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Moduli spaces of covers and the Hurwitz monodromy group

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§ 0. Introduction

We study the (course) moduli space of covers of the Riemann sphere of a given *Nielsen type* (§ 2). Properties of this space translate to statements about representations of the Hurwitz monodromy group. When the Nielsen type is of simple branching the moduli space is irreducible — a combinatorial result of Clebsch ([Cle]) that gave the first proofs of the irreducibility of the space of curves of a given genus. We give examples of Nielsen types for which the moduli space is reducible (§ 3). Testable necessary and sufficient conditions that the moduli space be a fine étale moduli space for covers of a specific Nielsen type appear in § 4. The study of § 5 of the "boundary" of the moduli space in terms of representations of the Hurwitz monodromy group relates to recent work of Harbater ([Har]). We now explain in detail.

Riemann's existence theorem lists the essentially distinct algebraic ways that a complex variable w can depend on a complex variable z: a satisfying translation of the fundamental ambiguity into a computation of the permutation representations of the fundamental group of the Riemann sphere, $\mathbb{P}^1$, with a finite number of points removed. There is, however, an ambiguity in Riemann's correspondence.

Consider a topologized family of algebraic relations of the form $f(z, w, \alpha)=0$, where α runs over the points of some parameter space P. This we may regard as a family of ramified covers of the Riemann sphere (z-sphere). As the branch points of this family move, the correspondence of a member of this family with a representation of a fundamental group $\pi_1(\mathbb{P}^1-\{z(1),\dots,z(r)\}, z(0))$ forces us to consider the effect of moving the branch points $z(1),\dots,z(r)$ *and* the base point $z(0)$. Although the fundamental groups of a fixed space computed with respect to two distinct base points are isomorphic, the isomorphism is noncanonical. Thus, even when the parameter space P is connected and when the branch points of $f(z, w, \alpha_1)=0$ and $f(z, w, \alpha_2)=0$ are the same for $\alpha_1, \alpha_2 \in P$, the algebraic relations between z and w may be essentially *distinct*. This ambiguity in Riemann's correspondence is expressed through a group, the *Hurwitz monodromy group*. Representations of this group are a source of information about families of algebraic relations.

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In detail: The existence theorem associates to a cover (of degree n), $X \xrightarrow{\varphi} \mathbb{P}^1$, of compact connected Riemann surfaces with r prescribed points of branching, an r-tuple $\boldsymbol{\sigma}=(\sigma(1),\dots,\sigma(r))$ of elements of S_n for which $G(\boldsymbol{\sigma})$, the group generated by $\sigma(1),\dots,\sigma(r)$ (*the monodromy group of the cover*), is transitive and the product $\sigma(1)\cdots\sigma(r)$ is the identity. Thus $\boldsymbol{\sigma}$ determines a permutation representation of the quotient of the free group on r generators $\Sigma_1,\dots,\Sigma_r$ by the minimal normal subgroup containing the product $\Sigma_1\cdots\Sigma_r$. This last group may be identified with the fundamental group of $\mathbb{P}^1$ with the points $z(1),\dots,z(r)$, over which we allow ramification of the cover, removed. Two covers $X_i \xrightarrow{\varphi_i} \mathbb{P}^1$, $i=1,2$, are *equivalent* if there exists an isomorphism $\Psi: X_1 \to X_2$ for which $\varphi_2\circ\Psi=\varphi_1$. Two r-tuples $\boldsymbol{\sigma}^{(1)}$ and $\boldsymbol{\sigma}^{(2)}$ are (*absolutely*) *equivalent* if there exists $\gamma \in S_n$ such that $\gamma^{-1}\cdot\sigma(i)^{(1)}\cdot\gamma=\sigma(i)^{(2)}$, $i=1,\dots,r$. The association is one-one between equivalence classes of covers and equivalences classes of r-tuples satisfying the properties just listed. The r-tuple $\boldsymbol{\sigma}$ is called a *description* of the branch cycles of the cover $X \xrightarrow{\varphi} \mathbb{P}^1$.

Since, however, most applications to algebraic and arithmetic geometry deal with covers, *not* one at a time, but in topologized families, we must adjust the ingredients of Riemann's correspondence to allow the branch points of a cover to move. To understand the families of covers containing a given cover $X \xrightarrow{\varphi} \mathbb{P}^1$ (or (X,φ)) we need a notion that delimits the covers we would expect to find in such a family.

The (*absolute*) *Nielsen class* $\mathrm{Ni}(\boldsymbol{\sigma})$ of $\boldsymbol{\sigma}$ contains all r-tuples of S_n which are descriptions of branch cycles of covers $X' \xrightarrow{\varphi'} \mathbb{P}^1$ for which (X',φ') has the same monodromy group as (X,φ) and for which (X',φ') and (X,φ) have "similar branching type" (§ 2: the $\bar{G}$-Nielsen class of $\boldsymbol{\sigma}$ is a finer notion that specializes to the Nielsen class when $\bar{G}$ is the normalizer of $G(\boldsymbol{\sigma})$ in S_n). If two covers are in a connected (flat) family of covers of $\mathbb{P}^1$, all having r distinct branch points, then their absolute Nielsen classes are the same ([Fr, 1]; § 3).

Let $\mathbb{P}^r$ denote projective r-space and D_r the discriminant locus of $\mathbb{P}^r$ (§ 1). The *Hurwitz monodromy group of degree r* is the fundamental group of $\mathbb{P}^r - D_r$, and it acts on the elements of $\mathrm{Ni}(\boldsymbol{\sigma})$ (§ 2) to produce an unramified cover $\mathscr{H}(n,r;\mathrm{Ni}(\boldsymbol{\sigma}))$ of $\mathbb{P}^r - D_r$ whose points are in one-one correspondence with the covers (X',φ') for which all possible descriptions of the branch cycles of (X',φ') fall in $\mathrm{Ni}(\boldsymbol{\sigma})$.

This paper examines the properties of the spaces $\mathscr{H}(n,r;\mathrm{Ni}(\boldsymbol{\sigma}))$ that are a key to an understanding of the families of covers that contain (X,φ): Is $\mathscr{H}(n,r;\mathrm{Ni}(\boldsymbol{\sigma}))$ irreducible? Is $\mathscr{H}(n,r;\mathrm{Ni}(\boldsymbol{\sigma}))$ naturally the parameter space for some total family of covers whose branch cycles are in $\mathrm{Ni}(\boldsymbol{\sigma})$? Answer to these two questions—not always (§ 3 and § 4). As n varies and $\boldsymbol{\sigma}$ runs over all allowable elements of S_n^r (for r fixed) do the spaces

$$\mathscr{H}(n,r;\mathrm{Br}(\boldsymbol{\sigma}^{\bar{G}},f)),$$

generalizing the irreducible components of $\mathscr{H}(n,r;\mathrm{Ni}(\boldsymbol{\sigma}))$, give a collection of covers of $\mathbb{P}^r - D_r$ that are cofinal in the family of unramified covers of $\mathbb{P}^r - D_r$ (end of § 2—a *congruence subgroup* problem)? This last problem arises as a generalization of the following observation of [Fr, 2]; p. 579—581 ([Fr, 5]; p. 152). There is an intimate connection between the *modular curve*, $Y_0(n)$, of level n, and the family of covers of $\mathbb{P}^1$ having a description of branch cycles given by $(\sigma(1),\sigma(2),\sigma(3),\sigma(4))$ as follows: $\sigma(i)$ is the linear transformation of $\mathbb{Z}/(n)$ given by $z \to -z+b(i)$ for some $b(i)\in\mathbb{Z}/(n)$; $\sigma(1)\cdots\sigma(4)$ is the identity; and $G(\boldsymbol{\sigma})$ is transitive.

In the case that one of the spaces $\mathscr{H}(n, r; \mathrm{Ni}(\boldsymbol{\sigma}))$ has several components, many applications (e.g., the arithmetic questions of [Fr, 1]) demand a method by which we may algebraically distinguish the properties of the covers associated to the points of one component of $\mathscr{H}(n, r; \mathrm{Ni}(\boldsymbol{\sigma}))$ from those associated to the points of the other components of $\mathscr{H}(n, r; \mathrm{Ni}(\boldsymbol{\sigma}))$. Geometrically, § 5 considers the way that these components lie on the "boundary" of other such spaces $\mathscr{H}(n, r'; \mathrm{Ni}(\boldsymbol{\sigma}'))$. The hope is that for each component $\mathscr{H}'$ of $\mathscr{H}(n, r; \mathrm{Ni}(\boldsymbol{\sigma}))$, there will exist $\boldsymbol{\sigma}'$ and r' for which $\mathscr{H}'$, and no other component of $\mathscr{H}(n, r; \mathrm{Ni}(\boldsymbol{\sigma}))$, is on the boundary of $\mathscr{H}(n, r'; \mathrm{Ni}(\boldsymbol{\sigma}'))$. In detail, § 5 rephrases this problem entirely in terms of representations of the Hurwitz monodromy groups of degree r' for all $r' \geqq r$. We apply these ideas to the examples of § 3 in relation to specific representations that arise from [Har].

A well-known problem: To prove algebraically (i.e., independent of the metric on $\mathbb{P}^1$) the properties of covers of $\mathbb{P}^1$ that derive from Riemann's existence theorem (e.g., [Gr] and [Har]). In [Fr, 3], following ideas close to those of § 5, we accomplish this through our construction of *inertia section families*, an algebraic replacement for the classical branch cycles of a cover. This paper contains much topological and combinatorical computation around the classical results of combinatorial group theory. Since the applications are to algebraic and arithmetic geometry, § 1 includes complete definitions of the various combinatorical groups and some tightening of the arguments and conclusions of [FB] to which we are in debt.

§ 1. The Artin braid group and the Hurwitz monodromy group

Let $\mathbb{A}_R^r$ and $\mathbb{A}_C^r$ be two copies of affine r-space, and consider the natural map $\mathbb{A}_R^r \xrightarrow{\Psi_r} \mathbb{A}_C^r$ that sends $(x_1, \ldots, x_r)$ to the r-tuple of symmetric functions

$$(y_1, \ldots, y_r) = \Big(\cdots, (-1)^i \Big(\sum_{j(1) < \cdots < j(i)} x_{j(1)} \cdots x_{j(i)}\Big), \cdots\Big).$$

The subscripts R and C denote (resp.) Roots and Coefficients. The cover $\mathbb{A}_R^r \xrightarrow{\Psi_r} \mathbb{A}_C^r$, the Noether cover, is Galois with group S_r. The variety $\mathbb{A}^r$ can be regarded as an affine subset of both $(\mathbb{P}^1)^r$ and of $\mathbb{P}^r$. Indeed, $\mathbb{A}^1 = \mathbb{P}^1 - \{\infty\}$ embeds $(\mathbb{A}^1)^r$ in $(\mathbb{P}^1)^r$, and $\mathbb{A}^r$ can be regarded as the subset of $\mathbb{P}^r$ represented by the $r+1$-tuples $(y_0, y_1, \ldots, y_r)$ with $y_0 = 1$. Then $(\mathbb{P}^1)^r$ and $\mathbb{P}^r$ are joined in a commutative diagram

$$\begin{tikzcd} \mathbb{A}_R^r \arrow[r, "\Psi_r"] \arrow[d] & \mathbb{A}_C^r \arrow[d] \\ (\mathbb{P}^1)^r \arrow[r, "\bar{\Psi}_r"] & \mathbb{P}^r \end{tikzcd} \tag{1.1}$$

where the vertical arrows are the respective identifications of $\mathbb{A}^r$ with subsets of $(\mathbb{P}^1)^r$ and $\mathbb{P}^r$ given above.

To see the nature of this diagram consider the set of nonzero polynomials in z

$$P_r = \left\{ \sum_{i=0}^{r} y_i \cdot z^i \,|\, (y_0, \ldots, y_r) \in \mathbb{C}^{r+1} - \{(0, \ldots, 0)\} \right\},$$

modulo the action of $\mathbb{C}^*$ that equivalences two polynomials if one is a non-zero multiple of the other. This set is then identified with $\mathbb{P}^r$, and the map $\bar{\Psi}_r$ maps $(x_1,\ldots,x_r)$ to $\prod_{i=1}^{r}(z-x_i)$; with the stipulation that if $x_i=\infty$, $z-x_i$ is replaced by the constant 1. Thus $\mathbb{P}^r$ can be regarded as the quotient of $(\mathbb{P}^1)^r$ by S_r. Finally, let Δ_r be the subset of $\mathbb{A}_R^r$ consisting of the points having two or more equal coordinates, and let D_r (the *discriminant locus* of the Noether cover) be the image of Δ_r under Ψ_r. By abuse we also denote by Δ_r (resp., D_r) the closure of Δ_r (resp., D_r) in $(\mathbb{P}^1)^r$ (resp., $\mathbb{P}^r$). We regard $\mathbb{P}^r-D_r$ as the collection of r *unordered distinct points of* $\mathbb{P}^1$.

The fundamental group of $\mathbb{A}_C^r-D_r=\mathbb{A}^r-D_r$, denoted $\pi_1(\mathbb{A}^r-D_r,q^{(0)})$, is called the (geometric) *Artin Braid Group*. Similarly, the fundamental group $\pi_1(\mathbb{P}^r-D_r,q^{(0)})$ is called the *Hurwitz monodromy group*. Let $F(\Sigma_1,\ldots,\Sigma_r)\stackrel{\text{def}}{=}F(r,\mathbf{\Sigma})$ be the free group on the r generators $\Sigma_1,\ldots,\Sigma_r$. Consider $\mathrm{Br}(F(r,\mathbf{\Sigma});\Sigma_1\cdots\Sigma_r)$, the group of automorphisms $\bar{Q}$ of $F(r,\mathbf{\Sigma})$ with these properties:

(1.2) a) $\bar{Q}$ maps $\Sigma_1\cdots\Sigma_r$ into itself; and

b) $\bar{Q}$ maps Σ_i to a conjugate of Σ_j for some j (dependent on i), $i=1,\ldots,r$.

Theorem 1.1 [ArE, 1, 2], [Bo], [Ni]). *The fundamental group* $\pi_1(\mathbb{A}^r-D_r,q^{(0)})$ *is isomorphic to* $\mathrm{Br}(F(r,\mathbf{\Sigma});\Sigma_1\cdots\Sigma_r)$. *This latter group has generators* $\bar{Q}_1,\ldots,\bar{Q}_{r-1}$ *subject only to these relations:* $\bar{Q}_i\cdot\bar{Q}_j=\bar{Q}_j\cdot\bar{Q}_i$ *for* $1\leq i<j\leq r-1$, $j\neq i+1$ *or* $i-1$; *and* $\bar{Q}_i\cdot\bar{Q}_{i+1}\cdot\bar{Q}_i=\bar{Q}_{i+1}\cdot\bar{Q}_i\cdot\bar{Q}_{i+1}$, $i=1,\ldots,r-2$. *In addition* ([FB]), *the natural map coming from the embedding of* $\mathbb{A}^r-D_r$ *in* $\mathbb{P}^r-D_r$ *induces an isomorphism of* $\pi_1(\mathbb{P}^r-D_r,q^{(0)})$ *with the quotient of* $\mathrm{Br}(F(r,\mathbf{\Sigma});\Sigma_1\cdots\Sigma_r)$ *by the minimal normal subgroup containing* $\bar{Q}(r)=\bar{Q}_1\cdot\bar{Q}_2\cdots\bar{Q}_{r-1}\cdot\bar{Q}_{r-1}\cdots\bar{Q}_2\cdot\bar{Q}_1$.

Indeed $\bar{Q}_i$ acts on the r-tuple $(\Sigma_1,\ldots,\Sigma_r)=\mathbf{\Sigma}$:

(1.3) $(\mathbf{\Sigma})\bar{Q}_i$ is equal to

$$(\Sigma_1,\ldots,\Sigma_{i-1},\Sigma_i\cdot\Sigma_{i+1}\cdot\Sigma_i^{-1},\Sigma_i,\Sigma_{i+2},\ldots,\Sigma_r),\ i=1,\ldots,r-1.$$

We also denote by $\bar{Q}_1,\ldots,\bar{Q}_{r-1}$ the images of $\bar{Q}_1,\ldots,\bar{Q}_{r-1}$ in $\pi_1(\mathbb{P}^r-D_r,q^{(0)})$.

Let $G(\Sigma_1,\ldots,\Sigma_r;\mathbf{\Sigma})\stackrel{\text{def}}{=}G(\mathbf{\Sigma})$ be the quotient of $F(r,\mathbf{\Sigma})$ by the minimal normal subgroup containing $\Sigma_1\cdots\Sigma_r$. Although this group is a free group on $r-1$ generators, the presentation given here most easily identifies it with the fundamental group of a sphere minus r points, as in § 5. Consider $A(r,\mathbf{\Sigma})$, the group of automorphisms $\bar{Q}$ of $G(\mathbf{\Sigma})$ given as follows: $\bar{Q}$ maps Σ_i to a conjugate of Σ_j for some j (dependent on i), $i=1,\ldots,r$. Note that this is the same statement as in expression (1.2) b), but *here* we mean conjugate within the group $G(\mathbf{\Sigma})$. The *mapping class group*, $M(r,\mathbf{\Sigma})$, is the quotient of $A(r,\mathbf{\Sigma})$ by the inner automorphisms of $G(\mathbf{\Sigma})$.

Theorem 1.2 ([KMS]; Theorem N9, [M]). *The natural map* $\mathrm{Br}(F(r,\mathbf{\Sigma});\Sigma_1\cdots\Sigma_r)$ *to* $M(r,\mathbf{\Sigma})$ *is surjective, and the kernel of this map is generated by the elements*

$$\tau_1=(\bar{Q}_2\bar{Q}_3\cdots\bar{Q}_{r-1})^{1-r},\ldots,$$
$$\tau_{l+1}=(\bar{Q}_1\cdots\bar{Q}_l)^{l+1}\cdot(\bar{Q}_{l+2}\cdots\bar{Q}_{r-1})^{l+1-r},\ldots,$$
$$\tau_{r-1}=(\bar{Q}_1\cdots\bar{Q}_{r-2})^{r-1},\quad \textit{and}\quad \tau=(\bar{Q}_1\cdots\bar{Q}_{r-1})^r.$$

We continue to denote by $\bar{Q}_1, \ldots, \bar{Q}_{r-1}$ the images of the generators of $\mathrm{Br}(F(r, \Sigma); \Sigma_1 \cdots \Sigma_r)$ in $M(r, \Sigma)$.

In § 2 we induce representations of the Hurwitz monodromy group through representations of the mapping class group. The distinction between these two groups becomes crucial in the calculations of § 3, and the next lemmas, of which partial like minded versions also appear in [FB], epitimize this distinction.

Definition 1. 3. For each of the groups $\pi_1(\mathbb{A}^r - D_r, q^{(0)})$, $\pi_1(\mathbb{P}^r - D_r, q^{(0)})$, and $M(r, \Sigma)$ there is a natural permutation representation of degree r, called the *Noether representation*, that maps $\bar{Q}_i$ to $\alpha_r(\bar{Q}_i) = (i\ i+1) \in S_r$.

Definition 1. 4. The *dihedral group* of degree n, and order $2 \cdot n$, is characterized as the unique group generated by two elements σ_1, σ_2 of order 2 for which $\sigma_1 \cdot \sigma_2$ is of order n. The *dicyclic group* of degree $2 \cdot n$, and order $4 \cdot n$, is characterized as the unique group generated by elements σ_1, σ_2 for which σ_1 is of order $2 \cdot n$; σ_2 of order 4; $\sigma_2^{-1} \cdot \sigma_1 \cdot \sigma_2 = \sigma_1^{-1}$; and σ_2^2 is in the group generated by σ_1.

Lemma 1. 5. *The group $\pi_1(\mathbb{P}^3 - D_3, q^{(0)})$ is isomorphic to the dicyclic group of degree* 6, *and $M(3, \Sigma)$ is isomorphic to S_3.*

Proof. From diagram (1. 1), the group $\pi_1(\mathbb{P}^3 - D_3, q^{(0)})$ is of order 12 once we have shown that the fundamental group of $(\mathbb{P}^1)^3 - \Delta_3$ is isomorphic to $\mathbb{Z}/(2)$. Let $\mathrm{SL}(2, C)$ be the group of 2×2 complex matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad - bc = 1$, and let $\mathrm{Möb}(\mathbb{C})$ be the group of complex Möbius transformations. Since, for any point $(z_1, z_2, z_3) \in (\mathbb{P}^1)^3 - \Delta_3$, there is a unique element $\beta \in \mathrm{Möb}(\mathbb{C})$ for which $\beta(z_1) = 0$, $\beta(z_2) = 1$, $\beta(z_3) = \infty$, $\mathrm{Möb}(\mathbb{C})$ is homeomorphic to $(\mathbb{P}^1)^3 - \Delta_3$. Further, $\mathrm{SL}(2, \mathbb{C})$, a simple group (with trivial fundamental group) is a degree 2 unramified cover of $\mathrm{Möb}(\mathbb{C})$. Thus the fundamental group of $(\mathbb{P}^1)^3 - \Delta_3$ is isomorphic to $\mathbb{Z}/(2)$.

Let $\bar{Q}_1$ and $\bar{Q}_2$ be the generators of $\pi_1(\mathbb{P}^3 - D_3, q^{(0)})$ described above. The elements $\bar{Q}_1 \cdot \bar{Q}_2$ and $\bar{Q}_1 \cdot \bar{Q}_2 \cdot \bar{Q}_1$ are also easily seen to be generators of $\pi_1(\mathbb{P}^3 - D_3, q^{(0)})$. From the relation $\bar{Q}_1 \cdot \bar{Q}_2 \cdot \bar{Q}_1 = \bar{Q}_2 \cdot \bar{Q}_1 \cdot \bar{Q}_2$ (Theorem 1. 1) we see that $(\bar{Q}_1 \cdot \bar{Q}_2)^3 = (\bar{Q}_1 \cdot \bar{Q}_2 \cdot \bar{Q}_1)^2$. Thus $\bar{Q}_1 \cdot \bar{Q}_2 \cdot \bar{Q}_1$ is of order 4, $\bar{Q}_1 \cdot \bar{Q}_2$ is of order 6, and $\pi_1(\mathbb{P}^3 - D_3, q^{(0)})$ is, following Definition 1. 2, the dicyclic group of degree 6.

