SINGULAR POINTS ON MODULI SPACES AND SCHINZEL'S PROBLEM

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Abstract. Many problems start with two (compact Riemann surface) covers $f : X \to \mathbb{P}^1_z$ and $g : Y \to \mathbb{P}^1_z$ of the Riemann sphere, $\mathbb{P}^1_z$, uniformized by a variable z. Some data problems have f and g defined over a number field K, and ask: What geometric relation between f and g hold if they map the values $X(\mathcal{O}_K/\boldsymbol{p})$ and $Y(\mathcal{O}_K/\boldsymbol{p})$ similarly for (almost) all residue classes of $\mathcal{O}_K/\boldsymbol{p}$. Variants on *Davenport's problem* interpret as relations between zeta functions [Fr12a, §7.3].

Schinzel's original problem was to describe expressions $f(x) - g(y)$, with $f, g \in \mathbb{C}(x)$ nonconstant, that are reducible. The easiest special cases are, as with both Davenport and Schinzel, when the f and g are polynomials (in $K[x]$). Then, when f is indecomposable, the solutions of both problems are related and they interpret using parameter (Hurwitz) spaces of r-branch point covers. The difficulty: Dropping indecomposability requires dealing with imprimitive groups. That requires group theory beyond the simple group classification.

To solve generalizations of the AGZ version of Schinzel's problem we must go beyond this limitation. Hurwitz spaces characterize the appropriate covers succinctly. Our main formula interprets covers *fixed* by a Möbius transformation in terms of branch cycles. This describes singular points on reduced Hurwitz spaces when $r > 4$, and, when $r = 4$, it interprets when the mysterious *moduli group* acts trivially.

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1. Goals of the paper

Suppose we have two compact Riemann surface covers $f : X \to \mathbb{P}^1_z$ (of degree $\deg(f) = n$) and $g : Y \to \mathbb{P}^1_z$ (of degree m), with respective (geometric) monodromy groups G_f and G_g. They give corresponding permutation representations T_f and T_g. In addition we will assume special conditions (1.1) for which the combination of (1.1a) and (1.1b) constitute our focus. See the opening paragraph of the proof of Cor. 1.11 for salient facts on $\mathrm{PGL}_2(\mathbb{C})$. Often our hypotheses give $\deg(f) = \deg(g)$.

(1.1a) $g = \alpha \circ f$ where $\alpha \in \mathrm{PGL}_2(\mathbb{C})$: (f, g) is a *Möbius pair*.
(1.1b) The Galois closures of f and g are the same.
(1.1c) The fiber product, $X \times_{\mathbb{P}^1_z} Y$ of f and g is reducible.
(1.1d) Special case of (1.1c): $X = \mathbb{P}^1_x$, $Y = \mathbb{P}^1_y$.
(1.1e) Special case of (1.1d): (f, g) is a polynomial pair.

§1.1 describes relations between these conditions, and the cases on which we concentrate. Our Main Thm. has several versions (§2.17). It includes a characterization of the combination (1.1a) (1.1b). Describing singularities of reduced Hurwitz spaces lies at one end of its uses: the case $T_f = T_g$. Schinzel's original problem is subsumed under (1.1e). We state that problem first in §1.1.2 before explaining its generalization in §1.1.3.

1.1. **Relating pairs of covers.**

Under condition (1.1a), the respective (geometric) monodromy G_f and G_g of f and g are the same, but the Galois closures are usually different. So, condition (1.1b) is much stronger than it. Condition (1.1c) reduces to (1.1b) (Prop. 1.3).

Whatever the branch points $\boldsymbol{z}$, for f, these produce conjugacy classes $\mathbf{C} = \mathrm{C}_1, \dots, \mathrm{C}_r$ in the geometric monodromy $G_f \le S_n$. Denote $\mathbb{P}^1_z \setminus \{\boldsymbol{z}\}$ by

$U_{\boldsymbol{z}}$. Usually, all conjugacy classes are nontrivial (not of the identity element). When they appear, we remark on the few exceptions to this.

