EFFECTIVE BRANCH CYCLE COMPUTATION

M. Fried - U.C. Irvine*

R. Whitley - U.C. Irvine

Abstract. Denote the Riemann sphere by $\mathbb{P}^1 = \mathbb{P}^1_z = \mathbb{C} \dot{\cup} \{\infty\}$ where z is a complex variable uniformizing the plane. The absolute Galois group, $G(\bar{\mathbb{Q}}(z))$, of the algebraic closure of $\bar{\mathbb{Q}}(z)$ is profree on a set of generators that are in one-one correspondence with the elements of $\bar{\mathbb{Q}}$ (Proposition 1.3).

We present a system of paths on $\mathbb{P}^1$ which allows an unambiguous assignation of branch cycles (in S_n) to each degree n extension $L/\bar{\mathbb{Q}}(z)$ (§1). Suppose that L is the quotient field of $\bar{\mathbb{Q}}[z,w]/(f(z,w))$ with f an irreducible polynomial. We give an algorithm for computing these branch cycles for L (§2). For special polynomials f - with emphasis on two practical examples - we analyze a computer program for computing these branch cycles (§3). In [Fr,3] we compare this with computation of the moduli-invariant called the Hurwitz class of f - a quantity that does depend on a choice of paths.

§1. Introduction and bouquet generators.

For any field F denote the algebraic closure of F by $\bar{F}$. Let $z' \in \bar{\mathbb{Q}}$ (resp., $z' = \infty$) and let $\bar{\mathbb{Q}}\{\{z - z'\}\}$ (resp., $\bar{\mathbb{Q}}\{\{1/z\}\}$) be the field of formal Laurent series in $z - z'$ (resp., $1/z$).

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Identify the respective algebraic closures of $\bar{\mathbb{Q}}\{\{z-z'\}\}$ and $\bar{\mathbb{Q}}\{\{1/z\}\}$ with $\bigcup_{e=1}^{\infty} \bar{\mathbb{Q}}\{\{(z-z')^{1/e}\}\}$ and $\bigcup_{e=1}^{\infty} \bar{\mathbb{Q}}\{\{z^{1/e}\}\}$ [SK;p.83] .

Let $L/\bar{\mathbb{Q}}(z)$ be a field extension of degree $n = [L:\bar{\mathbb{Q}}(z)]$. Then the extension $L \otimes \bar{\mathbb{Q}}\{\{z-z'\}\}$ is a sum of fields $L_1 \oplus \cdots \oplus L_k$ where $L_i = \bar{\mathbb{Q}}\{\{(z-z')^{1/e_i(z')}\}\}$ and $\sum_{i=1}^{k} e_i(z') = n$ [SK;p.86] . Excluding a finite number of values of z' the integers $e_i(z')$ are all 1 , $i = 1,\cdots k$. The exceptional values of z' , denoted $d(L/\bar{\mathbb{Q}}(z))$ are called the branch points of the extension. Let f be an irreducible polynomial in $\bar{\mathbb{Q}}[z,w]$, monic in w . In the case that L is the quotient field of $\bar{\mathbb{Q}}[z,w]/(f(z,w))$, the finite values of $d(L/\bar{\mathbb{Q}}(z))$ are contained in the set $d(f)$ consisting of the values z' for which this holds:

(1.1) $\left.\frac{\partial f}{\partial w}\right|_{z=z'}$ and $f(z',w)$ have a common factor in w .

If z' satisfies (1.1) and $\left.\frac{\partial f}{\partial z}\right|_{z=z'}$ has no common factor with $f(z',w)$ - that is, z' is not the z-coordinate of a singular point of $f(z,w) = 0$ - then z' certainly belongs to $d(L/\bar{\mathbb{Q}}(z))$ [SK;p.97]. The branch cycles of §2 give an ultimate test for distinguishing between $d(f)$ and $d(L/\bar{\mathbb{Q}}(z))$. The polynomial f also provides a test for whether ∞ is in $d(L/\bar{\mathbb{Q}}(z))$. With $m = \deg_z(f(z,w))$, let $g(z,w) = z^m \cdot f(1/z,w)$. It may not be monic. Let $g_o(z)$ be the coefficient of w^n . Rewrite $(g_o(z))^{n-1} \cdot g(z,w)$ as $g_1(z,g_o(z)\cdot w)$ where $g_1(z,w)$ is monic in w . Now apply the test of (1.1) to $g_1(z,w)$ with $z' = 0$. If $\infty \in d(L/\bar{\mathbb{Q}}(z))$ then (1.1) holds and if, in addition, $\left.\frac{\partial g_1}{\partial z}\right|_{z=o} \neq 0$, then $\infty \in d(L/\bar{\mathbb{Q}}(z))$.

Ex.1.1. Take $f(z,w) = h(w) - z$ with $h \in \bar{\mathbb{Q}}[w]$ a polynomial of degree n. Then $D(f)$ consists of the values $h(w')$ where w' runs over the zeros of $\frac{d}{dw}(h(w))$. Identify L, the quotient field of $\bar{\mathbb{Q}}[z,w]/(h(w) - z)$, with $\bar{\mathbb{Q}}(w)$. Then the finite values of $d(\bar{\mathbb{Q}}(w)/\bar{\mathbb{Q}}(z))$ are exactly the values of $D(f)$. In addition, let $g(z,w) = z\cdot(h(w) - 1/z) = z\cdot h(w) - 1$. If $h(w) = w^n + \sum_{i=o}^{n-1} a_i \cdot w^i$, then $z^{n-1}\cdot g(z,w) = g_1(z,z\cdot w)$ where $g_1(z,w) = w^n + (\sum_{i=o}^{n-1} a_i \cdot z^{n-1-i}\cdot w^i) - z^{n-1}$. The test above for $z' = 0$ indicates that $\infty \in d(L/\bar{\mathbb{Q}}(z))$. ■

Let $z(1), z(2), \cdots$ be an ordering of the elements of $\bar{\mathbb{Q}} \dot{\cup} \{\infty\}$. For simplicity assume that $z(1) = \infty$. Denote the normal subfield of $\overline{\mathbb{Q}(z)}$ consisting of the composite of field extensions $L/\bar{\mathbb{Q}}(z)$ with $d(L/\bar{\mathbb{Q}}(z)) \subseteq \{z(1),\cdots,z(r)\} = \{\underset{\sim}{z}_r\}$ by $N^{(r)}$. As above, $\mathbb{P}^1 = \mathbb{C} \dot{\cup} \{\infty\} = \mathbb{P}^1_z$. We choose $\beta(\underset{\sim}{z}_r) = \max_{2\le i\le r} (1 + \mathrm{Re}(z(i)))$ as basepoint for the fundamental group $\pi_1(\mathbb{P}^1 - \{\underset{\sim}{z}_r\}, \beta(\underset{\sim}{z}_r))$. Denote this group by $\pi_1^{(r)}$ and its profinite completion by $G^{(r)}$. Riemann's Existence theorem states that $G^{(r)}$ is topologically isomorphic to the Galois group $G(N^{(r)}/\bar{\mathbb{Q}}(z))$. Indeed, for $\sigma \in G^{(r)}$ and $\alpha \in N^{(r)}$ the action of σ on α goes as follows. Let N_α be the smallest (finite Galois extension of $\mathbb{Q}(z)$ containing α. There exists $\tau \in \pi_1(\mathbb{P}^1 - \{\underset{\sim}{z}_r\}, \beta(\underset{\sim}{z}_r))$ for which $\tau \equiv \sigma \bmod G(N^{(r)}/N_\alpha)$. Let P be a closed path in $\mathbb{P}^1 - \{\underset{\sim}{z}_r\}$ based at $\beta(\underset{\sim}{z}_r)$ for which $[P]$, the homotopy class of P, is τ. The monodromy theorem asserts that the function α' obtained from the analytic continuation of the Puiseux expansion for α at $\beta(\underset{\sim}{z}_r)$ depends only on $[P] \equiv \tau \bmod G(N^{(r)}/N_\alpha)$. The function α' is the result of the action of σ on α [Sp;Chap. 2]. Note in particular that this calculation depends on the base point $\beta(\underset{\sim}{z}_r)$.