Now consider the group $M(3, \Sigma)$. Since this group maps surjectively to S_3 via the Noether representation, it is isomorphic to S_3 if $(\bar{Q}_1 \cdot \bar{Q}_2)^3$ is the identity in this group. Consider the effect of $\bar{Q}_1 \cdot \bar{Q}_2$ on the 3-tuple $(\Sigma_1, \Sigma_2, \Sigma_3)$ consisting of the generators of $G(\Sigma)$:

(1. 4)

$$(\Sigma_1, \Sigma_2, \Sigma_3)(\bar{Q}_1 \cdot \bar{Q}_2) = (\Sigma_1 \cdot \Sigma_2 \cdot \Sigma_1^{-1}, \Sigma_1, \Sigma_3)\bar{Q}_2 = (\Sigma_1 \cdot \Sigma_2 \cdot \Sigma_1^{-1}, \Sigma_1 \cdot \Sigma_3 \cdot \Sigma_1^{-1}, \Sigma_1).$$

If we follow $\bar{Q}_1 \cdot \bar{Q}_2$ by conjugation by Σ_1, the total effect is to map the 3-tuple $(\Sigma_1, \Sigma_2, \Sigma_3)$ to $(\Sigma_2, \Sigma_3, \Sigma_1)$. It is now clear that, in $M(3, \Sigma)$, $(\bar{Q}_1 \cdot \bar{Q}_2)^3$ leaves $(\Sigma_1, \Sigma_2, \Sigma_3)$ invariant, and the lemma is concluded. ∎

Lemma 1. 6. *Let* $\mathbf{z}^{(0)} \in (\mathbb{P}^1)^r - \Delta_r$ *lie above* $q^{(0)} \in \mathbb{P}^r - D_r$. *The fundamental group* $\pi_1\left((\mathbb{P}^1)^r - \Delta_r, \mathbf{z}^{(0)}\right)$, *via diagram* (1. 1), *is identified with the kernel of the Noether representation* (*Definition* 1. 3). *In addition,* $\pi_1\left((\mathbb{P}^1)^r - \Delta_r, \mathbf{z}^{(0)}\right)$ *is the smallest normal subgroup of* $\pi_1(\mathbb{P}^r - D_r, q^{(0)})$ *containing* $(\bar{Q}_1)^2$. *In the special case that* $r = 4$, $(\mathbb{P}^1)^4 - \Delta_4$ *is homeomorphic to* $(\mathbb{P}^1 - \{0, 1, \infty\}) \times \left((\mathbb{P}^1)^3 - \Delta_3\right)$, *and the image of* $\pi_1\left((\mathbb{P}^1)^4 - \Delta_4, \mathbf{z}^{(0)}\right)$ *in* $M(4, \mathbf{\Sigma})$ *is generated by* $(\bar{Q}_1)^2$ *and* $\bar{Q}_1^{-1} \cdot (\bar{Q}_2)^2 \cdot \bar{Q}_1$.

Proof. For the statement on $\pi_1\left((\mathbb{P}^1)^r - \Delta_r, \mathbf{z}^{(0)}\right)$ see [Bo]; it is not difficult. We concentrate on the case $r = 4$. From Lemma 1. 5, $(\mathbb{P}^1)^3 - \Delta_3$ is homeomorphic to Möb$(\mathbb{C})$, with which we now identify it. Consider the map from $(\mathbb{P}^1 - \{0, 1, \infty\}) \times \text{Möb}(\mathbb{C})$ to $(\mathbb{P}^1)^4 - \Delta_4$ that sends (z, β) to $\left(\beta(z), \beta(0), \beta(1), \beta(\infty)\right)$ where $z \in \mathbb{P}^1 - \{0, 1, \infty\}$ and $\beta \in \text{Möb}(\mathbb{C})$. This map is clearly a complex analytic isomorphism.

The element $(\bar{Q}_2 \cdot \bar{Q}_3)^3$ is contained in $\pi_1\left((\mathbb{P}^1)^4 - \Delta_4, \mathbf{z}^{(0)}\right)$. Consider its image in $M(4, \mathbf{\Sigma})$. From the computation of the proof of Lemma 1. 5, the effect of $\bar{Q}_2 \cdot \bar{Q}_3$ on $(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4)$ is to send it to $(\Sigma_1, \Sigma_2 \cdot \Sigma_3 \cdot \Sigma_2^{-1}, \Sigma_2 \cdot \Sigma_4 \cdot \Sigma_2^{-1}, \Sigma_2)$; as in expression (1. 4). Inside of $G(\mathbf{\Sigma})$ this has the same effect as the automorphism that sends $(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4)$ to $(\Sigma_2^{-1} \cdot \Sigma_1 \cdot \Sigma_2, \Sigma_3, \Sigma_4, \Sigma_2)$. By analogy, $(\bar{Q}_2 \cdot \bar{Q}_3)^2$ sends $(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4)$ to

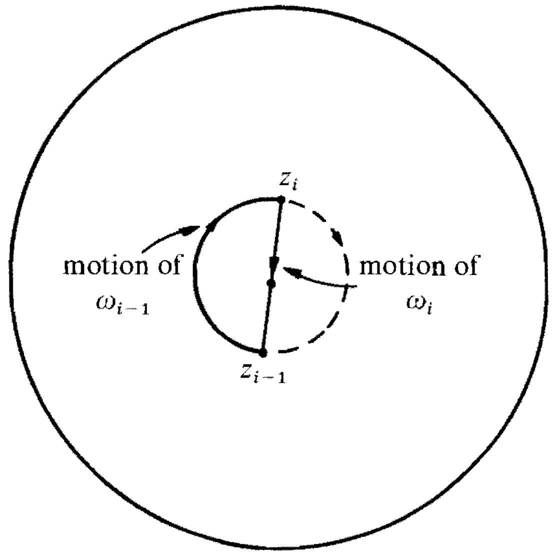
$$(\Sigma_3^{-1} \cdot \Sigma_2^{-1} \cdot \Sigma_1 \cdot \Sigma_2 \cdot \Sigma_3, \Sigma_4, \Sigma_2, \Sigma_3),$$

and $(\bar{Q}_2 \cdot \bar{Q}_3)^3$ sends $(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4)$ to $(\Sigma_4^{-1} \cdot \Sigma_3^{-1} \cdot \Sigma_2^{-1} \cdot \Sigma_1 \cdot \Sigma_2 \cdot \Sigma_3 \cdot \Sigma_4, \Sigma_2, \Sigma_3, \Sigma_4)$. Since, inside of $G(\mathbf{\Sigma})$, $\Sigma_1 \cdot \Sigma_2 \cdot \Sigma_3 \cdot \Sigma_4$ is the identity, we conclude that the image of $(\bar{Q}_2 \cdot \bar{Q}_3)^3$ inside of $M(4, \mathbf{\Sigma})$ is the identity. The natural projection that maps $(\mathbb{P}^1)^4 - \Delta_4$ onto the last three factors identifies (Lemma 1. 5) the image of $(\bar{Q}_2 \cdot \bar{Q}_3)^3$ with the generator of the second factor of

$$\pi_1\left((\mathbb{P}^1)^4 - \Delta_4, \mathbf{z}^{(0)}\right) = \pi_1\left(\mathbb{P}^1 - \{0, 1, \infty\}, z_0\right) \times \left((\mathbb{P}^1)^3 - \Delta_3, (z_1, z_2, z_3)\right).$$

In order to conclude the lemma we have only to find two generators of $\pi_1(\mathbb{P}^1 - \{0, 1, \infty\}, z_0)$ that are naturally identified with $(\bar{Q}_1)^2$ and $\bar{Q}_1^{-1} \cdot (\bar{Q}_2)^2 \cdot \bar{Q}_1$, respectively.

Figure 1. 7



The element $\bar{Q}_1$ (resp., $\bar{Q}_2$) of $\pi_1(\mathbb{P}^4 - D_4, q^{(0)})$ is represented by a path on $(\mathbb{P}^1)^4 - \Delta_4$ starting at (z_0, z_1, z_2, z_3) and ending at (z_1, z_0, z_2, z_3) (resp., starting at (z_0, z_1, z_2, z_3) and ending at (z_0, z_2, z_1, z_3)). Let ω_{i-1} be a uniformizing variable for the i-th coordinate of $(\mathbb{P}^1)^4$. The path $\bar{Q}_i$ is constant in all slots except the i-th and $i+1$-th: in the i-th slot ω_{i-1} moves from z_{i-1} to z_i; and in the $i+1$-th slot ω_i moves from z_i to z_{i-1}. Figure 1. 7 gives a representation of the motion of the coordinates ω_{i-1} and ω_i.

The motions of ω_{i-1} and ω_i together trace, clockwise on $\mathbb{P}^1$, the boundary of a half-disc.

Let $\bar{Q}_i^*$ be the path similar to $\bar{Q}_i$ except that in the i-th slot ω_{i-1} moves from z_i to z_{i-1} along the dotted arc of Figure 1. 7, and in the $i+1$-th slot ω_i moves from z_{i-1} back to z_i. In the notation of [ArE, 2] the path obtained by following $\bar{Q}_i$ and then $\bar{Q}_i^*$ (we write this as $\bar{Q}_i \circ \bar{Q}_i^*$) represents $(\bar{Q}_i)^2$ in $\pi_1(\mathbb{P}^4 - D_4, q^{(0)})$.

Figure 1. 8 a)

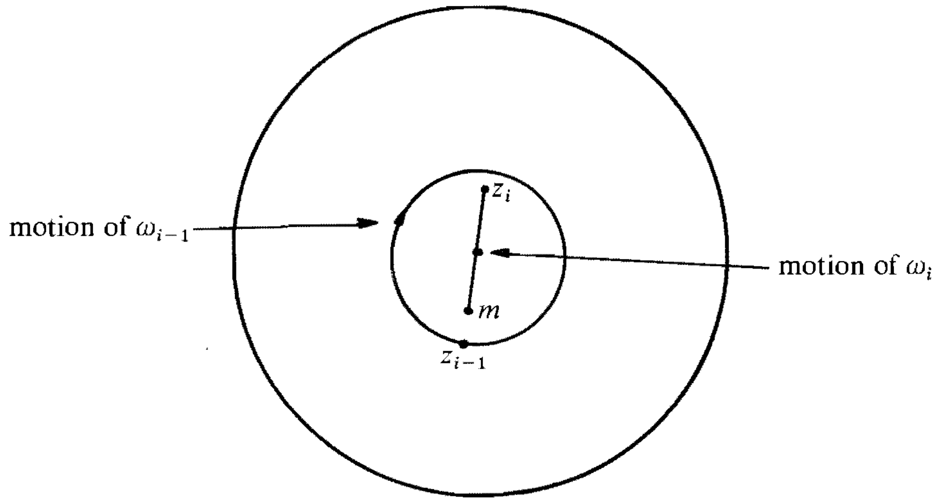
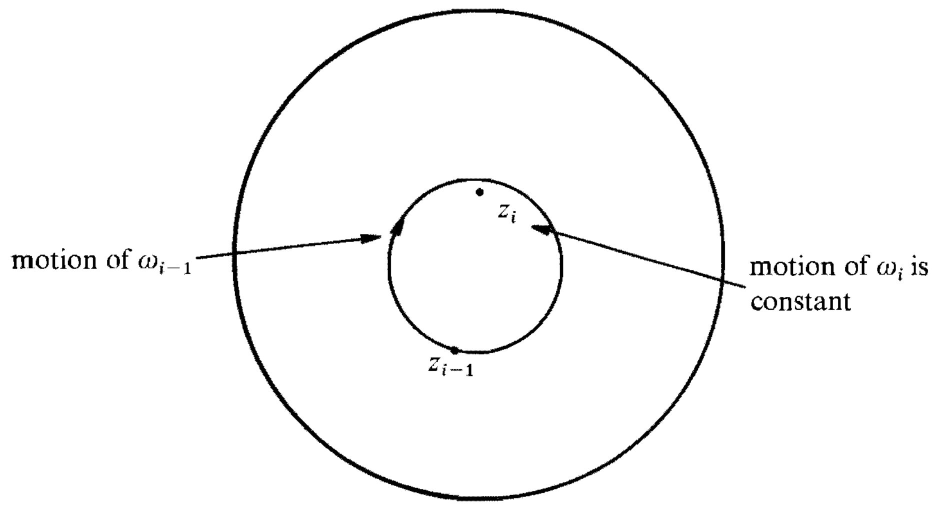
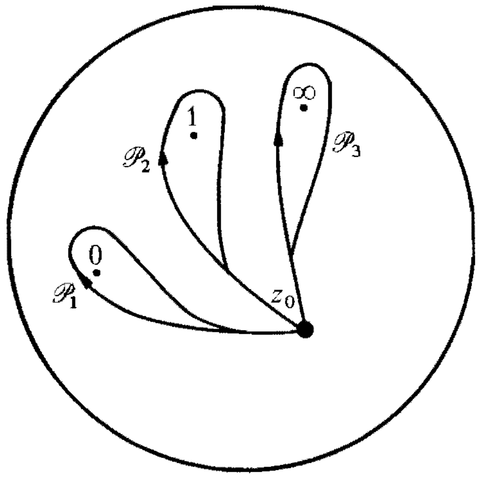


Figure 1. 8 b)



Let m be a point on the arc of the great circle "between" z_i and z_{i-1}. Let $\bar{Q}_i(m) \circ \bar{Q}_i^*(m)$ be the path that is constant in all slots except the i-th and $i+1$-th; in the i-th slot ω_{i-1} moves from z_{i-1} to z_{i-1} along the circle in Figure 1. 8 a) that goes "outside" of z_i; and in the $i+1$-th slot ω_i moves from z_i to m along the arc of the great circle, and then back to z_i. By letting m go to z_i we easily effect a homotopy in which we end up with a path that is non-constant only in the i-th slot, represented by Figure 1. 8 b). The fundamental group of $\mathbb{P}^1 - \{0, 1, \infty\}$ is generated by the class of the paths $\mathscr{P}_1$ and $\mathscr{P}_2$ as in Figure 1. 9. The class of the path $\mathscr{P}_3$ is the inverse of the product of the paths $\mathscr{P}_1$ and $\mathscr{P}_2$.

Figure 1. 9



The reader should have no trouble seeing now that the class of the path $\mathscr{P}_1$ (resp., $\mathscr{P}_2$), represented on $(\mathbb{P}^1 - \{0, 1, \infty\}) \times (0, 1, \infty) \subseteq (\mathbb{P}^1)^4 - \Delta_4$ is homotopic to the path representing $(\bar{Q}_1)^2$ (resp., $\bar{Q}_1^{-1} \cdot (\bar{Q}_2)^2 \cdot \bar{Q}_1$). This concludes the proof of the lemma. ■

§ 2. Nielsen classes, representations of the Hurwitz monodromy group and Hurwitz numbers

Let $\boldsymbol{\sigma} = (\sigma(1), \ldots, \sigma(r)) \in (S_n)^r$ have the following properties:

(2. 1) a) $\sigma(1), \ldots, \sigma(r)$ generate a transitive subgroup, denoted by $G(\boldsymbol{\sigma})$, of S_n; and

b) $\sigma(1) \cdots \sigma(r) = \mathrm{Id}$.

Let $\bar{G}$ be a subgroup of S_n containing $G(\boldsymbol{\sigma})$ and contained in the normalizer of $G(\boldsymbol{\sigma})$ in S_n. For $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ satisfying (2. 1) we say that $\boldsymbol{\sigma}$ is *$\bar{G}$-equivalent* to $\boldsymbol{\tau}$ if there exists $\gamma \bullet \bar{G}$ such that

$$\gamma^{-1} \cdot \boldsymbol{\sigma} \cdot \gamma = (\gamma^{-1} \cdot \sigma(1) \cdot \gamma, \ldots, \gamma^{-1} \cdot \sigma(r) \cdot \gamma)$$

is equal to $\boldsymbol{\tau}$. It is clear that the mapping class group $M(r, \boldsymbol{\Sigma})$, and therefore the Hurwitz monodromy group, acts on the collection of $\bar{G}$-equivalence classes of elements $\boldsymbol{\sigma}$ satisfying expression (2. 1) through substitution of the coordinates of $\boldsymbol{\sigma}$ for the elements $\Sigma_1, \ldots, \Sigma_r$ in the expression (1. 3). There are two other equivalence relations that arise naturally in this context. Denote the $\bar{G}$-equivalence class of $\boldsymbol{\sigma}$ by $\boldsymbol{\sigma}^{\bar{G}}$.

Let $\mathrm{Ni}(\boldsymbol{\sigma}^{\bar{G}})$ (the *$\bar{G}$-Nielsen class* associated to $\boldsymbol{\sigma}$) be the collection of $\bar{G}$-equivalence classes of elements represented by $\boldsymbol{\tau} \in (S_n)^r$ satisfying (2. 1) and for which there exists $\beta \in S_r$ with:

(2. 2) a) $G(\boldsymbol{\tau}) = G(\boldsymbol{\sigma})$; and

b) $\tau\big((i)\beta\big)$ is conjugate to $\sigma(i)$ in $G(\boldsymbol{\sigma})$, $i=1,\ldots,r$.

Again, it is clear that the Hurwitz monodromy group acts on the collection of $\bar{G}$-equivalence classes in $\mathrm{Ni}(\boldsymbol{\sigma}^{\bar{G}})$.

Definition 2. 1. The *$\bar{G}$-Hurwitz number* of $\boldsymbol{\sigma}$, denoted $\mathrm{Hur}(\boldsymbol{\sigma}^{\bar{G}})$, is the number of orbits of $\pi_1(\mathbb{P}^r - D_r, q^{(0)})$ on $\mathrm{Ni}(\boldsymbol{\sigma}^{\bar{G}})$. The *$\bar{G}$-Braid class* (or *$\bar{G}$-Hurwitz class*), denoted $\mathrm{Br}(\boldsymbol{\sigma}^{\bar{G}})$, associated to $\boldsymbol{\sigma}$ consists of the $\bar{G}$-equivalence classes in the orbit of the $\bar{G}$-equivalence class of $\boldsymbol{\sigma}$ under the action of $\pi_1(\mathbb{P}^r - D_r, q^{(0)})$. From the theory of the fundamental group, the transitive representation of $\pi_1(\mathbb{P}^r - D_r, q^{(0)})$ on the $\bar{G}$-Braid class of $\boldsymbol{\sigma}$ corresponds to an equivalence class of unramified covers, denoted

(2. 3) $\Psi_{\mathscr{H}}: \mathscr{H}\big(n, r; \mathrm{Br}(\boldsymbol{\sigma}^{\bar{G}})\big) \to \mathbb{P}^r - D_r$ where $\mathscr{H}\big(n, r; \mathrm{Br}(\boldsymbol{\sigma}^{\bar{G}})\big)$ is called the *Hurwitz parameter space* associated to $\boldsymbol{\sigma}^{\bar{G}}$.

In the case that $\bar{G}$ is the normalizer of $G(\boldsymbol{\sigma})$ in S_n, we drop the $\bar{G}$ notation and speak, for example, about the (absolute) Braid classes, or the (absolute) Hurwitz number, etc.

Lemma 2. 2. *In the special case that the coordinates of $\boldsymbol{\sigma}$ are pairwise non-conjugate in $\bar{G}$, the morphism $\Psi_{\mathscr{H}}$ factors through $(\mathbb{P}^1)^r - \Delta_r$ giving rise to a commutative diagram*

$$\begin{tikzcd} \mathscr{H}\big(n, r; \mathrm{Br}(\boldsymbol{\sigma}^{\bar{G}})\big) \arrow[r] \arrow[dr, "\Psi_{\mathscr{H}}"'] & (\mathbb{P}^1)^r - \Delta_r \arrow[d, "\Psi_r"] \\ & \mathbb{P}^r - D_r \end{tikzcd} \tag{2. 4}$$

Proof. Let $\bar{Q} \in \pi_1(\mathbb{P}^r - D_r, q^{(0)})$. Then $(\boldsymbol{\sigma})\bar{Q} = \boldsymbol{\tau}$ where there exists $\beta \in S_r$ such that $\tau\big((i)\beta\big)$ is conjugate to $\sigma(i)$ in $G(\boldsymbol{\sigma})$, $i=1,\ldots,r$. Thus, $(\boldsymbol{\sigma}^{\bar{G}})\bar{Q} = \boldsymbol{\tau}^{\bar{G}}$. Now suppose there exists $\tau' \in \boldsymbol{\tau}^{\bar{G}}$ and $\beta' \in S_r$ such that $\tau'\big((i)\beta'\big)$ is conjugate to $\sigma(i)$ in $G(\boldsymbol{\sigma})$, $i=1,\ldots,r$. If $\beta' \neq \beta$, then we deduce that two of the coordinates of $\boldsymbol{\sigma}$ are conjugate in $\bar{G}$. Thus $\beta = \beta'$ and the formula $(\boldsymbol{\sigma}^{\bar{G}})\bar{Q} = \boldsymbol{\tau}^{\bar{G}}$ uniquely determines $\beta \in S_r$ associated to $\bar{Q}$. In particular the representation of π_1 on $\bar{G}$-equivalence classes in $\mathrm{Br}(\boldsymbol{\sigma}^{\bar{G}})$ factors through a surjective homomorphism of $\pi_1(\mathbb{P}^r - D_r, q^{(0)})$ onto S_r corresponding to the natural map

$$(\mathbb{P}^1)^r - \Delta_r \xrightarrow{\Psi_r} \mathbb{P}^r - D_r.$$

From the theory of the fundamental group we obtain diagram (2. 4) as an immediate consequence. ∎

There is a more general representation of the mapping class group that is compatible with considerations of modular curves; especially when viewed from the perspective of that historical progenitor [FrK1] (for motivation see the section on modular curves in [Fr, 2]).

