4\,3),(1\,3\,5)$, and then conjugate by $\langle(2\,3),(1\,4\,5)\rangle$.

(5.3c) If g_1g_2 is (5). Then, product can't be of type (2)(3) (Riemann-Hurwitz), and have only to assure the (5) times g_3 doesn't fix anything. That means can't have $2 \mapsto 1$, $3 \mapsto 2$ or $1 \mapsto 3$. Also, since by conjugation by $\langle(4\,5),(1\,2\,3)\rangle$ can assume $(1\,5\,?\,?\,?)$ resulting in $(1\,5\,2\,4\,3)$ or $(1\,5\,3\,2\,4)$. Hint: For each of $(1\,5\,2\,4\,3)$ or $(1\,5\,3\,2\,4)$, we need to count all the ways to write this 5-cycle as a product of two 3-cycles. For $(1\,2\,3\,4\,5)$, assume the integer 1 is the common integer to the 3-cycles. So, $(g_1,g_2) = ((1\,2\,3),(1\,4\,5))$. Then, by conjugating by $\langle(1\,2\,3\,4\,5)\rangle$, gives the five cases where g_1 and g_2 have any desired integer in common.

(5.3d) Up to equivalence, there are exactly 4 covers from a), 6 covers from b) and 10 covers from c), or 20 total covers. Also, by applying powers of q_1 to case c) you get 10 total in two orbits of length five. Same for b), two orbits of length 3, and for a), two orbits of length two.

5.0.3. *Non-rigid A_n covers.* Consider $n \geq 5$, odd and squarefree. Let $\mathbf{C}$ be conjugacy classes of $(g_1, g_2, g_3) \in A_n^3$ with $g_1 = (1\,2)(3\,4)$, $g_2 = (1\,3\,5\,6\,7\ldots n)$ and $g_3 = (1\,2\ldots n)^{-1}$. Check: Geometric monodromy is A_n. Representatives for conjugation of S_n on $\mathrm{Ni}(S_n, \mathbf{C})/S_n$, $i = 3, \ldots, (n+1)/2$:

$$\boldsymbol{g}'_j = ((1\,2)(j\,j{+}1), (1\,3\,\ldots\,j\,j{+}2\,j{+}3\ldots n), g'_3).$$

RETURN

QUESTION 5.1. Exists $f : \mathbb{P}^1_y \to \mathbb{P}^1_z$ in $\mathbb{Q}[y]$?

If yes, derivative is $g(y) = (y-a)(y-b)y^{n-3} \in \mathbb{Q}[y]$. Conclude: a, b either in $\mathbb{Q}$ or conjugate over $\mathbb{Q}$. Further $f(x) =$:

$$y^n/n - (a{+}b)y^{n-1}/(n{-}1) + aby^{n-2}/(n{-}2) + d;$$

and $f(a) = f(b)$. With $d \in \mathbb{Q}$, $b/a = \alpha$, simplify: $(n-2)(\alpha^n - 1) = n(\alpha^{n-1} - \alpha)$. Divide by $\alpha - 1$:

$$h_n(\alpha) = (n-2)\alpha^{n-1} - 2(\alpha + \cdots + \alpha^{n-2}) + (n-2),$$

divisible by $(\alpha-1)^2$. Then, f over $\mathbb{Q}$ exists when $\in \mathbb{Q}[\alpha]$ of degree 2 divides $h_n(\alpha)/(\alpha-1)^2$. *Mathematica*: $h_n(\alpha)/(\alpha-1)^2$ irreducible over $\mathbb{Q}$ for odd $n \leq 31$.

5.1. Nontrivial components of fiber products.

5.2. Reduced Nielsen classes and mapping class orbits. Automorphisms of $\hat{\varphi}$ identify with the centralizer of G in $N_R(G, \mathbf{C})$. Point over z_0 gives $G(X/\mathbb{P}^1_z) \xrightarrow{\alpha} G$. A $\hat{\psi}$ (in (3.12)) is unique if G has no center: There exists a unique total family

$$\mathcal{T}^{\mathrm{in}}_{G,\mathbf{C}} = \mathcal{T}^{\mathrm{in}} \to \mathcal{H}^{\mathrm{in}} \times \mathbb{P}^1_z, \tag{5.4}$$

$\mathcal{H}^{\mathrm{in}}$ is a *fine moduli space*. The (minimal) field of definition of $\mathcal{T}^{\mathrm{in}}_{\boldsymbol{p}} \to \boldsymbol{p} \times \mathbb{P}^1_z$ is $\mathbb{Q}(\boldsymbol{p})$ [**FV91**].

PROPOSITION 5.2. *Get $(G, \mathbf{C})$ regular realizations over $\mathbb{Q}$ from $\boldsymbol{p} \in \mathcal{H}^{\mathrm{in}}(\mathbb{Q})$. Necessary: $\mathcal{H}^{\mathrm{in}}$ has a $\mathbb{Q}$ component ($\mathbf{C}$ is a rational union).*

5.2.1. *Reduced Nielsen classes.* Notation for M_4 generators: $\gamma_0 = q_1q_2$, $\gamma_1 = q_1q_2q_1$, $\gamma_\infty = q_2$. Product one:

$$1 = q_1q_2q_1q_1q_2q_1 = \gamma_1^2 = q_1q_2q_1q_2q_1q_2 = \gamma_0^3.$$

Compute Q_i s [**Fri90**]:

$$Q_1 = (2\,5\,3\,6\,4)(7\,9\,8),\ Q_2 = (1\,4\,9\,8\,5)(3\,6\,7),\ Q_3 = (2\,5\,3\,6\,4)(7\,9\,8).$$

Consider w-equivalence classes $\mathrm{Ni}(A_5, \mathbf{C}_{3^4})^{\mathrm{abs}}/\mathcal{Q}$ for $\mathrm{Ni}(A_5, \mathbf{C}_{3^4})^{\mathrm{abs}}$. Action of $M_4 = H_4/\mathcal{Q}$ produces a (ramified) cover $\mathcal{H}^{\mathrm{abs,rd}} \to \mathbb{P}^1_j \setminus \{\infty\}$. Compactify to $\bar{\varphi}^{\mathrm{abs,rd}} : \bar{\mathcal{H}}^{\mathrm{abs,rd}} \to \mathbb{P}^1_j$ with branch cycles $(\gamma'_0, \gamma'_1, \gamma_\infty)$ [**Fri99**, §7.4]:

$$\gamma'_0 = (2\,1\,4)(3\,7\,8)(5\,6\,9),\ \gamma'_\infty = (1\,4\,9\,8\,5)(3\,6\,7).$$

Note: The monodromy group is A_9. This is a cover with $(0, 1, \infty)$ as branch points; the *cusp widths* are 1, 3 and 5.

5.2.2. *Organizing braid orbits with the* **sh**-*incidence matrix.* First take $r = 4$. The **sh**-*incidence matrix* summarizes a *pairing* using **sh** on γ_∞ orbits.

For a general reduced Nielsen class, list γ_∞ orbits as $O_1, \dots, O_n$. The **sh**-incidence matrix $A(G, \mathbf{C})$ has (i, j) term $|(O_i)\mathbf{sh} \cap O_j|$. Since **sh** has order two on reduced Nielsen classes, this is a symmetric matrix. Equivalence $n \times n$ matrices A and $TA{}^{\rm t}T$ running over permutation matrices T (${}^{\rm t}T$ is its transpose) associated to elements of S_n. List γ_∞ orbits as

$$O_{1,1}, \dots, O_{1,t_1}, O_{2,1}, \dots, O_{2,t_2}, \dots, O_{u,1}, \dots, O_{u,t_u}$$

corresponding to $\bar{M}_4$ orbits. Choose T to assume $A(G, \mathbf{C})$ is arranged in blocks along the diagonal.

LEMMA 5.3. *If A_j is the jth block of $A(G, \mathbf{C})$, then A_j doesn't break into smaller blocks. So, $\bar{M}_4$ orbits form irreducible blocks in the* **sh**-*incidence matrix.*

PROOF. With no loss assume one $\bar{M}_4$ orbit and two blocks, with orbit listings as $O_1, \dots, O_k, O_{k+1}, \dots, O_t$. As, however, there is one orbit, for some $j \le k$, $|(O_i)\mathbf{sh} \cap O_j| \ne 0$ for some $i > k$. This contradicts there being two blocks. □

In practice it is difficult to list the γ_∞ orbits. So, we start with the H-M reps., apply **sh**, then complete the γ_∞ orbits and check $|(O_i)\mathbf{sh} \cap O_j|$. Sometimes we'll then be done. The case $(A_5, \mathbf{C}_{3^4})$ illustrates this. Denote (as above) the γ_∞ orbits of $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ by $O(5, 5; 1)$ and $O(5, 5; 2)$; γ_∞ orbits of

$$((5\,1\,3), (2\,4\,5), (1\,5\,4), (1\,2\,3)) \text{ and } ((3\,2\,4), (5\,1\,3), (1\,5\,4), (1\,2\,3))$$

by $O(3, 3; 1)$ and $O(3, 3; 2)$; and of $(\boldsymbol{g}_1)\mathbf{sh}$ by $O(1, 2)$.

TABLE 1. **sh**-Incidence Matrix for Ni_0

Orbit	$O(5,5;1)$	$O(5,5;2)$	$O(3,3;1)$	$O(3,3;2)$	$O(1,2)$
$O(5,5;1)$	0	2	1	1	1
$O(5,5;2)$	2	0	1	1	1
$O(3,3;1)$	1	1	0	1	0
$O(3,3;2)$	1	1	1	0	0
$O(1,2)$	1	1	0	0	0

5.2.3. *The* **sh**-*incidence matrix for general r.* For general r, denote $q_1 \cdots q_{r-1}$ at the Hurwitz monodromy level to be the shift $\mathbf{sh}_r$, so $\mathbf{sh}_4$ is what we call the *shift* above. Ideas for $r = 4$ generalize to indicate cusp geometry for general r.

The element $\mathbf{sh}_r$ plays the role of a shift in two ways. Consider an r-tuple $\bar{\boldsymbol{\sigma}} = (\bar{\sigma}_1, \dots, \bar{\sigma}_r)$ of free generators of F_r. The effect of $\mathbf{sh}_r$ on $\bar{\boldsymbol{\sigma}}$ is to give

$$(\bar{\boldsymbol{\sigma}})q_1 \cdots q_{r-1} = (\bar{\sigma}_1\bar{\sigma}_2\bar{\sigma}_1^{-1}, \dots, \bar{\sigma}_1\bar{\sigma}_r\bar{\sigma}_1^{-1}, \bar{\sigma}_1\bar{\sigma}_1\bar{\sigma}_1^{-1}).$$

In specializing to a Nielsen class the effect of $\mathbf{sh}_r$ is to the shift the Nielsen class representative entries by 1. Iterate this r times to see the effect of $\mathbf{sh}_r^r$ is conjugation on $\boldsymbol{\sigma}$ by the product $\bar{\sigma}_1 \cdots \bar{\sigma}_r$ of these generators.

Such a conjugation commutes with the action of the braid group. So we have an interesting interpretation for the action of *conjugating* by $\mathbf{sh}_r$ on the generators

$q_1, \dots, q_r$. Define q_0 to be $\mathbf{sh}_r^{-1} q_1 \mathbf{sh}_r$. Then, conjugation by $\mathbf{sh}_r$ on the left of the array $(q_0, q_1, \dots, q_{r-2}, q_{r-1})$ maps its entries to

$$\begin{aligned} \mathbf{sh}_r(q_0, \dots, q_{r-1})\mathbf{sh}_r^{-1} = &\quad (q_1, q_1q_2q_1q_2^{-1}q_1^{-1}, q_1q_2q_3q_2q_3^{-1}q_2^{-1}q_1^{-1}, \dots) \\ = &\quad (q_1, q_2, \dots, q_{r-1}, q_0). \end{aligned}$$

To see the effect of conjugation of $\mathbf{sh}_r$ on q_{r-1} use that $\mathbf{sh}_r^r$ is in the center of H_r (or of B_r). Then, $q_0 = \mathbf{sh}_r^r(q_0)\mathbf{sh}_r^{-r} = \mathbf{sh}_r q_{r-1} \mathbf{sh}_r^{-1}$.

Denote $q_{r-1}q_{r-2}\cdots q_1$ by $\mathbf{sh}_r'$. Notice $(\mathbf{sh}_r')^r$ has exactly the same effect on $\boldsymbol{\sigma}$ as does $\mathbf{sh}_r^r$. In H_r use that $q_1 \cdots q_r q_r \cdots q_1 = 1$ to see $\mathbf{sh}_r^r(\mathbf{sh}_r')^r = 1$, so $\mathbf{sh}_r^r = z$ has its square equal to 1. When $r = 4$ the group M_4 is exactly $H_r/\langle \mathbf{sh}_4^4 \rangle = H_r/\langle z \rangle$. An especially handy description of z in this case is $q_1 q_3^{-1}$. In general there is a $\mathbf{sh}_r$-incidence matrix. As in the case $r = 4$, it suffices to choose the image of q_v in $\bar{M}_r$ for some value of v. It doesn't make any difference which v, though for $r = 4$ it was convenient to take $v = 2$. Call the resulting element γ_∞. List the γ_∞ reduced orbits as $O_1, \dots, O_t$ and define $A(G, \mathbf{C})$ to be the matrix with (i, j) term $|(O_i)\mathbf{sh}_r \cap O_j|$. For general r it won't be symmetric.

6. Coordinates and covers

We will use covers of $\mathbb{P}_z^1$ as a record of an algebraic relation in one variable. Finding convenient ways to label such covers is necessary for applications. The test for the effectiveness of the labeling is how well it answers old questions and helps formulate new approaches to old topics. The first natural label to attach to a cover is its unordered set $\boldsymbol{z}$ of branch points. We know a lot about an r branch point cover if we know its Nielsen class and its branch points. Yet in practise that isn't enough information to answer questions that have guided 200 years of intensive work on genus 1 curves. The following points related to coordinates will occur in the remainder of this chapter.

(6.1a) Relation and implication of the definition of algebraic cover of $\mathbb{P}_z^1$ in Chap. 2 to that of cover in §2.

(6.1b) Genus 0 dihedral involution covers (of $\mathbb{P}_z^1$) correspond to rational functions $\mathbb{P}_w^1 \to \mathbb{P}_z^1$ and implications of this for explicitly presenting genus 0 and genus 1 covers of $\mathbb{P}_z^1$.

(6.1c)

(6.1d) Isomorphism classes of 1-dimensional complex tori correspond to $\mathrm{PGL}_2(\mathbb{C})$ equivaleces classes of points of U_r, and its analog for all Nielsen classes.

(6.1e) The universal covering space of $U_{\boldsymbol{z}}$ is the upper half plane and related examples of uniformization.

6.1. Algebraic covers and projective space. Chap. 2 (1.1) and (1.2) gave two definitions of analytic function $f(z)$ being algebraic function and related by an equation $m(w, z) = 0$ to z. From this Chap. 3 Prop. 3.12 preduced an unramified cover $\varphi^0 : X^0 \to U_{\boldsymbol{z}}$. Its key point is that there are $n = \deg_w(m)$ distinct values w' for which $m(w', z') = 0$ for $z' \in U_{\boldsymbol{z}}$.

EXAMPLE 6.1. From Chap. 2 §8.2 (see Chap. 3 4.3.3), a genus 1 degree 2 cover $\varphi : X \to \mathbb{P}_z^1$ is algebraic. The unique cover of $U_{\boldsymbol{z}}$ up to s-equivalence (§2.1.2) is the algebraic set $X_{\boldsymbol{z}}^0 = \{(z, w) \mid m(z, w) = 0\}$ with $m(z, w) = w^2 - \prod_{i=1}^4 (z - z_i)$. If one of the z_i s is ∞ replace $z - z_i$ by 1. Since the completion of X^0 to a ramified cover is unique, this also gives $\varphi : X \to \mathbb{P}_z^1$ up to s-equivalence. Abel's Theorem §7.6

shows they are also 1-dimensional complex tori. It further shows all 1-dimensional tori are algebraic. RETURN How can we use this to see all genus 1 surfaces are complex tori, and that all genus 0 curves are analytically isomorphic to $\mathbb{P}^1_z$?

DEFINITION 6.2 (General algebraic cover). Continue Def. 2.10. Let $\varphi : Y \to X$ be an analytic map of compact Riemann surfaces with $\boldsymbol{x}$ the branch points of φ. We say φ is *algebraic* if Y is an algebraic cover (with some unspecified map to $\mathbb{P}^1$) and there exists an analytic map $\psi : Y \to \mathbb{P}^1_w$ so that for some $x' \in X_{\boldsymbol{x}}$, ψ separates the points in the fiber $X_{z'}$.

PROPOSITION 6.3. *A cover $\varphi : X \to \mathbb{P}^1_z$ is algebraic if and only if $\mathbb{C}(X)$ has sufficient algebraic functions to separate all points on X. This implies that if X is algebraic and $X \to Y$ is a cover, then Y is also algebraic.*

A tentative definition of algebraic, called $\mathbb{P}^1$-algebraic, appears in Chap. 3 Def. 3.3. We start by explaining why it is typical to use the phrase *algebraic* on a manifold to mean it has an embedding in $\mathbb{P}^N$. Then, we consider the classical Luroth Theorem as a use of coordinates that leads us to note the complication in practical checking for decompositions of a cover.

6.1.1. *Invariants and automorphisms of* $(\mathbb{P}^1)^N$. Recall the definition of $\mathbb{P}^1$-algebraic from Chap. 3 (3.3). Ideas: Every algebraic manifold has a Galois cover by a $\mathbb{P}^1$-algebraic manifold, and every $\mathbb{P}^1$-algebraic manifold is algebraic (Segre embedding). Both $\mathbb{P}^N$ and $(\mathbb{P}^1)^N$ are simply connected [11.9d]. Yet, the spaces $\mathbb{P}^N$ are not $\mathbb{P}^1$-algebraic, for they have no analytic maps to $\mathbb{P}^1_z$ Chap. 3 [9.11e]. What do we get from algebraic that is better than uniformization?

6.2. Fields of definition, fields of moduli and Branch Cycle Lemma.

LEMMA 6.4. *If $\varphi : X \to \mathbb{P}^1_z$ has definition field K, then its branch points $\boldsymbol{z}$ form a K-set.*

What you need from an algebraic structure to define the field of moduli. What you need from a cover to define these two quantities. One thing we can define is the definition field of the branch points $\boldsymbol{z}$ over a cover $\varphi : X \to \mathbb{P}^1_z$.

LEMMA 6.5 (Branch point control). *Suppose $\varphi : X \to \mathbb{P}^1_z$ is algebraic, and the branch points $\boldsymbol{z}$ have K as a field of definition. Then, there is a cover $\varphi' : X' \to \mathbb{P}^1_z$ over $\bar{K}$ with these properties.*

(6.2a) *φ' is algebraic and has $\bar{K}$ as its definition field.*

(6.2b) *There exists analytic $\psi : X \to X'$ with $\varphi' \circ \psi = \varphi$.*

PROOF. □

Give a cover that has field of moduli $\mathbb{R}$, but not field of definition $\mathbb{R}$.

6.2.1. *Labeling Riemann surfaces and counting representatives of Nielsen classes.* Prop. 2.18 gives a precise way to label a particular Riemann surface: Show the branch points and cuts on $\mathbb{P}^1_z$, and give $\boldsymbol{g} \in \text{Ni}(G, \mathbf{C}, T_G)$. As we see in Chap. 5, if there is more than one s-equivalence class, equations for the cover won't have coefficients in functions of the branch points alone. That is very significant, and likely counter intuitive to the reader at this point. Many applications require finding a cover that has special coefficients (like over $\mathbb{Q}$). Inspecting actual equations for such a matter is rarely helpful if $r \geq 4$. So, what is the irrelevant information, and what to display?