The goal of the remainder of this subsection is to choose explicit paths whose homotopy classes generate $\pi_1(\mathbb{P}^1 - \{\underset{\sim}{z}_r\}, \beta(\underset{\sim}{z}_r))$ freely in such a way that the corresponding paths for $\pi_1(\mathbb{P}^1 - \{\underset{\sim}{z}_s\}, \beta(\underset{\sim}{z}_s))$, $s > r$, give a well defined map $\varphi_{s,r}: \pi_1^{(r)} \to \pi_1^{(s)}$. In addition the profinite limit over all r induces a profinite limit of $\{G^{(r)}\}_{r=1,2,\cdots}$ that is naturally isomorphic to $G(\overline{\mathbb{Q}(z)}/\overline{\mathbb{Q}}(z))$ (i.e., the absolute Galois group $G(\overline{\mathbb{Q}}(z))$) that shows it to be profree on generators $\{\sigma(z')\}_{z' \in \overline{\mathbb{Q}}}$.

We use the <u>lexicographical order</u> on $\mathbb{C}$: $z_1 < z_2$ if $\mathrm{Re}(z_1) \leq \mathrm{Re}(z_2)$ and if $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$, then $\mathrm{Im}(z_1) < \mathrm{Im}(z_2)$. Rename the elements of $\{z(2),\cdots,z(r)\}$ as $\{y(2),\cdots,y(r)\}$ so that $y(2) <\cdots< y(r)$. We use both names for a given point. Refer to Fig. 1.

Choose an integer $\ell \geq 7 : 2\pi/\ell < 1$. The vertical lines in Fig. 1 give a guide for positioning the points $a(2),\cdots,a(r)$: $a(i)$ lies in the region between the vertical line for $y(i)$ and the vertical line for $y(i+k+1)$, the next point <u>not</u> on the vertical line for $y(i)$.

Take γ_i to be the following path $\delta_i^{(1)} \cdot \delta_i^{(2)} \cdot \bar{\gamma}_i \cdot (\delta_i^{(2)})^{-1} \cdot (\delta_i^{(1)})^{-1}$: $\delta_i^{(1)}$ is the straight line from $\beta(\underset{\sim}{z}_r)$ to $a(i)$; $b(i)$ is the point of intersection of the boundary of a disc with center at $y(i)$ and the line from $a(i)$ to $y(i)$; $\delta_i^{(2)}$ is the straight line from $a(i)$ to $b(i)$; and $\bar{\gamma}_i$ is the clockwise path around an equalateral ℓ-gon with center $y(i)$ and starting at $b(i)$, $i = 2,\cdots,r$. Two further constraints:

(1.2) a) Excluding their initial points, $\gamma_2,\cdots,\gamma_r$ are non-intersecting; and

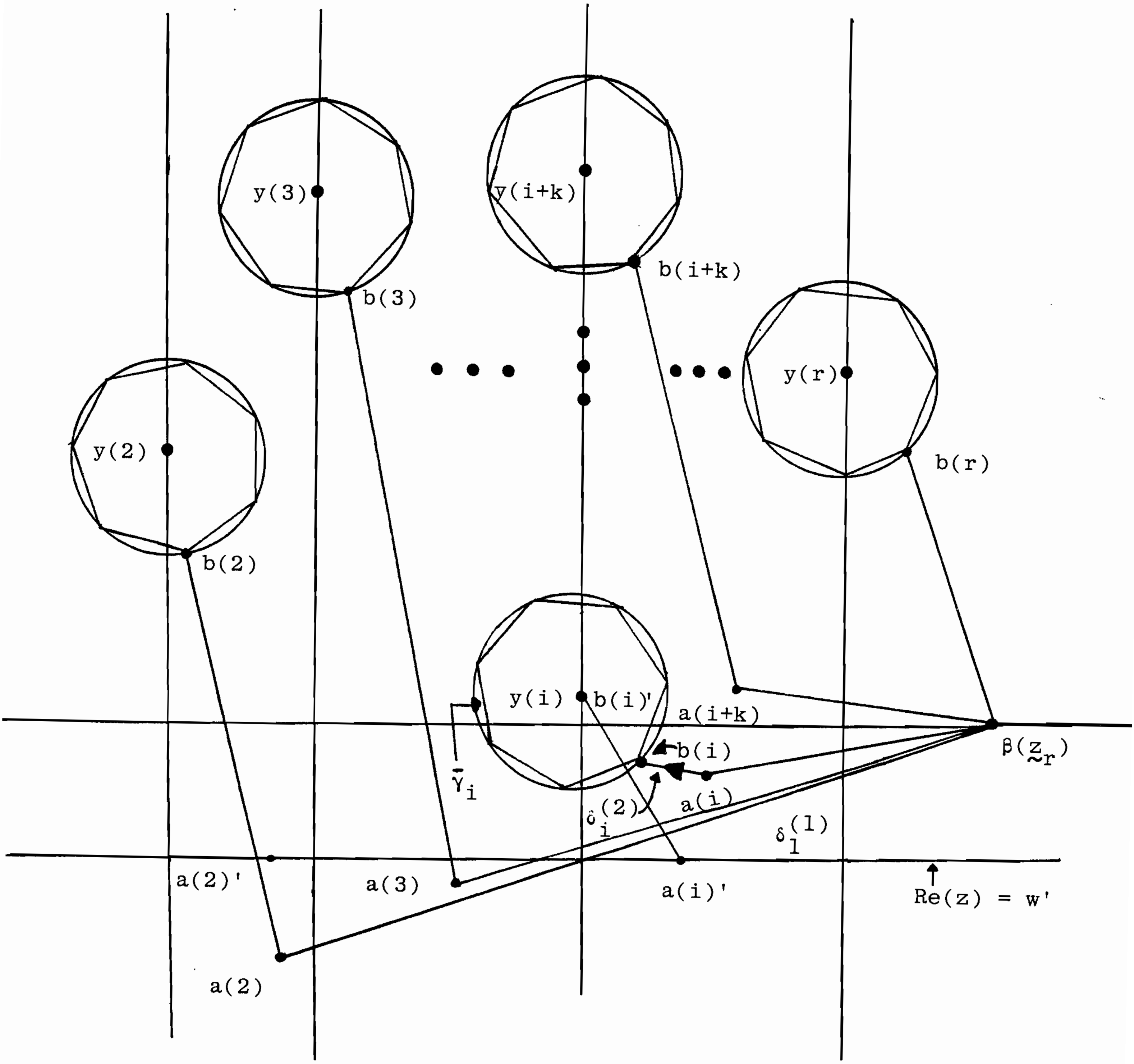

Fig. 1. 7-gon sample bouquet.

b) $\delta_2^{(1)},\cdots,\delta_r^{(1)}$ emanate in clockwise order from $\beta(\underset{\sim}{z}_r)$.

The next lemma follows from the technique of proof of [Ma;p.118].

LEMMA 1.2. The homotopy classes $[\gamma_2],\cdots,[\gamma_r]$ generate $\pi_1(\mathbb{P}^1 - \{\underset{\sim}{z}_r\},\beta(\underset{\sim}{z}_r))$ freely. Let $b(\infty)$ be a point on the real axis to the right of $\beta(\underset{\sim}{z}_r)$. Assume that $\bar{\gamma}_\infty$, the clockwise boundary of a disc on the Riemann sphere centered at ∞ and starting at $b(\infty)$, contains none of the points $y(2),\cdots,y(r)$. Let δ_∞ be the line segment from $\beta(\underset{\sim}{z}_r)$ to $b(\infty)$. Then $\gamma_\infty\cdot\gamma_2\cdots\gamma_r$ is homotopic on $\mathbb{P}^1 - \{\underset{\sim}{z}_r\}$ to the constant path where $\gamma_\infty = \delta_\infty\cdot\bar{\gamma}_\infty\cdot(\delta_\infty)^{-1}$. Finally, with the constraints above, $[\gamma_2],\cdots,[\gamma_r]$ do not depend on the choice of pairs $(a(2),b(2)),\cdots,(a(r),b(r))$.