Let $(\sigma(1),\ldots,\sigma(r))=\boldsymbol{\sigma}$ satisfy condition (2. 1). Assume also that the disjoint cycles of $\sigma(i)$ are given a labeling: $\sigma(i)=\beta(i,1)\cdots\beta(i,n(i))$, $i=1,\ldots,r$. If $\gamma\in\bar{G}$, then $\gamma^{-1}\cdot\boldsymbol{\sigma}\cdot\gamma$ is an r-tuple whose coordinates naturally inherit a labeling on their disjoint cycles. Let $f^{(i)}$ be *any* function (not necessarily one-one) from $\{1,2,\ldots,n(i)\}$ into $\mathbb{Z}$. And consider a new labeling of the disjoint cycles of $\sigma(i)$ given by: the integer $f^{(i)}(j)$ is associated to $\beta(i,j)$. With this labeling $\boldsymbol{\sigma}$ is now denoted by $(\boldsymbol{\sigma},f)$. For $\gamma\in\bar{G}$, $(\gamma^{-1}\cdot\boldsymbol{\sigma}\cdot\gamma,f_\gamma)$ is the inherited labeling on $\gamma^{-1}\cdot\boldsymbol{\sigma}\cdot\gamma$. For π any permutation of $\mathbb{Z}$ we may compose f_γ with π to obtain $(\gamma^{-1}\cdot\boldsymbol{\sigma}\cdot\gamma,\pi\circ f_\gamma)$. The equivalencing of $(\boldsymbol{\sigma},f)$ to $(\gamma^{-1}\cdot\boldsymbol{\sigma}\cdot\gamma,\pi\circ f_\gamma)$ generates a natural equivalence relation (also called $\bar{G}$-equivalence) on the allowable pairs $(\boldsymbol{\sigma},f)$. Act upon these finitely many equivalence classes with the mapping class group $M(r,\boldsymbol{\Sigma})$ as given by expression (1. 3) with the coordinates of $\boldsymbol{\Sigma}$ replaced by the coordinates of $\boldsymbol{\sigma}$, and the labeling of the element $\sigma(i)\cdot\sigma(i+1)\cdot\sigma(i)^{-1}$ given as above. The *$\bar{G}$-Braid class* (or *$\bar{G}$-Hurwitz class*), denoted $\mathrm{Br}(\boldsymbol{\sigma}^{\bar{G}},f)$, associated to $(\boldsymbol{\sigma},f)$ consists of the equivalence classes in the orbit of the $\bar{G}$-equivalence class of $(\boldsymbol{\sigma},f)$ under the action of $\pi_1(\mathbb{P}^r-D_r,q^{(0)})$ (induced through the action of the mapping class group). As in expression (2. 3) we obtain an equivalence class of unramified covers:

(2. 5) $\Psi_{\mathscr{H}(\boldsymbol{\sigma}^{\bar{G}},f)}:\mathscr{H}(n,r;\mathrm{Br}(\boldsymbol{\sigma}^{\bar{G}},f))\to\mathbb{P}^r-D_r$ where $\mathscr{H}(n,r;\mathrm{Br}(\boldsymbol{\sigma}^{\bar{G}},f))$ is called the *Hurwitz parameter space* associated to $(\boldsymbol{\sigma}^{\bar{G}},f)$. There are three special cases that stand out.

Case 1. $f^{(i)}(j)=1$, $j=1,\ldots,n(i)$, $i=1,\ldots,r$. In this case $\mathscr{H}(n,r;\mathrm{Br}(\boldsymbol{\sigma}^{\bar{G}},f))$ is the same as $\mathscr{H}(n,r;\mathrm{Br}(\boldsymbol{\sigma}^{\bar{G}}))$.

Case 2. $f^{(i)}(j)=i$, $j=1,\ldots,n(i)$, $i=1,\ldots,r$. In this case $\mathscr{H}(n,r;\mathrm{Br}(\boldsymbol{\sigma}^{\bar{G}},f))$ is called an *unsymmetrized Hurwitz parameter space* and it is isomorphic to a connected component of the fiber product $\mathscr{H}(n,r;\mathrm{Br}(\boldsymbol{\sigma}^{\bar{G}}))\times_{\mathbb{P}^r-D_r}((\mathbb{P}^1)^r-\Delta_r)$ coming from the natural maps of both components to $\mathbb{P}^r-D_r$.

Case 3. $f^{(i)}(j)=n(1)+\cdots+n(i-1)+j$, $j=1,\ldots,n(i)$, $i=1,\ldots,r$. In this case $\mathscr{H}(n,r;\mathrm{Br}(\boldsymbol{\sigma}^{\bar{G}},f))$ is called the *ramification Hurwitz parameter space*. It is clearly special in several ways; the simplest being (and we leave this as an easy exercise for the reader) that there is a natural diagram

$$\text{(2. 6)}\qquad \begin{tikzcd} \Psi_{\mathscr{H}(\boldsymbol{\sigma}^{\bar{G}},f)}:\mathscr{H}(n,r;\mathrm{Br}(\boldsymbol{\sigma}^{\bar{G}},f)) \arrow[r] \arrow[d, "\Lambda(g,f)"] & \mathbb{P}^r-D_r \\ \Psi_{\mathscr{H}(\boldsymbol{\sigma}^{\bar{G}},g)}:\mathscr{H}(n,r;\mathrm{Br}(\boldsymbol{\sigma}^{\bar{G}},g)) \arrow[ur] & \end{tikzcd}$$

for some covering map $\Lambda(g,f)$, and for g *any* allowable function as above.

Definition 2. 3. Denote by $T(\boldsymbol{\sigma}^{\bar{G}},f)$ the representation of $\pi_1(\mathbb{P}^r-D_r,q^{(0)})$ corresponding to $(\boldsymbol{\sigma}^{\bar{G}},f)$. We sometimes refer to the function f as (partial) *rigidifying data*.

We return to $T(\boldsymbol{\sigma}^{\bar{G}},f)$ in § 4, and we conclude this section with an analogue of the *congruence subgroup problem*.

Problem 2. 4. *Let $T:M(r,\boldsymbol{\Sigma})\to S_N$ be any transitive representation of $M(r,\boldsymbol{\Sigma})$ and let $H(T)$ be a subgroup of $M(r,\boldsymbol{\Sigma})$ for which T is equivalent to the representation of $M(r,\boldsymbol{\Sigma})$ on the right cosets of $H(T)$. Is it true that there exists a subgroup H' of $H(T)$ such that the representation $T_{H'}$, on the right cosets of H', is equivalent to the representation $T(\boldsymbol{\sigma}^{\bar{G}},f)$ for some pair $(\boldsymbol{\sigma}^{\bar{G}},f)$ as above?*

§ 3. An example with $G(\sigma)$-Hurwitz number 2; an example with absolute Hurwitz number 2

We continue the notation from the last section: $\boldsymbol{\sigma} \in (S_n)^r$; $\sigma(1) \cdots \sigma(r) = \mathrm{Id.}$; and $G(\boldsymbol{\sigma})$ is a transitive subgroup of S_n. Also let $N_{S_n}(G(\boldsymbol{\sigma}))$ be the normalizer of $G(\boldsymbol{\sigma})$ in S_n, and let $\bar{G}$ be a group between $G(\boldsymbol{\sigma})$ and $N_{S_n}(G(\boldsymbol{\sigma}))$. The two extreme cases are of the greatest significance: $\bar{G} = G(\boldsymbol{\sigma})$; and $\bar{G} = N_{S_n}(G(\boldsymbol{\sigma}))$. In this section we give an example where $\mathrm{Hur}(\boldsymbol{\sigma}^{G(\boldsymbol{\sigma})}) = 2$, but $\mathrm{Hur}(\boldsymbol{\sigma}^{N_{S_n}(G(\boldsymbol{\sigma}))}) = 1 = \mathrm{Hur}(\boldsymbol{\sigma})$ to show that the Hurwitz number is not always 1, and also that it can depend on the choice of $\bar{G}$. Finally we conclude with a different choice of $\boldsymbol{\sigma}$ for which $\mathrm{Hur}(\boldsymbol{\sigma}) = 2$. Note that $\mathrm{Hur}(\boldsymbol{\sigma}^{\bar{G}}) \geq \mathrm{Hur}(\boldsymbol{\sigma})$ for any allowable $\bar{G}$.

Let $G = G((\mathbb{Z}/(8))^*, 8)$ be the matrix group consisting of the collection

$$\left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \middle| \, a \in (\mathbb{Z}/(8))^*, \ b \in \mathbb{Z}/(8) \right\}.$$

We may regard G as a subgroup of S_8 through its action on the set $\mathbb{Z}/(8)$. Let $\gamma \in N_{S_8}(G)$ be such that γ normalizes the group generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ represents an 8-cycle in S_8, it is a simple calculation to conclude that $\gamma \in G$.

Lemma 3. 1. *The index of G in $N_{S_8}(G)$ is* 2.

Proof. Let $\gamma \in N_{S_8}(G)$. Then γ acts on the cyclic subgroups of G of order 8 by conjugation. From the observation just above, we have shown that the index of G in $N_{S_8}(G)$ is at most 2 if we show that there are only two cyclic subgroups of G of order 8. But it is easy to see that the groups generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 5 & 1 \\ 0 & 1 \end{pmatrix}$ are the only such groups.

The order of G is $4 \cdot 8 = 2^5$, and so G is contained in a 2-Sylow H of S_8. The order of H is $8 \cdot 2 \cdot 4 \cdot 2 = 2^7$, and since H is a nilpotent group, the normalizer of G in H is non-trivial. Thus, putting these two paragraphs together we conclude that the index of G in $N_{S_8}(G)$ is 2. ∎

The group G has an interesting *outer automorphism* denoted by α, that preserves conjugacy classes and is not represented by an element of S_8. Indeed

$$\alpha : \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \to \begin{cases} \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} & \text{if } a = 1 \text{ or } 3, \\[2ex] \begin{pmatrix} a & b+4 \\ 0 & 1 \end{pmatrix} & \text{if } a = 5 \text{ or } 7. \end{cases} \tag{3. 1}$$

Define $\boldsymbol{\tau}^{\alpha}$ to be the result of applying α to each coordinate of $\boldsymbol{\tau}$.

Example where the $G(\boldsymbol{\sigma})$-Hurwitz number is 2. Consider $\boldsymbol{\sigma} = (\sigma(1), \sigma(2), \sigma(3), \sigma(4))$ where

$$\sigma(1) = \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}, \quad \sigma(2) = \begin{pmatrix} 5 & 1 \\ 0 & 1 \end{pmatrix}, \quad \sigma(3) = \begin{pmatrix} 7 & 5 \\ 0 & 1 \end{pmatrix}, \quad \sigma(4) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then $G(\sigma)$ equals G. The conjugacy class of $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ in G consists of $\begin{pmatrix} a & b' \\ 0 & 1 \end{pmatrix}$ where b' runs over the elements of $\mathbb{Z}/(8)$ congruent to b modulo 2. Thus $\sigma(1), \sigma(2), \sigma(3), \sigma(4)$ are pairwise non-congruent and each element of the $G(\boldsymbol{\sigma})$-Nielsen class is uniquely represented by a 4-tuple $(\tau(1), \tau(2), \tau(3), \tau(4)) = \boldsymbol{\tau}$ with $\tau(1)\cdot\tau(2)\cdot\tau(3)\cdot\tau(4) = \mathrm{Id.}$, $\tau(i) = \begin{pmatrix} a_i & b_i \\ 0 & 1 \end{pmatrix}$ with these properties:

(3. 2) a) a_1, a_2, a_3, a_4 are a permutation of 1, 3, 5, 7,

b) if $a_i = 1$, then $b_i = 1$;

c) if $a_i = 3$, then $b_i = 1$; and

d) $b_i \equiv 1 \bmod 2$, for $i = 1, 2, 3, 4$.

Thus, $\mathrm{Ni}(\boldsymbol{\sigma}^{G(\boldsymbol{\sigma})})$ contains $(4\cdot 3\cdot 2\cdot 1)\cdot 4 = 96$ elements, and the Hurwitz space $\mathrm{Hur}(8, 4; \mathrm{Br}(\boldsymbol{\sigma}^{G(\boldsymbol{\sigma})}))$ is equipped with a natural map to $(\mathbb{P}^1)^4 - \Delta_4$ from Lemma 2. 2. We have a commutative diagram

(3. 3)
$$\begin{tikzcd} \mathrm{Hur}(8, 4; \mathrm{Br}(\boldsymbol{\sigma}^{G(\boldsymbol{\sigma})})) \arrow[r, "\Theta_{\mathscr{H}}"] \arrow[dr, "\Psi_{\mathscr{H}}"'] & (\mathbb{P}^1)^4 - \Delta_4 \arrow[d, "\Psi_4"] \\ & \mathbb{P}^4 - D_4 \end{tikzcd}$$

We now show that $\mathrm{Ni}(\boldsymbol{\sigma}^{G(\boldsymbol{\sigma})}) = \mathrm{Hur}(\boldsymbol{\sigma}^{G(\boldsymbol{\sigma})}) \cup \mathrm{Hur}((\boldsymbol{\sigma}^{\alpha})^{G(\boldsymbol{\sigma})})$. This follows if we show that the degree of $\Theta_{\mathscr{H}}$ is 2, and $\boldsymbol{\sigma}^{\alpha}$ is not contained in $\mathrm{Hur}(\boldsymbol{\sigma}^{G(\boldsymbol{\sigma})})$. To see this we have only, in the notation of Lemma 1. 6, to calculate the orbit of the group generated by $\bar{Q}_1^2$ and $\bar{Q}_1^{-1}\cdot\bar{Q}_2^2\cdot\bar{Q}_1$ on $\boldsymbol{\sigma}^{G(\boldsymbol{\sigma})}$.

Compute:

(3. 4) a)
$$(\boldsymbol{\sigma})(\bar{Q}_1)^2 = \left(\begin{pmatrix} 5 & 7 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 7 & 5 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right)\bar{Q}_1$$
$$= \left(\begin{pmatrix} 3 & 7 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 7 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 7 & 5 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right)$$

which is $G(\boldsymbol{\sigma})$-equivalent to $\left(\begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 7 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right)$;

b)
$$(\boldsymbol{\sigma})\bar{Q}_1^{-1}\cdot(\bar{Q}_2)^2\cdot\bar{Q}_1 = \left(\begin{pmatrix} 5 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 7 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 7 & 5 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right)(\bar{Q}_2)^2\cdot\bar{Q}_1$$
$$= \left(\begin{pmatrix} 5 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 7 & 5 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 7 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right)(\bar{Q}_2\cdot\bar{Q}_1)$$
$$= \left(\begin{pmatrix} 5 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 7 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 7 & 5 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right)\bar{Q}_1 = \boldsymbol{\sigma};$$

and

c) the results of a) and b) give a set stable under $\bar{Q}_1^2$ and $\bar{Q}_1^{-1} \cdot \bar{Q}_2^2 \cdot \bar{Q}_1$ and not containing

$$\boldsymbol{\sigma}^\alpha = \left(\begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 5 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 7 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right).$$

By the way, for the 4-tuple $\boldsymbol{\sigma} = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 7 & 1 \\ 0 & 1 \end{pmatrix} \right)$ all of the Hurwitz numbers are 1; a computation that should certainly be remembered when seeking examples where the Hurwitz number is greater than 1.

We know of no examples of $\boldsymbol{\sigma} \in (S_n)^r$ where $r = 3$ and $\mathrm{Hur}(\boldsymbol{\sigma}^{\bar{G}})$ (for allowable $\bar{G}$) is greater than 1. Here, however, is a way an example might occur. Let $G \subseteqq S_n$ be a transitive group having a conjugacy class preserving automorphism α which is not represented by conjugation by an element of $\bar{G}$, with $G \subset \bar{G} \subset N_{S_n}(G)$. Assume also that H is generated by $\sigma(1)$ and $\sigma(2)$ where

(3. 5) $\sigma(1)$, $\sigma(2)$, and $\sigma(3) = (\sigma(1) \cdot \sigma(2))^{-1}$ are pairwise non-conjugate in $\bar{G}$.

Lemma 3. 2. *The $\bar{G}$-Hurwitz number,* $\mathrm{Hur}(\boldsymbol{\sigma}^{\bar{G}})$, *of* $\boldsymbol{\sigma} = (\sigma(1), \sigma(2), \sigma(3))$ *given by expression* (3. 5) *is greater than* 1.

Proof. We show that $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^\alpha$ represent different $\bar{G}$-Hurwitz classes of $\mathrm{Ni}(\boldsymbol{\sigma}^{\bar{G}})$. Since $\sigma(1)$, $\sigma(2)$, and $\sigma(3)$ are pairwise non-conjugate, in order for $(\boldsymbol{\sigma})\bar{Q}'$ to be $\bar{G}$-equivalent to $\boldsymbol{\sigma}^\alpha$ we must have $\bar{Q}' \in \pi_1\left((\mathbb{P}^1)^3 - \Delta_3, z^{(0)}\right)$, as in Lemma 2. 2. From Lemma 1. 5, $\boldsymbol{\sigma}$ is mapped to a G-equivalent (and therefore $\bar{G}$-equivalent) 3-tuple under the action of $\pi_1\left((\mathbb{P}^1)^3 - \Delta_3, z^{(0)}\right)$. However, by hypothesis, $\boldsymbol{\sigma}^\alpha$ is *not* $\bar{G}$-equivalent to $\boldsymbol{\sigma}$. ■

Problem 3. 3. *Does there exist* $\boldsymbol{\sigma} \in (S_n)^3$ *where* $\boldsymbol{\sigma} = (\sigma(1), \sigma(2), \sigma(3))$, $G(\boldsymbol{\sigma})$ *is a transitive group,* $\sigma(1) \cdot \sigma(2) \cdot \sigma(3) = \mathrm{Id.}$, $\bar{G}$ *a group with* $G(\boldsymbol{\sigma}) \subseteqq \bar{G} \subseteqq N_{S_n}(G(\boldsymbol{\sigma}))$, *and* $\mathrm{Hur}(\boldsymbol{\sigma}^{\bar{G}}) \neq 1$?

Now we comment on $\boldsymbol{\sigma} = (\sigma(1), \ldots, \sigma(r))$ with $\sigma(i) = \begin{pmatrix} a_i & b_i \\ 0 & 1 \end{pmatrix} \in G\left((\mathbb{Z}/(8))^*, 8\right)$ and r arbitrary. Let X_1 (resp., X_3, X_5, X_7) be the collection of values b_i for which $a_i = 1$ (resp., 3, 5, 7). Let $|X_3| = m(3)$, $|X_5| = m(5)$, $|X_7| = m(7)$. The condition that $\sigma(1) \cdots \sigma(r)$ is the identity is equivalent to

(3. 6) a) $b_r \cdot a_1 \cdot a_2 \cdots a_{r-1} + b_{r-1} \cdot a_1 \cdot a_2 \cdots a_{r-2} + \cdots + b_1 = 0$; and

b) $(m(3) \bmod 2,\ m(5) \bmod 2,\ m(7) \bmod 2)$ is in the group generated by $(1, 1, 1)$ in $(\mathbb{Z}/(2))^3$.

In addition, $G(\boldsymbol{\sigma}) = G\left((\mathbb{Z}/(8))^*, 8\right)$ if and only if:

(3. 7) a) $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ for some b odd is in $G(\boldsymbol{\sigma})$; and

b) two of $m(3)$, $m(5)$, $m(7)$ are positive.

Let $X_i \bmod 2$ be the reduction of the elements of X_i modulo 2. Then expression (3. 7) a) is equivalent to either;

(3. 8) a) $X_i = \{0, 1\} \bmod 2$ for some $i = 3, 5$, or 7; or

b) $X_1 \bmod 2$ contains 1; or

c) $X_i = \{1\} \bmod 2$, $i = 3, 5$, and 7.

Problem 3. 4. *Are there infinitely many values of r for which there exist* $\boldsymbol{\sigma}$ *satisfying the conditions above, and for which* $\mathrm{Hur}(\boldsymbol{\sigma}^{G(\boldsymbol{\sigma})})$ *is different from* 1? See last lines of § 5 for potential examples.

Example where the absolute Hurwitz number is 2. Let $G = G((\mathbb{Z}/(8))^*, 8)$, as above. We continue to treat $\boldsymbol{\sigma} \in (S_n)^r$ satisfying the hypotheses at the beginning of the section, including $G(\boldsymbol{\sigma}) = G$: except that we are now interested in $\mathrm{Hur}(\boldsymbol{\sigma})$, the absolute Hurwitz number (i.e., $\bar{G} = N_{S_8}(G)$).

Lemma 3. 5. *For* $\boldsymbol{\sigma} \in (S_8)^4$ *the* 4*-tuple given above for which* $\mathrm{Hur}(\boldsymbol{\sigma}^{G(\boldsymbol{\sigma})}) = 2$, *we have* $\mathrm{Hur}(\boldsymbol{\sigma}) = 1$.