Notation that memorably labels the conjugacy classes can be helpful in displaying expectations from the Branch Cycle Lemma about the position of the branch points for a cover over $\mathbb{R}$, or the p-adics or over $\mathbb{Q}$. This is especially significant when two distinct conjugacy classes $\mathrm{C}_i, \mathrm{C}_j$ happen to be the same when extended to S_n through $T_G : G \to S_n$. A graphic using a particular $\boldsymbol{g}$ representative of the Nielsen class can be revealing if it displays the real points lying over the real line on $\mathbb{P}^1_z$. This is pretty much the game when $r = 3$, and it will be especially fruitful in applying to covers of the $\mathbb{P}^1_j$ when they are reduced Hurwitz spaces of some Nielsen class. Our most crucial cases have several H_r orbits. When we know how to do so, we might present select representatives of those orbits.

Often our conjugacy classes have special shapes that allow computing the number of s-equivalence classes in a Nielsen class directly. Sometimes, however, it is good to know there is a pure computation for this count coming directly from the structure constant formula for the group ring $\mathbb{Z}[G]$ (for example, [**Se92**, §7.2] or [**Vö96**, p. 54]). Let m be constant on conjugacy classes $\mathrm{C}_1, \dots, \mathrm{C}_r$ and $g \in G$. Value of m on C_i is $m(\mathrm{C}_i)$. Denote $\sum_{i=1}^r \sum_{u_i \in C_i} m(u_1 \cdots u_r g)$ by $I(m; \mathbf{C}, g)$. If $m = \chi$ is an irreducible character of G, $I(\chi; \mathbf{C}, g) =$

$$\sum_{i=1}^r \sum_{u_i \in C_i} \chi(u_1 \cdots u_r g) = \chi(y) \prod_{i=1}^r \chi(u_i)/\chi(1)^r.$$

Take $\chi_1, \dots, \chi_s$ the irreducible complex characters of G. Then, $I(m; \mathbf{C}, g) = \sum_i m_i I(\chi_i; \mathbf{C}, g)$. Write $m = m_i \chi_i$. Consider $\psi_G = \frac{1}{|G|} \sum_{i=1}^s \chi_i(1)\chi_i$: 1 at 1_G and 0 otherwise. So, $I(\psi_G; \mathbf{C}, g)$ counts solutions of $u_1, \dots, u_r g = 1$ with $u_i \in \mathrm{C}_i$:

$$N(\mathrm{C}_1, \dots, \mathrm{C}_r, g) = |G|^{r-1} \sum_{i=1}^s \prod_{j=1}^r \chi_i(\mathrm{C}_j)\chi_i(g).$$

6.3. Branch cycles for sequences of genus 0 covers. §2.4.3 describes the association of r rooted cuts with classical generators for $\pi_1(U_{\boldsymbol{z}}, z_0)$. We acknowledge [**MP93**] and [**CG95**] for their approach to the following computational problem.

PROBLEM 6.6. Starting from $\varphi : \mathbb{P}^1_u \to \mathbb{P}^1_z$ and given $\bar{g}_1, \dots, \bar{g}_r \in \pi_1(U_{\boldsymbol{z}}, z_0)$ classical generators, find an algorithm computing classical generators for $\mathbb{P}^1_u$ from among the following collection of elements.

(6.3) *RETURN*

Genus 0 and how to handle the appearance of such without knowing it is analytically OR topologically isomorphic to the sphere. The key point is to have that any oriented triangle bounds a disk, and then it is possible to replace a general tree with vertices by a collection of simple rooted cuts that then give a system of classical generators. Triangulate classical generators, and then consider lifts of the triangles to a cover using the cut version of Fig. 3 (§2.4.3).

6.3.1. *Luroth's Theorem.* Consider a rational function $f \in \mathbb{C}(w)$ of degree n. From Chap. 2 [9.4a],

$$\boldsymbol{z} = \left\{ z' \in \mathbb{C} \mid (f(x) - z', \frac{df}{dx}) = 1 \right\} \cup \{\infty\}.$$

Let $S = \{f^{-1}(\boldsymbol{z})\} \cup \{\infty\}$. From Ex. 3.14 (and Chap. 2 Thm. 6.4), $f : \mathbb{P}^1_w \setminus S \to U_{\boldsymbol{z}}$ is a cover that extends to a map of compact manifolds $\bar{f} : \mathbb{P}^1_w \to \mathbb{P}^1_z$. Suppose f is *indecomposable*: Not a composition $f_1(f_2(z))$ with $\deg(f_i) > 1$, $i = 1, 2$. Being indecomposable is equivalent to the Galois closure group G_f being a primitive subgroup of S_n. show exactly what permutations of points in the fiber of this cover extend to an automorphism of the cover (for most f, none) [9.21i].

6.4. Belyi's covers of $\mathbb{P}^1_z$. Suppose $\varphi : X \to \mathbb{P}^1_z$ is a map of compact Riemann surfaces with r branch points $\boldsymbol{z}$. Define the J_r-invariant to be the corresponding point $J(\varphi)$ of $U_r/\mathrm{PGL}_2(\mathbb{C}) = J_r$. Suppose $J(\varphi)$ is in K. The algebraic half of the Existence Theorem says X is w-equivalent to a cover over $\bar{K}$. In particular, two independent functions φ_1 and φ_2 on a compact Riemann surface X give a canonical embedding of X as a projective variety over $\bar{K}$. If $J(\varphi_1) \in \bar{K}$, then so is $J(\varphi_2)$. RETURN

6.4.1. *Covers defined over $\bar{\mathbb{Q}}$.* Special case of Lem. 6.5: 3 branch point covers strongly equivalent to covers over $\bar{\mathbb{Q}}$. From Lem. 6.5 we know that if any cover $\varphi : X \to \mathbb{P}^1_z$ has branch points over $\bar{\mathbb{Q}}$, then it is equivalent to a cover with definition field in $\bar{\mathbb{Q}}$. So, to construct a cover $\varphi : X \to \mathbb{P}^1_z$ equivalent to the given one over $\bar{\mathbb{Q}}$, with only three branch points, it suffices to find a map $h : \mathbb{P}^1_z \to \mathbb{P}^1_w$ that map the set $\boldsymbol{z}$ into the set $\{0, 1, \infty\}$.

PROPOSITION 6.7. *Each $\boldsymbol{z} \in \mathbb{P}^1_z(\bar{\mathbb{Q}})$ has*

$$\mathbb{P}^1_z \xrightarrow{f_1} \mathbb{P}^1_{w_1} \to \cdots \xrightarrow{f_u} \mathbb{P}^1_{w_u}$$

with $f_u \circ \cdots \circ f_1$ having branch points $0, 1, \infty$ and $\boldsymbol{z} \subset f^{-1}(0, 1, \infty)$. More generally, if $\boldsymbol{z} \in \mathbb{P}^1_z(K)$ with K of transendence dimension t over $\mathbb{Q}$, then there is a similarly result, with f having $t + 3$ (or fewer) branch points.

Main use: $G_{\mathbb{Q}}$ faithful on projective systems of 3 branch point covers. Use $g = g(X)$ for genus of Riemann surface X. Subtopics:

- List of 3 branch point $S_{n,g}$ covers
- Observations giving Belyi's result
- Compare $S_{n,g}$ covers with Belyi maps with Guralnick's genus g problem
- Some nonrigid 3 branch point covers

(6.3a) Find $g_1 : P^1_z \to P^1_{u_1}$ with $f_1(\boldsymbol{z})$ in branch point locus, branch points of g_1 in $\mathbb{Q}$.

(6.3b) With $g_1(\boldsymbol{z})$ in $\mathbb{Q}$, compose with $g_2 : P^1_{u_1} \to P^1_{u_2}$ so $g_2 \circ g_1$ has 3 branch points.

Induct on maximal degree of $\boldsymbol{z}$ support and r. Always assume $\{z_1, z_2, z_3\} = \{0, 1, \infty\}$.

Step a: Map by f, irreducible polynomial for branch point of maximal degree over Q. Adds branch points of f to list, but new branch points are zeros of $\frac{df}{dz}$: have lower degree.

Step b: Write $z_4 = a/b$, $a, b \in \mathbb{Z}$. Choose $\psi(z) = z^u(z-1)v$: logarithmic derivative is $u/z + v/(1-z)$. Assure z_4 is a branch point by choosing $u, v \in \mathbb{Z}$ so $u/z_4 + v = (1 - z_4)$. This reduces the branch points by one.

6.4.2. *Three branch point $S_{n,g}$ covers.* Call a cover $\varphi : X \to \mathbb{P}^1_z$ an $S_{n,g}$ cover if $g(X) = g$ and deg)$\psi) = n$.

LEMMA 6.8. *Fix g. There are infinitely many $n \geq 1$ for which there are three branch point covers $\varphi : X \to \mathbb{P}^1_z$, with $g(X) = g$ and having monodromy S_n.*

PROOF. Take $n = m_1 + \cdots + m_s$. Modify $S_{n,0}$ covers:

$$g_1 = (1 \ldots m_1) \cdots (m_1 + \cdots + m_{s-1} + 1 \ldots n).$$

Genus 0: Take

$$g_2 = \begin{array}{l}(m_1{-}1\ \dots\ 1)(m_1{+}m_2{-}1\ \dots\ m_1{+}1)\cdots\\ (n-1\ \dots\ n-m_s{+}1)(m_1\, m_1{+}m_2\ \dots\ n).\end{array}$$

Then, $\mathrm{ind}(g_1) = n - s$ and $\mathrm{ind}(g_2) = n - s - 1$. Compute $g_1g_2 = g_3$:

$$(m_1\, m_1 - 1\, m_1{+}m_2\, m_1{+}m_2 - 1\ \dots\ n\, n{-}1).$$

So, $\mathrm{ind}(g_3) = 2s - 1$. RET gives genus 0 covers. For S_n, select $m_1, \dots, m_s$ accordingly.

For $g = 1$ covers, switch 1 and 2 in g_1, but not in g_2. This changes nothing from conclusions, except adding 2 to index of g_3:

$$(m_1\, 2\, 1\, m_1{-}1\, m_1{+}m_2\, m_1{+}m_2 - 1\ \dots\ n\, n{-}1).$$

That concludes the proof. □

6.4.3. *Comparison with the genus g problem.* Belyi produces three branch point $\varphi : X \to \mathbb{P}^1_z$, usually composed of many maps between $\mathbb{P}^1$ s. The construction rarely provides covers that are primitive.

QUESTION 6.9. Does X, over $\bar{\mathbb{Q}}$ have a three branch point cover $\varphi : X \to \mathbb{P}^1_z$ with primitive monodromy.

Help from the literature. Fix g with $g > 6$. By [**GN95**], only $< \infty$ly many X of genus g have three branch point covers with solvable monodromy ([**Fri99**, §5-§6] for genus 0 problem).

CONJECTURE 6.10 (Guralnick). Genus g three branch point primitive covers of $\mathbb{P}^1_z$ with monodromy neither S_n, $\mathbb{Z}/2 \wr S_n$, A_n or $\mathbb{Z}/2 \wr A_n$ (for some n) are finite.

The hardest case will be the 3 branch point case, though the likely result is more precisely the following. There is a function $N(g)$, quadratic in g, and either $n \le N(g)$ or one of the following holding.

(6.4a) The Galois closure of $\varphi : X \to \mathbb{P}^1_z$ has genus at most 1 and G is cyclic, dihedral with n a prime or n is a prime or prime squared and the Galois closure is an elliptic curve (affine case).

(6.4b) $g = 0$ or 1 and the monodromy group G satisfies $G = A_n$, S_n, $A_m \wr S_2 \le G \le S_m \wr S_2$ $(n = m^2)$.

(6.4c) $g = 0$ or 1 and $G = A_m$ or S_m with $n = m(m-1)/2$. 4. $g > 1$ and $G = A_n$ or S_n.

Of course, if we have $f : X \to Y$ primitive and Y has genus $h > 0$, then for $h > 1$, $n \le g$, so only finitely many and if $h = 1$, then either n is bounded in terms of g or $G = A_n$ or S_n (these last two statements are in Guralnick-Neubauer from the first Seattle meeting and these hold in positive characteristic as well which is noted in my MSRI paper [the other volume].

The work left is in dealing with symmetric and alternating groups acting on k-sets with 3 or 4 branch points and in dealing with some product actions where the group is essentially $S_m \wr S_2$ acting with $n = (m(m-1)/2)^2$. Kay, Dan and I should have shortly a complete list of $g = 0$ aside from the open cases listed above (i.e. ones not involving symmetric groups) and John Shareshian and I are working on symmetric groups with possibly all of us working on the product case.

QUESTION 6.11 (Easier Question). Compatible with Lem. 6.8, are the $[X] \in \mathcal{M}_g$ having an $S_{n,g}$ cover of $\mathbb{P}^1_z$ dense?

6.4.4. *Non-rigid A_n covers.* Consider $n \geq 5$, odd and squarefree. Let $\mathbf{C}$ be conjugacy classes of $(g_1, g_2, g_3) \in A_n^3$ with $g_1 = (1\,2)(3\,4)$, $g_2 = (1\,3\,5\,6\,7\ldots n)$ and $g_3 = (1\,2\ldots n)^{-1}$. Check: Geometric monodromy is A_n. Representatives for conjugation of S_n on $\mathrm{Ni}(S_n, \mathbf{C})/S_n$, $i = 3, \ldots, (n+1)/2$:

$$\boldsymbol{g}'_j = ((1\,2)(j\,j{+}1), (1\,3\,\ldots\,j\,j{+}2\,j{+}3\ldots n), g'_3).$$

QUESTION 6.12. Exists $f : \mathbb{P}^1_y \to \mathbb{P}^1_z$ in $\mathbb{Q}[y]$?

If yes, derivative is $g(y) = (y-a)(y-b)y^{n-3} \in \mathbb{Q}[y]$. Conclude: a, b either in $\mathbb{Q}$ or conjugate over $\mathbb{Q}$. Further $f(x) =$:

$$y^n/n - (a{+}b)y^{n-1}/(n{-}1) + aby^{n-2}/(n{-}2) + d;$$

and $f(a) = f(b)$. With $d \in \mathbb{Q}$, $b/a = \alpha$, simplify: $(n-2)(\alpha^n - 1) = n(\alpha^{n-1} - \alpha)$. Divide by $\alpha - 1$:

$$h_n(\alpha) = (n-2)\alpha^{n-1} - 2(\alpha + \cdots + \alpha^{n-2}) + (n-2),$$

divisible by $(\alpha - 1)^2$. Then, f over $\mathbb{Q}$ exists when $\in \mathbb{Q}[\alpha]$ of degree 2 divides $h_n(\alpha)/(\alpha-1)^2$. *Mathematica*: $h_n(\alpha)/(\alpha-1)^2$ irreducible over $\mathbb{Q}$ for odd $n \leq 31$.

6.4.5. *Belyi's Theorem in positive characteristic.*

PROPOSITION 6.13. *Let K be an algebraically closed field of characteristic $p > 0$. A projective curve X over K has definition field K if there is a finite map $\varphi : X \to \mathbb{P}^1_z$ over K with only tame ramification and at most three branch points. Further, if a projective curve X has field of definition $\bar{\mathbb{F}}_p$, then it admits a finite map $\varphi : X \to \mathbb{P}^1_z$ with only tame ramification and at most three branch points if there is at least one finite map $\varphi' : X \to \mathbb{P}^1_z$ over $\bar{\mathbb{F}}_p$ with with only tame ramification.*

PROOF. The first statement follows from knowing that a deformation of tame covers that leaves the branch points fixed doesn't change the equivalence class of the cover. This is one half of Grothendieck's main theorem on the fundamental group of a cover. The second statement is very simple. The map $C_{p^n-1} : \mathbb{P}^1_z \to \mathbb{P}^1_w$ by $z \mapsto z^{p^n-1} - 1$ maps all elements of $\mathbb{F}^*_{p^n}$ to 1. The cover C_{p^n-1} is tamely ramified (and ramified only over 0 and ∞. So, by choosing n so that $\mathbb{F}_{p^n}$ contains all branch points, C_n maps these down to 0, 1, ∞. □

The hypothesis of existence of a tame cover $\varphi : X \to \mathbb{P}^1_z$ is Saïdi's; he noted more than the simple argument in the proof of Prop. 6.13. In characteristic $p \geq 3$, such tame covers exist by an analog of an argument of Lefschetz. This say you may project the curve X from a nonsingular projective embedding in $\mathbb{P}^3$ to get a simple branched cover over the algebraic closure of the finite field. (The arithmetic form of the argument of [**FJ78**, Lem. 2.1] applies since we may take the finite field cardinality to be arbitrarily large.) [**Schr02**, §6] raises all these points, then takes it one step further to deal with removing the hypothesis of existence of a tame cover when $p = 2$.

[**Schr02**, §5] allows going from showing the general curve of genus g over $\mathbb{C}$ has a presentation in the Nielsen class $\mathrm{Ni}(A_n, \mathbf{C}_{3^r})$ to finding an open algebraic set of curves of genus $g = r+1$ over $\bar{\mathbb{F}}_2$ in this Nielsen class. [**Schr02**, Cor. 5.3] proves the space of isomorphism classes of curves over $\bar{\mathbb{F}}_2$ in this Nielsen class has dimension at least $2g-3$. In the language of Chap. 5 §**??** and [**Fri89**], the *moduli dimension* of the Nielsen class $\mathrm{Ni}(A_n, \mathbf{C}_{3^r})$ is $2g-3$ for $r \geq n$. The expectation is that the moduli dimension is actually $3g-3$ ($r \geq n+1$): *full moduli*. Schroer's argument, though it quotes [**FKK01**], does not identify the two components —separated by a spin lifting invariant ([**Fri95a**, Ex. III.12], [**Fri96**] and Chap. 5 §**??**) —that occur in these two families. We expect full moduli for both components. Even if the Nielsen class $\text{Ni}(A_n, \mathbf{C}_{3^r})$ has full moduli dimension, unlike the case where $p \geq 3$, it is unclear (even over $\mathbb{C}$) if that implies each genus g curve has a representing cover with only odd order ramification.

6.5. Higher genus versions of Thm. 1.8 and uniformization. We now generalize Thm. 1.8 for any compact Riemann surface X. A proof along the lines of that theorem works, but with some technical difficulties (see [11.11]).

6.5.1. *Homology of a manifold and triangulations.* We tacitly assume a compact Riemann surface has a triangulation, say it is given by an analytic map to $\mathbb{P}^1_z$.