We refer to γ_i as a loop around $y(i)$ and to the collection $\gamma_2,\cdots,\gamma_r$ as a sample bouquet (for $\underset{\sim}{z}_r$) based at $\beta(\underset{\sim}{z}_r)$. From now on take $\ell = 7$, so that the prefix ℓ-gon will be unnecessary. In addition, our concern is with homotopy classes of paths in $\mathbb{P}^1 - \{\underset{\sim}{z}_r\}$. Thus the restrictions of (1.2) are excessive in practice. Here is a prescription for choosing satisfactory a's and b's without condition (1.2)a).

Let $w' = \min\{0,\mathrm{Im}(y(2)),\cdots,\mathrm{Im}(y(r))\} - 1$ and let

$$a(i)' = (\mathrm{Re}(y(i)) + \mathrm{Re}(y(i+k+1)))/2 + \sqrt{-1}\cdot w' \ , \ i = 2,\cdots,r-1$$

(as in Fig. 1). Take $a(r)' = \beta(\underset{\sim}{z}_r)$. Let the smaller of 1 and $\min_{1\le i<j\le r}\{|y(j)-y(i)|/2\}$ be b' . Finally, take $b(i)'$ to be the point along the line segment from $a(i)'$ to $y(i)$ that is a distance

of b' from $y(i)$, $i = 2,\cdots,r$.

Use $(a(i)', b(i)')$ to form a loop $\gamma(\underset{\sim}{z}_r)_i$ around $y(i)$ - just as we did with $(a(i),b(i))$, $i = 2,\cdots,r$ - with $\ell = 7$. Call the ordered collection $(\gamma(\underset{\sim}{z}_r)_2,\cdots,\gamma(\underset{\sim}{z}_r)_r)$ the <u>tied bouquet</u> associated to $\underset{\sim}{z}_r$. We use the tied bouquet from this point to give representatives of the paths that appear in Lemma 1.2. Let $F(r-1)$ denote the free group on generators $s_2,\cdots,s_r$. From Lemma 1.2 it is isomorphic to π_1 by the map that sends $[\gamma(\underset{\sim}{z}_r)_i]$ to s_i , $i = 2,\cdots,r$.

Consider any real point z' to the right of $\beta(\underset{\sim}{z}_r)$ along the real axis. Compare the groups $\pi_1(\mathbb{P}^1 - \{\underset{\sim}{z}_r\}, \beta(\underset{\sim}{z}_r))$ and $\pi_1(\mathbb{P}^1 - \{\underset{\sim}{z}_r\}, z')$ as follows. Write $\gamma(\underset{\sim}{z}_r)_i$ as $\delta_i^{(1)}\cdot\delta_i^{(2)}\cdot\bar{\gamma}_i\cdot(\delta_i^{(2)})^{-1}\cdot(\delta_i^{(1)})^{-1}$ relative to $(a(i)', b(i)')$ in the notation above. Then replace $\delta_i^{(1)}$ by $\delta_i'^{(1)}$, the straight line segment from z' to $a(i)'$, to obtain a path $\gamma_i' = \delta_i'^{(1)}\cdot\delta_i^{(2)}\cdot\bar{\gamma}_i\cdot(\delta_i^{(2)})^{-1}\cdot(\delta_i'^{(1)})^{-1}$, $i = 1,\cdots,r$. Take λ to be the path along the real line from $\beta(\underset{\sim}{z}_r)$ to z' . The map that takes a closed path P based at $\beta(\underset{\sim}{z}_r)$ to the path $(\lambda)^{-1}\cdot P\cdot\lambda$ based at z' induces an isomorphism $\psi(\beta(\underset{\sim}{z}_r),z') : \pi_1(\mathbb{P}^1 - \{\underset{\sim}{z}_r\}, \beta(\underset{\sim}{z}_r)) \to \pi_1(\mathbb{P}^1 - \{\underset{\sim}{z}_r\}, z')$. In addition, $\psi(\beta(\underset{\sim}{z}_r),z')$ takes $[\gamma(\underset{\sim}{z}_r)_i]$ to $[\gamma_i']$, $i = 1,\cdots,r$.

Now we add $z(r+1)$ to $\{\underset{\sim}{z}_r\}$. Form $\beta(\underset{\sim}{z}_{r+1})$ and identify $\pi_1(\mathbb{P}^1 - \{\underset{\sim}{z}_r\}, \beta(\underset{\sim}{z}_{r+1}))$ with $\pi_1(\mathbb{P}^1 - \{\underset{\sim}{z}_r\}, \beta(\underset{\sim}{z}_r))$ using $\psi(\beta(\underset{\sim}{z}_r), \beta(\underset{\sim}{z}_{r+1}))$. Let $\gamma(\underset{\sim}{z}_{r+1})$ be the tied bouquet corresponding to $\underset{\sim}{z}_{r+1}$. The inclusion of $\mathbb{P}^1 - \{\underset{\sim}{z}_{r+1}\}$ in $\mathbb{P}^1 - \{\underset{\sim}{z}_r\}$ induces a natural map

$$(1.3)\quad \pi_1^{(r+1)} = \pi_1(\mathbb{P}^1 - \{\underset{\sim}{z}_{r+1}\}, \beta(\underset{\sim}{z}_{r+1})) \to \pi_1(\mathbb{P}^1 - \{\underset{\sim}{z}_r\}, \beta(\underset{\sim}{z}_{r+1}))$$

that takes $[\gamma(\underset{\sim}{z}_{r+1})_i]$ to $[\gamma_i']$, $i = 1,\cdots,r$, and $[\gamma(\underset{\sim}{z}_{r+1})_{r+1}]$ to the identity. Define $\varphi_{r+1,r} : \pi_1^{(r+1)} \to \pi_1^{(r)}$ to be the composition of (1.3) and $\psi(\beta(\underset{\sim}{z}_{r+1}), \beta(\underset{\sim}{z}_r))^{-1}$. Then take

$\varphi_{s,s-1} \circ \varphi_{s-1,s-2} \circ \cdots \circ \varphi_{r+1,r}$ for $s > r$ to be $\varphi_{s,r}$. Since $\varphi_{t,s} \circ \varphi_{s,r} = \varphi_{t,r}$ for $t > s > r$, we may form the projective limit of the groups $\{\pi_1^{(r)}\}_{r=1,2,\cdots}$ to obtain a group isomorphic to $F(\infty)$: the free group on generators $s_2, s_3, \cdots$. In addition, this induces a projective limit of the groups $\{G^{(r)}\}_{r=2,3,\cdots}$ (the respective profinite completions of $\{\pi_1^{(r)}\}_{r=2,3,\cdots}$). From it conclude this folklore statement:

PROPOSITION 1.3. <u>The absolute Galois group</u>, $G(\bar{\mathbb{Q}}(z))$, <u>is profree on a set of generators that are in one-one correspondence with the elements of</u> $\bar{\mathbb{Q}}$. <u>The element</u> $s_2 \cdot s_3 \cdots = s_\infty$ <u>is the inverse of a natural inertial group generator for a place of</u> $\overline{\mathbb{Q}(z)}$ <u>lying over</u> $z = \infty$.

§2. Branch cycle computation.

Here we no longer assume that the elements of $\bar{\mathbb{Q}}$ are ordered. Rather, take $z(1),\cdots,z(r)$ to be the points of $d(f) \cup \{\infty\}$ (i.e., $z(1)=\infty$) where $f \in \bar{\mathbb{Q}}[z,w]$ is an irreducible polynomial, monic and of degree n in w.