1.1.1. *Notation for covers.* Here is how we put structure in this open-ended problem. Covers naturally belong in families. In characteristic 0 (over the complexes, $\mathbb{C}$), by moving the branch points of a cover, we can uniquely drag the cover along with that movement.

Once we have labeled a desired equivalence of covers, this defines the full family of covers attached to any one cover for which the monodromy group, G and r conjugacy classes $\mathbf{C} = \{\mathrm{C}_1, \dots, \mathrm{C}_r\}$, of G are assumed given. We refer to such families as a *Nielsen class*, for which we immediately use two types, *absolute* and *inner*. App. B briefly reviews them.

Both types consist, for some integer $r \geq 3$, of equivalence classes of r tuples $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_r) \in \mathbf{C}$ of elements in a finite group G, where $\in$ here means – in some order – the entries fall, with multiplicity, in $\mathbf{C}$. Occasionally we must consider $r = 2$. Those covers are Möbius equivalent to $x \mapsto x^n$. They are somewhat trivial, and require exceptions to our statements. So, we merely note their appearance when we must.

[Fr12a, §5.3.2] explains using *classical generators* of the fundamental group of $U_{\boldsymbol{z}}$. Our Main Theorem uses specifically created examples of them (as in App. C). The $\boldsymbol{\sigma} \in \mathbf{C}$ have these properties:

(1.2a) *Generation*: $\langle \sigma_i | i = 1, \dots, r \rangle = G_f \stackrel{\text{def}}{=} G \leq S_n$; and

(1.2b) *Product-one*: $\sigma_1 \cdots \sigma_r = 1$.

Those $\boldsymbol{\sigma} \in \mathbf{C}$ satisfying (1.2) is the *Nielsen class*, $\mathrm{Ni}(G, \mathbf{C})$, of $(G, \mathbf{C})$. Denote the subgroup of S_n normalizing G and permuting (with multiplicity) the classes in $\mathbf{C}$ by $N_{S_n}(G, \mathbf{C})$. The *absolute class* of $\boldsymbol{\sigma} \in \mathrm{Ni}(G, \mathbf{C})$ is

$$\{\alpha \boldsymbol{\sigma} \alpha^{-1} \mid \alpha \in N_{S_n}(G, \mathbf{C})\}.$$

Denote these equivalence classes, running over $\boldsymbol{\sigma} \in \mathrm{Ni}(G, \mathbf{C})$ by $\mathrm{Ni}(G, \mathbf{C})^{\mathrm{abs}}$.

§1.3.1 reminds of the coset representation, T_H, of a group G coming from any subgroup $H \leq G$. Up to conjugation by G_f, some subgroup defines the cosets of the representation T_f. We label one such as $G(T_f, 1)$: the elements of G_f that fix the integer 1 in the representation T_f. Similarly, any cover $f' : X' \to \mathbb{P}^1_z$ through which $\hat{f}$ factors corresponds to a coset representation (possibly not faithful) of G_f.

Remark 1.1. There is a significant distinction between the Galois closure of a cover over $\mathbb{C}$ and over a non-algebraically closed field. [Fr73] was sensitive to this. Reluctantly we here simplify by assuming we are over $\mathbb{C}$.

1.1.2. *Context for the AGZ version of Schinsel's problem.* Consider a polynomial $f \in \mathbb{C}[x]$. It produces a cover $\mathbb{P}^1_x \to \mathbb{P}^1_z$:

$$x' \in \mathbb{C} \cup \{\infty\} \mapsto f(x') \in \mathbb{C} \cup \{\infty\}.$$

Suppose $\deg(f) = n$, and $x_1, \dots, x_n$ are zeros of $f(x) - z$. Then, f has a (geometric) *monodromy* group, G_f. Its simplest description is the Galois group of $\Omega_f = \mathbb{C}(x_1, \dots, x_n)$ over $\mathbb{C}(z)$, together with the permutation representation of G_f on $x_1, \dots, x_n$.