Proof. Let $\beta \in S_8$ be a representative of the generator of the quotient $N_{S_8}(G(\boldsymbol{\sigma}))/G(\boldsymbol{\sigma})$ (see Lemma 3. 1). Represent $\gamma \in G$ as an element of S_8: if γ applied to i is j, where $i, j \in \mathbb{Z}/(8)$, then γ corresponds to the element in S_8 that takes $i+1$ to $j+1$. Then: $\begin{pmatrix} 5 & 1 \\ 0 & 1 \end{pmatrix}$ corresponds to $(1\ 2\ 7\ 8\ 5\ 6\ 3\ 4)$; $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ corresponds to $(1\ 2\ 3\ 4\ 5\ 6\ 7\ 8)$; $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ corresponds to $(24)\ (37)\ (68)$; and $\begin{pmatrix} 7 & 0 \\ 0 & 1 \end{pmatrix}$ corresponds to $(28)\ (37)\ (46)$. Finally we may take β to be $(1\ 3\ 5\ 7)$ and we compute that

$$(3.\,9)\quad \text{a)}\ \beta^{-1} \cdot \left(\begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 7 & 5 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) \cdot \beta$$
$$= \left(\begin{pmatrix} 3 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 7 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 7 & 7 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 7 \\ 0 & 1 \end{pmatrix} \right) = \boldsymbol{\sigma}^{\beta}.$$

In order to show that $\mathrm{Hur}(\boldsymbol{\sigma}) = 1$, from the computation of the discussion preceding expression (3. 4), we have only to show that there exists $\bar{Q} \in M(4, \boldsymbol{\Sigma})$ and $\gamma \in G(\boldsymbol{\sigma})$ for which $\gamma^{-1} \cdot (\boldsymbol{\sigma}^{\beta})\, \bar{Q} \cdot \gamma = \boldsymbol{\sigma}^{\alpha}$. The following chain of equivalences demonstrates the existence of $\bar{Q}$ and γ:

(3. 9) b) conjugation of $\boldsymbol{\sigma}^{\beta}$ by $\begin{pmatrix} 7 & 2 \\ 0 & 1 \end{pmatrix}$ gives $\left(\begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 7 & 5 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 1 \\ 0 & 1 \end{pmatrix} \right)$ and application of $\bar{Q}_3^{-1} \cdot \bar{Q}_2 \cdot \bar{Q}_3$ to this element gives

$$\left(\begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 7 & 7 \\ 0 & 1 \end{pmatrix} \right) \bar{Q}_2 \cdot \bar{Q}_3 = \left(\begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 5 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 7 & 7 \\ 0 & 1 \end{pmatrix} \right) \bar{Q}_3$$
$$= \left(\begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 5 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 7 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right),$$

which is $\boldsymbol{\sigma}^{\alpha}$. ■

In order to obtain an example where the absolute Hurwitz number is 2 we must consider $\boldsymbol{\sigma} \in (S_8)^6$. Indeed, let

$$\boldsymbol{\sigma} = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 5 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 7 \\ 0 & 1 \end{pmatrix} \right),$$

and

$$\boldsymbol{\tau} = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right).$$

We will show that $\mathrm{Ni}(\boldsymbol{\sigma}) = \mathrm{Br}(\boldsymbol{\sigma}) \cup \mathrm{Br}(\boldsymbol{\tau})$, and $\boldsymbol{\tau} \notin \mathrm{Br}(\boldsymbol{\sigma})$ in order to conclude that $\mathrm{Hur}(\boldsymbol{\sigma}) = 2$.

Definition 3. 6. Let $\boldsymbol{\tau}' \in \mathrm{Ni}(\boldsymbol{\sigma})$. Then $\boldsymbol{\tau}'$ is of *type* $(a_1, a_2, a_3, a_4, a_5, a_6) = \mathbf{a}$ if and only if $\tau(i)'$, the i-th coordinate of $\boldsymbol{\tau}'$ is $\begin{pmatrix} a_i & b_i \\ 0 & 1 \end{pmatrix}$ for some b_i, $i = 1, \ldots, 6$.

Let $T = \{(a_1, a_2, \ldots, a_6) \mid \text{there exists } \bar{\boldsymbol{\sigma}} \in \mathrm{Ni}(\boldsymbol{\sigma}) \text{ of type } (a_1, a_2, \ldots, a_6)\}$. Then S_6 acts transitively on T by permuting the coordinates of $\mathbf{a}$. The stabilizer of $(1, 3, 5, 5, 3, 1)$ is of order 8 generated by $(3\ 4)$, $(2\ 5)$, $(1\ 6)$. So $|T| = 6!/8 = 90$. Also, if $\bar{\boldsymbol{\sigma}} \in \mathrm{Ni}(\boldsymbol{\sigma})$ is of type $(a_1, \ldots, a_6)$, then (from proof of Lemma 3. 1) every element of $(\bar{\boldsymbol{\sigma}})^{N_{S_8}(G)}$ is of type $(a_1, \ldots, a_6)$ or of type $(a_1', a_2', \ldots, a_6')$ where

$$a_i' = \begin{cases} 1 & \text{if} \quad a_i = 5, \\ 3 & \text{if} \quad a_i = 3, \\ 5 & \text{if} \quad a_i = 1. \end{cases}$$

Finally, by conjugating by an element of G we may assume that there is a unique representative $\boldsymbol{\tau}' \in (\bar{\boldsymbol{\sigma}})^{N_{S_8}(G)}$ for which, if a_i' is the first 1, a_j' is the first 3, a_k' is the first 5, then $i < k$, $\tau(i)' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and $\tau(j)' = \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}$. Thus, for a fixed $\mathbf{a} \in T$, there are $1 \cdot 1 \cdot 4 \cdot 4 \cdot 4 \cdot 1 = 64$ elements of type $\mathbf{a}$ in $\mathrm{Ni}(\boldsymbol{\sigma})$, and the number of $N_{S_8}(G)$-equivalence classes making up $\mathrm{Ni}(\boldsymbol{\sigma})$ is $(|T|/2) \cdot 64 = 2{,}880$.

Recall now the homomorphism $\alpha_6 : \pi_1(\mathbb{P}^6 - D_6, q^{(0)}) \to S_6$ given by Definition 1. 3. Let $\langle (3\ 4), (2\ 5), (1\ 6) \rangle$ denote the subgroup of S_6 generated by $(3\ 4)$, $(2\ 5)$, and $(1\ 6)$, and let $H = \alpha_6^{-1}(\langle (3\ 4), (2\ 5), (1\ 6) \rangle)$. It is clear that if $\boldsymbol{\tau}'$ is of type $(1, 3, 5, 5, 3, 1)$ then $(\boldsymbol{\tau}')\bar{Q}$ is of type $(1, 3, 5, 5, 3, 1)$ iff $\bar{Q} \in H$ because $\alpha_6(\bar{Q})$ applied to the type $(1, 3, 5, 5, 3, 1)$ leaves it fixed if and only if $\alpha_6(\bar{Q}) \in \langle (3\ 4), (2\ 5), (1\ 6) \rangle$ (as above).

Lemma 3. 7. *The group H is generated by K, the kernel of α_6, and*

$$\bar{Q}_3, \bar{Q}_2 \cdot \bar{Q}_3 \cdot \bar{Q}_4 \cdot \bar{Q}_3 \cdot \bar{Q}_2 \quad \textit{and} \quad \bar{Q}_1 \cdot \bar{Q}_2 \cdot \bar{Q}_3 \cdot \bar{Q}_4 \cdot \bar{Q}_5 \cdot \bar{Q}_4 \cdot \bar{Q}_3 \cdot \bar{Q}_2 \cdot \bar{Q}_1.$$

Proof. Let H' be the group generated by K and the three given elements. There is a natural section $\beta : S_6 \to \pi_1(\mathbb{P}^6 - D_6, q^{(0)})$ (not a homomorphism) which maps (ij), $i < j$, to $\bar{Q}_i \cdot \bar{Q}_{i+1} \cdots \bar{Q}_{j-1} \cdot \bar{Q}_{j-2} \cdots \bar{Q}_i$, and for which $\beta \circ \alpha_6$ equals the identity map on S_6. Note, we are using a right-hand action for β and α_6. Now, if $\bar{Q} \in H$, then by definition of β, $(\bar{Q})\alpha_6 \circ \beta$ is in the group generated by the three elements in the statement of the lemma. Consider the element $k = ((\bar{Q})\alpha_6 \circ \beta)^{-1} \cdot \bar{Q}$. Since α_6 is a homomorphism we compute that $(k)\alpha_6 = \mathrm{Id}$. Thus $k \in K$ and $H \subseteq H'$. That H' is contained in H is obvious. This concludes the lemma. ■

Apply Lemma 1. 6 to obtain an explicit set of generators for H. These are in Table 1 of § 6.

Lemma 3. 8. *For* $\boldsymbol{\sigma}=(\sigma(1),\ldots,\sigma(r))\in(S_n)^r$ *with* $\sigma(1)\cdots\sigma(r)=\mathrm{Id.}$, *and* $\gamma\in G(\boldsymbol{\sigma})$, *there exists* $\bar{Q}\in\pi_1(\mathbb{P}^r-D_r,q^{(0)})$ *such that* $(\boldsymbol{\sigma})\bar{Q}=\gamma\cdot\boldsymbol{\sigma}\cdot\gamma^{-1}$.

Proof. We prove this by induction on the number of elements of $\{\sigma(1),\ldots,\sigma(r)\}$ needed to write γ as a product. To do this, let $\gamma=\gamma'\cdot\sigma(i)$ for some $\gamma'\in G(\boldsymbol{\sigma})$ and some integer i. Suppose that we have found $\bar{Q}'$ such that $(\boldsymbol{\sigma})\bar{Q}'$ is equal to $\boldsymbol{\tau}=\gamma'\cdot\boldsymbol{\sigma}\cdot(\gamma')^{-1}$ and suppose also that there exists $\bar{Q}''$ such that $(\boldsymbol{\tau})\bar{Q}''=\tau(i)\cdot\boldsymbol{\tau}\cdot\tau(i)^{-1}$. Then, $((\boldsymbol{\sigma})\bar{Q}')\bar{Q}''=\tau(i)\cdot\gamma'\cdot\boldsymbol{\sigma}\cdot(\gamma')^{-1}\cdot\tau(i)^{-1}$. But $\tau(i)\cdot\gamma'=\gamma'\cdot\sigma(i)\cdot(\gamma')^{-1}\cdot\gamma'=\gamma'\cdot\sigma(i)$. Thus, by the induction assumption we are returned to the case that $\gamma=\sigma(i)$, $i=1,\ldots,r$.

We have:

$$(3.10)\quad (((\boldsymbol{\sigma})\bar{Q}_i)\bar{Q}_{i+1})\cdots)\bar{Q}_{r-1}$$
$$=(\sigma(1),\ldots,\sigma(i-1),\sigma(i)\cdot\sigma(i+1)\cdot\sigma(i)^{-1},\ldots,\sigma(i)\cdot\sigma(r)\cdot\sigma(i)^{-1},\sigma(i)).$$

For $\boldsymbol{\tau}\in(S_n)^r$, $((((\boldsymbol{\tau})\bar{Q}_{r-1})\bar{Q}_{r-2})\cdots)\bar{Q}_1$ is equal to

$$(3.11)\quad \text{a)}\ (\beta\cdot\tau(r)\cdot\beta^{-1},\tau(1),\tau(2),\ldots,\tau(r-1))\ \text{ where }\ \beta=\tau(1)\cdots\tau(r-1).$$

In the case that $\tau(1)\cdots\tau(r)=\mathrm{Id.}$, expression (3. 11) a) becomes

$$(3.11)\quad \text{b)}\ (\tau(r),\tau(1),\ldots,\tau(r-1)).$$

Let $(\boldsymbol{\sigma})\bar{Q}^*$ be the r-tuple obtained in expression (3. 10), and let $((\boldsymbol{\sigma})\bar{Q}^*)\bar{Q}^{**}$ be the r-tuple obtained in expression (3. 11) a) by taking $\boldsymbol{\tau}=(\boldsymbol{\sigma})\bar{Q}^*$. Then

$$(((((\boldsymbol{\sigma})\bar{Q}^*)\bar{Q}^{**})\bar{Q}_1)\cdots)\bar{Q}_{i-1}$$

is equal to $\sigma(i)\cdot\boldsymbol{\sigma}\cdot\sigma(i)^{-1}$. ∎

Proposition 3. 9. *For* $\boldsymbol{\sigma}$ *the* 6-*tuple that appears just prior to Definition* 3. 6, *the absolute Hurwitz number of* $\boldsymbol{\sigma}$ *is* 2.

Proof. Let $A=\{(\boldsymbol{\sigma})\bar{Q}\mid\bar{Q}\in H\}$, $B=\{(\boldsymbol{\tau})\bar{Q}\mid\bar{Q}\in H\}$ where $\boldsymbol{\tau}$ is also given prior to Definition 3. 6. Let A' be the set of elements given in Table 2 of § 6: A' is a subset, consisting of 32 elements of $\{(\boldsymbol{\sigma})\bar{Q}\mid\bar{Q}\in\pi_1(\mathbb{P}^6-D_6,q^{(0)})\}$, chosen by a judicious process of applying elements of H to $\boldsymbol{\sigma}$. Table 3 of § 6 then contains the information to show that, as we had hoped, the set A' is stable under H. Thereby, Table 2 gives representatives of the 32 distinct $G(\boldsymbol{\sigma})$-equivalence classes in the orbit of $\boldsymbol{\sigma}^{G(\boldsymbol{\sigma})}$ under H.

Step 1. $\mathrm{Hur}(\boldsymbol{\sigma}^{G(\boldsymbol{\sigma})})>1$. By inspection we have seen that $\boldsymbol{\tau}$ is not in A. Suppose that $\gamma^{-1}\cdot(\boldsymbol{\sigma})\bar{Q}\cdot\gamma=\boldsymbol{\tau}$ for some $\gamma\in G(\boldsymbol{\sigma})$ and $\bar{Q}\in\pi_1(\mathbb{P}^6-D_6,q^{(0)})$. Let $(\boldsymbol{\sigma})\bar{Q}=\boldsymbol{\sigma}'$. From Lemma 3. 8 there exists $\bar{Q}^*\in\pi_1(\mathbb{P}^6-D_6,q^{(0)})$ for which $(\boldsymbol{\sigma}')\bar{Q}^*=\gamma^{-1}\cdot\boldsymbol{\sigma}'\cdot\gamma$, and since the type of $\boldsymbol{\sigma}'$ is conserved by conjugation by γ, we must have $\bar{Q}^*\in H$. So, $\boldsymbol{\tau}=(\boldsymbol{\sigma})\bar{Q}'$ for some $\bar{Q}'\in H$ contrary to $\boldsymbol{\tau}$ not being contained in A.

Step 2. $\mathrm{Hur}(\boldsymbol{\sigma})>1$. We show that $\boldsymbol{\tau}$ is not in $\mathrm{Br}(\boldsymbol{\sigma})$. Let $\bar{\beta}$ be any representative of the generator of $N_{S_8}(G)/G$ (as in Lemma 3. 1). From the argument of Step 1, if there exists $\bar{Q}\in\pi_1(\mathbb{P}^6-D_6, q^{(0)})$ for which $(\bar{\beta}^{-1}\cdot\boldsymbol{\sigma}\cdot\bar{\beta})\bar{Q}\in A$ we easily deduce that $\boldsymbol{\tau}$ is not in $\mathrm{Br}(\boldsymbol{\sigma})$. In the argument of Lemma 3. 5, in the embedding of G in S_8 we took $\beta=(1\ 3\ 5\ 7)$. Let

$$\bar{\beta}=\begin{pmatrix}3&0\\0&1\end{pmatrix}\begin{pmatrix}1&7\\0&1\end{pmatrix}^2\cdot\beta;\quad \text{so } \bar{\beta}=(3\ 7)(4\ 8)$$

in the embedding in S_8. Then

$$\bar{\beta}^{-1}\cdot\boldsymbol{\sigma}\cdot\bar{\beta}=\left(\begin{pmatrix}5&1\\0&1\end{pmatrix},\begin{pmatrix}3&1\\0&1\end{pmatrix},\begin{pmatrix}1&1\\0&1\end{pmatrix},\begin{pmatrix}1&7\\0&1\end{pmatrix},\begin{pmatrix}3&5\\0&1\end{pmatrix},\begin{pmatrix}5&3\\0&1\end{pmatrix}\right).$$

Apply $\bar{Q}_4^{-1}\cdot\bar{Q}_5^{-1}\cdot\bar{Q}_4^{-1}\cdot\bar{Q}_1^{-1}\cdot\bar{Q}_2^{-1}\cdot\bar{Q}_1^{-1}$ to this to get

$$\left(\begin{pmatrix}1&1\\0&1\end{pmatrix},\begin{pmatrix}3&1\\0&1\end{pmatrix},\begin{pmatrix}5&7\\0&1\end{pmatrix},\begin{pmatrix}5&7\\0&1\end{pmatrix},\begin{pmatrix}3&5\\0&1\end{pmatrix},\begin{pmatrix}1&1\\0&1\end{pmatrix}\right),$$

which is in A (see Table 2 of § 6).

Step 3. $\mathrm{Hur}(\boldsymbol{\sigma}^{G(\boldsymbol{\sigma})})=\mathrm{Hur}(\boldsymbol{\sigma})$. Let $\mathrm{Ni}(\boldsymbol{\sigma})^{\mathbf{a}}$ be the collection of elements of type $\mathbf{a}$ in $\mathrm{Ni}(\boldsymbol{\sigma})$. For $\bar{Q}\in\pi_1(\mathbb{P}^6-D_6, q^{(0)})$ let $A^{\bar{Q}}=\{(\boldsymbol{\tau}')\bar{Q}\mid\boldsymbol{\tau}'\in A\}$. Then we have shown that for all $\mathbf{a}\in T$ there exists $\bar{Q}\in\pi_1(\mathbb{P}^6-D_6, q^{(0)})$ for which

$$\mathrm{Ni}(\boldsymbol{\sigma})^{\mathbf{a}}=A^{\bar{Q}}\cup B^{\bar{Q}}\ (\text{where } A^{\bar{Q}}\cap B^{\bar{Q}}=\emptyset).$$

From Step 1, $\mathrm{Br}(\boldsymbol{\sigma}^{G(\boldsymbol{\sigma})})$ is the union of the $G(\boldsymbol{\sigma})$-equivalence classes in $A^{\bar{Q}}$ over all allowable $\bar{Q}$; and $\mathrm{Br}(\boldsymbol{\tau}^{G(\boldsymbol{\sigma})})$ is the union of the $G(\boldsymbol{\sigma})$-equivalence classes in $B^{\bar{Q}}$ over all allowable $\bar{Q}$. Thus $\mathrm{Hur}(\boldsymbol{\sigma}^{G(\boldsymbol{\sigma})})=2\geqq\mathrm{Hur}(\boldsymbol{\sigma})>1$ (from Step 2), and the result follows. ■

§ 4. On the existence of a fine moduli space corresponding to a given representation of the Hurwitz monodromy group

Again let $\boldsymbol{\sigma}\in(S_n)^r$, where $\sigma(1),\ldots,\sigma(r)$ generate a transitive subgroup $G(\boldsymbol{\sigma})$ of S_n, and $\sigma(1)\cdots\sigma(r)=\mathrm{Id}.$ as at the beginning of § 3. We consider in this section (for simplicity) only the case when $\bar{G}=N_{S_n}(G(\boldsymbol{\sigma}))$, the normalizer of $G(\boldsymbol{\sigma})$ in S_n. In particular we start with the representation of $\pi_1(\mathbb{P}^r-D_r, q^{(0)})$ on the $N_{S_n}(G(\boldsymbol{\sigma}))$-equivalence classes (or *absolute* equivalence classes) comprising $\mathrm{Ni}(\boldsymbol{\sigma}^{N_{S_n}(G(\boldsymbol{\sigma}))})=\mathrm{Ni}(\boldsymbol{\sigma})$ given by expression (1. 3) (and as explained prior to Definition 2. 1). The set $\mathrm{Ni}(\boldsymbol{\sigma})$ is a union $\bigcup_{k=1}^{l}\mathrm{Br}(\boldsymbol{\sigma}_k)$ of distinct absolute Braid classes, each of which corresponds to a *transitive* permutation representation T_k, $k=1,\ldots,l$, of $\pi_1(\mathbb{P}^r-D_r, q^{(0)})$. These representations are not necessarily distinct, and therefore the spaces $\{\mathscr{H}(n,r;\mathrm{Br}(\boldsymbol{\sigma}_k))\}_{k=1}^{l}$ (as in diagram (2. 3)) are not necessarily inequivalent as covers of $\mathbb{P}^r-D_r$. We define $\mathscr{H}(n,r;\mathrm{Ni}(\boldsymbol{\sigma}))$ to be the disjoint union $\bigcup_{k=1}^{l}\mathscr{H}(n,r;\mathrm{Br}(\boldsymbol{\sigma}_k))$ which is naturally presented as an unramified cover of $\mathbb{P}^r-D_r$.

The Nielsen type of a cover of $\mathbb{P}^1$. Let $X \xrightarrow{\varphi} \mathbb{P}^1$ be a cover of degree n of $\mathbb{P}^1$ by a nonsingular curve X for which $z_1, \ldots, z_r \in \mathbb{P}^1$ contain among them all the branch points of φ. By Riemann's existence theorem ([Fr, 1, 2]) X is given, up to equivalence as a cover of $\mathbb{P}^1$, by an absolute Hurwitz class represented by some $\boldsymbol{\sigma}' \in (S_n)^r$ (satisfying the standard properties given in the beginning sentence of this section). Define the *Nielsen type* of $X \xrightarrow{\varphi} \mathbb{P}^1$, to be $\mathrm{Ni}(\boldsymbol{\sigma}')$. As in [Fr, 3]; § 1.c), note that if X and φ are presented explicitly by algebraic equations in the variables of some projective space, then $\mathrm{Ni}(\boldsymbol{\sigma}')$ can be computed explicitly and algebraically (e.g., with no reference to the complex metric) from the coefficients of these algebraic equations. The point of introducing the space $\mathscr{H}(n, r; \mathrm{Ni}(\boldsymbol{\sigma}))$ is that it is a *course moduli space for covers of* $\mathbb{P}^1$ *of Nielsen type equal to* $\mathrm{Ni}(\boldsymbol{\sigma})$. That is, for each equivalence class of covers of $\mathbb{P}^1$ having its Nielsen type equal to $\mathrm{Ni}(\boldsymbol{\sigma})$ there is a unique naturally corresponding point of $\mathscr{H}(n, r; \mathrm{Ni}(\boldsymbol{\sigma}))$. In particular, suppose that we are given a *family of covers of type* $\mathrm{Ni}(\boldsymbol{\sigma})$ over the *parameter space* $\mathscr{P}$, represented by the symbol $(\mathscr{T}, \Phi, \mathscr{P})$, and consisting of the following data:

$$\begin{tikzcd} \mathscr{T} \arrow[r, "\Phi"] & \mathscr{P} \times \mathbb{P}^1 \arrow[r, "pr_1"] \arrow[dr, "pr_2"] & \mathscr{P} \\ & & \mathbb{P}^1 \end{tikzcd} \tag{4. 1}$$

where

a) Φ is a proper map of degree n;

b) $\mathscr{T}$ and $\mathscr{P}$ are complex manifolds; and

c) for each $\mathfrak{p} \in \mathscr{P}$, $pr_2 \circ \Phi$ presents the fiber $\mathscr{T}_{\mathfrak{p}} \stackrel{\mathrm{def}}{=} (pr_1 \circ \Phi)^{-1}(\mathfrak{p})$ as a cover of degree n of $\mathbb{P}^1$ of type $\mathrm{Ni}(\boldsymbol{\sigma})$.