For each $z' \in \bar{\mathbb{Q}} - d(f)$ there exist n distinct power series in $z - z'$, $w(z',v;z-z')$, $v = 1,\cdots,n$: each convergent in the open disc $D(z')$ of radius $r(z') = \min_{z''\in d(f)} \{|z - z'|\}$, and each giving a point $(z,w(z',v;z-z'))$ on $f(z,w) = 0$ in the ring $\mathcal{H}(D(z'))$ of functions holomorphic in $D(z')$. Recall the parameters $(a(i)',b(i)')$ for the ith loop of the tied bouquet $\underset{\sim}{\gamma}(\underset{\sim}{z}_r)$ (§1). Refer to Fig. 2.

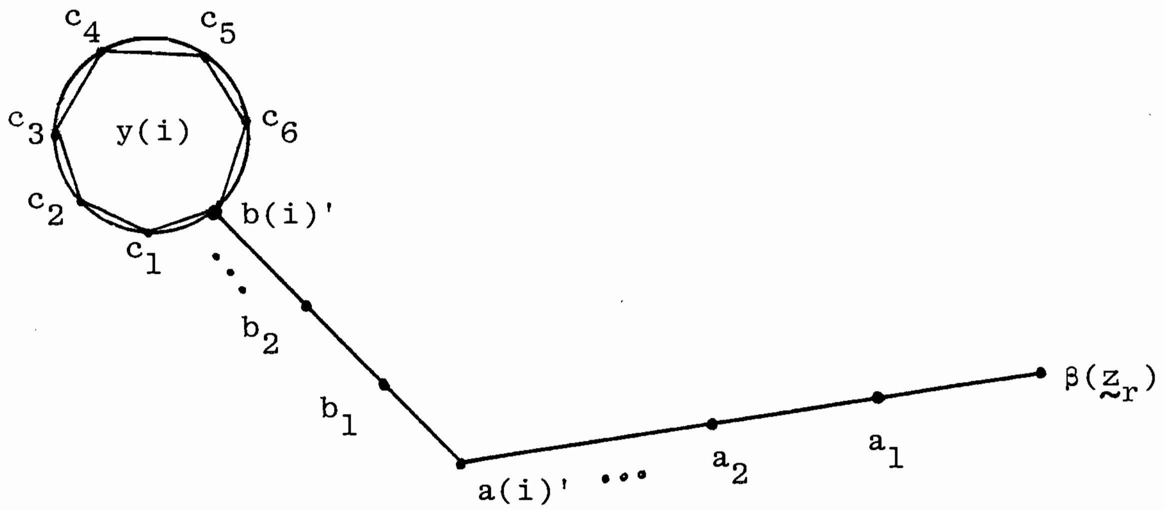

Fig. 2. Data for the ith branch cycle.

Select a_1 to be the point along the line segment from $\beta(\underset{\sim}{z}_r)$ to $a(i)'$ whose distance from $\beta(\underset{\sim}{z}_r)$ is

$$\min\{|a(i)' - \beta(\underset{\sim}{z}_r)| , (2/3)\cdot r(\beta(\underset{\sim}{z}_r))\} .$$

Then select a_2 to be the point along the line segment from a_1 to $a(i)'$ whose distance from a_1 is $\min\{|a(i)' - a_1|, (2/3)\cdot r(a_1)\}$. Continue inductively until there is a k for which $a_k = a(i)'$. Now use the same process with $a(i)'$ replacing $\beta(\underset{\sim}{z}_r)$ and $b(i)'$ replacing $a(i)'$ to find $b_1, b_2, \cdots, b_\ell$ along the line segment from $a(i)'$ to $b(i)'$ with $b_\ell = b(i)'$. Label the vertices of the 7-gon around $y(i)$ in clockwise order by $b(i)'$, $c_1, c_2, \cdots, c_6$.

Consider the following list of points: $\beta(\underset{\sim}{z}_r), a_1, a_2, \cdots, a_n$, $b_1, b_2, \cdots, b_\ell, c_1, c_2, \cdots, c_6, b_\ell, b_{\ell-1}, \cdots, b_1, a_k, a_{k-1}, \cdots, \beta(\underset{\sim}{z}_r)$. Rename them as $z_o, z_1, \cdots, z_t$ (i.e., $z_o = z_t = \beta(\underset{\sim}{z}_r)$ and $t = 2\cdot(k + \ell + 4)$) .

Now we <u>compare</u> the numbering of the n functions $\{w(z_j, v; z - z_j)\}_{v=1}^{n}$ with that of $\{w(z_{j+1}, v; z - z_{j+1})\}_{v=1}^{n}$ <u>through rearrangement of power series</u> [Sp;p.63-65] , $j = 0, \cdots, t-1$. Write $w(z_j, v; z - z_j)$ as a power series $\sum_{u=o}^{\infty} a(z_j, v, u)\cdot(z - z_j)^u$. A switch of summations in the expression $w(z_j, v; z - z_{j+1})$

$$w(z_j, v; z - z_j + 1) =$$

$$= \sum_{u=o}^{\infty} a(z_j, v, u)\cdot\left(\sum_{\ell=o}^{u} \binom{u}{\ell}\cdot(z_{j+1} - z_j)^{u-\ell}\cdot(z - z_{j+1})^{\ell}\right)$$

yields a new power series about z_{j+1} when we collect terms. Since $(z,w(z_j,v;z-z_{j+1}))$ is a point of $f(z,w) = 0$ with coordinates in $\mathcal{K}(D(z_{j+1}))$, there exists $\sigma_j \in S_n$ for which

$$(2.1) \qquad w(z_j,v;z-z_{j+1}) = w(z_{j+1},(v)\sigma_j;z-z_{j+1}) .$$

Let $\sigma(i) = \sigma_0 \cdot \sigma_1 \cdots \sigma_{t-1}$. This records the effect of analytically continuing the n functions $\{w(z_0,v;z-z_0)\}_{v=1}^{n}$ around the ith loop of the tied bouquet: $w(z_0,v;z-z_0)$ becomes $w(z_0,(v)\sigma(i);z-z_0)$, $v = 1,\cdots,n$. Form $\sigma(2),\cdots,\sigma(r)$ by following the same procedure for each $i = 2,\cdots,r$. Let $\sigma(1) = (\sigma(2)\cdots\sigma(r))^{-1}$.

Def. 2.1. The ordered r-tuple $\underset{\sim}{\sigma} = (\sigma(1),\cdots,\sigma(r)) \in (S_n)^r$ is the (set of) branch cycles of $f(z,w)$ relative to $\gamma(\underset{\sim}{z}_r)$ and the ordering of $\{w(z_0,v;z-z_0)\}_{v=1}^{n}$. A change in the latter by a permutation of the v's by $\tau \in S_n$ would result in the new branch cycles $(\tau^{-1}\cdot\sigma(1)\cdot\tau,\cdots,\tau^{-1}\cdot\sigma(r)\cdot\tau) = \tau^{-1}\cdot\underset{\sim}{\sigma}\cdot\tau$. If, however, instead of the loops of the tied bouquet we use other r-1-tuples of paths representing free generators of $\pi_1(\mathbb{P}^1 - \{\underset{\sim}{z}_r\},\beta(\underset{\sim}{z}_r))$, the set of resulting branch cycles has a more complicated description in terms of $\underset{\sim}{\sigma}$ [Fr,2;p.577-578]

The remainder of this section analyzes three problems that obstruct an effective determination of $\underset{\sim}{\sigma}$ from the coefficients of f:

i) We must compute $\{\underset{\sim}{z}_r\}$ to some reasonable approximation.

ii) Although we may inductively (and algebraically) find $a(z_j,v,u+1)$ in terms of $a(z_j,v,0),\cdots,a(z_j,v,u)$, we must compute $a(z_j,v,0)$ to some reasonable approximation.

iii) The coefficients of $w(z_j,v;z-z_{j+1})$ are power series in $z_{j+1}-z_j$, and to obtain approximate values for these we must estimate a reasonable point of truncation.