LEMMA 6.14. *Let X be a compact Riemann Surface and $\Gamma(X, \Omega_X)$ its space of global holomorphic differentials. With χ_X the Euler characteristic of X, the dimension $u \leq (2 - \chi_X)/2 = g_X$.*

THEOREM 6.15. *Let X be a compact Riemann surface. Let $\{x_1, \dots, x_r\} = \{\boldsymbol{x}\}$ be r distinct points on X. There is a number $g = g(X)$ such that for $x_0 \in X \setminus \{\boldsymbol{x}\} = X^0$, there are closed paths*

$$\alpha_1, \dots, \alpha_g;\ \beta_1, \dots, \beta_g;\ \gamma_1, \dots, \gamma_r \tag{6.5}$$

based at x_0 so their homotopy classes generate $\pi_1(X^0, x_0)$ with the one relation

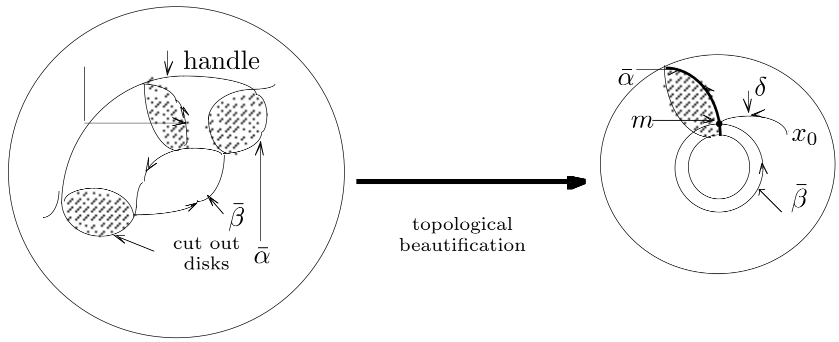
$$[\alpha_1][\beta_1][\alpha_1]^{-1}[\beta_1]^{-1} \cdots [\alpha_g][\beta_g][\alpha_g]^{-1}[\beta_g]^{-1}[\gamma_1] \cdots [\gamma_r]. \tag{6.6}$$

That is, Thm 6.15 says $\pi_1(X, x_0)$ is isomorphic to the quotient of the free group on (symbols given by) homotopy classes of the (6.5) paths, by the smallest normal subgroup containing expression (6.6). In addition, let (U, φ) be any coordinate neighborhood (with trivial fundamental group) in an atlas for X (Chap. 3 Def. 1.5) so that U contains $\{x_0, x_1, \dots, x_r\}$. Then we may take $\gamma_1, \dots, \gamma_r$ any set of paths that φ maps to a collection of classical generators relative to $(\varphi(x_1), \dots, \varphi(x_r))$ based at $\varphi(x_0)$ in $\varphi(U)$.

EXAMPLE 6.16 (A sphere with g handles—the case $r = 0$). Cut out 2g disjoint discs from the sphere; then join the boundaries of these discs in pairs by cylinders that slightly flare at the ends. We may embed any compact Riemann surface (having genus g) in $\mathbb{R}^3$ as a sphere with g handles. This is easy to prove for the complex torus of Chap. 3 Ex. 3.2.2 —it is homeomorphic to a sphere with one handle. The general fact, however, is more difficult [**Spr57**, Chap. 5]. If follows, however, quite easily from the uniformization of such a Riemann surface ($g \geq 2$) by the disk. PUT THE ARGUMENT OF A POLYGONAL DOMAIN HERE.

Nevertheless, assuming this, it is easy to draw paths representing generators of the fundamental group of a sphere with g handles (Fig. 9). Take g nonintersecting paths $\bar{\alpha}_1, \dots, \bar{\alpha}_g$ each of which goes like a bracelet around the handles; and g further nonintersecting paths $\bar{\beta}_1, \dots, \bar{\beta}_g$ where $\bar{\beta}_i$ travels along the ith handle from the unique point of intersection, m_i, of $\bar{\alpha}_i$ and $\bar{\beta}_i$ to the edge of the first hole cut in the sphere, then along the sphere to the other hole defining the handle, and, finally, back along the handle to m_i. In addition, these paths are chosen so that $\bar{\alpha}_i$ and $\bar{\beta}_j$ do *not* intersect for $i \neq j$, and the crossproduct $\boldsymbol{t}(\alpha_i) \times \boldsymbol{t}(\beta_i)$ of the tangent vectors to $\bar{\alpha}_i$ and $\bar{\beta}_i$ at m_i (§1.b) points outward.

FIGURE 9. A sphere with one handle (g=1) pretties itself



To obtain paths based at some specific point x_0 draw paths δ_i from x_0 to m_i and let $\alpha_i = \delta_i\bar{\alpha}_i(\delta_i)^{-1}$ (resp., $\beta_i = \delta_i\bar{\beta}_i(\delta_i)^{-1}$), $i = 1,\dots,g$. It is convenient to choose x_0 off the handles.

6.6. The Schwarz-Christoffel Transformation. Given a polygon P with vertices $A_1,\dots,A_n$ and clockwise interior angles $\pi\alpha_1,\dots,\pi\alpha_n$, we want to map the interior Π of this polygon on the Upper-half plane $\mathbb{H}$ in a one-one conformal fashion onto $\mathbb{H}$. We actually do the opposite, $f:\mathbb{H}\to\Pi$. Let $a_1 < a_2 < \cdots < a_n$ map respectively to $a_1,\dots,a_n$ by f. We can assign three of these points arbitrarily, and f is one-one on $\mathbb{H}$. [**Hil62**, Thm. 17.5.3] says: Let D be the interior of a simple closed curves C and $f : K \to D$ onto conformally. Then, f is continuous on $\bar{K}$ and the correspondence between $\bar{K}$ and $\bar{D}$ is one-one and bi-continuous.

Since such an f is locally one-one, f^{-1} is locally defined on D which is simply-connected. So, by the Monodromy Theorem (Chap. 3 Thm. 6.11) any branch of f^{-1} extends to D, and it is one-one. The next two subsections (based on [**?**, p. 372–374]) show f is the same as the function

$$F(z) = C_1\int_{a_1}^{z}(t-a_1)^{\alpha_1-1}\cdots(t-a_n)^{\alpha_n-1}\,dt + C_2. \tag{6.7}$$

To define F locally around any convenient point in $\mathbb{H}$ (say $z_0 = i$) as in Chap. 2 §3.4, use a branch $F_i(z)$ of $\log(z-a_i)$ around z_0. Then, interpret the integral for F to be a primitive (antiderivative) of $\prod_{i=1}^{n} e^{(\alpha_i-1)F_i(z)}$. Since $\mathbb{H}$ is analytically isomorphic to a disk, this choice of F extends analytically to all of $\mathbb{H}$ (Chap. 2 Prop. 3.6).

6.6.1. *Differential properties of F.* There are two components to the complement of a simple closed polygonal path [11.3a]. The interior is the component U_P that consists of points with nonzero winding number with respect to P. So, U_P is simply connected according to Chap. 2 §8.3. Apply the Riemann mapping theorem to show there exists an analytic one-one $f:\mathbb{H}\to U_P$. We want to identify f.

Consider the differential equation

$$\frac{d(\log(h'(z)))}{dz} = \frac{h''}{h'} = \sum_{j=1}^{n}\frac{\alpha_j-1}{z-a_j} \overset{\text{def}}{=} g(z). \tag{6.8}$$

Note that g is a linear fractional transformation with simple poles at a_j and corresponding residue $\alpha_j - 1$. Also, g vanishes at ∞ because the sum of the interior angles adds to 2π. Then, $F = h$ is a solution of (6.8). Then, To show that f is also, we show the logarithmic derivative $\frac{f''}{f'}$ of $f'(z)$ has the residues and poles of g and it vanishes at ∞. As the properties of g come from its being meromorphic on the whole plane, that first requires showing we can analytically continue f to anywhere in the whole plane minus $a_1, \dots, a_n$.

REMARK 6.17 (Point of the logarithmic derivative). The expression $\frac{h''}{h'}$ is invariant under composition by elements of $\mathrm{PGL}_2(\mathbb{C})$ having the form $z \mapsto ah(z) + b$ for $a \in \mathbb{C}^*$, $b \in \mathbb{C}$. Conversely, if $\mathcal{D}_{\mathcal{A}}(h) = \frac{h''}{h'} = \frac{h_1''}{h_1'}$, then $h_1(z) = ah(z) + b$: This differential equation $\mathcal{D}_{\mathcal{A}}(h) = \mathcal{D}\mathcal{A}(ah(z) + b)$ characterizes invariance of functions under the action of the affine group $\mathcal{A} = \mathbb{C} \times^s \mathbb{C}^*$ as in [11.25c].

6.6.2. *Schwarz's famous reflection principle.* Define f in the lower half plane by crossing (a_j, a_{j+1}) and reflecting its values across that line exactly as Schwartz did it [**Sc1890**, A paper from 1866]. Suppose h is any function on a domain D in the upper half plane that extends continuously to the line segment $(a, b) \subset \mathbb{R}$ (on the boundary of D). Denote the set of points of D reflected in the x-axis by $\bar{D}$. Further, assume h maps (a, b) to $L_{h(a),h(b)} \stackrel{\text{def}}{=} \{h(a) + t(h(b) - h(a)) \mid t \in (0, 1)\}$. Let $^{-\,L_{h(a),h(b)}}$ denote reflection of points $z \in \mathbb{C}$ in the line through $L_{h(a),h(b)}$. Then, $^{-\,L_{h(a),h(b)}} : z \mapsto A(\overline{A^{-1}(z)})$ with $A(z) = (h(b) - h(a)(\frac{(z-a)}{(b-a)}) + h(a)$.

LEMMA 6.18. *The formula* $h(w) \stackrel{\text{def}}{=} h(\bar{w})^{L_{h(a),h(b)}}$ *defines a function analytic on* $D \cup \bar{D} \cup (a, b)$ *and equal to* $h(z)$ *for* $z \in D$.

PROOF. If $A^{-1}(h(z))$ extends analytically to $D \cup \bar{D} \cup (a, b)$, then so does $h(z)$. This reverts us to the case h takes (a, b) to the line segment $(0, 1)$. [**Ahl79**, p. 172-173] emphasizes in the proof of this case, that it comes to extending the real and imaginary part of such an h to be harmonic. This he does by the formulas $\Re(h)(\bar{z}) = u(z)$ and $-\Im(h)(\bar{z}) = v(z)$ for $z \in D$. The tricky part is showing the points of (a, b) are also in the domain of harmonicity of the extending function $V(z)$: $V(z) = v(z)$ for $z \in D$, $-v(\bar{z})$ for $z \in \bar{D}$, and 0 on (a, b). For $t \in (a, b)$, this is an application of the *Poisson integral* [**Ahl79**, p. 168] P_V defined on any small disk D_t about t by the boundary values of V on that disk. The function $V - P_V$ vanishes on the intersection of D_t and the real line, and on the boundary of the disk in the upper half-plane. By the maximum principle for harmonic functions, $V - P_V$ is identically zero on the upper half of the disk, and similarly on the lower half of the disk, concluding the proof. □

Apply this to $D = \mathbb{H}$: We analytically continue f to the lower plane by extending $f = f^{(0)}$ to a function $f_i^{(0)}$ on $D \cup \bar{D} \cup (a_i, a_{i+1}) = U_i$ for some i. The effect is that we analytically continued along a path crossing (a_i, a_{i+1}) from the upper half plane to the lower half plane, and then took the unique function defined on the simply connected (even contractible) set U_i. From the reflection principle, the analytic function $f_i^{(0)}$ maps the lower half plane to the original (open) polygon reflected in the line through $f(a_i)$ and $f(a_{i+1})$. Now work with $f_i^{(0)}$ in the lower half plane, and apply the same principles. We can extend it back up into the upper half plane using any of the line segments (a_j, a_{j+1}). Denote $f_i^{(0)}$ by $f^{(1)}$. This analytically continues to the whole upper half plane (along a path) through (a_j, a_{j+1}). The result is a function $f_j^{(1)}$ on $\mathbb{H}$ that maps $\mathbb{H}$ to the interior of a new polygon.

By the reflection principle, the new polygon is just the result of reflecting the original polygon in two different lines. Note: Composition of two such reflections is an affine transformation: $M : \mathbb{C} \to \mathbb{C}$ by $z \mapsto a'z + b'$ for some $a' \in \mathbb{C}^*$ on the unit circle and $b' \in \mathbb{C}$. So the effect of the analytic continuations that end in the upper half plane is to give polygons congruent to the original polygon. On $\mathbb{H}$, since $f^{(1)}$ is composition with an affine transformation, we have $\mathcal{D}_{\mathcal{A}}(f) = \mathcal{D}_{\mathcal{A}}(f^{(1)}$ (see Rem. 6.17). The differential operator provides a function $\mathcal{D}_{\mathcal{A}}(f)$invariant under the different analytic continuations of f in the domain $\mathbb{C} \setminus \{a_1, \dots, a_n\}$.

6.6.3. *Singularities of analytic continuations of f.* The Scharz-Christoffel transformation that maps $\mathbb{H}$ onto the sector at angle $0 < \alpha\pi < \pi$ is $f(z) = z^\alpha$. We know f at a_j maps an interior angle of π radians to one of $\alpha_j\pi$ radians, and it analytically continues around a_j. The residue of $\frac{f''}{f'}$ for such a function f is well-defined up to a change of variables in f in a neighborhood of a punctured disk around a_j. With no loss, for the computation of the residue, f to be a branch of $(z-a_j)^\beta = e^{\beta\log(w)}$ with $w = z - a_j$ (Chap. 2 §8.2). Knowing that analytic continuation around the counterclockwise upper disk, gives a change of angle of $\alpha_j\pi$, shows β is α_j. So, the residue of $\frac{f''}{f'}$ at a_j is given by the residue of $\alpha_j(\alpha_j-1)z^{\alpha_j-1}/\alpha_j z^{\alpha_j} = (\alpha_j-1)1/z$, which is $\alpha_j - 1$. Conclude that $\frac{f''}{f'}$ has the right residues and is meromorphic everywhere. So, $f = F$ for some choice of C_1 and C_2.

6.7. Monodromy and hypergeometric functions. [**Ahl79**, p. 315–321] Starts with a discussion of homogeneous linear ordinary differential equations with meromorphic function coefficients of degree n:

$$\sum_{k=0}^{n} a_i(z)w^{(k)} = 0. \tag{6.9}$$

A meromorphic solution $w(z)$ in a neighborhood of z_0 is ordinary if $a_0(z_0) \neq 0$, and around such a point there are n linearly independent solutions. At an ordinary point if you also specify the values of $w^{(k)}(z_0)$, $k = 0, \dots, n-1$, then there will be a unique solution $w(z)$. The standard proof of this is given by writing a power series solution, solving for the coefficients inductively while establishing the series converges [**Ahl79**, p.310]. The conditions on the values of the first n terms in the powers series are linear, establishing that the space of solutions around z_0 is n-dimenional. While general (not necessarily linear) differential equations generalize our basic study of algebraic functions, linear such do not. For example, in studying algebraic equations attached to genus 1 curves, we treat the differential equation $(w')^2 = aw^3 + bw + c$ with $a, b, c \in \mathbb{C}$. Linear equations arise here in a different context.

6.7.1. *Monodromy from a differential equation.* Let $\boldsymbol{z}$ be the finite set of points at which at least one of the a_i s is not analytic or at which $a_0(z)$ has a zero. Then, as with any other analytic continuation situation we may analytically continue a basis $f_1, \dots, f_n$ of solutions at z_0 around any element of $\Pi(U_{\boldsymbol{z}}, z_0, z_1)$. At least for such linear differential equations, Riemann suggested establishing similar properties as for analytic continuation of meromorphic algebraic functions. Under certain conditions we can expect the monodromy action to determine the differential equation.

This, however, requires inspecting the nature of the differential equation near the support of $\boldsymbol{z}$ in a new way. By multiplying through by any denominators, we may with no loss assume the coefficients are analytic in $\mathbb{C}_z$, and that they have no common zero. In a neighborhood of z_0 we can analytically continue the solution, and since the space has dimension n, we are getting a representation of $\mathbb{Z}$ through a matrix M_{z_0}. Suppose $n = 2$ and the monodromy matrix has distinct eigenvalues $e^{2\pi i\alpha_1}$ and $e^{2\pi i\alpha_2}$. Then, the solutions would have local expressions as $z^{\alpha_1}h_1(z)$ and $z^{\alpha_2}h_2(z)$, with h_1 and h_2 analytic and nonzero at z_0. As usual, write $\ln(z - z_0) = m(z)$, a a branch of log in a disc about a point z' near z_0 to see the effect of a clockwise analytic continuation about z' about z_0: $e^{m(z)\alpha} \mapsto e^{(m(z)+2\pi i)\alpha}$. Looking at the term of highest order, gives the *indicial equation* for such α, which is the characteristic polynomial for the monodromy action matrix. A regular singular point is one for which the characteristic polynomial is computable from α satisfying the highest order solution from undetermined coefficients. State this condition in terms of the orders at z_0 of the coefficients of the differential equation. There are complications, however, if the solutions of that indicial equation have $e^{2\pi i)\alpha_1} = e^{2\pi i)\alpha_2}$. One possibility is that there are two values of α and they do give independent solutions. Another is that if you take the larger value (difference is positive integer) you get the solution $w_1(z)$ above, and you get another one by taking a solution of form $w_2(z) = Cw_1(z)\ln(z - z_0) + (z - z_0)^{\alpha_2}h_2(z)$, and finally if $\alpha_1 = \alpha_2$, this last situation does occur for certain. The general case is given by a similar Jordan canonical form.

We can also change variables and consider the possibility of having a regular singular point at ∞. If there are regular singular points everywhere (and only finitely many of them), then all coefficients are rational functions in z. When $n = 2$ this says $w'' - pw' - qw$ with p and q rational functions with q having at most a double pole, and p having at most a single pole anywhere in the finite plane. Then, at ∞ with $z = 1/u$, write $W(u) = w(1/u)$ and $\frac{dw}{dz}$ after as $-u^2\frac{dW(u)(}{du}$ and $\frac{d^2w}{dz^2} =$ after substitution $z \mapsto 1/u$ as $2u^3\frac{dW(u)(}{du} + u^4\frac{d^2W(u)}{d^2u}$. So, now we check if $u = 0$ is a regular singular point for this equation. A regular singular point is equivalent to $2/u + u^{-2}p(1/u)$ has a pole of order at most 1 at 0, and $u^{-4}q(1/u)$ has a pole of order at most 2 at 0. Bessel's equation: $zw'' + w' + zw$ therefore has regular singular points everywhere except at ∞.

6.7.2. *Regular singular points everywhere.* If there are but two regular singular points, put them at 0 and ∞. Then, $p(z) = A/z$ and $q = B/z^2$. Of course, in this case the global monodromy group is given by the action on $(z^{\alpha_1}, z^{\alpha_2}$ if the solutions of the indicial equation are distinct, and by $(z^\alpha, \ln(z)z^\alpha)$ if they are not.

Now suppose there are exactly three regular singular points, and place them at 0, 1, ∞.

6.7.3. *Application to integrals.* Let $a_1, \dots, a_r$ be any complex numbers and assume we have chosen $g_i(z)$ to be a branch of $(z - z_i)^{-a_i}$ in a neighborhood of z_0. Then, $\prod_{i=1}^r g_i\,dz \stackrel{\text{def}}{=} \omega_{\boldsymbol{z}}$ is a differential 1-form in a neighborhood of z_0 that analytically continues along any $\gamma \in \Pi_1(U_{\boldsymbol{z}}, z_0, z')$ to give a differential 1-form in the neighborhood of z'. Then, $\int_\gamma \stackrel{\text{def}}{=} I_{\boldsymbol{a}}(\gamma)$ makes sense. The monodromy theorem says this depends only on the homotopy class of the path. Further, if $\gamma_{i,j}$ is a piecewise differentiable path on $\mathbb{P}^1_z$ from z_i to z_j, then even $I_{\boldsymbol{a}}(\gamma_{i,j})$ makes sense by taking the integral to be $\lim \epsilon \mapsto 0 I_{\boldsymbol{a}}(I_{\boldsymbol{a}}(\delta_{e,\epsilon}) - \delta_{b,\epsilon})$ with with $\delta_{b,\epsilon}$ going from z_0 to $\gamma(\epsilon)$ in $U_{\boldsymbol{z}}$ and $\delta_{e,\epsilon}$ being the composite of $\delta_{b,\epsilon}$ and $\gamma_{[epsilon,1-\epsilon]}$.