Let $\Psi(\mathscr{H}, \mathscr{P})$ be the natural map $\mathscr{P} \longrightarrow \mathscr{H}(n, r; \mathrm{Ni}(\boldsymbol{\sigma}))$ that associates to $\mathfrak{p} \in \mathscr{P}$ the point of $\mathscr{H}(n, r; \mathrm{Ni}(\boldsymbol{\sigma}))$ representing the equivalence class of the cover $\mathscr{T}_{\mathfrak{p}} \longrightarrow \mathbb{P}^1$ given by (4. 1) c). Then the map $\Psi(\mathscr{H}, \mathscr{P})$ is a complex analytic map ([Fr, 1]; § 4, the proof of Proposition 5).

The notion of a fine moduli space for covers of $\mathbb{P}^1$ *of Nielsen type equal to* $\mathrm{Ni}(\boldsymbol{\sigma})$. Let $(\mathscr{T}, \Phi, \mathscr{P})$ and $(\mathscr{T}', \Phi', \mathscr{P})$ be two families of covers of type $\mathrm{Ni}(\boldsymbol{\sigma})$ over the same parameter space $\mathscr{P}$. We have two distinct notions of equivalence of such families of covers:

Definition 4. 1. The families $(\mathscr{T}, \Phi, \mathscr{P})$ and $(\mathscr{T}', \Phi', \mathscr{P})$ are *étale-equivalent* if there exists a finite unramified (surjective) map $\bar{\mathscr{P}} \xrightarrow{\Psi(\mathscr{P}, \bar{\mathscr{P}})} \mathscr{P}$ with the following properties. Let $\mathscr{T} \times_{\mathscr{P}} \bar{\mathscr{P}}$, the *fiber product* of $\mathscr{T}$ and $\bar{\mathscr{P}}$ over $\mathscr{P}$, be the complex manifold whose points are the pairs $(t, \bar{\mathfrak{p}})$ for which $pr_1 \circ \Phi(t) = \Psi(\mathscr{P}, \bar{\mathscr{P}})(\bar{\mathfrak{p}})$. There is a natural map $\mathscr{T} \times_{\mathscr{P}} \bar{\mathscr{P}} \xrightarrow{\bar{\Phi}} \bar{\mathscr{P}} \times \mathbb{P}^1$. Then there exists an analytic isomorphism $\Theta : \mathscr{T} \times_{\mathscr{P}} \bar{\mathscr{P}} \longrightarrow \mathscr{T}' \times_{\mathscr{P}} \bar{\mathscr{P}}$ for which: $\bar{\Phi}' \circ \Theta = \bar{\Phi}$. The families $(\mathscr{T}, \Phi, \mathscr{P})$ and $(\mathscr{T}', \Phi', \mathscr{P})$ are said to be *Zariski-equivalent* if we may take $\bar{\mathscr{P}} = \mathscr{P}$ in the definition above.

Suppose there exists a family of covers of type $\mathrm{Ni}(\boldsymbol{\sigma})$, say

$$(\mathscr{T}(\mathrm{Ni}(\boldsymbol{\sigma})), \Phi(\mathrm{Ni}(\boldsymbol{\sigma})), \mathscr{H}(n, r; \mathrm{Ni}(\boldsymbol{\sigma}))),$$

over the parameter space $\mathscr{H} = \mathscr{H}(n, r; \mathrm{Ni}(\boldsymbol{\sigma}))$ having the property that

(4. 2) $\mathscr{T}(\mathrm{Ni}(\boldsymbol{\sigma}))_{\mathfrak{p}} \to \mathbb{P}^1$ (as in (4. 1) c)) is a cover in the equivalence class of $\mathfrak{p} \in \mathscr{H}(n, r; \mathrm{Ni}(\boldsymbol{\sigma}))$.

For any other family $(\mathscr{T}, \Phi, \mathscr{P})$ of type $\mathrm{Ni}(\boldsymbol{\sigma})$ we may ask if the family

$$(\mathscr{T}(\mathrm{Ni}(\boldsymbol{\sigma})) \times_{\mathscr{H}} \mathscr{P}, \Phi(\mathrm{Ni}(\boldsymbol{\sigma})) \times (\Psi(\mathscr{H}, \mathscr{P}) \times \mathrm{Id.}), \mathscr{P}),$$

obtained by pullback over the map $\Psi(\mathscr{H}, \mathscr{P})$, is equivalent to the family $(\mathscr{T}, \Phi, \mathscr{P})$. The answer is given in the following lemma:

Lemma 4. 2. *Let* $\mathrm{Cen}_{S_n}(G(\boldsymbol{\sigma}))$ *be the subgroup of* S_n *consisting of the elements that centralize the group* $G(\boldsymbol{\sigma})$. *In the case that* $\mathrm{Cen}_{S_n}(G(\boldsymbol{\sigma})) = \{\mathrm{Id.}\}$, *the families* $(\mathscr{T}, \Phi, \mathscr{P})$ *and*

$$(\mathscr{T}(\mathrm{Ni}(\boldsymbol{\sigma})) \times_{\mathscr{H}} \mathscr{P}, \Phi(\mathrm{Ni}(\boldsymbol{\sigma})) \times (\Psi(\mathscr{H}, \mathscr{P}) \times \mathrm{Id.}), \mathscr{P})$$

are Zariski-equivalent. They are always étale-equivalent.

Proof. Contained in the proof of Proposition 5 of [Fr, 1]. ■

This lemma makes transparent the motivation for the following definition:

Definition 4. 3. The family $(\mathscr{T}(\mathrm{Ni}(\boldsymbol{\sigma})), \Phi(\mathrm{Ni}(\boldsymbol{\sigma})), \mathscr{H}(n, r; \mathrm{Ni}(\boldsymbol{\sigma})))$ is said to *present* $\mathscr{H}(n, r; \mathrm{Ni}(\boldsymbol{\sigma}))$ *as a fine étale-moduli space for covers of* $\mathbb{P}^1$ *of Nielsen type equal to* $\mathrm{Ni}(\boldsymbol{\sigma})$. And if $\mathrm{Cen}_{S_n}(G(\boldsymbol{\sigma})) = \{\mathrm{Id.}\}$, then it presents $\mathscr{H}(n, r; \mathrm{Ni}(\boldsymbol{\sigma}))$ as a *fine Zariski-moduli space.*

The main problem. *For which* $\boldsymbol{\sigma}$ *does there exist a family*

$$(\mathscr{T}(\mathrm{Ni}(\boldsymbol{\sigma})), \Phi(\mathrm{Ni}(\boldsymbol{\sigma})), \mathscr{H}(n, r; \mathrm{Ni}(\boldsymbol{\sigma})))$$

having property (4. 2).

Proposition 5 of [Fr, 1] gives conditions for this (e.g., if $\mathrm{Cen}_{S_n}(G(\boldsymbol{\sigma})) = \{\mathrm{Id.}\}$), but finding testable conditions on general $\boldsymbol{\sigma}$ for the existence of fine étale-moduli families is a difficult problem. From [Fr, 4] they do *not* exist when $r = 2$, but the case $r = 2$ has always been regarded as special: so, while of interest, it is not decisive. We now rephrase this problem entirely in terms of fundamental groups.

Combinatorial formulation of the main problem. Inside of S_{r+1} we identify S_r with those elements that fix the integer $r+1$. The space $\mathbb{P}^r \times \mathbb{P}^1$ fits in a diagram

$$(4.\ 3) \qquad (\mathbb{P}^1)^{r+1} \xrightarrow{\bar{\Psi}''_{r+1}} \mathbb{P}^r \times \mathbb{P}^1 \xrightarrow{\bar{\Psi}'_{r+1}} \mathbb{P}^{r+1}$$

with $\bar{\Psi}'_{r+1} \circ \bar{\Psi}''_{r+1} = \bar{\Psi}_{r+1}$ (as in (1. 1)), and $\bar{\Psi}''_{r+1}$ presenting $(\mathbb{P}^1)^{r+1}$ as a Galois cover of $\mathbb{P}^r \times \mathbb{P}^1$ with Galois group identified with S_r.

Let $\mathbb{P}^r \times \mathbb{P}^1 - \Delta'_{r+1}$ be the locus of $\mathbb{P}^r \times \mathbb{P}^1$ lying over $\mathbb{P}^{r+1} - D_{r+1}$. We need some convenient labeling of the basepoints on the various manifolds whose fundamental groups we shall now relate:

$$\mathbf{z}^{(r+1)} \in (\mathbb{P}^1)^{r+1} - \Delta_{r+1}, \mathbf{z}^{(r+1)} = (z_1, \ldots, z_{r+1});$$

and $\mathbf{z}^{(r)} = (z_1, \ldots, z_r)$ is a base point for $(\mathbb{P}^1)^r - \Delta_r$, the image of $(\mathbb{P}^1)^{r+1} - \Delta_{r+1}$ under the projection $pr^{(r)}$ onto the first r factors. Then, $\mathrm{q}^{(r+1)}$ (resp., $\mathrm{q}^{(r)}$) is the image of $\mathbf{z}^{(r+1)}$ (resp., $\mathbf{z}^{(r)}$) in $\mathbb{P}^{r+1} - D_{r+1}$ (resp., $\mathbb{P}^r - D_r$). The natural basepoint for $\mathbb{P}^r \times \mathbb{P}^1 - \Delta'_{r+1}$ is then $(\mathrm{q}^{(r)}, z_{r+1})$. The fiber map $\mathbb{P}^r \times \mathbb{P}^1 - \Delta'_{r+1} \to \mathbb{P}^r - D_r$ contains

$$\mathrm{q}^{(r)} \times \mathbb{P}^1 - \{(\mathrm{q}^{(r)}, z_i)\}_{i=1}^r$$

in the fiber over $\mathrm{q}^{(r)}$. This set is natural identified with $\mathbb{P}^1 - \{z_i\}_{i=1}^r$, and in this identification it is reasonable to denote the base point $(\mathrm{q}^{(r)}, z_{r+1})$ by z_0.

Thus we obtain the natural sequence of fundamental groups

$$(4.4)\quad 1 \to \pi_1(\mathbb{P}^1 - \{z_1, \ldots, z_r\}, z_0) \xrightarrow{(inj)^*} \pi_1(\mathbb{P}^r \times \mathbb{P}^1 - \Delta'_{r+1}, (\mathrm{q}^{(r)}, z_{r+1})) \xrightarrow{(pr^{(r)})^*} \pi_1(\mathbb{P}^r - D_r, \mathrm{q}^{(r)}) \to 1$$

which does *not* happen to be exact: take $r=2$ where the middle term (Lemma 1. 5) is a finite group, while the left end term is isomorphic to $\mathbb{Z}$. However we do have:

Lemma 4. 4. *For $r \geqq 3$ the sequence* (4. 4) *is exact.*

This result is stated (the reader must do some unraveling of notation, however, to see this) on [FB]; p. 256. Indeed, following the discussion on [FB]; p. 244, 245, 255 and using the main result of [Ch] we may describe generators for the groups of the sequence (4. 4) in terms of generators of $(\mathbb{P}^{r+1} - D_{r+1}, \mathrm{q}^{(r+1)})$. To avoid abuses of notation denote these generators by $\bar{Q}_1^{(r+1)}, \ldots, \bar{Q}_r^{(r+1)}$, compatible (excluding the superscripts) with the notation of § 1. Then the generators of $\pi_1(\mathbb{P}^r \times \mathbb{P}^1 - \Delta'_{r+1}, (\mathrm{q}^{(r)}, z_{r+1}))$ appear in the following list:

$$(4.5)\quad \begin{aligned} &\bar{Q}_r^{(r+1)} \cdot \bar{Q}_{r-1}^{(r+1)} \cdots \bar{Q}_2^{(r+1)} \cdot (\bar{Q}_1^{(r+1)})^2 \cdot (\bar{Q}_2^{(r+1)})^{-1} \cdots (\bar{Q}_r^{(r+1)})^{-1}, \ldots, \\ &\bar{Q}_r^{(r+1)} \cdot (\bar{Q}_{r-1}^{(r+1)})^2 \cdot (\bar{Q}_r^{(r+1)})^{-1}, (\bar{Q}_r^{(r+1)})^2, \\ &(\bar{Q}_1^{(r+1)})^{-1}, (\bar{Q}_2^{(r+1)})^{-1}, \ldots, (\bar{Q}_{r-1}^{(r+1)})^{-1}. \end{aligned}$$

In this list the first r generators represent generators of $\pi_1(\mathbb{P}^1 - \{z_1, \ldots, z_r\}, z_0)$ which may be identified in order with $\Sigma_1, \ldots, \Sigma_r$, the generators of $G(\mathbf{\Sigma})$ (following expression (1. 3)). Note that the relation $\Sigma_1 \cdots \Sigma_r = \mathrm{Id.}$ becomes a conjugate of the relation $\bar{Q}(r+1)$ of Theorem 1. 1. In addition, the last $r-1$ generators under $(pr^{(r)})^*$ get mapped, respectively, to the natural generators $(\bar{Q}_1^{(r)})^{-1}, \ldots, (\bar{Q}_{r-1}^{(r)})^{-1}$ of $\pi_1(\mathbb{P}^r - D_r, q^{(r)})$.

Lemma 4. 5. *The action of conjugation by $\bar{Q}_1^{(r+1)}, \ldots, \bar{Q}_{r-1}^{(r+1)}$ on the first r generators in the list* (4. 5) *is naturally identified with the action of $\bar{Q}_1, \ldots, \bar{Q}_{r-1}$ on $(\Sigma_1, \ldots, \Sigma_r)$ in expression* (1. 3).

Proof. As a convenience for our notation we show that conjugation by $(\bar{Q}_i^{(r+1)})^{-1}$ is identified with the action of $\bar{Q}_i^{-1}$ on $(\Sigma_1, \ldots, \Sigma_r)$. That is, we must show that

$$\bar{Q}_i^{(r+1)} \cdot (\bar{Q}_r^{(r+1)} \cdots \bar{Q}_{j+1}^{(r+1)} \cdot (\bar{Q}_j^{(r+1)})^2 \cdot (\bar{Q}_{j+1}^{(r+1)})^{-1} \cdots (\bar{Q}_r^{(r+1)})^{-1}) \cdot (\bar{Q}_i^{(r+1)})^{-1}$$
$$= \bar{Q}_i^{(r+1)} \cdot \Sigma_j^{(r+1)} \cdot (\bar{Q}_i^{(r+1)})^{-1}$$

is equal to

(4. 6) a) $\bar{Q}_r^{(r+1)} \cdots \bar{Q}_{j+1}^{(r+1)} \cdot (\bar{Q}_j^{(r+1)})^2 \cdot (\bar{Q}_j^{(r+1)})^{-1} \cdots (\bar{Q}_r^{(r+1)})^{-1}$ if $j \neq i,\ i+1$; and

b) $\bar{Q}_r^{(r+1)} \cdots \bar{Q}_{j+2}^{(r+1)} \cdot (\bar{Q}_{j+1}^{(r+1)})^2 \cdot (\bar{Q}_{j+2}^{(r+1)})^{-1} \cdots (\bar{Q}_r^{(r+1)})^{-1}$ if $j = i$.

Consider expression (4. 6) b). Then $\bar{Q}_j^{(r+1)} \cdot \Sigma_j^{(r+1)} \cdot (\bar{Q}_j^{(r+1)})^{-1}$ is equal to

(4. 7) $\bar{Q}_r^{(r+1)} \cdots \bar{Q}_{j+2}^{(r+1)} (\bar{Q}_j^{(r+1)}) \cdot \bar{Q}_{j+1}^{(r+1)} \cdot (\bar{Q}_j^{(r+1)})^2 \cdot (\bar{Q}_{j+1}^{(r+1)})^{-1} \cdot (\bar{Q}_j^{(r+1)})^{-1} \cdots (\bar{Q}_r^{(r+1)})^{-1}$

from relations of Theorem 1. 1.

Also, by using $((\bar{Q}_j^{(r+1)})^{-1} \cdot \bar{Q}_{j+1}^{(r+1)} \cdot (\bar{Q}_j^{(r+1)}))^2 = (\bar{Q}_{j+1}^{(r+1)} \cdot \bar{Q}_j^{(r+1)} \cdot (\bar{Q}_{j+1}^{(r+1)})^{-1})^2$ in the middle of expression (4. 7) we easily obtain expression (4. 6) b). Now consider expression (4. 6) a). Since it follows trivially from the relations of Theorem 1. 1 in the case that $i < j$, we assume that $i > j$. Thus, $\bar{Q}_i^{(r+1)} \cdot \Sigma_j^{(r+1)} \cdot (\bar{Q}_i^{(r+1)})^{-1}$ is equal to

$$(4.8) \qquad \bar{Q}_r^{(r+1)} \cdots \bar{Q}_i^{(r+1)} \cdot \bar{Q}_{i+1}^{(r+1)} \cdot \bar{Q}_i^{(r+1)} \cdot \bar{Q}_{i-1}^{(r+1)} \cdots (\bar{Q}_j^{(r+1)})^2$$
$$\cdots (\bar{Q}_i^{(r+1)})^{-1} (\bar{Q}_i^{(r+1)} \cdot \bar{Q}_{i+1}^{(r+1)} \cdot \bar{Q}_i^{(r+1)})^{-1} \cdots (\bar{Q}_r^{(r+1)})^{-1}.$$

In using the relation $\bar{Q}_i^{(r+1)} \cdot \bar{Q}_{i+1}^{(r+1)} \cdot \bar{Q}_i^{(r+1)} = \bar{Q}_{i+1}^{(r+1)} \cdot \bar{Q}_i^{(r+1)} \cdot \bar{Q}_{i+1}^{(r+1)}$, expression (4. 8) becomes

$$(4.9) \qquad \bar{Q}_r^{(r+1)} \cdots \bar{Q}_{i+1}^{(r+1)} \cdot \bar{Q}_i^{(r+1)} \cdot \bar{Q}_{i+1}^{(r+1)} \cdot \bar{Q}_{i-1}^{(r+1)} \cdots (\bar{Q}_j^{(r+1)})^2$$
$$\cdots (\bar{Q}_{i-1}^{(r+1)})^{-1} \cdot (\bar{Q}_{i+1}^{(r+1)})^{-1} \cdot (\bar{Q}_i^{(r+1)})^{-1} \cdots (\bar{Q}_r^{(r+1)})^{-1}.$$

Now we can pass the inner appearances of $\bar{Q}_{i+1}^{(r+1)}$ and $(\bar{Q}_{i+1}^{(r+1)})^{-1}$ right down to the center where they cancel each other to see that (4. 6) a) holds. ∎

Now we state a problem containing the main problem of this section. Let H' (resp., H'') be a subgroup of finite index of $\pi_1(\mathbb{P}^1 - \{z_1, \ldots, z_r\}, z_0)$ (resp., $\pi_1(\mathbb{P}^r - D_r, \mathfrak{q}^{(r)})$). Denote by $T_{H'}$ (resp., $T_{H''}$) the corresponding permutation representation of

$$\pi_1(\mathbb{P}^1 - \{z_1, \ldots, z_r\}, z_0) \text{ (resp., } \pi_1(\mathbb{P}^r - D_r, \mathfrak{q}^{(r)})).$$

Problem 4. 6. *Describe the pairs (H', H'') for which there exists a subgroup H of $\pi_1(\mathbb{P}^r \times \mathbb{P}^1 - \Delta'_{r+1}, (\mathfrak{q}^{(r)}, z_{r+1}))$ fitting in the exact sequence*

$$(4.10) \qquad 1 \to H' \xrightarrow{(inj)^*} H \xrightarrow{(pr^{(r)})^*} H'' \to 1$$

via restriction of the maps in expression (4. 4).

Of course, H, even if it exists, is not unique. For a given H, the cover of

$$\mathbb{P}^r \times \mathbb{P}^1 - \Delta'_{r+1}$$

is called a *neighborhood of the cover of* $\mathbb{P}^1 - \{z_1, \ldots, z_r\}$ corresponding to H'. We are most interested in the special case of the problem that arises from one of the pairs $(\boldsymbol{\sigma}, f)$ of § 2.

After some preliminaries we conclude this section with a detailed analysis of the case $r=3$ which was partially considered in [Fr, 1]; § 3, Example 2. Let $N(H')$ (resp., $N(H'')$, $N(H)$) be the normalizer of H' (resp., H'', H) in

$$\pi_1(\mathbb{P}^1 - \{z_1, \ldots, z_r\}, z_0) \text{ (resp., } \pi_1(\mathbb{P}^r - D_r, \mathfrak{q}^{(r)}), \pi_1(\mathbb{P}^r \times \mathbb{P}^1 - \Delta'_{r+1}, (\mathfrak{q}^{(r)}, z_{r+1}))).$$

Suppose that H_1 and H_2 *both* fit in an exact sequence in place of H of expression (4. 10). Then, for each $h'' \in H''$ let $h_i \in H_i$ be such that $(pr^{(r)})^*(h_i) = h''$. Then $h_1^{-1} \circ h_2 = \bar{h}$ is contained in $\pi_1(\mathbb{P}^1 - \{z_1, \ldots, z_r\}, z_0)$ and it normalizes H' since everything in the group generated by H_1 and H_2 normalizes H'. The image of $\bar{h}$ in $N(H')/H'$ depends only on h'', and we obtain a natural map

$$H'' \xrightarrow{tw^{(r)}(H_1, H_2)} N(H')/H'. \tag{4. 11}$$

The reader might recognize this as the start of an obstruction theory approach to the classification of the groups H that fit in the diagram (4. 10). It would, however, lead us a bit afield to continue this. As an aside, and a warning, *starting with the group* H_1, the classical theory interprets the possible groups H_2 fitting in the diagram (4. 10) (up to equivalence given by conjugation inside of $\pi_1(\mathbb{P}^r \times \mathbb{P}^1 - \Delta'_{r+1}, (\mathfrak{q}^{(r)}, z_{r+1}))$) as the elements of a Čech cohomology *set* $\check{H}^1(\mathscr{H}(H''), \mathscr{A}\mathscr{U}\mathscr{T})$. Here: $\mathscr{H}(H'')$ is the cover of $\mathbb{P}^r - D_r$ corresponding to the group H''; and $\mathscr{A}\mathscr{U}\mathscr{T}$ is a sheaf of groups whose stalks are isomorphic to $N(H')/H'$. However, there is a complication in that $\mathscr{A}\mathscr{U}\mathscr{T}$ may *not* be a constant sheaf.