Let $\{a(z_j,v,0)'\}$ be a collection of approximations to the respective elements of $\{a(z_j,v,0)\}$, $j = 0,\cdots,t-1$, $v = 1,\cdots,n$, and let $m(j)$, $j = 0,\cdots,t-1$ be a series of integers. Form a polynomial approximation, $w(z_j,v;z-z_j)'$, of degree $m(j)$ to $w(z_j,v;z-z_j)$ by using $a(z_j,v,0)'$ in the formula for $a(z_j,v_,u)$ in place of $a(z_j,v,0)$, $u = 1,\cdots,m(j)$.

Then rearrange $w(z_j,v;z-z_j)'$ about z_{j+1} to obtain $w(z_j,v;z-z_{j+1})'$. Denote its constant term by $a(z_{j+1},v,0)''$. Our goal is to choose the collections $\{a(z_j,v,0)'\}$ and $\{m(z_j)\}$ so that we uniquely determine the element σ_j through the following formula:

$$(2.2)\qquad \min_{1\le s\le n}\{|a(z_{j+1},v,0)''-a(z_{j+1},s,0)'|\} =$$

$$|a(z_{j+1},v,0)''-a(z_{j+1},(v)\sigma_j,0)'|\ ,\ v=1,\cdots,n\ .$$

Let $\epsilon_1 = \max_{j,v}\{|a(z_j,v,0)-a(z_j,v,0)'|\}$. Let $a(z_{j+1},v,0)^*$ be the constant term of $w(z_j,u;z-z_{j+1})$ (i.e., $a(z_{j+1},v,0)^* = a(z_{j+1},(v)\sigma_j,0)$) and let $\epsilon_2 = \max_{j,v}\{|a(z_{j+1},v,0)^*-a(z_{j+1},v,0)''|\}$. For simplicity the next result assumes that the coefficients of the polynomial $f(z,w)$ are explicit elements of $\mathbb{Q}(\sqrt{-1})$.

THEOREM 2.2. <u>There is an effective procedure for calculating</u> $d(f)\cup\{\infty\}$ <u>and the branch cycles</u> $\underset{\sim}{\sigma}$ <u>(up to conjugation by an element of</u> S_n<u>)</u> <u>of an irreducible polynomial</u> $f\in\mathbb{Q}(\sqrt{-1})[z,w]$, <u>monic and</u>

of degree n in w, relative to the tied bouquet computed from d(f).

Proof. Use the notation above. The result follows from an effective computation of σ_j, $j = 1,\cdots,t-1$. Let ϵ be less than $(2/3)\cdot \min_{j,k,\ell} \{|a(z_j,k,0) - a(z_j,\ell,0)|\}$, $1 \leq k \leq \ell < n$, $j = 0,1,\cdots,t-1$. Choose $\{a(z_j,v,0)'\}$ so that ϵ_1 and ϵ_2 are less than $\epsilon/2$. Then expression (2.2) determines σ_j uniquely, $j = 0,\cdots,t-1$. Thus, we are done if we give an effective procedure for finding $\{a(z_j,v,0)\}_{j,v},\epsilon,\epsilon_1$ and ϵ_2. We use Newton's method [Sm; see §3] throughout the proof whenever we need approximations to the zeros of a polynomial $g(x) \epsilon \mathbb{C}[x]$ whose coefficients are given explicitly as elements of $\mathbb{Q}(\sqrt{-1})$. The remainder of the proof divides into parts. One subtlety: We don't know the elements of d(f) (Part 2 below) except by approximation. Thus we must form our tied bouquet from approximations to these elements, rather than from the elements themselves.

Part 1. Root-difference algorithm. For $g(x) \epsilon \mathbb{Q}(\sqrt{-1})[x]$ apply Euclid's algorithm to g, $\frac{d}{dx}(g)$, $\frac{d^2}{dx^2}(g),\cdots$ to find the multiplicity of the roots of g, and to factor $g(x)$ into $\prod_{i=1}^{\ell} g_i(x)$ where $g_i(x) \epsilon \mathbb{Q}(\sqrt{-1})[x]$, g_{i+1} divides g_i, $i = 1,\cdots,\ell-1$ and all zeros of g_1 are of multiplicity one. Replace g by g_1, which we assume to be of degree n'. Compute the discriminant of g_1, $d(g_1)$ [Wae;p.87]. If $x_1,\cdots,x_{n'}$ are the zeros of g_1, then $d(g_1) = (\prod_{i<j} (x_i - x_j))^2$. With no loss assume that g_1 is monic so that a bound for the absolute values of the roots of g_1 is given by 1 plus the sum of the absolute values of the coefficients of g_1: denote this by A. Then

$$(2.3) \quad \min_{1\leq i<j\leq n'} \{|x_i - x_j|\} > \sqrt{d(g_1)/(2\cdot A)^{n'\cdot(n'-1)-2}} = A'(g) .$$

Part 2. Choose data for Fig. 2. For $f \in \mathbb{Q}(\sqrt{-1})[z,w]$ given in the statement of the theorem, first compute approximations to the values of $d(f)$. The set $d(f)$ is the set of zeros of $h(z)$, the discriminant of f regarded as an element of $\mathbb{Q}(\sqrt{-1},z)[w]$. Compute $h(z)$ explicitly from [Wae;p.87]. Use Part 1 to compute $A'(h)$, a lower bound on the absolute values of the differences of the roots of $h(z)$; and to compute $h_1(z)$, a polynomial whose zeros are of multiplicity one and exactly the same as the zeros of h.

Approximate all the zeros of $h_1(z)$ to within $A'(h)/12$ by elements $\{z(2)',\cdots,z(r)'\}$. Let $y(2)',\cdots,y(r)'$ be the lexicographical reordering of $z(2)',\cdots,z(r)'$ (i.e., $y(2)' <\cdots< y(r)'$). Now form the data of Fig. 2 with $y(i)'$ replacing $y(i)$, $i=2,\cdots,r$. Our parameters guarantee that the paths of the tied bouquet $\underset{\sim}{\chi}(\underset{\sim}{z}_r')$ give the same homotopy classes in $\pi_1(\mathbb{P}^1 - \{\underset{\sim}{z}_r\}, \beta(\underset{\sim}{z}_r))$ (via the map $\psi(\beta(\underset{\sim}{z}_r'),\beta(\underset{\sim}{z}_r))$, or its inverse) as do the paths of the tied bouquet $\underset{\sim}{\chi}(\underset{\sim}{z}_r)$ given by the actual elements of $d(f)$. Therefore the branch cycles computed from f with respect to $\underset{\sim}{\chi}(\underset{\sim}{z}_r')$ will be the same as those computed with respect to $\underset{\sim}{\chi}(\underset{\sim}{z}_r)$ - up to conjugation by an element of S_n.

Part 3. Computation of ϵ, ϵ_1 and ϵ_2. Here we assume that the points $z_o,\cdots,z_{t-1}$ (as in expression (2.1)) come from Fig. 2 for the $\{z(2)',\cdots,z(r)'\}$ data. Take approximations to $z_1,\cdots,z_t$ in $\mathbb{Q}(\sqrt{-1})$ that are within $A'(h)/12$ of their actual values. Let $g_j(w) = f(z_j,w)$. Choose ϵ less than one-half of the minimum of the absolute value of the difference of the roots of $g_j(w)$, $j=0,\cdots,t-1$ A standard majorant argument [Hi; p.94] allows us to express the bound ϵ_2 for the absolute value of the difference of $a(z_{j+1},v,0)^*$ and

$a(z_{j+1},v,0)$" in terms of $m(j)$ and a bound, say ϵ_1', on the absolute value of the difference between $a(z_j,v,0)$ and $a(z_j,v,0)'$. Choose $m(j)$ and ϵ_1' appropriately, according to this relation so that ϵ_2 is at most $\epsilon/2$. Now let $\epsilon_1 = \min(\epsilon/2, \epsilon_1')$. With these choices the proof is complete. ■

§3. Computer computation of branch cycles.