Notice, unless all the a_i s are the same, this is placing an ordering on them to form $\omega_{\boldsymbol{a}}$. Suppose $\varphi : U_{\boldsymbol{z}} \to U_{\boldsymbol{z}}$ is a diffeomorphism that fixes each of the points in $\boldsymbol{z}$. To see the effect of the diffeomorphisms it suffices to take the case $(z_1, z_2, z_2) = (0, 1, \infty)$, and let $z_4 = \lambda$.

7. Abel's contributions and modular curves

This culminated in Abel's beautiful characterization of analytic functions, the precise form that the Riemann-Roch theorem takes, on a complex torus. Fine though it is in its classical form, we now take a nontraditional view that soon will reveal *modular curves* and some of their far-reaching generalizations. Our approach motivates the whole theory of moduli spaces of Riemann surfaces. We start with finding the essential parameters that characterize these integrals up to algebraic transformations. Functions that naturally live on a (1-dimensional) complex torus are algebraically related to inverses of special cases of functions given by integrals in (6.7). Compatible with other lessons about coordinates, the functions that live on one complex torus cannot be *elementarily* related to those on an analytically nonisomorphic torus.

RETURN If you have a function that separates points at one fiber, why does it separate at all but finitely many fibers? Answer: Suppose $g : X \to \mathbb{P}^1_w$, and for infinitely many $z' \in \mathbb{P}^1_z$, there are $w_1(z'), w_2(z') \in \varphi^{-1}(z')$ such that $g(w_1(z')) = g(w_2(z'))$.

7.1. Integrals of primitives. In §6.7 we have the special case appearing in Chap. 2 (6.6): $\int_\gamma \frac{dz}{(z^3+cz+d)^{\frac{1}{2}}}$. with $c, d \in \mathbb{C}$. Use the notation $\mathcal{A}_h(U_{\boldsymbol{z}})$ for the analytic continuations of h based at z_0 (Chap. 2 §4.5).

LEMMA 7.1. *Suppose $h(z) \in \mathcal{E}(U_{\boldsymbol{z}}, z_0)$ is an algebraic function of degree n. Form $H(z)_\lambda = \mathrm{Int}(h(z))_\gamma$ to mean the analytic continuation of a primitive $H(z)$ for $h(z)$ along $\gamma \in \pi_1(U_{\boldsymbol{z}}, z_0)$ (*Chap. 2 *§4.3). Define G_H to be the monodromy group of this action: The collection of permutations on $\mathcal{A}_H(U_{\boldsymbol{z}})$. Then, there is a natural map $G_H \to G_h$ given by taking the derivative: $\frac{dH_\gamma}{dz} \mapsto h_\gamma$. Denote by $G_h(1)$ the stabilizer of h in G_h and let L_h be the pullback of $G_h(1)$ in G_H. Then, L_h is an abelian group and, $L_h = L_{h_\gamma}$ for $\gamma \in \pi_1(U_{\boldsymbol{z}}, z_0)$. This identifies G_H as a subgroup of the wreath product $L_h \wr G_f = (L_h)^n \times^s G_f$ (§8.4).*

PROOF. The effect of operating by $\gamma^* \in \pi_1(U_{\boldsymbol{z}}, z_0)$ on $\{\mathrm{Int}(h(z))_\gamma\}_{\gamma \in \pi_1(U_{\boldsymbol{z}}, z_0)}$ maps to the action on $\{h(z)_\gamma\}_{\gamma \in \pi_1(U_{\boldsymbol{z}}, z_0)}$ by taking the derivative: $h_\gamma = \frac{dH_\gamma}{dz}$. Further, suppose $h_\gamma = h$: γ is in the pullback of $G_h(1)$. Then, H_γ is a primitive of $h_\gamma = h$, and so $H_\gamma(z) = H(z) + c_\gamma$ with $c_\gamma \in \mathbb{C}$. Since c_γ is $\int_\gamma h(z)dz$, for $\gamma_1, \gamma_2 \in \ker(\Gamma)$, $c_{\gamma_1} + c_{\gamma_2} = c_{gamma_1 + \gamma_2}$. Use the notation L_h for this abelian group of integration constants.

Consider a conjugate h_γ of h given by $\gamma \in \pi_1(U_{\boldsymbol{z}}, z_0)$. Then, $\int_{\gamma^*} h_\gamma \, dz$ is the same as $\int_{\gamma \cdot \gamma^*} h \, dz$. As $[\gamma]\pi_1(U_{\boldsymbol{z}}, z_0) = \pi_1(U\boldsymbol{z}, z_0)$, the final set of integrals around closed paths is the same. Finally, let us embed G_H in $L_h \wr G_f = (L_h)^n \times^s G_f$. Let $\{h_i\}_{i=1}^n$ be represent the distinct elements of $\mathcal{A}_h(U_{\boldsymbol{z}})$. It has a map to G_h and the kernel of that map fixes each of L_{h_γ}, each of which we can identify. The argument is now the same as in Chap. 3 8.14 that this gives a wreath product. □

EXAMPLE 7.2 (Branch cycles in $S_n \setminus A_n$). Assume each class in $\mathbf{C}$ is in $S_n \setminus A_n$. Let $\boldsymbol{p} \in \mathcal{H}(G, \mathbf{C})^{\text{in,rd}}$ lie over $\boldsymbol{z}$. Take the regular representation of G as giving a map $G \to S_{|G|}$. The cover $\varphi_{\boldsymbol{p}} : X_{\boldsymbol{p}} \to \mathbb{P}^1_z$ naturally factors through $E_{\boldsymbol{z}} \to \mathbb{P}^1_z$: Quotient $X_{\boldsymbol{p}}$ by $G \cap A_{|G|}$. (This works for any even r; $E_{\boldsymbol{z}}$ is then hyperelliptic.)

Suppose $r = 4$ and $\boldsymbol{g} \in \text{Ni}(G, \mathbf{C})$ with $G \le A_n$. Choose $h_1, h_2 \in S_n \setminus A_n$. Then, $(h_1 g_1, g_2 h_2, h_2^{-1} g_3, g_4 h_1^{-1})$ satisfies the product-one condition. It produces a Nielsen class (for some new group) with moduli problem directly recognizing the j-line as parameterizing elliptic curves.

Write out the covering property for $w \mapsto h(w)$ and $y \mapsto y^2$. Then, take the fiber product over $\mathbb{P}^1_z$. Use the path lifting property for each separately, then join them and ask how you can understand this to be like an exponential map. Problem: The set is more complicated, and not looking like an open subset of the plane. Then, consider the integral $F(w) = \int_\gamma \frac{dw}{\sqrt{h(w)}}$ in analogy with the discussion right after Prop. 3.5.

Let $H(w)$ be the inverse function of $F(w)$: $G(F(w)) = w$. Conclude this uniformization of

$$E_g = \{(w, y) \mid y^2 - h(w)\} : \ z \mapsto (G(z), G'(z)). \tag{7.1}$$

LEMMA 7.3. *Periods of $H(w)$ and why they have rank two. FINISH THIS*

7.2. Starting Abel's Theorem. We start with an interpretation of Abel's the basic problem which was about the nature of antiderivatives. Analyze elementary antiderivatives, like the watershed example $\int \frac{dx}{\sqrt{x^3+ax+b}}$. Specifically, what is the dependence of these antiderivatives on the parameters a and b?

Here $m(z, w) = w^2 - (z^3 + az + b)$. Let $\boldsymbol{z}$ be ∞ together with the zeros of $z^3 + az + b$. Write $G(z) = \frac{1}{\sqrt{z^3+az+b}}$, a branch of this square root defined in a neighborhood of $z_0 \notin \boldsymbol{z}$ (Chap. 2 §8.2). Consider $F(z) = F_{a,b}(z)$, an antiderivative of $G(z)$, locally. As in Chap. 2 §4.3, it has analytic continuations along elements of $\Pi(U_{\boldsymbol{z}}, z_0)$ and it depends only on the homotopy class of the path. These continuations produce an *abelian group* of periods (Chap. 2). Chap. 4 shows the group is $\mathbb{Z} \times \mathbb{Z}$. Further, its fit with the analytic continuations of $G(z)$ appears in the semidirect product $\mathbb{Z} \times \mathbb{Z} \times^s \{\pm 1\}$ (§8). Let D_n be the dihedral group of order $2n$.

7.2.1. *The antiderivatives: when are they equivalent by substitutions?* This was Abel's version of the problem. He already had experience with showing functions might be new: The new functions that produced solutions to the general quintic. His formulation of old included taking compositions of rational functions, multiplication by constants, various other functions regarded as known, and the conceptual addition of

(7.2) functional inverse.

We have learned in Chap. 2 that there really are but two elementary functions if you use (7.2): as functions of a complex variable they are z and $\log(z)$. Also, you may profitably consider the integral locally as a function of z by regarding it as as an antiderivative in z. By allowing this you agree that you will stay within elementary functions. Still, we sceptically scrutinize the antiderivative operation, for we are also asking when that applied to elementary functions takes us out of them.

Example 1: . Example: $z^{1/n} = e^{z/n}$.

7.3. Substitutions by elementary functions. We may take a branch of inverse of a locally one-one analytic function Chap. 2 §6.1. We did that in §6.6 and found that Substitutions in the antiderivative variable $w = F_{a,b}(z)$ become compositions of the inverse function $F^{-1}_{a,b}(w) = z$. So Abel took a more restricted version of the problem by equivalencing by rational function substitutions and derivatives in w.

PROBLEM 7.4. When is $L_{a,b,c} \stackrel{\text{def}}{=} \mathbb{C}(F^{-1}_{a,b,c}(w), \frac{d\,F^{-1}_{a,b,c}}{dw}, \dots)$ isomorphic to $L_{a',b',c'}$? Only one derivative needed by Abel's use of the chain rule. The fields $L_{a,b,c}$ are called function fields. We can describe every element of $L_{a,b,c}$.

7.4. Explicit functions for dihedral covers. We have applied the Existence Theorem in §4.3 to see how many different involution dihedral covers of $U_{\boldsymbol{z}}$ we get when $r = 4$. Now we consider another approach to dihedral covers. The two approaches are complementary. In §4.3 we have a degree p^{k+1} function to $\mathbb{P}^1_u$ (to retain the classical z variable for Abel's approach, we switch the variable in Riemann's Existence Theorem to u). In Abel's approach, the connection to the u variable is nonobvious.

Let $L(\omega_1, \omega_2) \stackrel{\text{def}}{=} L_{\boldsymbol{\omega}} = \{m_1\omega_1 + m_2\omega_2 \mid m_1, m_2 \in \mathbb{Z}\}$ be a lattice in $\mathbb{C}$. First observation: With no loss we may assume $\tau = \omega_2/\omega_1$ has positive imaginary part. We say τ lies in the *upper half plane*. *Elliptic functions* $f(z)$ on $L_{\boldsymbol{\omega}} = L$ have all elements of L as *periods*. In the next lemma we consider $f(z)$ in a fundamental domain for L.

THEOREM 7.5. *Residues of an elliptic function sum to 0. Conclude: A nonconstant elliptic function has as many zeros as it has poles. Finally, if* $a_1, \dots, a_n$ *are the zeros of* f, *and* $b_1, \dots, b_n$ *are its poles, then* $\sum_{j=1}^n a_j - \sum_{j=1}^n b_j \in L$.

PROOF. Take B as the clockwise boundary of the fundamental domain. Let $f(z)$ be an elliptic function for L. The sum of the residues is therefore $\frac{1}{2\pi i}\int_B f(z)\,dz$. Integrals in opposite directions on opposite sides of the parallelogram cancel. Therefore, this is 0. Apply this to $f'(z)/f(z)$, also an elliptic function. For the last sentence, compute $\frac{1}{2\pi i}\int_B z f'(z)/f(z)\,dz$ carefully. ☐

Here is the Weierstrass $\wp$-function:

$$\wp(z;\omega_1,\omega_2) = \frac{1}{z^2} + \sum_{\omega\neq 0}\left(\frac{1}{(z-\omega)^2} - \frac{1}{(\omega)^2}\right). \tag{7.3}$$

This actually converges, except at $z = 0$. The observation is there exists $u > 0$ with $|m_1\omega_1 + m_2\omega_2| \geq u(|m_1| + |m_2|)$. Clearly, $\wp(z;\omega_1,\omega_2)$ is an even function. It's derivative $\frac{d\wp}{dz}(z;\omega_1,\omega_2) = \wp'(z) = -2\sum_\omega \frac{1}{(z-\omega)^3}$ is an odd doubly periodic function. Conclude that $\wp$ is also doubly periodic.

7.5. The function $\zeta(z)$. Since $\wp$ has residues equal to 0, there exists $\zeta(z)$ whose derivative is $\wp$. We see easily:

$$\zeta_(z) = \frac{1}{z} + \sum_{\omega\neq 0}\left(\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2}\right).$$

Thus, $\zeta(z+\omega_1) = \zeta(z) + \eta_1$ and $\zeta(z+\omega_2) = \zeta(z) + \eta_2$ for some constants η_1 and η_2. Integration of ζ around B shows

$$\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i. \tag{7.4}$$

Further, take $\sigma(z)$ to be a function whose logarithmic derivative is $\zeta(z)$:

$$\sigma_\tau(z) = \sigma(z) = z \prod_{\omega \neq 0} \left(1 - \frac{z}{\omega}\right) e^{z/\omega + \frac{1}{2}(z/\omega)^2}. \tag{7.5}$$

Clearly, $\sigma(z)$ is an odd function. Also, $\sigma(z + \omega_1) = -\sigma(z)e^{\eta_1(z+\omega_1/2)}$, etc. The values $z = \omega_i/2$ already play a special role.

7.6. Abel's construction of functions when $g = 1$.

THEOREM 7.6 (Abel, 1837). *An elliptic function with periods ω_1 and ω_2, zeros $a_1, \ldots, a_n$ and poles $b_1, \ldots, b_n$ is a constant times $\prod_{k=1}^n \frac{\sigma(z-a_k)}{\sigma(z-b_k)}$* [**Ahl79**, p. 267].

The functions $\wp(z)$ and $\wp'(z)$ satisfy this simple equation:

$$\wp'(z)^2 = 4\wp(z)^3 - 60G_2\wp(z) - 140G_3 \tag{7.6}$$

with $G_k = \sum_{\omega \neq 0} \frac{1}{\omega^k}$. The formula is important, but the principle easy: create a doubly periodic function with no poles.

The point: $(\wp(z), \wp'(z))$ maps complex analytically from the complex torus to the points of the cubic equation (7.6). Thus, the complex torus and the points on the cubic equation represent the same Riemann surface. A most important step is to interpret ideas on the complex torus through (7.6).

7.7. The unique θ function with odd characteristic. On the complex torus $\mathbb{C}/L_{\boldsymbol{\omega}}$, we may characterize $\wp(z)$ among elliptic functions with a pole of order 2 at the origin, and no other pole up to a linear change of variable. Let $W(z)$ be another, and normalize by linear change so $\wp(z) - W(z)$ has no poles at all and expansion of $W(z)$ around ∞ looks like $\frac{1}{z^2} + a_1 z +$ higher terms. Then, $\wp(z) - W(z)$ is identically zero. The same argument shows $\wp(z)$ is an even function: $\wp(z) - \wp(-z) \equiv 0$.

Then, $\zeta(z)$ is the unique antiderivative that is odd: $\zeta(z) = -\zeta(-z)$ for $z \in \mathbb{C}$. Finally, $\sigma_\tau(z)$ has the property its logarithmic derivative equals $\zeta(z)$. Thus, it is defined up to multiplicative constant, and it must be an odd function. Conclusion:

THEOREM 7.7. *Up to a multiplicative constant, there is a unique odd function $\sigma_\tau(z)$ that plays the role of a θ function on $X_\tau = \mathbb{C}/L_\tau$. Suppose $D_{\boldsymbol{a},\boldsymbol{b}} = \sum_{j=1}^n a_j - \sum_{j=1}^n b_j$ represents any degree 0 divisor on X_τ. Then,*

$$\omega_{\boldsymbol{a},\boldsymbol{b}} = \sum_{j=1}^n \zeta(z - a_i) - \sum_{j=1}^n \zeta(z - b_j)dz$$

is a differential form on X_τ with these properties.

(7.7a) *It is a logarithmic differential with polar divisor $D_{\boldsymbol{a},\boldsymbol{b}}$.*
(7.7b) *Its periods are pure imaginary.*

PROOF. We've shown everything except the properties of $\omega_{\boldsymbol{a},\boldsymbol{b}}$. We have $\zeta(z + \omega_i) = \zeta(z) + \eta_i$ for some complex number η_i, $i = 1, 2$. Therefore, $\sum_{j=1}^n \zeta(z - a_i) - \sum_{j=1}^n \zeta(z - b_j)$ is already a well-defined elliptic function on X_τ.
DO WE KNOW ITS PERIODS ARE PURE IMAGINARY. □

7.8. Uniformization and the j-line. Addition on $\mathbb{C}/L$ is obvious.

7.8.1. *The cubic equation for the torus.* How do we express this in coordinates from the cubic equation? There is this formula [**Ahl79**, Ex. 2–7, p. 269]:

$$\det \begin{vmatrix} \wp(z) & \wp'(z) & 1 \\ \wp(u) & \wp'(u) & 1 \\ \wp(u+z) & -\wp'(u+z) & 1 \end{vmatrix} = 0. \tag{7.8}$$

Consider the zeros of the cubic on the right side of (7.6). Call these e_1, e_2, e_3. Since $\wp'(z)$ is an *odd* function, we find these e_i's are $\wp(\frac{\omega_1}{2})$, $\wp(\frac{\omega_2}{2})$ and $\wp(\frac{\omega_1+\omega_2}{2})$. Since $\wp(z)$ assumes each value with multiplicity 2 (on a fundamental domain), the e_i's are distinct. Thus, $\lambda(\tau) = \frac{e_3-e_2}{e_1-e_2}$ is a well-defined function of τ.

We want a well-defined function of L: equivalently, a function preserved under unimodular transformations of τ. Notice these properties of $\lambda(\tau)$.

(7.9a) Unimodular transformations of (ω_1, ω_2), represented by

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \bmod 2 \tag{7.10}$$

leave $\lambda(\tau)$ invariant.

(7.10b) Unimodular transformations don't satisfying (7.9a) don't leave $\lambda(\tau)$ invariant.

(7.10c) $\lambda(\tau)$ takes all values except 0 or 1 for τ in the upper half plane.

(7.10d) Each value $\lambda(\tau)$ takes on, it assume locally with multiplicity 1.

Define $j(\tau)$ to be $\frac{4}{27}\left(\frac{e_1e_2+e_2e_3+e_3e_1}{(e_1-e_2)^2(e_2-e_3)^2(e_3-e_1)^2}\right)$. Consider the subfield of $\mathbb{C}(\lambda(\tau))$ invariant under the full unimodular group (as in (7.9b)). Then, $j(\tau)$ generates this field [**Ahl79**, Ex. on p. 274].

7.8.2. *The function* $\lambda(\tau)$. Finally, from (7.9c) and (7.9d) we have Picard's *big* theorem.