The main point of the above discussion is to motivate the role of the group $N(H')/H'$. We comment further on a necessary condition on the group H'' in order that a sequence (4. 10) exists. Assume that the generators $\Sigma_1, \ldots, \Sigma_r$ (as in the discussion following expression (4. 5)) of $\pi_1(\mathbb{P}^1 - \{z_1, \ldots, z_r\}, z_0)$ are mapped, respectively, to $\sigma(1), \ldots, \sigma(r) \in S_n$ via the representation of $\pi_1(\mathbb{P}^1 - \{z_1, \ldots, z_r\}, z_0)$ on the right cosets of H' (with $n = [\pi_1(\mathbb{P}^1 - \{z_1, \ldots, z_r\}, z_0) : H']$). Let $H''(\boldsymbol{\sigma})$ be the subgroup of

$$\pi_1(\mathbb{P}^r - D_r, \mathfrak{q}^{(r)})$$

corresponding to the cover $\mathscr{H}(n, r; \mathrm{Br}(\boldsymbol{\sigma}))$ as in expression (2. 3).

Lemma 4. 7. *In order, for a given* H' *and* H'', *that an exact sequence* (4. 10) *exists, we must have* H'' *contained in some conjugate of* $H''(\boldsymbol{\sigma})$.

Proof. The group $H''(\boldsymbol{\sigma})$ is defined (up to conjugation) to be the stabilizer of the equivalence class of $\boldsymbol{\sigma}$ in the action of $\pi_1(\mathbb{P}^r - D_r, \mathfrak{q}^{(r)})$ on the elements of $\mathrm{Br}(\boldsymbol{\sigma})$. From Lemma 4. 5, we may identify this action through the induced action of $\bar{Q}_1^{(r+1)}, \ldots, \bar{Q}_{r-1}^{(r+1)}$ on $\mathrm{Br}(\boldsymbol{\sigma})$ via conjugation on $\pi_1(\mathbb{P}^1 - \{z_1, \ldots, z_r\}, z_0)$. Let $h \in H$. Then $(pr^{(r)})^*(h)$ maps the equivalence class of $\boldsymbol{\sigma}$ to the equivalence class of $(T_{H'}(h^{-1} \cdot \Sigma_1 \cdot h), \ldots, T_{H'}(h^{-1} \cdot \Sigma_r \cdot h))$ where $T_{H'}: \pi_1(\mathbb{P}^1 - \{z_1, \ldots, z_r\}, z_0) \to S_n$ is the right coset representation coming from the group H'. However, since $h^{-1} \cdot H' \cdot h = H'$, the resulting r-tuple is equivalent to $\boldsymbol{\sigma}$. This proves that $H'' \subset H''(\boldsymbol{\sigma})$, and concludes the lemma. ■

Finally we note that the existence of a family presenting $\mathscr{H}(n, r; \mathrm{Ni}(\boldsymbol{\sigma}))$ as a fine étale-moduli space is equivalent to the existence of H in the sequence (4. 10) in the case that $H'' = H''(\boldsymbol{\sigma})$.

The Case $r=3$. The basic diagram that we must consider is

(4. 12)

$$\begin{array}{ccccccccc} 1 \to \pi_1(\mathbb{P}^1-\{0,1,\infty\},z_0) & \to & \pi_1\big((\mathbb{P}^1)^4-\Delta_4,\mathbf{z}^{(4)}\big) & \to & \pi_1\big((\mathbb{P}^1)^3-\Delta_3,\mathbf{z}^{(3)}\big) \to 1 \\ \| & & \downarrow & & \downarrow \\ 1 \to \pi_1(\mathbb{P}^1-\{0,1,\infty\},z_0) & \to & \pi_1\big(\mathbb{P}^3\times\mathbb{P}^1-\Delta_4',(\mathrm{q}^{(3)},z_4)\big) & \to & \pi_1(\mathbb{P}^3-D_3,q^{(3)}) \to 1. \end{array}$$

Of course, from Lemma 1. 6, the top row splits in the nicest way possible: the middle term is the direct product of the end terms, and the right end term is $\mathbb{Z}/(2)$.

Let $\boldsymbol{\sigma}=(\sigma(1),\sigma(2),\sigma(3))\in(S_n)^3$ be such that $\sigma(1),\sigma(2),\sigma(3)$ generate a transitive subgroup of S_n, and $\sigma(1)\cdot\sigma(2)\cdot\sigma(3)=\mathrm{Id}$. In terms of the discussion above, let $H'(\boldsymbol{\sigma})$ be a subgroup of $\pi_1(\mathbb{P}^1-\{0,1,\infty\},z_0)$ giving a right coset representation equivalent to the representation obtained by mapping the generators $\Sigma_1,\Sigma_2,\Sigma_3$ (as in the discussion following expression (4. 5)), respectively, to $\sigma(1),\sigma(2),\sigma(3)$. Let $H''_{un}=\pi_1\big((\mathbb{P}^1)^3-\Delta_3,\mathbf{z}^{(3)}\big)$, and further let $H_{un}=H'(\boldsymbol{\sigma})\times H''_{un}\subseteq\pi_1\big((\mathbb{P}^1)^4-\Delta_4,\mathbf{z}^{(4)}\big)$. Thus we obtain an affirmative answer to Problem 4. 6 for the pair $(H'(\boldsymbol{\sigma}),H''_{un})$. In terms of the notation of § 2, H''_{un} corresponds to the pair $(\boldsymbol{\sigma},f)$ where f is the function given by Case 2 (following expression (2. 5)). If we let $H''(\boldsymbol{\sigma})$ be the subgroup of $\pi_1(\mathbb{P}^3-D_3,q^{(3)})$ corresponding to Case 1, then an affirmative answer to Problem 4. 6 for the pair $(H'(\boldsymbol{\sigma}),H''(\boldsymbol{\sigma}))$ is easily seen to give an affirmative answer for *any* pair $(H'(\boldsymbol{\sigma}),H'')$ where $H''\subseteq H''(\boldsymbol{\sigma})$; and thus an affirmative answer for *all* the H'' allowed by Lemma 4. 7. There are 3 cases:

(4. 13) a) $\sigma(1),\sigma(2),\sigma(3)$ are pairwise *non-conjugate* in $G(\boldsymbol{\sigma})$;

b) $\sigma(1),\sigma(2),\sigma(3)$ are pairwise *conjugate* in $G(\boldsymbol{\sigma})$; and

c) *two* of the elements $\sigma(1),\sigma(2),\sigma(3)$ are conjugate in $G(\boldsymbol{\sigma})$, but b) does *not* hold.

From the proof of Lemma 1. 5, especially expression (1. 4),

$$H''(\boldsymbol{\sigma})=\pi_1\big((\mathbb{P}^1)^3-\Delta_3,\mathbf{z}^{(3)}\big)$$

if (4. 13) a) holds, and $H''(\boldsymbol{\sigma})$ *contains* a subgroup conjugate to the subgroup of index 3 in $\pi_1(\mathbb{P}^3-D_3,\mathrm{q}^{(3)})$ generated by $\bar{Q}_1^{(3)}$ if (4. 13) c) holds.

Note. Generally we would "expect" $H''(\boldsymbol{\sigma})$ to be a subgroup of index 3 (rather than just "contains") in this latter case. Likewise we would "expect" $H''(\boldsymbol{\sigma})$ to equal $\pi_1(\mathbb{P}^3-D_3,\mathrm{q}^{(3)})$ in case (4. 13) b) holds. However, it is easy to give examples that show that this expectation does not hold in general.

We first consider a special quest. Find $H(\boldsymbol{\sigma})$ with these properties:

$$H(\boldsymbol{\sigma})\subseteq\pi_1\big(\mathbb{P}^3\times\mathbb{P}^1-\Delta_4',(\mathrm{q}^{(3)},z_4)\big);$$

(4. 14) a) $H(\boldsymbol{\sigma})\cap\pi_1\big((\mathbb{P}^1)^4-\Delta_4,\mathbf{z}^{(4)}\big)=H_{un}$; and

b) $1\to H'(\boldsymbol{\sigma})\to H(\boldsymbol{\sigma})\to H''(\boldsymbol{\sigma})\to 1$ is exact.

Now, using the normalizer notation preceding expression (4. 11), consider the sequence (not necessarily exact on the right)

(4. 15) $$1\to N(H'(\boldsymbol{\sigma}))/H'(\boldsymbol{\sigma})\to N(H_{un})/H_{un}\xrightarrow{pr^{**}}N(H''_{un})/H''_{un}.$$

Theorem 4. 8. *A group $H(\boldsymbol{\sigma})$ satisfying expression* (4. 14) *exists if and only if the exact sequence*

$$(4.\,16)\quad 1 \to N\big(H'(\boldsymbol{\sigma})\big)/H'(\boldsymbol{\sigma}) \to (pr^{**})^{-1}\big(H''(\boldsymbol{\sigma})/H''_{un}\big) \to H''(\boldsymbol{\sigma})/H''_{un} \to 1$$

splits.

The sequence (4. 16) *splits if the order of $H''(\boldsymbol{\sigma})/H''_{un}$ (which divides* 6) *is relatively prime to the order of $N\big(H'(\boldsymbol{\sigma})\big)/H'(\boldsymbol{\sigma})$.*

Proof. In order to show exactness of the sequence (4. 16) we have only to show that the right hand map is onto.

Let $\bar{Q}^{(3)} \in \pi_1(\mathbb{P}^3 - D_3, \mathrm{q}^{(3)})$ represent an element of $H''(\boldsymbol{\sigma})/H''_{un}$. Then $\bar{Q}^{(3)}$ is the image of $\bar{Q}^{(4)}$ where $\bar{Q}^{(4)} \in \pi_1\big(\mathbb{P}^3 \times \mathbb{P}^1 - \Delta'_4, (\mathrm{q}^{(3)}, z_4)\big)$ is in the subgroup generated by $Q_1^{(4)}, \ldots, Q_3^{(4)}$. In addition the action of $\bar{Q}^{(3)}$ on $\boldsymbol{\sigma}$ is given, according to the prescription of Lemma 4. 5, by forming $\big(\cdots, T_H\big((\bar{Q}^{(4)})^{-1} \cdot \Sigma_i \cdot \bar{Q}^{(4)}\big), \ldots\big) = \boldsymbol{\tau}$ (as in the proof of Lemma 4. 7). Since $\bar{Q}^{(3)}$ represents an element of $H''(\boldsymbol{\sigma})/H''_{un}$, $\boldsymbol{\tau}$ is equivalent to $\boldsymbol{\sigma}$. This means that $\bar{Q}^{(4)}$ normalizes $H'(\boldsymbol{\sigma})$, or equivalently, $\bar{Q}^{(4)} \in N(H_{un})$. Thus, the sequence (4. 16) is exact.

Suppose sequence (4. 16) splits. So there exists a group H with: $H_{un} \subseteq H$; and $H/H_{un} \xrightarrow{pr^{**}} H''(\boldsymbol{\sigma})/H''_{un}$ a one-one map. Thus, we may take $H(\boldsymbol{\sigma})$ equal to H. Conversely, if $H(\boldsymbol{\sigma})$ exists, then the induced map $H(\boldsymbol{\sigma})/H_{un} \xrightarrow{pr^{**}} H''(\boldsymbol{\sigma})/H''_{un}$ is one-one and we obtain a splitting of the sequence (4. 16).

The final statement of the Theorem is the Schur-Zassenhaus lemma that guarantees splitting of a sequence of finite groups under the given conditions. ■

We next consider the problem of finding examples of allowable triples $\boldsymbol{\sigma}$ for which the sequence (4. 16) does not split. Of course, it would always split if

(4. 17) the lower row of the diagram (4. 12) splits.

Such a splitting would necessitate the existence of elements of order 4 and 6 inside of $\pi_1\big(\mathbb{P}^3 \times \mathbb{P}^1 - \Delta'_4, (\mathrm{q}^{(3)}, z_4)\big)$ (Lemma 1. 5). These elements must generate the dicyclic group of degree 6 and map onto the elements of $\pi_1(\mathbb{P}^3 - D_3, \mathrm{q}^{(3)})$ labeled as $\bar{Q}_1^{(3)} \cdot \bar{Q}_2^{(3)} \cdot \bar{Q}_1^{(3)}$ and $\bar{Q}_1^{(3)} \cdot \bar{Q}_2^{(3)}$.

From [FB], the element $\alpha = \bar{Q}_1^{(4)} \cdot \bar{Q}_2^{(4)} \cdot \bar{Q}_3^{(4)}$ has order 8, and, of course, we have the defining relation $\bar{Q}_1^{(4)} \cdot \bar{Q}_2^{(4)} \cdot \bar{Q}_3^{(4)} \cdot \bar{Q}_3^{(4)} \cdot \bar{Q}_2^{(4)} \cdot \bar{Q}_1^{(4)}$ equal to the identity. Consider the chain of equalities:

$$\begin{aligned}
(4.\,18)\quad &\text{a)}\ (\bar{Q}_1^{(4)} \cdot \bar{Q}_2^{(4)} \cdot \bar{Q}_3^{(4)}) \cdot (\bar{Q}_1^{(4)} \cdot \bar{Q}_2^{(4)} \cdot \bar{Q}_3^{(4)}) \cdot \alpha^2 = \\
&\text{b)}\ \bar{Q}_1^{(4)} \cdot \bar{Q}_2^{(4)} \cdot (\bar{Q}_1^{(4)} \cdot \bar{Q}_2^{(4)} \cdot \bar{Q}_3^{(4)} \cdot \bar{Q}_2^{(4)}) \cdot \alpha^2 = \\
&\text{c)}\ (\bar{Q}_1^{(4)} \cdot \bar{Q}_2^{(4)})^2 \cdot (\bar{Q}_3^{(4)} \cdot \bar{Q}_2^{(4)} \cdot \bar{Q}_1^{(4)} \cdot \bar{Q}_2^{(4)} \cdot \bar{Q}_3^{(4)}) \cdot \alpha = \\
&\text{d)}\ (\bar{Q}_1^{(4)} \cdot \bar{Q}_2^{(4)})^2 \cdot (\bar{Q}_1^{(4)}(\bar{Q}_3^{(4)} \cdot \bar{Q}_2^{(4)} \cdot \bar{Q}_3^{(4)})\,\bar{Q}_1^{(4)}) \cdot \alpha = \\
&\text{e)}\ (\bar{Q}_1^{(4)} \cdot \bar{Q}_2^{(4)})^3 \cdot \bar{Q}_3^{(4)} \cdot \bar{Q}_2^{(4)} \cdot \bar{Q}_1^{(4)} \cdot \alpha = (\bar{Q}_1^{(4)} \cdot \bar{Q}_2^{(4)})^3 .
\end{aligned}$$

Thus, from (4. 18), $\bar{Q}_1^{(4)} \cdot \bar{Q}_2^{(4)}$ is an element of order 6 that maps to $\bar{Q}_1^{(3)} \cdot \bar{Q}_2^{(3)}$. Also $(\bar{Q}_1^{(4)} \cdot \bar{Q}_2^{(4)} \cdot \bar{Q}_1^{(4)})^2 = (\bar{Q}_1^{(4)} \cdot \bar{Q}_2^{(4)})^3$ (via the relation $\bar{Q}_1^{(4)} \cdot \bar{Q}_2^{(4)} \cdot \bar{Q}_1^{(4)} = \bar{Q}_2^{(4)} \cdot \bar{Q}_1^{(4)} \cdot \bar{Q}_2^{(4)}$). Thus $\bar{Q}_1^{(4)} \cdot \bar{Q}_2^{(4)} \cdot \bar{Q}_1^{(4)}$ is an element of order 4 that maps to $\bar{Q}_1^{(3)} \cdot \bar{Q}_2^{(3)} \cdot \bar{Q}_1^{(3)}$. From the characterization of the dicyclic group in Definition 1. 4 we obtain the desired splitting if and only if $\bar{Q}_1^{(4)} \cdot \bar{Q}_2^{(4)} \cdot \bar{Q}_1^{(4)}$ conjugates $\bar{Q}_1^{(4)} \cdot \bar{Q}_2^{(4)}$ into $(\bar{Q}_2^{(4)})^{-1} \cdot (\bar{Q}_1^{(4)})^{-1}$. This amounts to the relation

(4. 19) a) $(\bar{Q}_1^{(4)} \cdot \bar{Q}_2^{(4)}) \cdot (\bar{Q}_1^{(4)} \cdot \bar{Q}_2^{(4)} \cdot \bar{Q}_1^{(4)}) \cdot (\bar{Q}_1^{(4)} \cdot \bar{Q}_2^{(4)}) = \bar{Q}_1^{(4)} \cdot \bar{Q}_2^{(4)} \cdot \bar{Q}_1^{(4)}$, or

b) $\mathrm{Id.} = \bar{Q}_2^{(4)} \cdot \bar{Q}_1^{(4)} \cdot \bar{Q}_1^{(4)} \cdot \bar{Q}_2^{(4)}$, which does *not* hold.

We draw one positive conclusion from this:

Corollary 4. 9. *The sequence* (4. 16) *splits if* $H''(\boldsymbol{\sigma})$ *is a cyclic group (since it is then of order* 2). *Thus, the only possibility that the sequence* (4. 16) *does not split occurs when* $(\boldsymbol{\sigma})\bar{Q}$ *is absolutely equivalent to* $\boldsymbol{\sigma}$ *for all* $\bar{Q} \in \pi_1(\mathbb{P}^3 - D_3, \mathrm{q}^{(3)})$ *(in particular,* (4. 13) b) *holds).*

Finally we drop condition (4. 14) a). Analogous to Theorem 4. 1 we easily obtain

Theorem 4. 10. *A group* $H(\boldsymbol{\sigma})$ *satisfying expression* (4. 14) b) *exists if and only if the exact sequence*

(4. 20) $$1 \to N(H'(\boldsymbol{\sigma}))/H'(\boldsymbol{\sigma}) \to (pr^{**})^{-1}(H''(\boldsymbol{\sigma})) \to H''(\boldsymbol{\sigma}) \to 1$$

splits. In particular, if $\mathrm{Ni}(\boldsymbol{\sigma}) = \bigcup_{k=1}^{l} \mathrm{Br}(\boldsymbol{\sigma}_k)$, *the splitting of* (4. 20) *with* $\boldsymbol{\sigma}$ *replaced by* $\boldsymbol{\sigma}_k$, $k = 1, \ldots, l$ *is equivalent to the following statement*: $\mathscr{H}(n, r; \mathrm{Ni}(\boldsymbol{\sigma}))$ *is a fine étale-moduli space for covers of* $\mathbb{P}^1$ *of Nielsen type* $\mathrm{Ni}(\boldsymbol{\sigma})$.

Example 4. 11. *An elementary rephrasing of the nonexistence of a fine étale-module space in the case* $r = 3$. Following [Fr, 1]; Lemma 2. 1 we identify $N(H'(\boldsymbol{\sigma}))/H'(\boldsymbol{\sigma})$ with the subgroup of S_n, $\mathrm{Cen}_{S_n}(G(\boldsymbol{\sigma}))$, that centralizes $G(\boldsymbol{\sigma})$ with $\boldsymbol{\sigma} = (\sigma(1), \sigma(2), \sigma(3))$ as above. Suppose also that there exist elements $\alpha(1)$ and $\alpha(2) \in S_n$ such that

$$\alpha(i) \cdot \boldsymbol{\sigma} \cdot (\alpha(i))^{-1} = (\boldsymbol{\sigma})\bar{Q}_i, \quad i = 1, 2.$$

Let G^* be the subgroup of S_n generated by $\alpha(1)$, $\alpha(2)$ and $\mathrm{Cen}_{S_n}(G(\boldsymbol{\sigma}))$. Consider the natural map from G^* to S_3 (i.e., $\alpha(1) \to (12)$, $\alpha(2) \to (23)$). Then there *is* an étale moduli space if

(4. 21) $$\mathrm{Cen}_{S_n}(G(\boldsymbol{\sigma})) \to G^* \to S_3$$

splits. Indeed, let G^{**} be the group that fits in a cartesian diagram

(4. 22) $$\begin{array}{ccc} G^{**} & \to & D_6 \\ \downarrow & & \downarrow \\ G^* & \to & S_3 \end{array}$$

where the right vertical column is the natural map of the dihedral group of degree 6 (Def. 1. 4 to S_3). Diagram (4. 22) gives rise to the exact sequence

(4. 23) $$\mathrm{Cen}_{S_n}(G(\boldsymbol{\sigma})) \to G^{**} \to D_6,$$

and there exists an étale moduli space over $\mathscr{H}(n, 3; \mathrm{Ni}(\boldsymbol{\sigma}))$ if and only if this sequence splits.