We have written a computer program for computing branch cycles for a polynomial $f(z,w)$ in §2 of the form $F(w) - z$. This simplification hides none of the real difficulties, which we shall analyze below using the following two examples:

Example 1. [Fr,1;§1] $F(w) = w^5/5 - (c+d)\cdot w^4/4 + (c\cdot d - 2)\cdot w^3/3 + (c+d)\cdot w^2 - 2\cdot c\cdot d\cdot w$ where $c\cdot d = -2/5$, $F(c) \neq F(d)$ and $c + d \in \mathbb{Q}$.

Example 2. [Fr,2; p.593]

$$F(w) = w^7 - 7\cdot\lambda\cdot t\cdot w^5 + (4-\lambda)\cdot t\cdot w^4 + (14\cdot\lambda - 35)\cdot t^2\cdot w^3$$

$$- (8\cdot\lambda + 10)\cdot t^2\cdot w^2 + [(3-\lambda)\cdot t^2 + 7\cdot(3\cdot\lambda + 2)\cdot t^3]\cdot w - t^3/3 ,$$

where $\lambda = (1-\sqrt{-7})/2$ and $t \in \mathbb{Q}$.

Note how practical examples force us to consider coefficients outside of $\mathbb{Q}(i)$. Examples of this type arise in many arithmetic geometry problems. In particular the $F(w)$ in Example 1 have the property that $\{z_0 \in \mathbb{Z} \mid F(w) - z_0$ is reducible, but has no linear factor$\}$ is infinite. For polynomials $F(w)$ that cannot be written as a composition of nonlinear lower degree polynomials (indecomposable) this can only happen if $\deg(F) = 5$ - a consequence of the classification of finite simple groups. Similarly, the $F(w)$ in Example 2 have the property that each of them has the same range as their complex conjugates as functions on all the residue class fields of $\mathbb{Z}[\sqrt{-7}]$, even though they are not obtained from one another by linear change of the variable. Such examples of nonlinearly related polynomial pairs F and G for indecomposable polynomials $F(w)$ (and some ring of integers of a number field K) occur only if deg(F) = 7,11,13,15,21 and 31 and not at all if $K = \mathbb{Q}$ (again, a consequence of the classification of finite simple groups).

Part 2 of the proof-algorithm of Theorem 2.2 involves finding, for a given polynomial F , the zeros $w_1, \cdots, w_{n-1}$ of $F'(w) = 0$ and the resulting branch points $F(w_1), \cdots, F(w_{n-1})$ (called $z(2), \cdots, z(r)$ in the proof) . For a polynomial of moderate degree, this is usually a routine calculation. To solve $F'(w) = 0$ apply Muller's method to the polynomial deflated by the previously found zeros, and follow with Newton's method applied to $F'(w) = 0$ to compensate for inaccuracy induced by the deflation [AK;2.9-2.10].

Wavrik [W] points out that the Euclidean algorithm, as it appears in Part 1 of the proof of Theorem 2.2, for computing the exact multiplicity of the zeros of a polynomial with integer coefficients presents severe computer difficulties. We can bypass this, since we know a'priori that both examples have 3 finite branch points, and thus we can decide the multiplicity of specific branch points. But, at present our program relies on this deus ex machina for computing multiplicity of branch points.

In Example 1, the choice of the parameters $c = 2$ and $d = -1/5$ leads to two branch points (-.41192 and 8) of multiplicity 1 and one branch point (9) of multiplicity two.

In Example 2 the choice $t = 1$ is unsatisfactory: two of the branch points (3.16667 + 25.9223i and 3.16667 + 25.8783i) are so close as to effectively limit the accuracy of the computation to about three decimal places in the most critical part of the continuation around the branch points. (Note that with branch points as close as these, it is reassuring to know that there are 3 branch points.) The choice $t = 2$ separates the branch points nicely, but the coefficients of the resulting polynomial are large enough to interfere with our later need to accurately evaluate F and its derivatives at other points (e.g., the coefficient of w is 206 - 216.952i). We chose $t = 1.2$ by hand.

The most interesting part of the computation involves the rearrangement of power series to compute the branch cycles that arise from analytic continuation around the tied bouquet (Fig. 2). Choose the base point $\beta = \beta(\underset{\sim}{z}_r')$ as in §1, where $\underset{\sim}{z}_r'$ are the approximations to the branch points that we have just computed. To determine the n distinct values of the algebraic functions that satisfy $F(w) = z$ at $z = \beta$, solve for w in $F(w) = \beta$ to get $w_1(\beta),\cdots,w_n(\beta)$ using the same algorithm which applied to find the branch points. It is mathematically trivial, but computationally significant that one need only continue from the base point to the "stem" where the straight line intersects the circle about a given branch point $(b(i)'$ in Fig.2), continue around the circle back to the stem, and then compare the values of the w's. Although we must begin at the base point in order to compare the permutations about the various branch points, it is unnecessary to continue all the way back to the base point.

Follow the argument of §2 up to (2.1) to continue $w_1(z),\cdots,w_n(z)$ as functions of z up to the "stem point', $s = b(i)'$ of the ith branch point to get $w_1(s),\cdots,w_n(s)$, $i = 2,\cdots,r$. Then continue around the circle back to s to obtain $\tilde{w}_1(s),\cdots,\tilde{w}_n(s)$. The permutation $\sigma(i)$ has the property that $(j)\sigma(i)$ is for each j the integer k that minimizes $|w_j(s) - \tilde{w}_k(s)|$.

The analytic continuation repeats the following process: Given a point p_1 on the path (of Fig. 2) and a value $w(p_1)$ satisfying

$F(w(p_1)) = p_1$, find the values $w(p_2)$ at the next point p_2 on the path. This is a classical problem in inversion of power series. From the power series (polynomial) $F(w) = z$, find the inverse series $w(p_1;z-p_1) = w(p_1) + \sum_{u=1}^{\infty} a(p_1,u)\cdot(z-p_1)^u$ and rearrange this series about p_2 to find $w(p_2)$.

The first 7 coefficients $a(p_1,1),\cdots,a(p_1,7)$ appear in [AbSt;p.18] as a function of the coefficients of F (the first 13 in [0]) . For these computations, however, it is more efficient to use Knuth's power series inversion algorithm S [Kn;p.448-9]. Knuth calls algorithm S <u>sequential</u>: one can compute m coefficients in the inverse series without specifying m in advance. But there are asymptotically superior algorithms [BKu] .

With an example of any complexity - rather than, say, $w^2 = z$ - it is necessary to <u>vary the step size</u> along the path according to the majorant Part 3 of the proof of Theorem 2.2 in order to guarantee reasonably rapid converge of the series as we approach the branch points. But the majorant estimate is generally too conservative and repetitious calculation of the majorant terms is inefficient. Therefore we proceed as follows.

Given p_1 and $w_j(p_1)$, $j = 1,\cdots,n$, compute M_1 , the minimum distance from p_1 to a branch point, $M_2 = \min\{|F'(w_j(p_1))| : 1 \le j \le n\}$, and choose the next point p_2 so that $|p_2 - p_1| \le R\cdot\min\{M_1,M_2\}$ with the convergence factor R , $0 < R \le 1/2$, chosen below. The M_2 constant results from a linear change of variables to put $F(w) = z$ in a form so that $F(0) = 0$ and the inverse series for w_j at p_2 is of the form $\tilde{z} + \tilde{a}_2\cdot\tilde{z}^2 + \cdots$ with $\tilde{z} = (p_2 - p_1)/F'(w_j(p_1))$.

Begin with $R = 1/2$. This often gives rapid convergence of the series for $w_1(p_2),\cdots w_n(p_2)$; using the convergence criterion that the last term in the series has modulus at most a small fraction of the w_j values between which we are trying to distinguish (say, at most $(1/100)\cdot\min\{|w_j(p_1)-w_k(p_1)| : j \neq k\}$) . If this criterion is not satisfied by the first twenty terms of the series, reset R to R/2 , compute a new p_2 , and try again. Since halving the size of R roughly doubles the computation time, for any $R < 1/2$ we try to satisfy the convergence criterion with 2R every fifth point of the path.