Theorem 7.8. *Suppose $f(z)$ is entire and it omits at least two values. Then, $f(z)$ is constant.*

Proof. Suppose $f(z)$ omits at least two values. With no loss, take these to be 0, 1. For any other value of f, say, $f(z_0)$, (3) shows there is τ_0 in the upper half plane with $\lambda(\tau_0) = f(z_0)$. Apply (4). The complex version of the implicit function theorem gives an analytic function $h(z)$ defined in a neighborhood of z_0. It has these properties: $\lambda(h(z)) = f(z)$ for z close to z_0; and $h(z_0) = \tau_0$. Since the complex plane is simply connected, the *monodromy theorem* says h, defined locally for each z_0, is the restriction of one entire function $H(z)$. Note, however, $H(z)$ takes its values in the upper half plane. This violates the maximum modulus principle: $e^{iH(z)}$ has absolute value at most 1. □

7.8.3. *Uniformization from the Existence Theorem.*

Corollary 7.9. *Uniformization of $U_{\boldsymbol{z}}$, $r \geq 3$, by a disk following the deformation talk at MSRI.*

7.8.4. *Involution dihedral covers again.* Let $f(z) = (z-1)\cdots(z-r)$, let D be an open connected set in $\mathbb{C} - \{1, \ldots, r\}$ and let $A_1, \ldots, A_k$ be the connected components of $\mathbb{P}^1_z \setminus D$. Suppose A_k is the component that contains ∞. Now take $r = 3$. Assume that γ is a path defined on $[0,1]$. We can define $g_\gamma(t)$, a branch of $\log((z-1)(z-2)(z-3))$ along γ, by the following formula:

$$\frac{1}{2\pi i}\Big(\int_\gamma \frac{dz}{z-1} + \int_\gamma \frac{dz}{z-2} + \int_\gamma \frac{dz}{z-3}\Big).$$

Let $a_i(\gamma)$ be the index of γ with respect to $i = 1, 2, 3$. Thus $e^{g_\gamma(t)/n}$ is independent of γ if and only if $a_1 + a_2 + a_3 \equiv 0 \bmod n$ for every closed path in D. For example, if $n = 3$, then $g(z)$ exists if $D = \mathbb{C} \setminus [1, 3]$, but it doesn't exist if $D = \mathbb{C} \setminus \{1\} \cup [2, 3]$.

Consider closed paths γ based at 0 in the case that $n = 2$ and $D = \mathbb{C} \setminus \{1, 2, 3\}$. If a closed path has index 1 with respect to 1 and index 0 with respect to 2 and 3, we must have $\int_\gamma f(z)^{\frac{1}{2}}\, dz \neq 0$. Otherwise there would be an analytic function $F(z)$ defined on $D = \mathbb{C} \setminus [1, 3]$ such that $\frac{dF}{dz}$ would be a branch of $(f(z))^{\frac{1}{2}}$, contrary to the above argument.

8. Algebraic coordinates

8.1. Points about algebraic varieties. 2. Segre embedding of two projective varieties. P. 66

For prevariety, I need to talk about the patching maps.

3. Variety: p. 68: A variety is a prevariety X (finite cover of affine varieties – irreducible and have a coordinate ring that is an integral domain) together with the property that if $f : Y \to X$ and $g : Y \to X$, $Y \times_X Y \cap \Delta$ is closed in $Y \times Y$. The case when f and g are the projections is just the case Δ_X is closed in $X \times X$. The general case follows from this case because it is also factors through $Y \times : (f, g) \to X \times X$ and the set in question is just the pullback of Δ_X, which being closed has its inverse closed.

Prop. 5 (p. 71): If X is a prevariety and any two points are in an affine open piece, then X is a variety. Reason: Get Hausdorff in the affine pieces, and so this applies to projective varieties.

Prop. 6: X a variety, $U = \text{Spec}(R)$ and $V = \text{Spec}(S)$, then $U \cap V$ has coordinate ring $R \cdot S$ with the composite in the function field. $K(X)$,

p. 86: Local set-theoretic complete intersection: If Z has codimension r there is an open set U of any given $z \in Z$ so that there are r functions in the coordinate ring that set-theoretically describes the set Z around z.

p. 89: Suppose a morphism $\mathbb{P}^n \xrightarrow{f} \mathbb{P}^m$ with closed image (§9 says this is always true), then the image is a point or has dimension n. Reason: If the image has dimension less than n, then we can find n hypersurfaces $H_1, \ldots, H_n$ in $\mathbb{P}^m$ defined by homogeneous polynomials so that the intersection of these with the image is trivial. Pull these back to the $\mathbb{P}^n$ to conclude you have n hyperplanes in $\mathbb{P}^n$ that intersect trivially.

Equations may not give us what we expect, so it is necessary to have simple criteria to assure that what is essential from some particular type of equation compactification does not depend on the particular choice of compactification.

8.2. Completion of the fundamental group.

8.3. Functions on the universal covering space. We follow the treatment of [**BL92**, Appendix].

Let X be a complex manifold, $\tilde{X}$ its universal covering space. Then, $\pi_1(X)$ acts on $\tilde{X}$ on the left and on $\mathbb{C}(\tilde{X})$ on the right. Identify $H^1(\pi_1(X), \mathbb{C}(\tilde{X})^*)$ with factors of automorphy. For $f \in H^1(\pi_1(X), \mathbb{C}(\tilde{X})^*)$, $f(g_1 g_2) = (f(g_1))g_2 f(g_2)$.

LEMMA 8.1. *Each element of $H^1(\pi_1(X), \mathbb{C}(\tilde{X}))$ defines a line bundle on X.*

PROOF. Note: $f \in H^1(\pi_1(X)), \mathbb{C}(\tilde{X})^*)$ determines a function h from $\pi_1(X) \times \tilde{X} \to \mathbb{C}^*$ by $h(g, x) = f(g)(x)$, and $h(g_1 g_2, x) = h(g_1, g_2(x))h(g_2, x)$. Define an action of $\pi_1(X)$ on $\tilde{X} \times \mathbb{C} by g(x,t) = (g(x), f(g(x))t)$. The action is free because $f(g(x))$ is nonvanishing. Also, the quotient of this action is a bundle $\mathcal{L}_f$.

To obtain a trivialization of $\mathcal{L}_f$, choose an open covering $\mathcal{U}_{\alpha \in I}$ such that on each open U of the set, the universal cover has the form $U \times \pi_1(X)$. Choose a connected W_α in $\pi^{-1}(U_\alpha)$ respecting the naturalprojection $\pi : \tilde{X} \to X$. For W_1 covering U_1 and W_2 covering U_2, there exists $g_{1,2} \in \pi_1(X)$ such that $g_{1,2}$ takes the point $w(x)$ of W_1 over x to a point of W_2 over $x \in U_1 \cap U_2$. We take the transition functions as $s_{1,2}(x) = f(g_{1,2})w_2$. Check the cocycle condition holds. □

8.4. Some comparisons with [Har77] and [Mu66]. [**Spr57**] has no standard notation for transition functions on a complex manifold. Apparently such a standard notation, such as that for transition functions $\psi_{\beta,\alpha} = \varphi_\beta \circ \varphi_\alpha^{-1}$ (as in Lem. 2.2) seems to have awaited the sheaf formulation of manifolds. Early places where US students could see this notation include [**Gun66**, p. 15]. As in Def. 3.6 we tend to record our analytic or meromorphic functions on U_α as those on $\varphi_\alpha(U_\alpha)$. What, however, is the precise ring of analytic functions on the overlap $U_\alpha \cap U_\beta$, of two coordinate charts $(U_\alpha, \varphi_\alpha)$. Using the ring $\mathcal{H}_\mathcal{U}(U_\alpha) = \{f \circ \varphi_\alpha \mid f \in \mathcal{H}(\varphi_\alpha(U_\alpha))\}$ exactly solves this problem of having a ring directly defined on the open set U_α. For each open set $U \subset X$ we define the analytic functions on U by sying they are functions on U whose restriction to each $U \cap U_\alpha$ is the pullback by φ_α of a function analytic on $\varphi_\alpha(U \cap U_\alpha)$. This gives us the essential properties of the *sheaf of holomorphic functions*, $\mathcal{H}_X$.

For example, in [**Mu66**, p. 33] the maps defining a presheaf for a chain of open sets $U_1 \subset U_2 \subset U_3$ are called *restriction* and the cocycle condition $\psi_{\gamma,\beta} \circ \psi_{\beta,\alpha} = \psi_{\gamma,\alpha}$ for transition functions takes the form $\mathrm{rest} U_2, U_1 \circ \mathrm{rest} U_3, U_2 = \mathrm{rest} U_3, U_1$. That is, restriction is from the bigger set to the smaller, and the direction is said this way in the subscriptions. This is at the ring level. At the point level it goes the other way.

8.5. The lemmas of Noether and Chow. In response to your questions about $U_r = \mathbb{P}^r \setminus D_r$, here are the relevant answers. Include the definition of normal variety.

PART A. D_r is an algebraic variety: There are two ways to see that. First way: There is an equation that describes it. The start of that equation comes from its basic description.

Write $\mathbb{P}^r$ as the collection of nonzero polynomials (up to multiplication by a nonzero constant) of degree at most r. If a polynomial has degree less than r, regard ∞ as being a zero of it. Then, D_r is the locus of polynomials with two or more zeros that are equal. If you write the condition for a polynomial $f(z)$ to have repeated roots, it is that the gcd of f and f' have a common factor. The Euclidean algorithm allows you to write the gcd of f and f' as $hf + gf'$ for some relatively prime polynomials h and g. Finally, the condition that there are such relatively prime h and g giving a nontrivial linear combination can be written as a condition on a matrix (formed from the coefficients of f and f') having a nonzero determinant. So, D_r is a polynomial equation in the coefficients of f formed from setting a determinant expression to 0.

Here is how that relates to Chap. 3 and Chap. 4. In Chap. 3 I talk about the concept of P^1-algebraic (Def. 3.10). I've always wanted to put that discussion in a book, for $\mathbb{P}^1$-algebraic is a naive definition of algebraic that is close to being algebraic but not quite. The definition – which you would never have seen before – is equivalent to a compact manifold having an embedding in $(\mathbb{P}^1)^N$ for some N. The definition of algebraic is that there is an embedding in $\mathbb{P}^N$ and §4.1.2 discusses the difference between $\mathbb{P}^1$-algebraic and being algebraic. I've left the full motivation – as I say in §4.1.2 – for the definition of algebraic for Chap. 4. See in Part C why I wait until Chap. 4 to do this. Also, I mention a second more abstract reason why D_r is algebraic there.

PART B. The complex structure on U_r is from the complex structure on $\mathbb{P}^r$ (§4.2.2). An open subset of a complex manifold is a complex manifold. PART C. Your question "A big question we have is how Hurwitz spaces are algebraic varieties."

ANSWER: How about an easier question? Suppose you have a cover X of $\mathbb{P}^1\setminus\boldsymbol{z} = U_{\boldsymbol{z}}$ (Riemann sphere minus a finite set of points). Why is X algebraic? Here is one answer: Because you can compactify X in a unique way to a compact Riemann surface (this is at the beginning of Chap. 4), and all compact Riemann surfaces are algebraic. Better yet, you can see its algebraic structure by relating it to the algebraic structure of $\mathbb{P}^1$. All of this is Riemann's Existence Theorem, the topic of Chap. 4. This is not an easy theorem. Understanding the significance and how to use this are the main topics of the book. The question you are asking is a generalization of this: Any cover of $\mathbb{P}^r$ minus an algebraic subset is an open subset of an algebraic variety. The result is due to Grauert and Remmert and it plays a big role in my first big paper on relating the Inverse Galois Problem to Hurwitz spaces in 1977. So, you must wait to Chap. 4 for a full discussion.

The simple relation between $(\mathbb{P}^1)^r$ and $\mathbb{P}^r$ is that the latter is the quotient by an S_r action of the former. At first I intended that to be in Chap. 3. Late in the game I saw it made a more coherent discussion to be in Chap. 4. Once, however, you know that, then you have that D_r is the quotient by of the fat diagonal on $(\mathbb{P}^1)^r$. A valuable lemma is this. Let X be an algebraic variety and let G be a finite group acting algebraically on X. Then X/G is also an algebraic variety. Indeed, showing the value of having this fact is what is my motivation to someone reading these chapters for expanding beyond $\mathbb{P}^1$-algebraic to the definition involving $\mathbb{P}^N$.

Additional Point: Chow's Lemma – which I'm sure Mirroslav will do in his class – says that any complex analytic subset of an algebraic variety is also algebraic. This is NOT deep, though conceptually very valuable.

8.6. The Branch Cycle Lemma. If a cover $\varphi : X \to \mathbb{P}^1_z$ in this family has definition field K, then the Galois closure of the cover has Galois (*arithmetic monodromy*) group a subgroup of $N_{S_n}(G, \mathbf{C})$.

8.6.1. *BRANCH CYCLE SETUP.*

QUESTION 8.2 ($(G, \mathbf{C})$-cover?). Quest. A: Does $\boldsymbol{z}$ and $\varphi_{\boldsymbol{g}} : X_{\boldsymbol{g}} \to \mathbb{P}^1_z$ exist with branch points $\boldsymbol{z}$, $\langle \boldsymbol{g} \rangle = G$ and $\boldsymbol{g} \in \mathbf{C}$ ($\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C})$) over $\mathbb{Q}$?

Quest. B: A $(G, \mathbf{C})$-Galois cover over $\mathbb{Q}$?

Q. A or B requires $\boldsymbol{z}$ is a $\mathbb{Q}$ set. For $z_0 \in \mathbb{P}^1_z(\mathbb{Q})$: $\sigma \in G_{\mathbb{Q}}$ acts on $\gamma \in \pi_1(U_{\boldsymbol{z}}, z_0)$ through what the result does to $f \in \mathcal{E}(U_{\boldsymbol{z}}, z_0)^{\mathrm{alg}}$:

$$f \mapsto f_{\sigma^{-1} \circ \gamma \circ \sigma} = f_{\gamma^\sigma}.$$

Extend $\pi_1(U_{\boldsymbol{z}'}, z_0) \to G$ to $\psi_{\boldsymbol{z}', z_0} : \pi_1(U_{\boldsymbol{z}'}, z_0)^{\mathrm{alg}} \to G$. As a profinite group, $\pi_1(U_{\boldsymbol{z}'}, z_0)^{\mathrm{alg}}$ is also free on r (topological) generators modulo a product-one relation.

Notation: $\sigma \in G_K$ maps to $n_\sigma \in \hat{\mathbb{Z}}^* = G(\mathbb{Q}^{\mathrm{cyc}}/\mathbb{Q})$. For any $K \le \mathbb{C}$ (like $\mathbb{R}$ or $\mathbb{Q}_p$) consider

$$\pi_1(U_{\boldsymbol{z}}, z_0)^{\mathrm{ar}} \stackrel{\mathrm{def}}{=} \pi_1(U_{\boldsymbol{z}}, z_0) \times^s G_K.$$

BRANCH CYCLE ARGUMENT: [**Fri73**] or [**Fri77**]

LEMMA 8.3 (Branch Cycle Argument). *For Q. B: ψ must extend to a homomorphism $\psi : \pi_1(U_{\boldsymbol{z},z_0})^{\mathrm{ar}} \to G$.*

Find $\pi \in S_r$ to satisfy $z_i^\sigma = z_{(i)\pi}$. Then, affirmative to Q. B requires

$$C_{(i)\pi}^{n_\sigma} = C_i,\ i = 1, \dots, r. \tag{8.1}$$

Affirmative for Q. A requires a choice of Galois closure group $\hat{G}$ between G and $N_{S_n}(G, \mathbf{C})$. For each such $\hat{G}$, replace (8.1) *by*

$$g_\sigma C_{(i)\pi}^{n_\sigma} g_\sigma^{-1} = C_i,\ i = 1, \dots, r,\ \textit{for some } g_\sigma \in \hat{G}.$$

Let $G = A_5$, C_5^+ the class of $(1\,2\,3\,4\,5)$, C_5^- the class of $(1\,3\,5\,2\,4)$, C_3 the class of 3-cycles. Example four branch point covers:

(8.1a) $\mathbf{C}_{5_+^2 3^2}$: No for Q. B, yes for Q. A.
(8.1b) $\mathbf{C}_{5_+ 5_- 3^2}$: Yes for Q. A and B.
(8.1c) $\mathbf{C}_{5_+^2 5_-^2}$: Yes for Q. A and B.

8.6.2. *Field of moduli.* Ingredients for a *field of moduli* (over $\mathbb{Q}$):

- A collection of algebraic objects $\mathcal{P}$ over $\bar{\mathbb{Q}}$ invariant under $G_\mathbb{Q}$.
- A $G_\mathbb{Q}$ equivalence relation $\mathcal{E}$ on $\mathcal{P}$.

For $\boldsymbol{p} \in \mathcal{P}$ let $\mathcal{E}_{\boldsymbol{p}}$ be its equivalence class: $H_{\boldsymbol{p}}^{\mathcal{E}}$ is the subgroup of $G_\mathbb{Q}$ stabilizing $\mathcal{E}_{\boldsymbol{p}}$; $K_{\boldsymbol{p}}^{\mathcal{E}}$ is the fixed field of $H_{\boldsymbol{p}}^{\mathcal{E}}$.

FACT 8.4. *For each $\boldsymbol{p}' \in \mathcal{E}_{\boldsymbol{p}}$, any field of definition of $\boldsymbol{p}'$ contains $K_{\boldsymbol{p}}^{\mathcal{E}}$.*

From a Nielsen class $\mathrm{Ni}(G, \mathbf{C}, T)^{\mathrm{abs}}$: $\mathbf{C}$ a rational union of classes and $\boldsymbol{z} \in \mathbb{P}^r \setminus D_r(\mathbb{Q})$.

(8.1a) $\mathcal{E}_{\boldsymbol{z}}^{\mathrm{abs}}(G, \mathbf{C}, T)$: s-classes in $\mathrm{Ni}(G, \mathbf{C}, T)$ with branch points $\boldsymbol{z}$.
(8.1b) Same as (a) except $\mathcal{E}_{\boldsymbol{z}}^{\mathrm{abs,rd}}(G, \mathbf{C}, T)$ is w-equivalence classes.
(8.1c) As in (b) except drop T; replace $^{\mathrm{abs}}$ by $^{\mathrm{in}}$.
(8.1d) As in the previous, except drop $\boldsymbol{z}$.

COURSE MODULI:

QUESTION 8.5. For an equivalence class $\mathcal{E}$, and $\boldsymbol{p} \in \mathcal{P}$ how to compute $K_{\boldsymbol{p}}^{\mathcal{E}}$? Does $\mathcal{E}_{\boldsymbol{p}}$ contain something over $K_{\boldsymbol{p}}^{\mathcal{E}}$? What is the lattice $\mathcal{L}_{\boldsymbol{p}}/K_{\boldsymbol{p}}^{\mathcal{E}}$ of definition fields for elements of $\mathcal{E}_{\boldsymbol{p}}$?

Most algebraically defined equivalence classes, including those defined by Nielsen classes have a reasonable (course) *moduli space* $\mathcal{H}_{\mathcal{E}}$.

FACT 8.6. *In the covering space equivalences, $\mathcal{H}_{\mathcal{E}}$ is an affine algebraic variety with a well-defined field of definition — as a moduli space. If $\mathbf{C}$ is a rational union it is $\mathbb{Q}$.*

PROPOSITION 8.7. *The Branch Cycle Lemma gives the correct field of definition for moduli spaces defined by Nielsen classes.*

Meaning: For any family of covers in this equivalence class, you can define a natural map to the moduli space so the space is locally — for the étale topology — the pullback of a family over the given space.

8.6.3. *How Hurwitz spaces arise.* Given $\varphi : X \to \mathbb{P}^1$, branch points $\boldsymbol{z}_0$, from RET: What if you wanted equations for it?

- Would you regard $\boldsymbol{z}$ as variable?
- Express equations in $\boldsymbol{z} \in U_r = \mathcal{P}^r \setminus D_r$?