Unfortunately, at this writing, we have no example of an appropriate $\boldsymbol{\sigma} \in S_n^3$ for which (4. 21) does not split—but, certainly (?), such an example must exist. ■

§ 5. Harbater's representation and distinguishing Hurwitz classes in a given Nielsen class

Recall the definition of the Nielsen type $\mathrm{Ni}(\boldsymbol{\sigma})$ of a cover $X \xrightarrow{\varphi} \mathbb{P}^1$, as given at the beginning of § 4. Recall also that, from [Fr, 3]; § 1. 2) from an explicit presentation of X^φ and φ through algebraic equations, we can compute $\mathrm{Ni}(\boldsymbol{\sigma})$ explicitly. In a manner entirely analogous we may define, for $G(\boldsymbol{\sigma}) \subseteqq \bar{G} \subseteqq N_{S_n}(G(\boldsymbol{\sigma}))$, the *$\bar{G}$-Nielsen type*, $\mathrm{Ni}(\boldsymbol{\sigma}^{\bar{G}})$, and the $\bar{G}$-Hurwitz type $\mathrm{Br}(\boldsymbol{\sigma}^{\bar{G}})$, of the cover $X \xrightarrow{\varphi} \mathbb{P}^1$. Of course, if (§ 2)

$$\mathrm{Ni}(\boldsymbol{\sigma}^{\bar{G}}) = \mathrm{Ni}(\boldsymbol{\sigma}) = \mathrm{Br}(\boldsymbol{\sigma}^{\bar{G}}) = \mathrm{Br}(\boldsymbol{\sigma}),$$

then the points of the irreducible space $\mathscr{H}(n, r; \mathrm{Ni}(\boldsymbol{\sigma}))$ are in a natural one-one correspondence with the space of deformations of one particular cover; and very little data is needed to reconstruct $\mathscr{H}(n, r; \mathrm{Ni}(\boldsymbol{\sigma}))$ as the space of deformations of some cover.

If, however, for example, $\mathrm{Ni}(\boldsymbol{\sigma}) = \sum_{k=1}^{l} \mathrm{Br}(\boldsymbol{\sigma}_k)$ with $l \geqq 2$, we may wonder in what essential and algebraically recognizable way the covers associated to the points of $\mathscr{H}(n, r; \mathrm{Br}(\boldsymbol{\sigma}_u))$ and $\mathscr{H}(n, r; \mathrm{Br}(\boldsymbol{\sigma}_v))$ differ for $u \neq v$.

We list two of the disparate possibilities concluding with the one of most interest to this section. Discussion of $\bar{G}$-Nielsen classes (rather than just absolute Nielsen classes) is similar—we drop the $\bar{G}$ notation for simplicity until the last example in the section.

Conjugation almost equivalence of covers. We abstract the situation that actually occurred in the last example of § 3.

Consider finite groups H_1, H_2, H and G with these properties:

(5. 1) a) $H_1, H_2 \subset H$ and $H \lhd G$;

b) $\alpha \cdot H_1 \cdot \alpha^{-1} = H_2$ for some $\alpha \in G$;

c) H_1 and H_2 are nonconjugate subgroups of H; and

d) there exist $\boldsymbol{\sigma} = (\sigma(1), \ldots, \sigma(r)) \in H^r$ whose coordinates generate H and for which $\sigma(1) \cdots \sigma(r) = \mathrm{Id}.$, and the coordinates of $\alpha \cdot \boldsymbol{\sigma} \cdot \alpha^{-1}$, in some order, are conjugate in H to the coordinates of $\boldsymbol{\sigma}$.

Let $T_{H_1}: G \longrightarrow S_n$ be the right coset representation corresponding to the subgroup H_1. Assume this representation restricted to H is faithful. For $z(1), \dots, z(r)$ distinct points of $\mathbb{P}^1$ and $\Sigma_1, \dots, \Sigma_r$ generators of $\pi_1(\mathbb{P}^1 - \{z(1), \dots, z(r)\})$ as in the discussion prior to Theorem 1. 2, the coordinates of $T_{H_1}(\boldsymbol{\sigma})$ give rise to a representation of

$$\pi_1(\mathbb{P}^1 - \{z(1), \dots, z(r)\})$$

and therefore a cover $X_1 \xrightarrow{\varphi_1} \mathbb{P}^1$ for which $T_{H_1}(\boldsymbol{\sigma})$ is a description of the branch cycles. Similarly $T_{H_2}(\boldsymbol{\sigma})$ is a description of the branch cycles of a cover $X_2 \xrightarrow{\varphi_2} \mathbb{P}^1$.

Definition 5. 1. Let $\mathbb{P}^1_i$, $i = 1, 2$, be two copies of $\mathbb{P}^1$. Consider a commutative diagram of covers

(5. 2)
$$\begin{tikzcd}
X_1 \arrow[r, "\varphi_1"] \arrow[dr, "\Psi_1"'] & \mathbb{P}^1_1 \arrow[r, "\varphi_2"] \arrow[d, "\gamma"] & X_2 \arrow[dl, "\Psi_2"] \\
& \mathbb{P}^1_2 &
\end{tikzcd}$$

where $X_1 \xrightarrow{\Psi_1} \mathbb{P}^1_2$ and $X_2 \xrightarrow{\Psi_2} \mathbb{P}^1_2$, are equivalent covers. We say that $X_1 \xrightarrow{\varphi_1} \mathbb{P}^1_1$ and $X_2 \xrightarrow{\varphi_2} \mathbb{P}^1_1$ are *almost equivalent covers.*

Lemma 5. 2. *The covers $X_i \xrightarrow{\varphi_i} \mathbb{P}^1 = \mathbb{P}^1_1$ whose branch cycles derive from the conditions of expression* (5. 1) *are almost equivalent covers where the cover $\mathbb{P}^1_1 \xrightarrow{\gamma} \mathbb{P}^1_2$ is a cyclic group whose generator corresponds to the element α of* (5. 1) b).

Proof. With no loss, in expression (5. 1) a), we may assume that α and H generate G. Denote by $\mathbb{C}(X_1)$ the function field of X_1 and let $\widehat{\mathbb{C}(X_1)}$ be the Galois closure of the field extension $\mathbb{C}(X_1)/\mathbb{C}(\mathbb{P}^1_1)$. Then $G((\widehat{\mathbb{C}(X_1)}/\mathbb{C}(\mathbb{P}^1_1)) = H$. By hypothesis $\mathbb{C}(X_2) \subset \widehat{\mathbb{C}(X_1)}$. Let $\operatorname{Aut}(\widehat{\mathbb{C}(X_1)})$ be the automorphism group of the field $\widehat{\mathbb{C}(X_1)}$. By hypothesis $\operatorname{Aut}(\widehat{\mathbb{C}(X_1)})$ contains G, a group generated by H and an automorphism α that maps $\mathbb{C}(X_1)$ to $\mathbb{C}(X_2)$ as in (5. 1) b). Let K be the fixed field of G. By the Galois correspondence $K = \mathbb{C}(\mathbb{P}^1_2)$ where the extension $\mathbb{C}(\mathbb{P}^1_1)/\mathbb{C}(\mathbb{P}^1_2)$ is isomorphic to G/H. The rest of the lemma is clear. ∎

Definition 5. 3. Two covers $X_1 \xrightarrow{\varphi_1} \mathbb{P}^1$ and $X_2 \xrightarrow{\varphi_2} \mathbb{P}^1$ are said to be *conjugation almost equivalent* if they have branch cycle descriptions given by expression (5. 1) d). In this case we say, also, that the branch cycles are conjugation almost equivalent. Warning! Although conjugation almost equivalence is an equivalence relation (being dependent only on the conjugation of subgroups of a group H by elements of the normalizer of H) almost equivalence is *not* an equivalence relation.

Problem 5. 4. *Find $\boldsymbol{\sigma}, \boldsymbol{\tau} \in (S_n)^r$ for which $\boldsymbol{\tau} \in \mathrm{Ni}(\boldsymbol{\sigma})$, $\mathrm{Br}(\boldsymbol{\sigma}) \neq \mathrm{Br}(\boldsymbol{\tau})$ and no pair of representatives of $\mathrm{Br}(\boldsymbol{\sigma})$ and $\mathrm{Br}(\boldsymbol{\tau})$ are conjugation almost equivalent.*

Boundary separation of covers. The spaces $\mathscr{H}(n, r; \mathrm{Ni}(\boldsymbol{\sigma}))$ and $\mathscr{H}(n, r; \mathrm{Br}(\boldsymbol{\sigma}))$ are unramified covers of $\mathbb{P}^r$ (expression (2. 3)). Thus, there are (unique) normal algebraic sets $\bar{\mathscr{H}}(n, r; \mathrm{Ni}(\boldsymbol{\sigma}))$ (resp., $\bar{\mathscr{H}}(n, r; \mathrm{Br}(\boldsymbol{\sigma}))$) and a finite morphism

(5. 3) $$\bar{\Psi}: \bar{\mathscr{H}}(n, r; \mathrm{Ni}(\boldsymbol{\sigma})) \longrightarrow \mathbb{P}^r$$

for which restriction of $\bar{\Psi}$ to $\bar{\mathscr{H}}(n, r; \mathrm{Ni}(\boldsymbol{\sigma}))|_{\mathbb{P}^r - D_r}$ gives a cover of $\mathbb{P}^r - D_r$ equivalent to expression (2. 3).

It seems reasonable that the points of $\bar{\mathscr{H}}(n, r; \mathrm{Ni}(\boldsymbol{\sigma}))$ in the complement of $\mathscr{H}(n, r; \mathrm{Ni}(\boldsymbol{\sigma}))$ ought, in some way, to correspond to covers of $\mathbb{P}^1$; and they do. Indeed, showing this is the chief qualitative task of [Har] (further explanation in [Fr, 3]; § 2).

We now list the possible descriptions of the branch cycles for covers of $\mathbb{P}^1$ corresponding to points in the fiber $\bar{\Psi}^{-1}(\mathfrak{h})$ for $\mathfrak{h}$ a point of D_r. For the next discussion we need not assume that $G(\boldsymbol{\sigma})$ is transitive.

Let $1 \leqq j(1) < j(2) < \cdots < j(t-1) < r$ be a sequence of positive integers between 1 and r. Define the *coelescing operator* $P(\mathbf{j})$ corresponding to $\mathbf{j} = (j(1), \ldots, j(t-1))$ to be the operator that sends $\boldsymbol{\sigma}$ to

$$\text{(5. 4)} \qquad \boldsymbol{\sigma}^{P(\mathbf{j})} = \big(\sigma(1) \cdots \sigma(j(1)), \sigma(j(1)+1) \cdots \sigma(j(2)), \ldots, \sigma(j(t-1)+1) \cdots \sigma(r)\big) \in (S_n)^t.$$

Finally we have the *coelescing correspondence* $C(\mathbf{j})$ that associates to $\boldsymbol{\sigma}$ the set of $G(\boldsymbol{\sigma})$-equivalence classes (Lemma 3. 8) of t-tuples $\boldsymbol{\sigma}^{C(\mathbf{j})} = \{((\boldsymbol{\sigma})\bar{Q})^{P(\mathbf{j})}\}_{\bar{Q} \in M(r, \boldsymbol{\Sigma})}$ where $M(r, \boldsymbol{\Sigma})$ is the mapping class group of § 1. The following lemma follows easily from the definitions.

Lemma 5. 5. *If $\boldsymbol{\tau} \in \boldsymbol{\sigma}^{C(\mathbf{j})}$ then the set of equivalence classes of elements of the form $(\boldsymbol{\tau})\bar{Q}^{(t)}$ is contained in $\boldsymbol{\sigma}^{C(\mathbf{j})}$ as $\bar{Q}^{(t)}$ runs over the elements of $M(t, \boldsymbol{\Sigma})$. That is,* $\mathrm{Br}(\boldsymbol{\tau}) \subseteqq \boldsymbol{\sigma}^{C(\mathbf{j})}$ *in the notation following Definition* 2. 1.

Suppose now that $\mathfrak{h} \in D_r$ lies below a point $\mathfrak{p} \in \Delta_r$ where $\mathfrak{p} = (z_1, \ldots, z_r) \in (\mathbb{P}^1)^r$ has this property: the first j_1 coordinates of $\mathfrak{p}$ are equal; the next $j_2 - j_1$ coordinates are equal, but distinct from the previous j_1 coordinates; the next $j_3 - j_2$ coordinates are equal, but distinct from the previous coordinates; etc. Then the covers of $\mathbb{P}^1$ corresponding to the points of the fiber $\bar{\Psi}^{-1}(\mathfrak{h})$ arise, up to equivalence, from the representations of

$$\pi_1(P^1 - \{z_1, z_{j(1)+1}, z_{j(t-1)+1}\})$$

obtained by mapping the usual t-tuple of generators to the respective t-tuples of $\boldsymbol{\sigma}^{C(\mathbf{j})}$.

Now suppose that $\boldsymbol{\tau}$ represents an element of $\mathrm{Ni}(\boldsymbol{\sigma})$ but $\boldsymbol{\tau}$ does *not* represent an element of $\mathrm{Br}(\boldsymbol{\sigma})$ (i.e., $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ represent the same Nielsen class, but distinct Hurwitz classes). Let s be a (possibly large) positive integer and let $\boldsymbol{\sigma}^* \in (S_n)^s$ where

$$\sigma(1)^* \cdots \sigma(s)^* = \mathrm{Id}.$$

Let $1 \leqq j(1) < \cdots < j(r-1) < s$. We say that *the pair* $(\boldsymbol{\sigma}^*, \mathbf{j})$ *separates the* Hurwitz classes of $\boldsymbol{\sigma}$ *and* $\boldsymbol{\tau}$ if

$$\text{(5. 5)} \qquad \boldsymbol{\sigma} \text{ is in } (\boldsymbol{\sigma}^*)^{C(\mathbf{j})} \text{ but } \boldsymbol{\tau} \text{ is not.}$$

For example, the pair $(\boldsymbol{\sigma}, \mathbf{j})$ where $j(i) = i$, $r = s$, separates $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$.

Definition 5. 6. We say that $\boldsymbol{\sigma}^* \in (S_n)^s$ is *algebraically distinguished* (or just distinguished) if there exists $\bar{Q}^{(s)} \in M(s, \boldsymbol{\Sigma})$ for which $(\boldsymbol{\sigma}^*)\bar{Q}^{(s)}$ is of the form

$$\big(\gamma(1), \gamma(1)^{-1}, \gamma(2), \gamma(2)^{-1}, \ldots, \gamma(l), \gamma(l)^{-1}\big)$$

(i.e., $2 \cdot l = s$).

Problem 5. 7 (Recognition of Hurwitz classes). *Let* $\boldsymbol{\sigma}=(\sigma(1),\ldots,\sigma(r))\in(S_n)^r$ *satisfy*: $\sigma(1)\cdots\sigma(r)=\mathrm{Id}.$, *and the group* $G(\boldsymbol{\sigma})$ *generated by the coordinates of* $\boldsymbol{\sigma}$ *is a transitive subgroup of* S_n. *Does there exist* $\boldsymbol{\sigma}^*\in(S_n)^{2\cdot l}$ *for some* l *with these properties*: $\boldsymbol{\sigma}^*$ *is algebraically distinguished*; *and for some* $\mathbf{j}=(j(1),\ldots,j(r-1))$ *with* $1\leqq j(1)<\cdots<j(r-1)<s$, $(\boldsymbol{\sigma}^*,\mathbf{j})$ *separates the* $\bar{G}$*-Hurwitz class* $\mathrm{Br}(\boldsymbol{\sigma}^{\bar{G}})$ *from the* $\bar{G}$*-Hurwitz class* $\mathrm{Br}(\boldsymbol{\tau}^{\bar{G}})$ *for each* $\boldsymbol{\tau}$ *representing a* $\bar{G}$*-Hurwitz class distinct from the* $\bar{G}$*-Hurwitz class of* $\boldsymbol{\sigma}$?

Finally, consider the special case of Problem 5. 7 where we ask about $(\boldsymbol{\sigma}^*,\mathbf{j})$ if the conclusion holds in the case $l=r-1$ and

(5. 5) a) $\boldsymbol{\sigma}^*=(\sigma(1),\sigma(1)^{-1},\sigma(2),\sigma(2)^{-1},\ldots,\sigma(r-1),\sigma(r-1)^{-1})$; and

b) $\mathbf{j}=(1,2,3,\ldots,r-1)$.

The answer is *not always* as we show below.

Condition (5. 5) is especially recommended from considering [Har].

The reader can easily find $\bar{Q}$ in the $2\cdot(r-1)$ degree Artin Braid group for which $(\boldsymbol{\sigma}^*)\bar{Q}=(\sigma(1),\sigma(2),\ldots,\sigma(r-1),\bar{\sigma}(r),\ldots,\bar{\sigma}(2\cdot(r-1)))$ for some

$$\bar{\sigma}(i)\in S_n,\ i=r,\ldots,2\cdot(r-1).$$

Thus $((\boldsymbol{\sigma}^*)\bar{Q})^{P(\mathbf{j})}=\boldsymbol{\sigma}$. Call the representation of the $2\cdot(r-1)$ degree Hurwitz monodromy group on the $\bar{G}$-equivalence classes of the $\bar{G}$-Nielsen class associated to $\boldsymbol{\sigma}^*$ (as in § 3) the *$\bar{G}$-Harbater Representation of the $2\cdot(r-1)$ degree Hurwitz monodromy group associated to* $\boldsymbol{\sigma}\in(S_n)^r$.

An **example** coming from the examples of § 3. Following the above notation, let $\bar{G}=G$ where

$$G=G((\mathbb{Z}/(8))^*,8)=\left\{\begin{pmatrix}a&b\\0&1\end{pmatrix}\middle|\,a\in(\mathbb{Z}/(8))^*,\ b\in\mathbb{Z}/(8)\right\},$$

as in § 3, regarded as a subgroup of S_8. Take

$$\bar{\boldsymbol{\sigma}}=\left(\begin{pmatrix}1&1\\0&1\end{pmatrix},\begin{pmatrix}3&1\\0&1\end{pmatrix},\begin{pmatrix}5&1\\0&1\end{pmatrix},\begin{pmatrix}7&5\\0&1\end{pmatrix}\right),$$

and

$$\bar{\boldsymbol{\sigma}}^*=\left(\begin{pmatrix}1&1\\0&1\end{pmatrix},\begin{pmatrix}1&7\\0&1\end{pmatrix},\begin{pmatrix}3&1\\0&1\end{pmatrix},\begin{pmatrix}3&5\\0&1\end{pmatrix},\begin{pmatrix}5&1\\0&1\end{pmatrix},\begin{pmatrix}5&3\\0&1\end{pmatrix}\right),$$

in place of $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^*$ in the discussion just above expression (5. 5). From the first example of § 3 the G-Hurwitz number of $\bar{\boldsymbol{\sigma}}$ is 2, and we want to know if $(\bar{\boldsymbol{\sigma}}^*,(1,2,3))$ separates these two G-Hurwitz classes. Since the other G-Hurwitz class in the G-Nielsen class of $\bar{\boldsymbol{\sigma}}$ is represented by $\bar{\boldsymbol{\sigma}}^{\alpha}=\left(\begin{pmatrix}1&1\\0&1\end{pmatrix},\begin{pmatrix}3&1\\0&1\end{pmatrix},\begin{pmatrix}5&5\\0&1\end{pmatrix},\begin{pmatrix}7&1\\0&1\end{pmatrix}\right)$ we may phrase our problem quite simply. Using Lemma 3. 8 we ask if there exists $\bar{Q} \in M(2\cdot(r-1), \Sigma)$ with the first three coordinates of $(\bar{\sigma}^*)\bar{Q}$, in order, equal to

$$(5.10) \qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 5 \\ 0 & 1 \end{pmatrix}.$$

If there is such a $\bar{Q}$, then $(\bar{\sigma}^*, (1, 2, 3))$ does *not* separate these two G-Hurwitz classes, and otherwise it does. We could dispense with the problem immediately, and negatively, if the G-Hurwitz number of $\bar{\sigma}^*$ were 1; but one easily sees from the second example of § 3 that it is two. Surprisingly, the answer is negative: indeed item 11 in Table 2 of § 6 reveals the existence of just such a $\bar{Q}$. Thus, the G-Harbater representation does *not* succeed in distinguishing between the two G-Hurwitz classes of $\mathrm{Ni}(\bar{\sigma})$.

We have failed to answer Problem 5. 7 in the affirmative in this serious special case. It would be most interesting to consider the case where $\bar{\sigma}$ is replaced by $\bar{\sigma}^*$ so as to contemplate separating ability of the G-Harbater representation in a substantial example. In this case, however, we would be forced to consider the Hurwitz number of an element in $(S_8)^{10}$, and this is approaching the limit of even computer capability. We need a general representation theory approach to answer Problem 5. 7 one way or the other.

§ 6. Tables

Table 1

Generators of $\alpha_6^{-1}\{\langle(3\ 4), (2\ 5), (1\ 6)\rangle\}$

1. $\bar{Q}_1^2$
2. $\bar{Q}_1^{-1}\bar{Q}_2^2\bar{Q}_1$
3. $\bar{Q}_1^{-1}\bar{Q}_2^{-1}\bar{Q}_3^2\bar{Q}_2\bar{Q}_1$
4. $\bar{Q}_1^{-1}\bar{Q}_2^{-1}\bar{Q}_3^{-1}\bar{Q}_4^2\bar{Q}_3\bar{Q}_2\bar{Q}_1$
5. $\bar{Q}_1^{-1}\bar{Q}_2^{-1}\bar{Q}_3^{-1}\bar{Q}_4^{-1}\bar{Q}_5^2\bar{Q}_4\bar{Q}_3\bar{Q}_2\bar{Q}_1$
6. $\bar{Q}_2^2$
7. $\bar{Q}_2^{-1}\bar{Q}_3^2\bar{Q}_2$
8. $\bar{Q}_2^{-1}\bar{Q}_3^{-1}\bar{Q}_4^2\bar{Q}_3\bar{Q}_2$
9. $\bar{Q}_2\bar{Q}_3\bar{Q}_4\bar{Q}_3\bar{Q}_2$
10. $\bar{Q}_2^{-1}\bar{Q}_3^{-1}\bar{Q}_4^{-1}\bar{Q}_5^2\bar{Q}_4\bar{Q}_3\bar{Q}_2$
11. $\bar{Q}_3^{-1}\bar{Q}_4^2\bar{Q}_3$
12. $\bar{Q}_3^{-1}\bar{Q}_4^{-1}\bar{Q}_5^2\bar{Q}_4\bar{Q}_3$
13. $\bar{Q}_4^2$
14. $\bar{Q}_4^{-1}\bar{Q}_5^2\bar{Q}_4$
15. $\bar{Q}_5^2$
16. $\bar{Q}_3$
17. $\bar{Q}_1\bar{Q}_2\bar{Q}_3\bar{Q}_4\bar{Q}_5\bar{Q}_4\bar{Q}_3\bar{Q}_2\bar{Q}_1$

Table 2

Representatives of the elements in $\mathrm{Br}(\boldsymbol{\sigma}^{G(\boldsymbol{\sigma})})$ *of type* $(1, 3, 5, 5, 3, 1)$ *as explained in the proof of Proposition* 3. 9.