Appendix 1 gives the program output for a computation of the branch cycles of Example 1.

For Example 2 this program gives answers which are interesting and incorrect. Here is the result of continuation about the third branch point, which uses 138 points.

Values of w at the stem:

	Before continuation	After continuation
w #1	- .630237 + .711137i	- .871076 + .963909i
w #2	1.64127 - 1.25681i	1.87107 - 1.18989i
w #3	- .896993 + .974962i	- .616436 + .708794i
w #4	1.87896 - 1.18538i	1.62158 - 1.28227i
w #5	- .911095 + 1.04259i	- .610312 + .68854i
w #6	1.59555 - 1.287i	1.88399 - 1.18467i
w #7	- 2.49507 + .956932i	- 7.49507 + .956962i

This is, of course, impossible since the resulting "permutation" has both 2 and 6 going to 4 and both 3 and 5 going to 1.

The problem is that the base point is 6.472, the second branch point is 5.472 + 45.4337i , and the third branch point is 5.472 + 46.5716i . The straight line which joins the base point to the third branch point is very close to the second branch point, and so, in spite of repeatedly halving R and using 138 points, some of the values of the w_j's come too close together to be distinguishable as we pass the second branch point. We resolve this specific problem by moving the base point; see Appendix 2 for the program output which gives the branch cycles of Example 2.

The point of this discussion of the above problem is not why it arises and how it may be resolved, in hindsight that is all clear, but that it is possible to numerically continue w and get a value which is wrong even though it is plausibly close to one of the possible values. Note that the values given are obtained by repeated approximate numerical continuation of the series and that there is only one application of Newton's method; the initial application at the base point. The program in this way deviates from the proof of Theorem 2.2 so that in the final results one can see the cumulative error due to the sequence of analytic continuations. Any application of Newton's method would cover up these errors. Nevertheless the error contribution to the final calculati of that initial application of Newton's method cannot be dismissed. Thus, for future applications, it is worth analyzing the effect of [Sm] - to which we now digress - on an algorithm like that of Theorem 2.2.

We must find all roots of a sequence of polynomials $h_1(z)$, $g_j(w)$, $j=0,\cdots,t-1$ where the polynomials themselves derive algorithmically from a single polynomial $f \in \mathbb{Q}(\sqrt{-1})[z,w]$. The zeros of these polynomials are all of multiplicity one. The polynomials $g_0(w),\cdots,g_{t-1}(w)$ depend on the zeros of $h_1(z)$ and on f . Of course, there are $r-1 = |d(f)|$ independent such calculations. In order to discuss theoretical bounds on the running time of our algorithm we need a condition that bounds r and the coefficients of f . Assume that n and m are given: $\deg(f) \leq m$ (along with $\deg_w(f) = n$) ; and all coefficients of f are bounded in absolute value by 1 (along with f monic in w) . In particular this gives computable bounds for the quantities r (e.g., $r \leq m^{2\cdot(n-1)}$) and the absolute value of the coefficients of $h_1(z)$. We can give a bound for t in terms of an estimate of the maximum of the differences of the absolute values of the zeros of $h_1(z)$ along with $A'(h_1)$ (expression (2.3)). And the computation easily produces values $z_o,\cdots,z_{t-1}$, for each value of j, $2 \leq j \leq r$ (as given in Part 2 of the proof of Theorem 2.2). From Part 1 of the proof of Theorem 2.2, bounds on $z_o,\cdots,z_{t-1}$ give bounds on the coefficients of $g_o,\cdots,g_{t-1}$ and therefore a bound away from 0 on the absolute values of the differences of the roots of each of the g_j's, $j = 0,\cdots,t-1$. Thus there is an estimate for ϵ in terms of n and m alone.

At this point, according to the algorithm of Theorem 2.2 we have an explicit bound on the allowable error for the approximations to the roots of $h_1(z)$ and $g_j(w)$, $j = 0,\cdots,t-1$. This bound can be given in terms of n and m alone. In order to estimate the "running time" for the algorithm of Theorem 2.2 we need now to estimate the number of iterations required from Newton's algorithm to make an estimate within the allowed tolerance - denoted $\epsilon'(n,m)$ - for the collection of zeros of $h_1(z)$, $g_j(w)$, $j = 0,\cdots,t-1$.

Newton's method, however, has a quixotic dependence on the initial approximation to a zero of a polynomial $p(x)$, and on the location of the zeros of $\frac{d}{dx}(p(x))$. Smale [Sm] defines an <u>approximate zero</u> of $p(x)$ to be a value x_o for which, starting with x_o the uth iterate of Newton's method gives a value x_u for which $|p(x_n)| < (1/2)^u \cdot |p(x_o)|$. In the space of monic polynomials of degree n' with coefficients bounded by 1 we induce a measure on subsets of the space by identifying them with subsets of the "unit cube" in $\mathbb{R}^{n'}$. The whole unit cube is assumed to have measure 1, and therefore we regard the space of monic polynomials of degree n' with coefficients bounded by 1 as a probability space. Then [Sm;p.31] shows that for $p(x) \in \mathbb{Q}(\sqrt{-1})[x]$ in this space and for $0 < \mu < 1$, with probability $1 - \mu$ Newton's method starting at $x = 0$ arrives at an approximate zero in $s = [100\cdot(n' + 2)^9/\mu^7]$ steps. Here [ ] denotes the "greatest integer" function. A true analysis of any program's running time would necessarily need to incorporate such estimates for Newton's algorithm.

One measure of the complexity of these calculations is the time it takes to run the examples. The program was written in BASIC and run on a personal computer (a TRS-80 Model I, 48K). In interpreted BASIC, Example 1 took a half hour. Consequently it was useful, particularly for debugging, to compile the program using ACCEL 3 (ACCEL 3, A Compiler for TRS-80 BASIC, (copyright 1982), Southern Software, Box 11721, San Francisco, CA 94101). After compilation, Example 1 took 10 minutes, Example 2 took 17 minutes (and Example 2, with the unfortunate choice $\beta = 6.572$, took one hour).

A program listing can be obtained by writing to the authors.

Appendix 1

```
PROGRAM ENCORE:
ANALYTIC CONTINUATION AROUND A TIED BOUQUET.
JULY 12  1983


F(W) IS A POLYNOMIAL OF DEGREE  5  WITH COEFFICIENTS:
J                   COEFFICIENT OF THE J-TH POWER OF W
 0                   0 + 0 i
 1                   4 + 0 i
 2                   9 + 0 i
 3                  -4 + 0 i
 4                  -2.25 + 0 i
 5                   1 + 0 i


THE BRANCH POINTS  ARE:
MULTIPLICITY        BRANCH POINT
 1                  -.41192 + 0 i
 1                   8 + 0 i
 2                   9 + 0 i


THE BASE POINT P(0) OF THE BOUQUET IS  10 + 0 i

 AT THIS POINT THE W'S HAVE THE INITIAL VALUES:
W NUMBER            W(P(0))
 1                   1.38833 +-.364865 i
 2                   1.38833 + .364865 i
 3                  -1.43039 + .182042 i
 4                  -1.43039 +-.182042 i
 5                   2.33411 + 0 i


RESULTS FROM CONTINUATION AROUND THE BRANCH POINT  S( 1 ) = -.41192 + 0 i
USING SERIES WITH  35  POINTS.

VALUES OF W, AT THE 'STEM' OF THE CIRCLE,BEFORE AND AFTER CONTINUATION:
W # 1
 .0259550 +-.0234236 i
-.418575 +-8.15218E-03 i

W # 2
 2.25207 + .607027 i
 2.25537 + .604094 i

W # 3
-.419491 + .022795 i
 .0284884 + 7.7339E-03 i

W # 4
-1.86382 +-1.88483E-03 i
-1.86354 + 1.08247E-03 i

W # 5
 2.25462 +-.603966 i
 2.25083 +-.606257 i


THE PERMUTATION IS:
```

1	2	3	4	5
3	2	1	4	5

RESULTS FROM CONTINUATION AROUND THE BRANCH POINT S(2) = 8 + 0 i
USING SERIES WITH 19 POINTS.