Why concentrate on $\boldsymbol{z}_0$? Want $\varphi_{\boldsymbol{z}}$ for *all* $\boldsymbol{z}$. If possible, then analytically continuing $\varphi_{\boldsymbol{z}}$ around $\mathcal{P} \in \pi_1(U_r, \boldsymbol{z}_0)$ returns to $\varphi_{\boldsymbol{z}_0}$.

Homotopy class of $\mathcal{P}$ is $Q_{\mathcal{P}} \in H_r$: Hurwitz monodromy group. At end of $\mathcal{P}$ the cover has branch cycle description $(\boldsymbol{g})Q_{\mathcal{P}}$. Compute with starting classical generators of $\pi_1(U_{\boldsymbol{z}_0})$. So, $\varphi_{\boldsymbol{z}}$ valid for all $\boldsymbol{z}$ requires $(\boldsymbol{g})Q_{\mathcal{P}}$ be $\boldsymbol{g}$ (modulo conjugation by G or closely related). Check: Is $(\boldsymbol{g})Q$ essentially $\boldsymbol{g}$ for all $Q \in H_r$.

EXAMPLE 8.8. Consider $\boldsymbol{g} =$

$$((1\,2\,3), (3\,2\,1), (1\,4\,5), (1\,5\,4)) \in \mathrm{Ni}(A_5, \mathbf{C}_{3^4}).$$

My next talk computes $(\boldsymbol{g})Q_{\mathcal{P}}$.

Find: $\varphi_{\boldsymbol{z}}$ coefficients have coordinates for nontrivial U_r cover. What cover?

Explain why nonsingular conics in $\mathbb{P}^n$ are isomorphic to $\mathbb{P}^n$ over $\mathbb{C}$ (use Lem. 4.13), but not over $\mathbb{Q}$.

9. Using algebraic coordinates and higher monodromy

The complements are topics where we are incomplete with corroborating proofs, giving, however, enough to use them in Chap. 5.

9.1. Complements on algebraic coordinates.

9.2. Fundamental groups from branch cycles and higher monodromy.

9.2.1. *Computing the fundamental group from branch cycles.* Suppose $\varphi : X \to \mathbb{P}^1_z$ has X of genus 0 and $\boldsymbol{z}$ as branch points. Then §**??** has a procedure for computing classical generators for $X \setminus \{\varphi^{-1}(\boldsymbol{z})\}$ from those for $U_{\boldsymbol{z}}$. In particular, given $Y \to X \to \mathbb{P}^1_z$, this gives a uniform procedure for computing branch cycles for the cover $Y \to X$, and thus expresses these branch cycles from those for φ. A fairly simple procedure allows computing

9.2.2. *Action on the fundamental group and homology of a fiber.*

9.2.3. *Action on periods of integrals.*

9.3. Flat bundles and complete reducibility. Check out of [**Gr70**, §3 and §4] on complete reducibility in the action on flat bundles.

9.3.1. *The genus 0 problem.* Suppose $f(w)$ is rational function in w. It maps points on the w-sphere to the z-sphere. The Galois closure group of the splitting field of $f(w) - z$ over $\mathbb{C}(z)$ (monodromy group of f) is special. That is the gist of the genus 0 problem over $\mathbb{C}$. (The same qualitative statement holds for any fixed genus.) Guralnick and Thompson's original version is this. With finitely many exceptions, the simple composition factors of the monodromy group of such a map must be alternating or cyclic groups. The solution of this left three big problems.

(9.1a) What are the precise monodromy groups, with finitely many exceptions, of *indecomposable* rational functions? Guralnick's 0-Conjecture: We only get alternating groups, symmetric groups, cyclic and dihedral groups. These should come only with special degree representations.

(9.1b) What groups must one add for rational functions over fields of positive characteristic? Guralnick's p-Conjecture: In characteristic p add Chevalley groups over extensions of $\mathbb{F}_p$ to alternating and cyclic groups.

(9.1c) Mumford's Question: What function fields in one variable over $\mathbb{C}$ have uniformizations by the Galois closure field of a rational map?

[**Fri99**] discusses all of these and the history from the Davenport Problem motivations to the complete resolution of the genus 0 problem. Further, Davenport's problem in positive characteristic corroborates Guralnick's inspired p-conjecture. So does the voluminous work of Abhyankar toward his exponent mantra for producing Chevalley groups over $\mathbb{F}_p$ from genus 0 covers. A gem from 1995 is Müller's listing of the monodromy groups of polynomials [**Mül95**].

Like the genus 0 problem, Mumford's question has several forms. For example: Any curve defined by a separated variables equation $f(w) - g(u)$ would have its function field in the composite of splitting fields over $\mathbb{C}(z)$ of functions $f(w)-z$ and $g(u)-z$. That includes all hyperelliptic curves. Directly, the description of modular curves as moduli of genus 0 curves [**Fri78**] produces elliptic curves from systems of rational function Galois closures, no composite required. Mumford's question has no representation in this volume. It remains untouched in that no function field has been excluded from the genus 0 closure field.

9.4. Unramified Frattini covers.

9.4.1. *Projectives in the category of profinite group covers.* Projectives in the category of profinite group covers of a given group, and here projectives that are Frattini exist. Considered covers with kernel pro-p and kernel elementary p. $\mathcal{C}_{p^\infty}(G)$, $\mathcal{C}_p(G)$. Other category $K[G]$ modules $\mathcal{C}_{k[G]}(M)$ covering M. Porjective profinite objects exist and are unique up to isomorphism. The name for the last is $\mathbb{P}(M)$. Frattini subgroup corresponds to radical of M.

$\hat{F}_2(2)\times^s\mathbb{Z}/3$ has an embedding of $F_2\times^s\mathbb{Z}/3$ in it, though this will not always be the case. Actually the character of the abelian quotient of ${}_2\tilde{A}_5$ is not $\mathbb{Q}-rational$, so this doesn't have a corresponding lattice. Holt and PLeskin:

Gruenberg 1970s: $\mathcal{C}(G) \equiv \mathcal{C}_{\mathbb{F}_p[G]}(\omega)$ with ω Aug. ideal. $\mathrm{Ext}^1(\omega, M) = \mathrm{Ext}^2(\mathbf{1}, M) = H^2(G, M)$. $M \to N \to \omega$ is equivalent to $0 \to M \to H \to G \to 1$. It works with longer sequences too.

Corollary 9.1 (Gaschutz). $M_0 = \Omega^2(\mathbb{F}_p)$.

Draw the conclusion that if $\dim(M_n) > 1$, then the exact trivial action part of G_n (in G_0) is the $O_{p'}(G)$. To see the nonobvious direction, restriction to a p-element g in this action, and conclude that g would be acting trivially on a projective.

If $\dim(M_n) > 1$, $Z(G_n) \cap O_{p'}(G) = 1$, G_n is p-perfect $\implies Z(G_{n+1}) = 1$, and G_{n+1} is p-perfect and $\dim(M_{n+1}) > 1$.

Proposition 9.2 (Griess-Schmid). $\dim(M_n) = 1$ *if and only if G is p-super solvablle with cyclic p-Sylow.*

9.5. Equations, coordinates and cryptography. Sometimes the intense studies that engage mathematicians hide the forest behind the trees.

Much of physical science started with the study of relations between theoretically measurable quantities. The applications came because some of these quantities were practically measurable or had in-hand control (like the force and angle of a rocket launch) while the other was more theoretical (like the final destination of the rocket). Further, there are so many physical systems and so many possible trajectories of different characteristic, that once the idea of analytic coordinates was available, it was inescapable to ask when one system could be transformed to another and how one would test for that. Any one of the following topics has characteristics that resonate into an area of investigation.

(9.2a) Deciding if there are substitutions of variables that change one integral into another.
(9.2b) Finding normal forms for collections of trajectories of a dynamical system.
(9.2c) Creating large isolated systems of interacting particles that emulate logical computer functions.
(9.2d) Finding uniformization situations that allow effectively encoding data in special ways in the set of solutions of equations.

10. A piece of the historical record

We start with some historical comments, first personal then some gleaned from [**Ne81**] and other sources.

10.1. The career view. When I was a student 1964–1967, I roamed the library at University of Michigan, gleaning from a great store of mathematics, the topics that interested me greatly. These, of course, would have inundated me in my quest to get done, and get done quickly, despite my background as an electrical engineer (who worked for three years in aerospace companies). While my primary interest at the time was in algebraic number theory, there was a quickening surge when I saw the papers of Abel and Galois in the beginning volumes of Crelle's journal. What I immediately saw was that Galois' historical legend was of a different nature than that supported by those of his works Crelle published.

Books on the algebraic theory of curves didn't appeal to me much, for they tended to handle one curve at a time, in intricate detail. There wasn't much at the time on Riemann surfaces in English, except [**Spr57**]. Further, I saw that the only way to escape the excessive reliance on special forms was through moduli. Moduli formulations of problems were instituted in the early 1960's by Grothendieck and his school. In few 21st century libraries can a student roam through the beginning volumes of Crelle's journal. The difference between being able to order them upon demand and having them there for the curious roaming student (includes curious faculty) is akin to the difference between having snow in California's mountains when you want it, and having God bring it to your home.

In looking back I have the memory that few other topics interested me. Partly that was to prevent the domination of the library volumes calling "Read me, read me!" Those books were real to me, and the urgency of all that mathematics was an insistent plea, that called for my resistence. I let my intuition guide me. So, on certain topics very important to the last fifteen years of my work, I learned little at all, sometimes thinking – as with profinite groups – the subjects weren't that tough. Other topics, related to Lie algebras, differential geometry and partial differential equations seemed to have hordes already committed to them. Even if I avoided their study, others assured they weren't neglected.

These personal comments clarify —though I was commonly praised for my openness to all kinds of mathematics —that my education went through a many year process before my desire to see, and add to, the connectedness between topics grew. As, however, that happened, the resources for feeing that desire were being removed from the libraries in the late seventies on. Further, the tremendous growth of mathematics was leading mathematicians — and all other academicians — into a personal contest whose criterion for success was dominated by a count of numbers of papers.

Suppose for example, one mathematician in her/his career writes 70 papers, each of roughly 40 pages. Contrast that with another who writes 560 papers, each on average five pages. Without even looking at the papers, my vision of the first writer is that s/he built upon projects that defined underappreciated portions of extant areas, or even created new additions to these. I would expect the ratio of theorems to definitions and supporting examples to be slightly smaller in the former, though I would also expect much less starting and stopping from paper to paper to present a common setting. Two other points I would expect, without seeing the papers. The journals, departments and peer reviews of the former career would have many more hesitancies about the evaluation of that career; while the mathematical community would have more focused comments on the nature of that career. It seems significant, however, as to whether these two alternative careers are conscious decisions or merely manifest of the talent and organizational contingencies faced by the mathematician. Are they what you can do, or what you would do?

10.2. Influences on Riemann. We list some significant events in the theory of complex variables.

- A.-L. Cauchy 1789–1857: By 1825 had command of the definite integral between complex limits and presented the Cauchy Integral Theorem.
- P.A. Laurent (1813–1854): In 1843 discovered the Laurent expansion of an analytic function in the deleted neighborhood of an isolated singularity.
- J. Liouville (1809–1882): Formulated many theorems in the theory of elliptic functions.
- V. Puiseux (1820–1883): Investigated analytic continuation in studies of the behaviour of algebraic functions in the neighborhood of one of their branch points.
- C.A. Briot (1817–1882) and J.C. Bouquet (1819–1885): Assembled the previous topics in a series of articles in 1856, bringing them together in the influential [**BB1856**].

[**Ne81**] says it took from 1814 to 1846 to expand a special case of Cauchy's Theorem to integration over a general closed path. Cauchy didn't recognize the significance of the Cauchy-Riemann equation until 1851. Like Weiertrass, Ch. Méray (1835–1911, a Briot and Bouquet disciple), emphasized that continuity is insufficient. As expected, he emphasized the need for a theory based on Taylor series.

B. Riemann (1826–1866) from his thesis 1851 and his 1857 articles on abelian functions, used the Cauchy-Riemann equations exclusively. He basing many of his proofs on potential theory. [**Ne81**, p. 89]: It was Gauss' (1777–1855) writings that the young Riemann studied wth special zeal. From these he drew signficiant inspirations for his [doctoral] thesis. He wrote his father how he found these papers. What he especially appreciated was Gauss' contributions to conformal mapping using essentially a Dirichlet principle.

According to Betti, Riemann said he got the idea of *cuts* from conversations with Gauss (1777–1855; see §2.4) [**Ne81**, p. 90]. Letters of Klein and Schering attest to Gauss' influence on Riemann's theory of hypergeometric series. This influence came partly from Gauss' papers. Still, it is striking to consider the over 70 year old Gauss sketching plans for such an etherial construction to the very young Riemann.

During his time in Berlin (1847–1849) P.G. Dirichlet (1805–1859), G. Eisenstein (1823–1852) and C.G.J. Jacobi (1804–1851) especially influenced Riemann. He attended Dirichlet's lectures on partial differential equations, and Eisenstein and Jacobi lectures on elliptic integrals. Riemann read Cauchy and Legendre on elliptic functions. [**Ne81**, p. 91]:

> Riemann was suitable, as no other German mathematician then was, to effect the first synthesis of the "French" and "German" approaches in general complex function theory.

His introductory lectures started with these topics: the Cauchy integral formulae; operations on infinite series; the Laurent series; and analytic continuation by power series. [**Ne81**, p. 92] includes a photocopy of a famous picture on analytic continuation from Riemann's own hand. Picard and Lefschetz both used this picture (from Riemann's collected works) in autobiographies of what influenced critical theorems of theirs. Riemann also lectured on the argument principle, the product represention of an entire fuction with arbitrarily prescribed zeros and the evaluation of definite integrals by residues. His most advanced lectures were from his published papers solving the Jacobi inversion problem (§**??**).

10.2.1. *Competition between Riemann and Weierstrass.* [**Ne81**, p. 93]: K. Weierstrass (1815–1897) himself stressed above all the great influence of N.H. Abel (1802–1829) on him. At first Weierstrass was an unknown. Only after his 1856 paper on abelian functions did he get his position in Berlin. It was in 1856 that the competition between Riemann and Weierstrass became intense, around the solution of the Jacobi Inversion problem.

[**Ne81**, p. 93]: May 18 and July 2, 1857, Riemann submitted his two part solution to Jacobi's general inversion problem with these carefully measured words:

> Jacobi's inversion problem, which is settled here, has already been solved for the hyperelliptic integrals in several ways through the persistent and regally successful work of Weierstrass, of which a survey has been communicated in Vol. 47 of the Journ für Math. (p. 289). Until now, however, only a part of these investigations has been fully worked out and published (vol. 52, p. 285), namely the part that was outlined in §1 and §2 of the earlier paper and in the first half of §3, pertaining to elliptic functions. Only after the full publication of the promised paper shall we be able to tell to what extent the later parts of the presentation agree with my article not only in results but also in the methods leading to them.

Weierstrass consequently withdrew the 3rd installment of his investigations, which he had in the meantime finished and submitted to the Berlin Academy. He explained this (much later) in his collected works as follows.

> I withdrew [the 1857 manuscript] for, a few weeks later, Riemann published an article on the same topic, [...] on entirely different foundations from mine and did not make immediately clear that it agreed completely with mine in its results. The proof

> for it entailed some investigations of chiefly an algebraic nature, whose execution was not altotether easy for me ... But after this difficulty was overcome it seemed to me that a thorough going overhaul of my paper was necessary. ... I could only toward the end of 1869 give to the solution of the general inversion problem that form in which I have treated it from then on in my lectures.

10.2.2. *Soon after Riemann died.* Publicly they seemed to have gotten along [**Ne81**, p. 95]. Professionally the mutual influence was unquestionably great. It would be entirely conceivable that the general systematic construction of the Weierstrassian function theory, achieved around 1860, could have been inspired by the works of Riemann perstaining to the same set of ideas.

[**Ne81**, p. 96]: After Riemann's death, Weierstrass attacked his methods quite often and even openly. July 14, 1870 was when he read his now famous critique on the Dirichlet Principle before the Royal Academy in Berlin. Weierstrass showed there did not always exist a function among those admitted [in variation problems] whose expression in question attained the lower bound, as Riemann had assumed. A letter to H. A. Schwarz on Oct. 3, 1875 says:

> The more I think about the principles of function theory, the firmer becomes my conviction this must be based on the foundation of algebraic truths, and that, consequently, it is not the right way if instead of building on simple and fundamental algebraic theorems, one appeals to the "transcendental" [by which Riemann has discovered so many of the most important properties of aglebraic functions].

During its heydey (1870–1890), the Weierstrassian school took over nearly every position in Germany. For instance, Schwarz was at Göttingen.

[**Ne81**, p. 98] asserts it was the Goursat part of Cauchy's theorem that renovated Riemann's approach, starting around 1900. [**Ahl79**, p. 111] with no precise citation, refers to Goursat's contribution as,

> This beautiful proof, which could hardly be simpler is due to É. Goursat, who discovered that the classical hypothesis of a continuous $f'(z)$ is redundant.

Curiously, there is precisely one reference [**Ahl79**, p. 121] in *all* of [**Ahl79**]. This is a footnote:

> Without use of integration R. L. Plunkett proved the continuity of the derivative (BAMS **65**, 1959). E. H. Connell and P. Porcelli proved the existence of all derivatives (BAMS **67**, 1961). Both proofs lean on a topological theorem due to G. T. Whyburn.

That unique quote suggests Ahlfors supports the significance of Goursat to Riemann's renovation. Yet, there is a complication in analyzing Neuenschwanden's thesis. Wow would one document that this event turned mathematicians to the geometric/analytic view of Riemann? Historically it seems sensible to investigate the span from [**AG1895**] to [**Wey55**] as a shift from genus 1 to higher genus. Yet, that period is clearly insufficient to deal with an aspect of the true shift, from moduli of genus 1 curves (including modular curves) to general moduli. Theories toward the latter include *Teichmüller theory* (analytic) and *geometric invariant theory* (algebraic) or expedient precursors of the Hurwitz space approach like the

Schiffer-Spencer deformation theories of varying the complex structure around a single point of a Riemann surface.

I suspect Goursat's theorem is a simple explanation that first year graduate students can follow. Likely, however, serious applications and resonant questions required understanding the variation of structures on a Riemann surfaces with the variation of the surface itself. My experiences are that not only do these issues confound graduate students, often specialists in complex variables struggle with these. Both technically and conceptually handling the hidden monodromy considerations (see Chap. 1 §5.4.3) is a tough topic. One practical approach to this topic, hidden in combinatorial actions of the braid group (et. al.) in this chapter, appears undiluted in Chap. 5. The only tool flexible enough to handle the complexity of the structure was that of Riemann. If that is right, then it is the documentation of these applications and questions that would illuminate on the story of the resurrection of Riemann's work. This makes it all look like slow continual progress. When, however, we come to Galois, the story has a different nature. We see it through modular curves which still to this day herald those works that accrue the most prestige.

10.3. The place of Galois. One thing is certain: Mathematicians often use Galois' name. By contrast, the most often told stories of the circumstances of his death appear unlike the essence of Galois.

I give the gist of what [**Rig96**] says about Galois' suicide. Galois, despondent from the suicide of his father, and the rejection of his papers by the Academy of Sciences, primarily from the negligence of Cauchy, committed a heroic suicide. She says: "offering his body against the politics of the Bourbon restoration." His rejection by Stéphanie Poterin-Dumotel exacerbated his despondency. She was the daughter of a doctor, Jean-Louis Poterin-Dumotel, who lived on the same street where Galois was transferred during a parole from prison for his major political escapade. She wasn't, in anyone's eyes, a "tart."