1. $$\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}, \begin{pmatrix}3&1\\0&1\end{pmatrix}, \begin{pmatrix}5&1\\0&1\end{pmatrix}, \begin{pmatrix}5&3\\0&1\end{pmatrix}, \begin{pmatrix}3&5\\0&1\end{pmatrix}, \begin{pmatrix}1&7\\0&1\end{pmatrix}\right) = \boldsymbol{\sigma}$$

2. $$\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}, \begin{pmatrix}3&1\\0&1\end{pmatrix}, \begin{pmatrix}5&3\\0&1\end{pmatrix}, \begin{pmatrix}5&1\\0&1\end{pmatrix}, \begin{pmatrix}3&3\\0&1\end{pmatrix}, \begin{pmatrix}1&5\\0&1\end{pmatrix}\right) = (\boldsymbol{\sigma})\bar{Q}_1^2$$

3. $$\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}, \begin{pmatrix}3&1\\0&1\end{pmatrix}, \begin{pmatrix}5&3\\0&1\end{pmatrix}, \begin{pmatrix}5&7\\0&1\end{pmatrix}, \begin{pmatrix}3&7\\0&1\end{pmatrix}, \begin{pmatrix}1&7\\0&1\end{pmatrix}\right) = (\boldsymbol{\sigma})\bar{Q}_2^2$$

4. $$\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}, \begin{pmatrix}3&1\\0&1\end{pmatrix}, \begin{pmatrix}5&1\\0&1\end{pmatrix}, \begin{pmatrix}5&5\\0&1\end{pmatrix}, \begin{pmatrix}3&3\\0&1\end{pmatrix}, \begin{pmatrix}1&7\\0&1\end{pmatrix}\right) = (\boldsymbol{\sigma})\bar{Q}_4^2$$

5. $$\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}, \begin{pmatrix}3&1\\0&1\end{pmatrix}, \begin{pmatrix}5&1\\0&1\end{pmatrix}, \begin{pmatrix}5&3\\0&1\end{pmatrix}, \begin{pmatrix}3&3\\0&1\end{pmatrix}, \begin{pmatrix}1&5\\0&1\end{pmatrix}\right) = (\boldsymbol{\sigma})\bar{Q}_5^2$$

6. $$\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}, \begin{pmatrix}3&1\\0&1\end{pmatrix}, \begin{pmatrix}5&3\\0&1\end{pmatrix}, \begin{pmatrix}5&1\\0&1\end{pmatrix}, \begin{pmatrix}3&5\\0&1\end{pmatrix}, \begin{pmatrix}1&7\\0&1\end{pmatrix}\right) = (\boldsymbol{\sigma})\bar{Q}_3$$

7. $$\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}, \begin{pmatrix}3&1\\0&1\end{pmatrix}, \begin{pmatrix}5&7\\0&1\end{pmatrix}, \begin{pmatrix}5&5\\0&1\end{pmatrix}, \begin{pmatrix}3&5\\0&1\end{pmatrix}, \begin{pmatrix}1&7\\0&1\end{pmatrix}\right) = (\boldsymbol{\sigma})\bar{Q}_2\bar{Q}_3\bar{Q}_4^2\bar{Q}_3\bar{Q}_2$$

8. $$\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}, \begin{pmatrix}3&1\\0&1\end{pmatrix}, \begin{pmatrix}5&1\\0&1\end{pmatrix}, \begin{pmatrix}5&5\\0&1\end{pmatrix}, \begin{pmatrix}3&1\\0&1\end{pmatrix}, \begin{pmatrix}1&5\\0&1\end{pmatrix}\right) = (\boldsymbol{\sigma})\bar{Q}_1^2\bar{Q}_2^2$$

9. $$\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}, \begin{pmatrix}3&1\\0&1\end{pmatrix}, \begin{pmatrix}5&3\\0&1\end{pmatrix}, \begin{pmatrix}5&7\\0&1\end{pmatrix}, \begin{pmatrix}3&5\\0&1\end{pmatrix}, \begin{pmatrix}1&5\\0&1\end{pmatrix}\right) = (\boldsymbol{\sigma})\bar{Q}_2^2\bar{Q}_5^2$$

10. $$\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}, \begin{pmatrix}3&1\\0&1\end{pmatrix}, \begin{pmatrix}5&7\\0&1\end{pmatrix}, \begin{pmatrix}5&3\\0&1\end{pmatrix}, \begin{pmatrix}3&7\\0&1\end{pmatrix}, \begin{pmatrix}1&7\\0&1\end{pmatrix}\right) = (\boldsymbol{\sigma})\bar{Q}_2^2\bar{Q}_3$$

11. $$\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}, \begin{pmatrix}3&1\\0&1\end{pmatrix}, \begin{pmatrix}5&5\\0&1\end{pmatrix}, \begin{pmatrix}5&1\\0&1\end{pmatrix}, \begin{pmatrix}3&3\\0&1\end{pmatrix}, \begin{pmatrix}1&7\\0&1\end{pmatrix}\right) = (\boldsymbol{\sigma})\bar{Q}_4^2\bar{Q}_3$$

12. $$\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}, \begin{pmatrix}3&1\\0&1\end{pmatrix}, \begin{pmatrix}5&7\\0&1\end{pmatrix}, \begin{pmatrix}5&1\\0&1\end{pmatrix}, \begin{pmatrix}3&5\\0&1\end{pmatrix}, \begin{pmatrix}1&3\\0&1\end{pmatrix}\right) = (\boldsymbol{\sigma})\bar{Q}_2\bar{Q}_3\bar{Q}_4\bar{Q}_1^2\bar{Q}_4\bar{Q}_3\bar{Q}_2$$

13. $$\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}, \begin{pmatrix}3&1\\0&1\end{pmatrix}, \begin{pmatrix}5&1\\0&1\end{pmatrix}, \begin{pmatrix}5&1\\0&1\end{pmatrix}, \begin{pmatrix}3&3\\0&1\end{pmatrix}, \begin{pmatrix}1&3\\0&1\end{pmatrix}\right) = (\boldsymbol{\sigma})\bar{Q}_4\bar{Q}_5^2\bar{Q}_4$$

14. $$\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}, \begin{pmatrix}3&1\\0&1\end{pmatrix}, \begin{pmatrix}5&3\\0&1\end{pmatrix}, \begin{pmatrix}5&3\\0&1\end{pmatrix}, \begin{pmatrix}3&7\\0&1\end{pmatrix}, \begin{pmatrix}1&3\\0&1\end{pmatrix}\right) = (\boldsymbol{\sigma})\bar{Q}_3\bar{Q}_4\bar{Q}_5^2\bar{Q}_4\bar{Q}_3$$

15. $$\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}, \begin{pmatrix}3&1\\0&1\end{pmatrix}, \begin{pmatrix}5&3\\0&1\end{pmatrix}, \begin{pmatrix}5&5\\0&1\end{pmatrix}, \begin{pmatrix}3&3\\0&1\end{pmatrix}, \begin{pmatrix}1&1\\0&1\end{pmatrix}\right) = (\boldsymbol{\sigma})\bar{Q}_1\bar{Q}_2^2\bar{Q}_1$$

16. $$\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}, \begin{pmatrix}3&1\\0&1\end{pmatrix}, \begin{pmatrix}5&1\\0&1\end{pmatrix}, \begin{pmatrix}5&1\\0&1\end{pmatrix}, \begin{pmatrix}3&1\\0&1\end{pmatrix}, \begin{pmatrix}1&1\\0&1\end{pmatrix}\right) = (\boldsymbol{\sigma})\bar{Q}_1\bar{Q}_2\bar{Q}_1^2\bar{Q}_2\bar{Q}_1$$

17. $$\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}, \begin{pmatrix}3&1\\0&1\end{pmatrix}, \begin{pmatrix}5&3\\0&1\end{pmatrix}, \begin{pmatrix}5&3\\0&1\end{pmatrix}, \begin{pmatrix}3&5\\0&1\end{pmatrix}, \begin{pmatrix}1&1\\0&1\end{pmatrix}\right) = (\boldsymbol{\sigma})\bar{Q}_1\bar{Q}_2\bar{Q}_4^2\bar{Q}_2\bar{Q}_1$$

18. $$\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}, \begin{pmatrix}3&1\\0&1\end{pmatrix}, \begin{pmatrix}5&3\\0&1\end{pmatrix}, \begin{pmatrix}5&5\\0&1\end{pmatrix}, \begin{pmatrix}3&5\\0&1\end{pmatrix}, \begin{pmatrix}1&3\\0&1\end{pmatrix}\right) = (\boldsymbol{\sigma})\bar{Q}_1\bar{Q}_2\bar{Q}_5^2\bar{Q}_2\bar{Q}_1$$

19. $$\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}, \begin{pmatrix}3&1\\0&1\end{pmatrix}, \begin{pmatrix}5&5\\0&1\end{pmatrix}, \begin{pmatrix}5&1\\0&1\end{pmatrix}, \begin{pmatrix}3&1\\0&1\end{pmatrix}, \begin{pmatrix}1&5\\0&1\end{pmatrix}\right) = (\boldsymbol{\sigma})\bar{Q}_1\bar{Q}_2\bar{Q}_3\bar{Q}_1^2\bar{Q}_3\bar{Q}_2\bar{Q}_1$$

Table 2 (continued)

20. $$\left(\begin{pmatrix}1&1\\0&1\end{pmatrix},\ \begin{pmatrix}3&1\\0&1\end{pmatrix},\ \begin{pmatrix}5&5\\0&1\end{pmatrix},\ \begin{pmatrix}5&3\\0&1\end{pmatrix},\ \begin{pmatrix}3&3\\0&1\end{pmatrix},\ \begin{pmatrix}1&1\\0&1\end{pmatrix}\right)=(\boldsymbol{\sigma})\bar{Q}_1\bar{Q}_2^2\bar{Q}_1\bar{Q}_3$$

21. $$\left(\begin{pmatrix}1&1\\0&1\end{pmatrix},\ \begin{pmatrix}3&1\\0&1\end{pmatrix},\ \begin{pmatrix}5&1\\0&1\end{pmatrix},\ \begin{pmatrix}5&7\\0&1\end{pmatrix},\ \begin{pmatrix}3&5\\0&1\end{pmatrix},\ \begin{pmatrix}1&3\\0&1\end{pmatrix}\right)=(\boldsymbol{\sigma})\bar{Q}_2\bar{Q}_3\bar{Q}_4\bar{Q}_1^2\bar{Q}_4\bar{Q}_3\bar{Q}_2\bar{Q}_3$$

22. $$\left(\begin{pmatrix}1&1\\0&1\end{pmatrix},\ \begin{pmatrix}3&1\\0&1\end{pmatrix},\ \begin{pmatrix}5&7\\0&1\end{pmatrix},\ \begin{pmatrix}5&3\\0&1\end{pmatrix},\ \begin{pmatrix}3&5\\0&1\end{pmatrix},\ \begin{pmatrix}1&5\\0&1\end{pmatrix}\right)=(\boldsymbol{\sigma})\bar{Q}_4^2\bar{Q}_3\bar{Q}_1^2$$

23. $$\left(\begin{pmatrix}1&1\\0&1\end{pmatrix},\ \begin{pmatrix}3&1\\0&1\end{pmatrix},\ \begin{pmatrix}5&5\\0&1\end{pmatrix},\ \begin{pmatrix}5&7\\0&1\end{pmatrix},\ \begin{pmatrix}3&5\\0&1\end{pmatrix},\ \begin{pmatrix}1&7\\0&1\end{pmatrix}\right)=(\boldsymbol{\sigma})\bar{Q}_4^2\bar{Q}_3\bar{Q}_4^2$$

24. $$\left(\begin{pmatrix}1&1\\0&1\end{pmatrix},\ \begin{pmatrix}3&1\\0&1\end{pmatrix},\ \begin{pmatrix}5&5\\0&1\end{pmatrix},\ \begin{pmatrix}5&7\\0&1\end{pmatrix},\ \begin{pmatrix}3&3\\0&1\end{pmatrix},\ \begin{pmatrix}1&5\\0&1\end{pmatrix}\right)=(\boldsymbol{\sigma})\bar{Q}_4^2\bar{Q}_3\bar{Q}_4^2\bar{Q}_5^2$$

25. $$\left(\begin{pmatrix}1&1\\0&1\end{pmatrix},\ \begin{pmatrix}3&1\\0&1\end{pmatrix},\ \begin{pmatrix}5&5\\0&1\end{pmatrix},\ \begin{pmatrix}5&5\\0&1\end{pmatrix},\ \begin{pmatrix}3&3\\0&1\end{pmatrix},\ \begin{pmatrix}1&3\\0&1\end{pmatrix}\right)=(\boldsymbol{\sigma})\bar{Q}_4^2\bar{Q}_3\bar{Q}_4^3\bar{Q}_5^2\bar{Q}_4$$

26. $$\left(\begin{pmatrix}1&1\\0&1\end{pmatrix},\ \begin{pmatrix}3&1\\0&1\end{pmatrix},\ \begin{pmatrix}5&7\\0&1\end{pmatrix},\ \begin{pmatrix}5&7\\0&1\end{pmatrix},\ \begin{pmatrix}3&7\\0&1\end{pmatrix},\ \begin{pmatrix}1&3\\0&1\end{pmatrix}\right)=(\boldsymbol{\sigma})\bar{Q}_4^2\bar{Q}_3\bar{Q}_4^2\bar{Q}_3\bar{Q}_4\bar{Q}_5^2\bar{Q}_4\bar{Q}_3$$

27. $$\left(\begin{pmatrix}1&1\\0&1\end{pmatrix},\ \begin{pmatrix}3&1\\0&1\end{pmatrix},\ \begin{pmatrix}5&7\\0&1\end{pmatrix},\ \begin{pmatrix}5&1\\0&1\end{pmatrix},\ \begin{pmatrix}3&3\\0&1\end{pmatrix},\ \begin{pmatrix}1&1\\0&1\end{pmatrix}\right)=(\boldsymbol{\sigma})\bar{Q}_4^2\bar{Q}_3\bar{Q}_4^2\bar{Q}_1\bar{Q}_2^2\bar{Q}_1$$

28. $$\left(\begin{pmatrix}1&1\\0&1\end{pmatrix},\ \begin{pmatrix}3&1\\0&1\end{pmatrix},\ \begin{pmatrix}5&5\\0&1\end{pmatrix},\ \begin{pmatrix}5&5\\0&1\end{pmatrix},\ \begin{pmatrix}3&1\\0&1\end{pmatrix},\ \begin{pmatrix}1&1\\0&1\end{pmatrix}\right)=(\boldsymbol{\sigma})Q_4^2\bar{Q}_3\bar{Q}_4^2\bar{Q}_1\bar{Q}_2\bar{Q}_1^2\bar{Q}_2\bar{Q}_1$$

29. $$\left(\begin{pmatrix}1&1\\0&1\end{pmatrix},\ \begin{pmatrix}3&1\\0&1\end{pmatrix},\ \begin{pmatrix}5&7\\0&1\end{pmatrix},\ \begin{pmatrix}5&7\\0&1\end{pmatrix},\ \begin{pmatrix}3&5\\0&1\end{pmatrix},\ \begin{pmatrix}1&1\\0&1\end{pmatrix}\right)=(\boldsymbol{\sigma})\bar{Q}_4^2\bar{Q}_3\bar{Q}_4^2\bar{Q}_1\bar{Q}_2\bar{Q}_4^2\bar{Q}_2\bar{Q}_1$$

30. $$\left(\begin{pmatrix}1&1\\0&1\end{pmatrix},\ \begin{pmatrix}3&1\\0&1\end{pmatrix},\ \begin{pmatrix}5&1\\0&1\end{pmatrix},\ \begin{pmatrix}5&7\\0&1\end{pmatrix},\ \begin{pmatrix}3&3\\0&1\end{pmatrix},\ \begin{pmatrix}1&1\\0&1\end{pmatrix}\right)=(\boldsymbol{\sigma})\bar{Q}_4^2\bar{Q}_3\bar{Q}_4^2\bar{Q}_1\bar{Q}_2^2\bar{Q}_1\bar{Q}_3$$

31. $$\left(\begin{pmatrix}1&1\\0&1\end{pmatrix},\ \begin{pmatrix}3&1\\0&1\end{pmatrix},\ \begin{pmatrix}5&5\\0&1\end{pmatrix},\ \begin{pmatrix}5&3\\0&1\end{pmatrix},\ \begin{pmatrix}3&5\\0&1\end{pmatrix},\ \begin{pmatrix}1&3\\0&1\end{pmatrix}\right)=(\boldsymbol{\sigma})\bar{Q}_4^2\bar{Q}_3\bar{Q}_4^2\bar{Q}_2\bar{Q}_3\bar{Q}_4\bar{Q}_1^2\bar{Q}_4\bar{Q}_3\bar{Q}_2\bar{Q}_3$$

32. $$\left(\begin{pmatrix}1&1\\0&1\end{pmatrix},\ \begin{pmatrix}3&1\\0&1\end{pmatrix},\ \begin{pmatrix}5&7\\0&1\end{pmatrix},\ \begin{pmatrix}5&5\\0&1\end{pmatrix},\ \begin{pmatrix}3&3\\0&1\end{pmatrix},\ \begin{pmatrix}1&5\\0&1\end{pmatrix}\right)=(\boldsymbol{\sigma})\bar{Q}_4^2\bar{Q}_3\bar{Q}_4^2\bar{Q}_1^2$$

Table 3

For $1 \leqq i < j \leqq 6$, $A_{ij} = \bar{Q}_i^{-1} \bar{Q}_{i+1}^{-1} \cdots \bar{Q}_{j-2}^{-1} \bar{Q}_{j-1}^2 \bar{Q}_{j-2} \cdots \bar{Q}_{i+1} \bar{Q}_i$ and $A_{ii+1} = \bar{Q}_i^2$.
$B = \bar{Q}_2 \bar{Q}_3 \bar{Q}_4 \bar{Q}_3 \bar{Q}_2$

Generators of $\alpha_6^{-1}\{\langle(3, 4), (2, 5), (1, 6)\rangle\}$

	1	2	3	4	5	6	7	8	9
Elements in Table 2	A_{12}	A_{13}	A_{14}	A_{15}	A_{16}	A_{23}	A_{24}	A_{25}	B
1	2	21	31	32	1	3	11	1	7
2	1	15	27	23	2	8	22	2	5
3	8	14	26	19	3	1	7	3	4
4	9	13	25	22	4	6	23	4	3
5	6	30	20	7	5	9	19	5	2
6	5	18	12	24	6	4	10	6	23
7	24	12	18	5	7	11	3	7	1
8	3	16	28	10	8	2	24	8	22
9	4	17	29	11	9	5	32	9	19
10	19	26	14	8	10	23	6	10	11
11	22	25	13	9	11	7	1	11	10
12	20	7	6	30	12	25	14	12	21
13	17	4	11	29	13	18	31	13	14
14	16	3	10	28	14	21	12	14	13
15	21	2	32	31	15	16	29	15	30
16	14	8	19	26	16	15	20	16	29
17	13	9	22	25	17	30	27	17	28
18	30	6	7	20	18	13	26	18	31
19	10	28	16	3	19	32	5	19	9
20	12	24	5	18	20	29	16	20	27
21	15	1	23	27	21	14	25	21	12
22	11	29	17	4	22	24	2	22	8
23	32	31	21	2	23	10	4	23	6
24	7	20	30	6	24	22	8	24	32
25	29	11	4	17	25	12	21	25	26
26	28	10	3	16	26	31	18	26	25
27	31	32	2	21	27	28	17	27	20
28	26	19	8	14	28	27	30	28	17
29	25	22	9	13	29	20	15	29	16
30	18	5	24	12	30	17	28	30	15
31	27	23	1	15	31	26	13	31	18
32	23	27	15	1	32	19	9	32	24

Table 3 (continued)

Generators of $\alpha_6^{-1}\{\langle(3,4),(2,5),(1,6)\rangle\}$

	10	11	12	13	14	15	16	17
Elements in Table 2	A_{26}	A_{35}	A_{36}	A_{45}	A_{46}	A_{56}	$\bar{Q}_3$	$\bar{Q}_1 B \bar{Q}_1$
1	24	10	31	4	21	5	6	6
2	7	19	27	9	25	6	5	32
3	22	23	26	6	14	9	10	4
4	19	7	25	1	13	8	11	3
5	23	22	20	8	30	1	2	24
6	32	11	12	3	18	2	1	1
7	2	4	18	10	12	32	23	23
8	11	32	28	5	16	4	19	19
9	10	24	29	2	17	3	22	22
10	9	1	14	7	26	22	3	11
11	8	6	13	23	25	19	4	10
12	15	13	6	26	7	27	21	21
13	28	12	11	21	4	16	13	26
14	29	31	10	18	3	17	14	25
15	12	28	32	17	2	18	20	15
16	25	27	19	30	8	13	16	16
17	26	20	22	15	9	14	17	17
18	27	25	7	14	6	15	31	31
19	4	2	16	24	28	11	8	8
20	21	17	5	28	24	31	15	20
21	20	26	23	13	1	30	12	12
22	3	5	17	32	29	10	9	9
23	5	3	21	11	31	24	7	7
24	1	9	30	19	20	23	32	5
25	16	18	4	31	11	28	25	14
26	17	21	3	12	10	29	26	13
27	18	16	2	29	32	12	30	27
28	13	15	8	20	19	25	28	28
29	14	30	9	27	22	26	29	29
30	31	29	24	16	5	21	27	30
31	30	14	1	25	23	20	18	18
32	6	8	15	22	27	7	24	2

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