VALUES OF W, AT THE 'STEM' OF THE CIRCLE,BEFORE AND AFTER CONTINUATION:
W # 1
1.07561 +-.0876477 i
1.10219 +-.0775605 i

W # 2
1.81131 + .161285 i
2.17408 +-.0659881 i

W # 3
-1.23032 + .0537482 i
-1.24617 + .0468737 i

W # 4
-1.57167 +-.0380733 i
-1.55982 +-.034118 i

W # 5
2.16636 +-.089079 i
1.77896 + .134702 i

THE PERMUTATION IS:

1	2	3	4	5
1	5	3	4	2

RESULTS FROM CONTINUATION AROUND THE BRANCH POINT S(3) = 9 + 0 i
USING SERIES WITH 13 POINTS.

VALUES OF W, AT THE 'STEM' OF THE CIRCLE,BEFORE AND AFTER CONTINUATION:
W # 1
1.39908 +-.266209 i
1.36385 + .260401 i

W # 2
1.39908 + .266209 i
1.43446 +-.267171 i

W # 3
-1.42244 + .130355 i
-1.40408 +-.131211 i

W # 4
-1.42244 +-.130355 i
-1.44036 + .126805 i

W # 5
2.2967 + 0 i
2.29544 + .0116205 i

THE PERMUTATION IS:

1	2	3	4	5
2	1	4	3	5

Appendix 2

```
PROGRAM ENCORE:
ANALYTIC CONTINUATION AROUND A TIED BOUQUET.
JULY 14  1983

F(W) IS A POLYNOMIAL OF DEGREE  7  WITH COEFFICIENTS:
J               COEFFICIENT OF THE J-TH POWER OF W
 0              -.576 + 0 i
 1               45.936 +-46.0996 i
 2              -20.16 + 15.2395 i
 3              -40.32 +-26.6692 i
 4               4.2 + 1.58745 i
 5              -4.2 + 11.1122 i
 6               0 + 0 i
 7               1 + 0 i

THE BRANCH POINTS  ARE:
MULTIPLICITY    BRANCH POINT
 2               5.472 +-21.1417 i
 2               5.472 + 45.4337 i
 2               5.472 + 46.5716 i

THE BASE POINT P(0) OF THE BOUQUET IS  6.472 + 46 i

 AT THIS POINT THE W'S HAVE THE INITIAL VALUES:
W NUMBER        W(P(0))
 1              -.559284 + .760862 i
 2               1.42917 +-1.29506 i
 3              -.966508 + .843631 i
 4               1.70893 +-1.34709 i
 5              -.984696 + 1.21809 i
 6               1.86655 +-1.13948 i
 7              -2.49416 + .959045 i
ELBOW AND STEM FOR BRANCH POINT #  1

E1 = 6.472 + 46 i  AND E2 =  5.48047 +-20.5728 i

SMALLER R9 =  .25  AT POINT #  23
LARGER R9 =  .5  AT POINT #  27

RESULTS FROM CONTINUATION AROUND THE BRANCH POINT  S( 1 ) =  5.472 +-21.1417 i
USING SERIES WITH  32  POINTS.

VALUES OF W, AT THE 'STEM' OF THE CIRCLE,BEFORE AND AFTER CONTINUATION:
W #  1
 .422759 +-.245576 i
 .565795 +-.357644 i

W #  2
 .566546 +-.355008 i
 .421042 +-.249458 i

W #  3
-2.10998 + .925282 i
-2.18444 + .929389 i
```

```
W # 4
 1.65338 +-1.87578 i
 1.65333 +-1.87579 i

W # 5
-.645793 + 1.75316 i
-.645729 + 1.75315 i

W # 6
 2.30072 +-1.12787 i
 2.30072 +-1.12792 i

W # 7
-2.19978 + .931244 i
-2.09182 + .924103 i


THE PERMUTATION IS:


 1       2       3       4       5       6       7

 2       1       7       4       5       6       3
ELBOW AND STEM FOR BRANCH POINT #  2

E1 =  6.472 + 46 i  AND E2 =  5.96708 + 45.7141 i


RESULTS FROM CONTINUATION AROUND THE BRANCH POINT  S( 2 ) =  5.472 + 45.4337 i
USING SERIES WITH  14  POINTS.

VALUES OF W, AT THE 'STEM' OF THE CIRCLE,BEFORE AND AFTER CONTINUATION:
W # 1
-.588677 + .726854 i
-.600956 + .73221 i

W # 2
 1.46487 +-1.29305 i
 1.67307 +-1.30483 i

W # 3
-.947286 + .915628 i
-.989558 + 1.17415 i

W # 4
 1.66689 +-1.3209 i
 1.4664 +-1.30571 i

W # 5
-.97547 + 1.17868 i
-.922276 + .913756 i

W # 6
 1.87334 +-1.16515 i
 1.86678 +-1.16945 i

W # 7
-2.49366 + .957938 i
-2.49395 + .95776 i


THE PERMUTATION IS:


 1       2       3       4       5       6       7

 1       4       5       2       3       6       7
```

ELBOW AND STEM FOR BRANCH POINT # 3

E1 = 6.472 + 46 i AND E2 = 5.96595 + 46.2893 i

RESULTS FROM CONTINUATION AROUND THE BRANCH POINT S(3) = 5.472 + 46.5716 i
USING SERIES WITH 14 POINTS.

VALUES OF W, AT THE 'STEM' OF THE CIRCLE,BEFORE AND AFTER CONTINUATION:
W # 1
-.606706 + .776723 i
-.89084 + .823733 i

W # 2
1.44327 +-1.32374 i
1.43556 +-1.32733 i

W # 3
-.891024 + .848563 i
-.602707 + .793713 i

W # 4
1.7213 +-1.29928 i
1.83507 +-1.15002 i

W # 5
-1.01447 + 1.19739 i
-1.01974 + 1.20563 i

W # 6
1.84249 +-1.15769 i
1.73653 +-1.30486 i

W # 7
-2.49486 + .958052 i
-2.49516 + .958204 i

THE PERMUTATION IS:

1	2	3	4	5	6	7
3	2	1	6	5	4	7

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Professors
M. Fried - U.C. Irvine
R. Whitley

Department of Mathematics
University of California
Irvine, California 92717
U.S.A.

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" Effective Branch Cycle Computation"

Karl Strambach

(The Editors)

Prof. Dr. Karl Strambach

Referee's Report on

"Effective Branch Cycle Computations" by M. Fried and R. Whitley (submitted to Geometriae Dedicata)

In this article the authors investigate the problem of explicitely determining (by computer) the monodromy of branched coverings of $\mathbb{C}\mathbb{P}^1$. This is certainly an interesting question. However, the authors present only little new information in too many pages. I do not think that the content justifies a publication of this lenght. My suggestion would be that the authors rewrite the paper in a very condensed form and then incorporate it in a paper on the mathematics of this type of examples.

My criticism is based on, for instance, p. 1.4-1.8, p.3.5-3.8, the presentation of the output, and the comments in §3 referring back to §2. The details on p. 1.4-1.8 are so obvious, that one does not have to spell it out for the human reader; the reader is here almost treated as a non-understanding computer. The discussion of what went wrong in one of the tests of the program (on p. 3.5-3.6) is not worth on page; the mistake is too obvious. The story of the complexity and running time on p.3.6-3.8 is too vague to really estimate something. The lay-out of p. 3.9-3.13 is ridiculous; things can be tabulated and grouped together much more efficiently, so that in printed form it would take maybe only one page. Finally, look at theorem (2.2): the statement is so obvious that for the pure math side a proof is superfluous; the authors give a proof which also contains an algorithm for computing, but then in §3 they remark that some of the steps cannot be used for an actual computer program; so what then is the use of §2?