10.3.1. *Removing the ethereal from what happened.* [**Rig96**, p. 112] has the description of Galois' sacrifice, the morning of May 30, 1832. For shear detail, it takes your breath away. It's so solid about the climate of his sacrifice by comparison with the legend. I've never quite seen how most mathematician's credit the *dual* story as a romance. It lacks much of the drama of Rigatelli's analysis. Next, my own words try to capture the essense of about 50 pages from [**Rig96**].

While still on parole, Galois could have joined Auguste and Michel Chevalier in the Saint-Simonian community at Ménilmontant. Still, the rules imposed by Bezard and Enfantin, the leaders of the movement, would have taken away his independence. The larger picture requires some familiarity with court politics of the time. Marie-Corline, duchesse de Berry, had recently returned to France. (She didn't know Galois personally, yet we see she plays a real role in what Galois was about.) As the widow of Charles X's son, at the time of his assassination, she was expecting an heir. The boy, now 12, was living in exile in Prague, under the guidance of Cauchy. Yes, that is *our* Cauchy from §10.2. He was was demonstrating his devotion to the Bourbon dynasty. Galois offered himself as a hero sacrifice for the necessity of taking up dramatic arms for the republic (not the monarchy). Rigatelli poses that he arranged a dual with his friend L. D. (is that all we know of him?). A complicated piece in the tragedy, was that only his opponant's pistol would be loaded. He left several letters plausibly corroborating the dual [**Rig96**,

p. 109]. To accomplish certain aims his group, *Friends of the People*, needed only to spread the news the duel was actually a police ambush.

He did not tell Chevalier of his plot. Rather, he said only that the rejection of Stéphanie was devastating. Rigatelli emphasizes the skillful writing in these letters disguising the true situation. These letters gave rise to all the legends. They spoke with certainty of his death. Rigatteli suggests this was a sure sign of contrivance. The newspaper *Le Précurseur* actually told the story as it happened:

> At point blank range, each [of these friends] was given a pistol and fired. Only one of the pistols was loaded [**Rig96**, p. 113].

The following day, at midday, 3,000 people were present at the cemetary of Montparnasse. The plan was to attack the police, when the coffin lowered into the grave. While Plaignol and Pinel, leaders of the *Friends of the People*, were delivering the eulogy in honour of Galois, word passed that General Lamarque had died. They decided this second funeral would attract a much larger, more emotionally involved, crowd. A swift decision brought Galois' funeral to a hasty, silent end.

The National Academy rejected Galois' famous memoire, on his solvability criterion for construction by radicals had been rejected. It was only published 14 years later. A few mathematicians conceive it, beyond doubt, as the foundation of modern algebra. Many, however, do not. Reason: It is common to view it as sketching some general abstract idea of group and permutation representations. Reality, again, is emphatically more mathematically precise and problem oriented. Reality takes account of someone having to understand its contents.

10.3.2. *Group theory highlights in Galois' works.* [**Rig96**, p. 133] One of Galois' results was that primitive representations of solvable groups must have prime power degree. He saw that degree p (prime) equations have group of type $\mathbb{Z}/p \times^s (\mathbb{Z}/p)^*$ (the elementary semi-direct product). He thought to also do this for general primitive representations. He did this by considering the (Galois) field of order p^v and he looked at the roots as listed by the congruences mod$p$ in this field two subscripts equal if they are given by $k^{p^v} \equiv k \bmod p$. So, he hoped to show the roots were permuted to take x_k to $x_{(ak+b)^{p^r}}$ with $a^{p^v-1} \equiv 1$, $b^{p^v} \equiv b \bmod p$ and r an integer. He was trying to say that if you take the roots to be $x_{i_1,i_2,\dots,i_v}$ (vector space designation over the field $\mathbb{Z}/p$ of dimension v), then the group is in affine transformations augmented by the Frobenius. Rigatelli says *if and only if*, though certainly these are not usually solvable. She says Galois realized this later.

10.3.3. *Solvability criterion and Modular curves.* [**Rig96**][p. 137]: The third memoire is in the letter to Chevalier. In this he starts by considering integrals of the three kind. He understood (from having read Legendre and Abel —as he says in his papers) that if g is the number of integrals of 1st kind then the number of periods is $2g$ [the number of global holomorphic differentials is half the rank of the first homology: §6.5]. He was considering the monodromy on the periods, though Rigatelli does not note this. The equation giving the division of periods into p equal parts has degree $p^{2n}-1$, and its group is $\mathrm{GL}_{p^{2n}}$. [**BAD62**, p. 162–165] starts with this quote from Galois' letter to Chevalier:

> La condition que j'ai indiquée dans le bulletin de Ferussac pour que l'equation soit soluble par radicaux ist trop restreinte. Il y a peu d'exceptions, mais il y en a.

My English translation: The condition that I have indicated in the bulletin of Ferussac for the equation to be solvable by radicals is too restrictive. There are few exceptions, but there are some.

After explaining the idea of primitive equations he says the following (to Chevalier). We may, however, thank historians for struggling for us with the meaning of difficult language translation combined with archaic mathematical formulations.

(10.1a) For an equation of prime degree to be solvable, it is necessary and sufficient that from any two known roots, the others are rational functions of them.

(10.1b) If an equation of degree m is solvable by radicals, then m is a prime power.

(10.1c) Further, in the case of prime-power degree, the equation is solvable if any two roots rationally give the others.

Rule (10.1c) overlooks the particular cases $m = 9$ and 25, $m = 4$ and generally that where u^t is a divisor of $\frac{p^v-1}{p-1}$ with u prime and $\frac{(p^v-1)v}{u^t(p-1)} \equiv p \bmod u^t$. Galois asserts in his letter that this case, nevertheless, deviates very little from the general rule. It must always be that with two of the roots known, the others are deduced from them, by means of a number of radicals of degree p, equal to the number of divisors of the type u^t satisfying the equations above. Galois says all these results come from his theory of permutations.

Finally, he says, let k by the modulus of an elliptic function, $p > 3$ a prime. In order that the equation of degree $p+1$ that gives the diverse modules of functions transformed relatively to the prime number p, be solvable by radicals, it is necessary from two choices, either one of the roots is rationally known, or each is a rational functions from any other. It does not matter here, of course, what are the particular values of the modulus k. It is evident that this does not hold in general. He also says it is remarkable that the general modular equation of degree 6, corresponding to the prime 5, may be reduced to a fifth degree equation. This does not hold for any higher degree modular equations.

11. Exercises

11.1. Topology of covers. Let $\varphi : X \to Y$ be an unramified cover.

(11.1a) Suppose Y has a countable basis for its topology. Show the same holds for X: X is *second countable*.

(11.1b) Now assume Y is a second countable Riemann surface and X is also a Riemann surface where φ a ramified cover. Show X is second countable. Show X is compact if and only it has the limit point property: An infinite sequence has a convergent subsequence.

The following shows a strong equivalence $\psi : Y^1 \to Y^2$ between two finite ramified covers (Y^i, ψ^i) of X (as in §3.2.2), $i = 1, 2$, is automatically an analytic isomorphism.

(11.2a) Restrict both covers to $X \setminus D(\psi')$: show that $\psi : Y^1_{X\setminus D(\psi^1)} \to Y^2_{X\setminus D(\psi^2)}$ is analytic.

(11.2b) Use Riemann's removable singularities theorem [**Ahl79**, p. 129] to extend the map of a) to an analytic map including the discrete set $Y^1_{D(\psi^1)}$.

We now investigate the Jordan curve theorem.

(11.3a) Let γ' be a simple closed polygonal path on $\mathbb{P}^1 = \mathbb{P}^1_z$ and let U be a connected component of $\mathbb{P}^1 \setminus \gamma'$. Show the points of γ' in the boundary of U are both open and closed. Let z_0 be in the range of γ'. Thus show that

$\mathbb{P}^1 \setminus \gamma'$ has at most two connected components, told apart as the points that connect in a neighborhood on the left of z_0 versus those that connect to a neighborhood on the right of z_0. The component connected to ∞ consists of points that relative to γ have winding number 0. So, it suffices to show there is a point of $\mathbb{P}^1_z \setminus \gamma'$ with nonzero winding number relative to γ'. See [**Ahl79**, p. 118].

(11.3b) Let γ' be a simple simplicial closed path in $\mathbb{P}^1$ and let W be the interior of γ'. Let γ'' be a simple simplicial path that meets γ' only at the end points, x_0 and x_1, of γ'. Note that $\gamma' \setminus \{x_0, x_1\}$ consists of two connected components. Conclude from the Jordan curve theorem that each component together with γ'' defines a simple closed curve whose interior consists of one of the two components of $W \setminus \gamma''$. Each component consists of the points path connected to any given point of the component.

Relate back to Chap. 2 [9.17] to complete a discussion of orientability of complex manifolds. Also Def. 2.21.

(11.4a) Use the complex structure to get the orientation from the sign of the expression $dx \wedge dy$. That works by substituting variables along the paths. Show that if $Y \to X$ is a finite ramified cover, an orientation of X gives an orientation of Y, and if X is a complex manifold, so is Y.

(11.4b) Show $d(f(z)dz) = 0$.

The *Brouwer separation theorem* [**?**, p. 11-21] states that a compact differentiable 2-dimensional manifold M in $\mathbb{R}^3$ *separates*. That is, $\mathbb{R}^3 \setminus M$ consists of two components, an *inside* that is bounded, and an *outside* that is unbounded.

(11.5a) Use the separation theorem to show that there is a continuously varying vector $\boldsymbol{w}_m$ of length 1 for each $m \in M$ such that the *dot product* $\boldsymbol{w}_m \cdot \boldsymbol{v}_m = 0$ for every vector $\boldsymbol{v}_m$ tangent to some path on M through m (i.e., $\boldsymbol{w}_m$ is *normal* to M at m; §1.c).

(11.5b) Use the notation of Chap. 3 §2.21. Show that a compact differentiable manifold in $\mathbb{R}^3$ is orientable. Hint: Restrict to coordinate charts that at each point have the RETURN

(11.5c) Orientation for higher dimensional manifolds.

(11.5d) Show that there is an unramified cover $\mathbb{P}^1 \to X$ of degree 2 for which X is not orientable. Conclude that such an X cannot be embedded in $\mathbb{R}^3$. Hint: $\mathbb{P}^1$ is homeomorphic to the 2-sphere, S^2, in $\mathbb{R}^3$. Make the set whose points consist of pairs of endpoints of diameters of S^2 into a manifold.

The next exercise series shows how a little combinatorics of triangles reveals that the Euler characteristic of a Riemann surface being genus 0 or 1 shows it is topologically a sphere or a torus. This continues the discussion of Rem. 2.19. Suppose two Riemann surfaces X_i, $i = 1, 2$, have triangulations have respective triangulations T_i, $i = 1, 2$. Suppose there is a numbering of the simplices, edges and vertices in both so that the numbering for one is exactly the same as the numbering for the other. Call the triangulations *equivalent*.

(11.6a) Suppose for the triangulation of X_c in Prop. 2.18 the Euler characteristic is 2 ($g_{\boldsymbol{g}} = 0$). Lay out a triangulation on the sphere equivalent to this triangulation. Conclude an Euler characteristic 2 implies the surface is homeomorphic to the sphere.

(11.6b) Do the same for concluding about X_c when its genus is 1 that it is homeomorphic to a torus.

11.2. Artin braids and Hurwitz monodromy. Notation is from Def. 1.1.

(11.7a) Show any group requiring at most $|S|$ generators is a quotient of $F(S)$.

(11.7b) Let $[G,G]$ denote the *commutator subgroup* of a group G: Elements $g_1g_2g_1^{-1}g_2^{-1}$ for all $g_1, g_2 \in G$ generate $[G,G]$. Let $A(S) = F(S)/[F(S), F(S)]$ (or A_n if $|S| = n$) be the *free abelian group* on S. Show any abelian group requiring at most $|S|$ generators is a quotient of $A(S)$.

(11.7c) Show $A(S) = A_n \cong \mathbb{Z}^n$ if $S = \{s_1, \dots, s_n\}$.

(11.7d) Let $F(S) = F_{2r}$ and let $\bar{R}(S)$ be the normal subgroup generated by

$$s_1 s_{r+1} s_1^{-1} s_{r+1}^{-1} s_2 s_{r+2} s_2^{-1} s_{r+2}^{-1} \cdots s_r s_{2r} s_r^{-1} s_{2r}^{-1}.$$

Show that $G = F_{2r}/\bar{R}(S)$ is *not* a free group. Hint: $G/[G,G] = A_{2r}$.

Refer to the properties in Prop. 3.3.

(11.8a) other automorphisms of F_r not included in the braid group.

(11.8b)

(11.8c) Define the straight (or pure) braids to be the elements of B_r in the kernel of $\Psi_{r,*}$.

(11.8d) Let $\mathrm{Inn}(F_r)$ be the normal subgroup of $\mathrm{Aut}(F_r)$ generated by conjugations of elements of F_r on itself. The *mapping class group* (of degree r) is the image in $\mathrm{Aut}(F_r)/\mathrm{Inn}(F_r)$ of automorphisms of $F_r/\bar{R}$ induced by the action of H_r (or B_r) on F_r. Denote this group by M_r. Show

$$\begin{aligned} \tau_1 = (Q_2Q_3 \cdots Q_{r-1})^{1-r}, \dots, \tau_{\ell+1} &= \\ (Q_1 \cdots Q_\ell)^{\ell+1}(Q_{\ell+2} \cdots Q_{r-1})^{\ell+1-r}, \dots, \tau_{r-1} &= (Q_1 \cdots Q_{r-2})^{r-1} \end{aligned}$$

and $\tau = (Q_1 \cdots Q_{r-1})^r$ are in the kernel of the natural map $H_r \to M_r$.

(11.8e) Show there is a unique group (the *dihedral group* of degree n) of order $2n$ and generated by two elements σ_1, σ_2 of order 2 for which $\sigma_1\sigma_2$ is of order n. Similarly, show there is a unique group (the *dicyclic group* of degree $2n$) of order $4n$ and generated by σ_1, σ_2 of respective orders $2n$ and 4, and for which $\sigma_2^{-1}\sigma_1\sigma_2 = \sigma_1^{-1}$ and σ_2^2 is in the group generated by σ_1.

(11.8f) Show H_3 is isomorphic to the dicyclic group of degree 6, and that M_3 is isomorphic to S_3. Hint: Q_1Q_2 and $Q_1Q_2Q_1$ are also generators of H_3.

11.3. Seifert-van Kampen theorem and fiber products.

(11.9a) Give an example to show why $U \cap V$ must be connected for Thm. refthm7 to hold. Hint: Look at Fig 2.1; but it's not the easiest example.

(11.9b) Show that if $\pi_1(U \cap V, x_0)$ is trivial in Thm. 1.5, then $\pi_1(X, x_0)$ is the free product of $\pi_1(U, x_0)$ and $\pi_1(V, x_0)$. Conclude in this case that if $\pi_1(U, x_0)$ and $\pi_1(V, x_0)$ are, respectively, isomorphic to F_r and F_s (Ex. 2.5) then $\pi_1(X, x_0)$ is isomorphic to F_{r+s}.

(11.9c) If $\pi_1(V, x_0)$ is trivial, show that $i(U,X)_* : \pi_1(U, x_0) \to \pi_1(X, x_0)$ is surjective with kernel the smallest normal subgroup N of $\pi_1(U, x_0)$ containing the image of $\pi_1(U \cap V, x_0)$ by the map $i(U \cap V, V)_*$. Hint: Take H to be $\pi_1(U, x_0)/N$. Let $\beta(U) : \pi_1(U, x_0) \to H$ be the natural map ($\beta(V)$ is the trivial homomorphism). Conclude that the kernel of $\beta(X)$ is N.

(11.9d) Show $\pi_1(\mathbb{P}^n)$ is trivial for $n \geq 0$. Hint: It is a union of pieces with trivial fundamental group. There is another approach. If V is a projective manifold, and V_1 is a codimension 1 subvariety, then the natural map $\pi_1(V \setminus V_1) \to \pi_1(V)$ is surjective. Now use that $\mathbb{C}^n$ is contractible.

Consider fiber products and pushouts in the category of finite groups covering a given group. This is entirely different than the similar categorical notions that appear in Chap. 3 [9.3].

(11.10a) Show the free product of groups $G_1, \dots, G_t$ in Lem. 1.4 is a pushout in the category of groups and homomorphisms by taking the map of $\{1\}$ into each of these groups. Note: (1.2) is therefore a sum, rather than a product. Product would be a group G with maps to all the G_i s so that any H that maps to all the G_i s would map to G.

(11.10b) Show uniqueness of pushouts is general in categories of groups. Apply this to Thm. 1.5 to see why this defines $\pi_1(X, x_0)$ uniquely from the maps $\pi_1(U \cap V, x_0) \to \pi_1(U, x_0)$ and $\pi_1(U \cap V, x_0) \to \pi_1(V, x_0)$.

11.4. Residues and uniformization for covers of curves of genus 1. Thm. 6.15 gives the fundamental group of an r punctured Riemann surface X of genus g. Thm. 2.6 started with a statement about the fundamental group of $U_{\boldsymbol{z}}$.

(11.11a) Genus 1 Curve.

(11.11b) This also applies to any Riemann surface uniformized by a disc or by the complex plane.

(11.11c) State and prove a generalization of Thm. 2.6 and Cor. 2.6 characterizing the ramified covers of X ramified over a finite subset $\boldsymbol{x} = (x_1, \dots, x_r)$ of X, where X is any compact Riemann surface. Use $X^0_{\boldsymbol{x}}$ for $X \setminus \boldsymbol{x}$.

(11.11d) Consider the case X has genus 1. Suppose $\varphi : Y \to X$ is a ramified cover, with $\boldsymbol{x}$ its branch points and $\langle a, b, \bar{g}_1, \dots, \bar{g}_r \rangle$ classical generators of $\pi_1(X^0_{\boldsymbol{x}}, x_0)$.

(11.11e) HELP Assume N is odd, and label a conjugacy class in S_N as of type 2^k if it is the conjugacy class of k disjoint 2-cycles. Let $\mathbf{C}_{3\cdot n, n-1, 1}$ be the conjugacy classes of type $(2^n, 2^n, 2^n, 2^{n-1}, 2)$ in S_N with $n = (N-1)/2$. Recall also the Nielsen class notation $\boldsymbol{g} \in \mathbf{C}_{3\cdot n, n-1, 1}$ to mean in some order the entries of $\boldsymbol{g} = (g_1, \dots, g_5)$ are in the respective conjugacy classes.

11.5. Reducible fiber products. Assume $\varphi : Y_i \to X$ are ramified covers of compact Riemann surfaces, and let $W = Y_1 \times_X Y_2$ be the fiber product with its topology coming from it being a subspace of $Y_1 \times Y_2$. Chap. 3 [9.11c] gives simple situations where fiber products W are reducible. We develop more substantial examples by applying RET to the groups of Chap. 3 [9.20].

(11.12a) Finish the proof of Prop. 3.4 by considering g_k, an e_k cycle in S_{e_k}, $k = 1, 2$. Show $(g_1, g_2) \in S_{e_1} \times S_{e_2}$, acting on $\{(i, j)\}_{1 \leq i \leq e_1, 1 \leq j \leq e_2}$ is a product of (e_1, e_2) disjoint $[e_1, e_2]$ cycles.

(11.12b) How does a) describe the irreducible components of the (normal) fiber product of $\mathbb{P}^1_{w_i} \to \mathbb{P}^1_z$ be $w_i \mapsto w_i^{e_i}$, $i = 1, 2$?

(11.12c)

Galois correspondence and primitivity II: Consider the components of $Y \times_X Y \setminus \Delta$ of form $Y' \times_X Y' \setminus \Delta$. Each of these attests to a decomposition of $f : Y \to X$

according to Chap. 3 [9.22c]. We point out how the use of coordinates gives a more practical test.