Indecomposability of a rational function f is equivalent to *primitivity* of its monodromy: $f : \mathbb{P}^1_x \to \mathbb{P}^1_z$ doesn't factor through two lower degree maps. When f is indecomposable, a corollary to the solution of Davenport's Problem, [Fr73], solved Schinzel's problem by showing that

(1.3) Schinzel pairs must have $\deg(f) = 7, 11, 13, 15, 21$ or 31.

We understand the families of such pairs (f, g) ([Fr99, §9.2] or [Fr12a, §5.3]).

So, we start by assuming $f = f_1 \circ f_2$, and $\deg(f_i) > 1$, $i = 1, 2$: f *decomposes*. For Schinzel's Problem consider these extensions of what is a trivial relation between f and g (allowing a switch of f and g).

(1.4a) Composite reducibility: $f_1(x) - g(y)$ factors.

(1.4b) A particular case of composition reducibility: $g = f_1 \circ g_2$,

[Fr87, Def. 2.1] calls an example reducible $f(x) - g(y)$ *newly reducible* – nontriviality for Schinzel's Problem – if composite reducibility (1.4a) does not hold. We call the corresponding (f, g) a *Schinzel pair*.

Problem 1.2. Describe the Schinzel pairs (f, g) in the case $g = \alpha \circ f$, $\alpha \in \mathrm{PGL}_2(\mathbb{C})$: a Schinzel Möbius pair, à la (1.1a).

1.1.3. *Schinzel's problem and Galois closure.* Prop. 1.3 compares the Galois closure condition (1.1b) with (1.1c): The fiber product, $X \times_{\mathbb{P}^1_z} Y$, of two covers $f : X \to \mathbb{P}^1_z$ and $g : Y \to \mathbb{P}^1_z$ has more than one component . We stay within the category of compact Riemann surface covers by replacing the set theoretic fiber product of f and g by its (nonsingular; still projective) normalization (see for example [Fr12a, §2.1]). We always do that, unless otherwise said. This way components of the fiber product have no intersection points, and we can associate branch cycles each component.

Generalizing §1.1.2, we expand on imprimitivity as it arises in covers. Suppose $f : X \to Z$ is a finite (separable) map of (normal) algebraic varieties and it factors through $f' : X' \to Z$ with $\deg(f) > \deg(f') > 1$. Then the Galois correspondence implies representation T_f on the Galois closure group G_f is imprimitive: f' defines a subgroup properly between $G(T_f, 1)$ and G_f. Conversely, imprimitivity produces such an f'. So, the permutation representation $T_{f'}$ on the Galois closure of f' extends naturally to the coset representation on G_f from pullback of $G(T'_f, 1)$.

Proposition 1.3. *As above, respectively assume f and g factor through $f' : X' \to \mathbb{P}^1_z$ and $g' : Y' \to \mathbb{P}^1_z$. Then, there is a pair (f', g') with both $\deg(f')$ and $\deg(g')$ maximal among those pairs with the following two properties.*

(1.5a) *Their Galois closures $\hat{f}' : \hat{X}' \to \mathbb{P}^1_z$ and $\hat{g}' : \hat{X}' \to \mathbb{P}^1_z$ are equivalent as Galois covers.*

(1.5b) *Components of $X \times_{\mathbb{P}^1_z} Y$ map one-one (and on)to components of $X' \times_{\mathbb{P}^1_z} Y'$.*

Condition (1.5a) implies f' and g' have exactly the same branch points. Count the components on $X' \times_{\mathbb{P}^1_z} Y'$ as the orbits of $G_{f'}(T_{f'}, 1)$ under $T_{g'}$.