(11.13a) Reference p. 37, R. Rosario, Thesis: Unirational Fields, Univ. of Cantabria: $(x-\alpha)\prod_{j=1}^{k} P_{i_j}$ with these being the factors of $P(x)$ over $K(\alpha)$. If the coefficients generate a proper subfield then imprimitive. Looking for zeros forming a set of imprimitivity.

This exercise considers the branch point and reduced branch point maps $\Psi_r : (\mathbb{P}^1_z)^r \to \mathbb{P}^r$ and $\Psi_r^{\mathrm{rd}} : (\mathbb{P}^1_z)^r \to J_r$.

(11.14a)

(11.14b) Do the discriminant locus D_r.

(11.14c)

Look back at Ex. 4.1, and do Schur conjecture.

(11.15a) The $\mathbb{Q}$ absolute r-equivalence classes are represented by their branch points $\{0,\infty\}$ and the collection $\{z_1, z_2 \mid z_1 = \sqrt{m}, z_2 = -\sqrt{m}$, m a square-free integer.

(11.15b) Do the setup for the Schur conjecture.

(11.15c) Finish the Schur conjecture.

11.6. Cuts, tangential base points and symbolic pictures. Consider the use of the fundamental group $\pi_1(U_{\boldsymbol{z}}, D_{\boldsymbol{v}})$ in the proof of Thm. 2.26.

(11.16a) Show that if $D_{\boldsymbol{v}'}$ is another tangential disk to 0, giving an isomorphism between $\pi_1(U_{\boldsymbol{z}}, D_{\boldsymbol{v}})$ and $\pi_1(U_{\boldsymbol{z}}, D_{\boldsymbol{v}'})$ depends on how you regard $D_{\boldsymbol{v}}$ connected to $D_{\boldsymbol{v}'}$.

(11.16b) Use as a base point for cuts one of the branch points.

This exercise builds from Chap. 3 [7.2.3]. The point is that we often need notation to differentiate between more subtle appearance of conjugacy classes in S_n. Use notation of §2.4 for discussing cuts.

(11.17a) For the situation of one cut, complete the proof of Lem. 2.15.

(11.17b)

11.7. Alternating group conjugacy classes. We first finish considering the rationality of conjugacy classes in alternating groups.

(11.18a) Assume $\boldsymbol{g} \in S_n^r$, $n \geq 3$ and $\langle \boldsymbol{g} \rangle$ is transitive. Show $\langle \boldsymbol{g} \rangle \geq A_n$ if $\boldsymbol{g}$ contains a 3-cycle. Hint: Show $\langle \boldsymbol{g} \rangle$ must be primitive and imitate Chap. 3 [9.15e].

(11.18b) Let C be an A_n conjugacy class with $\mathrm{C}_{S_n} = (m_1)\cdots(m_t)$ and all m_i s distinct and odd. Write $m = [m_1, \ldots, m_t]$ as $\prod_{i=1}^{s} q_i^{v_i}$; q_i s distinct primes and the v_i s positive. Suppose $q_i^{v_{i,j}}$ exactly divides m_j by $v_{i,j}$. Denote $(v_{i,1}, \ldots, v_{i,t})$ by $\boldsymbol{v}_i$. Define $\mu : \mathbb{Z}^t \to \mathbb{Z}/2$ by $(a_1, \ldots, a_t) \mapsto \sum_{j=1}^{t} a_j \mod 2$. Follow Prop. 3.19 using k, generating $(\mathbb{Z}/q_i^{v_i})^*$, acting on $\oplus_{j=1}^{t} \mathbb{Z}/q_i^{v_{i,j}}$. Identify k with a permutation $\tau_k \in S_{V_i}$, with $V_i = \oplus_{j=1}^{t} q_i^{v_{i,j}}$. Show $\tau_k \in A_n$ if and only if $\mu(\boldsymbol{v}_i) = 0$. Hint: τ_k comes from the product of the actions of k on $\mathbb{Z}/q_i^{v_{i,j}}$.

(11.18c) With C as in b), show M_{C} is nontrivial if and only if $\mu(\boldsymbol{v}_i)$ is nonzero for some i between 1 and s. Let J be those i with $\mu(\boldsymbol{v}_i) \neq 0$. Denote $\sqrt{\prod_{i\in J}(-1)^{q_i-1)/2} q_i}$ by α_{C}. Show M_{C} is $\mathbb{Q}(\alpha_{\mathrm{C}})$.

(11.18d) Consider the 12 pairs C of conjugacy classes C of A_{25} for which $\mathrm{C}_{S_{25}} \neq \mathrm{C}$. Imitate Ex. 3.21 to show that of these the only pair consisting of rational conjugacy classes is that with $\mathrm{C}_{S_{25}} = (25)$. Show that for C with $\mathrm{C}_{S_{25}} = (1)(9)(15)$ (resp. $(1)(3)(21)$), $M_{\mathrm{C}} = \mathbb{Q}(\sqrt{-3 \cdot 5})$ (resp. $M_{\mathrm{C}} = \mathbb{Q}(\sqrt{-7})$).

Subtler issues about conjugacy classes in A_n.

(11.19a) Consider $\boldsymbol{g} \in S_n^r$, $n \geq 4$, that consist of products of two disjoint 2-cycles. If n is even show that there are examples with $\langle \boldsymbol{g} \rangle$ transitive, but not primitive. If $n = 7$, show $\langle \boldsymbol{g} \rangle$ could be $\mathrm{PSL}_3(\mathbb{Z}/2)$ instead of A_7.

(11.19b) Let $g_1 = (1\,n)$, $g_2 = (2 \cdots n)$, $g_3 = (1\,2 \cdots n)^{-1}$, $r = 3$ and $H = A_n$ in Corollary 2.17. Show that $\psi_H : Y_H \to \mathbf{P}^1$ has a description of its branch cycles given by $((1\,2), (1\,2))$. Find $f(y) \in \mathbf{Q}[y]$ such that $\psi : \mathbb{P}^1_y \to \mathbb{P}^1_z$ by $y \mapsto f(y) = z$ has $(\sigma_1, \sigma_2, \sigma_3)$ as a description of its branch cycles.

Take $r = 4$ and $G = S_5$. Let C_1 and C_3 be the conjugacy classes of 2-cycles in S_5, C_2 the conjugacy class of a 3-cycle and C_4 the conjugacy class of a 5-cycle. Consider the Nielsen class $\mathrm{Ni}(S_5, \mathbf{C})/S_5 = \mathrm{Ni}^+$:

$$\{\boldsymbol{g} = (g_1, \ldots, g_4) \mid g_1 g_2 g_3 g_4 = 1, \langle \boldsymbol{g} \rangle = S_5 \text{ and } \boldsymbol{g} \in \mathbf{C}\}/S_5.$$

(11.20a) How many elements are in Ni^+?

(11.20b) Let $\psi : \pi_1(U_{\boldsymbol{z}}) \to S_5$ map a fixed set $\bar{g}_1, \ldots, \bar{g}_4$ into some element of Ni^+. Why is the cover corresponding to such a homomorphism a genus 0 compact Riemamn surface minus a finite set of points?

(11.20c) Represent S_5 on the 10 unordered distinct pairs of integers from $\{1, \ldots, 5\}$: $T : S_5 \to S_10$. Example: $(1\,2\,3\,4\,5)$ has two orbits on these 10 pairs. What are the lengths of the disjoint cycles of T applied to an element of the conjugacy class of a 3-cycle in S_5?

(11.20d) Compose ψ with T to get $T \circ \psi = \psi' : \pi_1(U_{\boldsymbol{z}})$. What is the genus of the curve at the top of the corresponding cover $X = X_\psi \to \mathbb{P}^1_z$?

(11.20e) Does the isomorphism class of X_ψ depend on ψ (assuming ψ is in the Nielsen class Ni^+)?

Now we discuss the genus of the corresponding degree 10 covers. Let $\boldsymbol{g}$ be a branch cycle description of the cover from Ni^+ in [11.20]. Compute the genus g of ${}_{+}\mathcal{T}^{(2)}_{\boldsymbol{p}}$ from Riemann-Hurwitz:

$$2(10 + g - 1) = \sum_{i=1}^{4} \mathrm{ind}(R(g_i)). \tag{11.21}$$

Suppose g_1 and g_3 are 2-cycles from S_5. Then, $R(g_i)$ has shape $(2)(2)(2)$ in the representation R, $i = 1, 3$. Similarly, if g_2 is a 3-cycle, $R(g_3)$ has shape $(3)(3)(3)$. Finally, $R(g_4)$ has shape $(5)(5)$. Thus, the total contribution to the right side of (11.21) is $2 \cdot 3 + 6 + 2 \cdot 4 = 20$ and $g = 1$.

Next: Compute Ni^+ modulo conjugation by S_5. Choose S_5 representatives with g_4 equal $g_\infty = (1\,2\,3\,4\,5)^{-1}$. Divide Ni^+ into two sets T_1 and T_2: $\boldsymbol{g} \in T_1$ has g_1 and g_2 with no integers of common support, and $\boldsymbol{g} \in T_2$ has g_1 and g_2 with one integer of common support. Conjugate by a power of $\boldsymbol{g}_\infty$ to assure elements of T_1 have $g_1 = (1\,j)$ with $j = 2$ or 3. Similarly, elements in T_2 have 1 as common support of g_1 and g_2. From this, list $\mathrm{Ni}^{+,\mathrm{abs}}$.

Now we consider some genus covers with group A_5 and branch cycles having the following type. $(3)(3)(3)(5)$: Suppose $g_3 = (1\,2\,3)$.

(11.22a) Ramification: g_1g_2 is (2)(2), assume missing integer is 1, so to get product a 5-cycle: may assume g_1g_2 is $(2\,5)(3\,4)$. Now everthing is fixed and need only count number of ways to write g_1g_2 is a product of two three cycles. Hint: Products of two 3-cycles giving $(2\,5)(3\,4)$: You get one element from $(4\,2\,5)(2\,3\,4)$. Now conjugate the pair $((4\,2\,5),(2\,3\,4))$ by the centralizer of $(2\,5)(3\,4)$, the group $\langle(2\,5)(3\,4),(2\,4)(3\,5)\rangle$.

(11.22b) If g_1g_2 is (3), then conjugate by $\langle g_3\rangle$ to assume common integer is 1, and $g_1g_2 = (1\,4\,5)$. Hint: Take $(g_1, g_2) = ((1\,4\,3),(1\,3\,5)$, and then conjugate by $\langle(2\,3),(1\,4\,5)\rangle$.

(11.22c) If g_1g_2 is (5). Then, product can't be of type (2)(3) (Riemann-Hurwitz), and have only to assure the (5) times g_3 doesn't fix anything. That means can't have $2 \mapsto 1$, $3 \mapsto 2$ or $1 \mapsto 3$. Also, since by conjugation by $\langle(4\,5),(1\,2\,3)\rangle$ can assume $(1\,5\,?\,?\,?)$ resulting in $(1\,5\,2\,4\,3)$ or $(1\,5\,3\,2\,4)$. Hint: For each of $(1\,5\,2\,4\,3)$ or $(1\,5\,3\,2\,4)$, we need to count all the ways to write this 5-cycle as a product of two 3-cycles. For $(1\,2\,3\,4\,5)$, assume the integer 1 is the common integer to the 3-cycles. So, $(g_1, g_2) = ((1\,2\,3),(1\,4\,5))$. Then, by conjugating by $\langle(1\,2\,3\,4\,5)\rangle$, gives the five cases where g_1 and g_2 have any desired integer in common.

(11.22d) Up to equivalence, there are exactly 4 covers from a), 6 covers from b) and 10 covers from c), or 20 total covers. Also, by applying powers of q_1 to case c) you get 10 total in two orbits of length five. Same for b), two orbits of length 3, and for a), two orbits of length two.

11.8. Differentials and differential equations. The space of holomorphic differentials has dimension bounded by g.

(11.23a) Finish the pairing with homology classes.

(11.23b) Show df in (2.13) is a 2-form.

Let $\mathcal{H}(D)$ be functions analytic on a domain D. A *differential equation* on D comes from $m \in \mathcal{H}(D)[w_0, w_1, \dots, w_k]$: a polynomial with coefficients in $\mathcal{H}(D)$. Solutions of the equation are functions $f(z)$, analytic in some disk in D with $m(f(z), \frac{df}{dz}, \dots, \frac{d^kf}{dz^k}) \equiv 0$ on this disk. Especially interesting are equations defined by m linear in the variables $w_0, w_1, \dots, w_k$, with coefficients in $\mathbb{C}[z]$: linear, algebraic differential equations. Examples: $m_1(w_0, w_1) = w_0 - w_1$, $m_2(w_0, w_1) = w_0 - zw_1$ and $m_3(w_0, w_1) = zw_0 - w_1$.

(11.24a) Suppose $m \in \mathbb{C}[z][w_0, \dots, w_k]$ defines a linear algebraic differential equation. Let $f(z)$, analytic on D, solve the equation. Show there exists a finite set $\boldsymbol{z} \subset \mathbb{P}^1_z$ satisfying (1.1a). Hint: Let $\frac{d}{dz}$ act on functions analytic on D. Produce a matrix operator $\mathcal{D}$ on $\mathcal{H}(D)^{k+1}$:

$$(f_0, f_1, \dots, f_k) \mapsto (m(f_0, f_1, \dots, f_k), f_1 - \frac{df_0}{dz}, \dots, f_k - \frac{df_{k-1}}{dz}).$$

Find $\boldsymbol{z}$ from the determinant of $\mathcal{D}$.

(11.24b) Show the vector space of solutions (analytic in a disk centered at $z_0 \in \mathbb{P}^1_z \setminus \boldsymbol{z}$) of the differential equation m has dimension k.

(11.24c) Consider the case $m = P_1(z)w_0 - P_2(z)w_2$. where $g = P_1/P_2$ is a rational function satisfying some well know conditions that permit its solutions to be continued over every path on the z-sphere. When the denominator of g has degree $2p + 2$ then there are $2p - 1$ unknown conditions that prevent the unique specification of the numerator - the goal being that the ratio of a pair of independent solutions of the equation should yield a conformal equivalence of the branched universal covering surface for the $2p+2$ punctured sphere with the unit disk (at least for $p>1$). Thus it is really the moduli problem for hyperelliptic curves.

11.9. Schwartzian and Beltrami equations. Riemann and Schwarz used functional equations to characterize the nature of many functions. The most famous after Riemann's application to θ functions is the Schwarzian derivative. Call $\mathcal{D}$ a *Schwarzian* for a subgroup $G \le \mathrm{PGL}_2(\mathbb{C})$ if $\mathcal{D}(g(h(z))) = \mathcal{D}(h(z))$ for $g \in G$ and any meromorphic function h, and conversely, the set of $g \in \mathrm{PGL}_2(\mathbb{C})$ for which this holds exactly for relevant meromorphic h defines G. Use notation from Chap. 2 [9.14].

(11.25a) Show the translation group $\mathcal{T}$ has Schwarzian $\mathcal{D}_{\mathcal{T}}$ given by $h \mapsto \frac{dh}{dz}$.

(11.25b) Show the group of multiplications $\mathcal{M}$ has Schwarzian $\mathcal{D}_{\mathcal{M}}$ given by $h \mapsto \frac{h'}{h}$, the logarithmic derivative.

(11.25c) Show the affine group $\mathbb{C} \times^s \mathbb{C}^* = \mathcal{A}$ has Schwarzian $\mathcal{D}_{\mathcal{A}} = \mathcal{D}_{\mathcal{M}} \circ \mathcal{D}_{\mathcal{T}}$.

(11.25d) Any element of $\mathrm{PGL}_2(\mathbb{C})$ has either the form $z \mapsto az+b$ or $z \mapsto a+b/(z-c)$. Conclude (by changing $h(z)$ to $h(z)-c$): $\mathrm{PGL}_2(\mathbb{C})$ has a Schwarzian if there is a differential operator $\mathcal{D}_\tau$ ($\tau : z \mapsto 1/z$) with

$$\mathcal{D}_\tau(\frac{h''}{h'}) = \mathcal{D}_\tau(\frac{(1/(h)''}{(1/(h)'})$$

for any meromorphic h. Compute that $\mathcal{D}_\tau(g) = g' - \frac{1}{2}g^2$ works: $\mathrm{PGL}_2(\mathbb{C})$ has a Schwarzian $\mathcal{D}_\tau \circ \mathcal{D}_{\mathcal{A}}$.

The Beltrami Equation: Irwin, P. 35 of your book. This is a loaded page, in which you take a Riemannian structure and turn it into a quasiconformal structure. I am doing exercises (Chap. 3 of a book) in which I took my own path to the Beltrami equation to suit a theory of uniformization I'm using. I wanted to do something along the lines you are doing. You have, however, the statement: "the most nontrivial ... is the verification that $\mu f_z = f_{\bar{z}}$ has homeomorphic solutions."

I looked in the rest of your book, and couldn't find a proof that $\mu f_z = f_{\bar{z}}$ has homeomorphic solutions if μ is bounded by 1 in the neighborhood of a given point. Is it somewhere there, and if so, what is the easiest solution of this?

11.10. Frattini covers and half-canonical classes.

(11.26a)

For the particular situation defining a half-canonical class from a differential satisfying Chap. 3 (5.11), the choices are defined up to chains with values in ± 1. Now consider the number of boundary equivalent half-canonical classes. Every set of transition functions has a global meromorphic section (how hard would that be to prove in this case). So, each boundary equivalence class has a differential giving it by situation (5.11). So, it is enough to use the setup of Chap. 3 (5.11) to define all boundary equivalent half-canonical classes.

(11.27a) Suppose $\{h_{\beta,\alpha}\}$ and $\{h'_{\beta,\alpha}\}$ are two sets of transition functions defining half-canonical classes by the rule (5.11). Let h_α/h'_α be the corresponding ratios of the functions from (5.11). Their squares form a function on X and they give a homomorphism $\pi_1(X) \to \mathbb{Z}/2$ by the following rule. For any closed path, form the analytic continuation of h_α/h'_α around the path.

This works if the formula for analytic continuation applies on a manifold which it does by §6.2. So, given any two their ratio defines a cover of X, and conversely.

(11.27b) Now, suppose the cover $\hat{Y} \to \hat{X}$ is unramified. We already have a homomorphism $\pi_1(U_{\boldsymbol{z}}, z_0) \to \hat{A}_n$ defining this. Further, the kernel of this to A_n factors through a map $\pi_1(\hat{X}) \to \mathbb{Z}/2$. Note: If you take the divisor of $d\hat{\varphi}$, it is A_n invariant.

(11.27c)

11.11. Differential forms, orientation, area and the Laplacian. Why is an orientation forced in order to integrate a form? How would we generalize length to get area? Define $\wedge$ multiplication.

(11.28a) Pythagorian formula for area.

Show with any Riemannian manifold X replacing $\mathbb{R}^2$, with $ds^2 : \mathbb{T}_X \times \mathbb{T}_X \to \mathcal{C}_X^\infty$ the nondegenerate symmetric 2-tensor.

(11.29a) Consider an open set U in $\mathbb{R}^n$. Call a differential 1-form ω *integrable* if it has the form df for some $f \in \mathcal{C}_U^\infty$. The *integrability condition* is that $d\omega = 0$. $f \mapsto df \mapsto T_{df}$ maps to the (1,1) tensor $\Delta(T_{df})(T, \omega) = D_T(T_{df}) \otimes \omega$. Now contract back to a function.

Take a basis of differentials and recall the inner product $\langle \varphi_1, \varphi_2 \rangle = \int_X d\varphi_1 * d\bar{\varphi}_2$.

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