Proof. The minimal simultaneous Galois cover of $\hat{X}$ and $\hat{Y}$ fits in the following commutative diagram, as $\hat{h} : \hat{W}_{f,g} \to \mathbb{P}^1_z$:

(1.6)
$$\begin{array}{ccccccc} & \overset{\mathrm{pr}_{\hat{Y}}}{\nearrow} & \hat{Y} & \overset{\mathrm{pr}_{g,\hat{V}}}{\searrow} & & & \\ \hat{W}_{f,g} & & & & \hat{V}_{f,g} & \xrightarrow{e} & \mathbb{P}^1_z \\ & \underset{\mathrm{pr}_{\hat{X}}}{\searrow} & \hat{X} & \underset{\mathrm{pr}_{f,\hat{V}}}{\nearrow} & & & \end{array}$$

In this diagram $\hat{V}_{f,g}$ is the maximal Galois cover through which both $\hat{f}$ and $\hat{g}$ factor. With G_e the group of the cover e, "restriction" gives natural maps $\mathrm{pr}^*_{f,\hat{V}} : G_f \to G_e$ and $\mathrm{pr}^*_{g,\hat{V}} : G_g \to G_e$ to G_e. Then, $G_{\hat{h}}$ naturally identifies with the fiber product

$$G_f \times_{G_e} G_g \overset{\mathrm{def}}{=} \big\{(\sigma_1, \sigma_2) \in G_f \times G_g \mid \mathrm{pr}^*_{f,\hat{V}}(\sigma_1) = \mathrm{pr}^*_{g,\hat{V}}(\sigma_2)\big\}.$$

The statement on the existence of (f', g') is from [Fr73, Prop. 2] (with some extra comments on its generality in [Fr12a, Lem. 4.2]). Let $\hat{X}'$ be the Galois closure cover of f'. A $z' \in \mathbb{P}^1_z$ is a branch point if some $1 \neq \sigma \in G_{f'}$ fixes $\hat{x}' \in \hat{X}'$ lying over z'. The Galois closures of f' and g' are the same and so therefore are their branch points. This concludes the proposition. □

Corollary 1.4. *If (f, g) is a Schinzel pair then the Galois closure of the f and g covers are identical.*

Conversely, if the Galois closures of f and g are the same, then the fiber product of f and g is reducible if

(1.7) *$G_f(T_f, 1)$ is intransitive in T_g.*

Assume (1.7) holds. Then, (f, g) is a Schinzel pair if

(1.8) *$G_f(T_f, 1)$ is transitive in the coset representation T_H for any subgroup $H \leq G_f$ properly containing $G_f(T_g, 1)$.*

In Prop. 1.3, $e : \hat{V}_{f,g} \to \mathbb{P}^1_z$ factors through the common Galois cover $\hat{X}' = \hat{Y}'$, but Ex. 1.5 shows they might not be equal.

Example 1.5 (Comparing Galois covers)**.** Take two, inequivalent, degree $n \geq 3$ simple-branched covers of $\mathbb{P}^1_z$ with the same branch point locus $\boldsymbol{z}$. For $r \geq 2(n-1)$, there are always several. (Count elements in the Nielsen class $\mathrm{Ni}(S_n, \mathbf{C}_{2^r})^{\mathrm{abs}}$.) For simple-branched covers, 2-cycles generate the group, so it is S_n, a primitive group, and $\hat{X}' = \hat{Y}' = \mathbb{P}^1_z$. Then, $\hat{V}_{f,g}$, in this case, is the degree 2 "discriminant" cover of $\mathbb{P}^1_z$ for branching at $\boldsymbol{z}$.

1.2. **Nielsen version of Schinzel's Problem.** Prop. 1.3 reverts Schinzel's problem to considering cover pairs (f, g) with the same Galois closures. That is, we have absolute Nielsen classes with distinct permutation representations defining abs. That requires notation using distinct, faithful transitive representations, T_1, T_2, of G. We sometimes shorten this to

$$\mathrm{Ni}(G, \mathbf{C}, T_i) \stackrel{\mathrm{def}}{=} \mathrm{Ni}(G, \mathbf{C})^{\mathrm{abs}_i};\ \text{in place of } N_{S_n}(G, \mathbf{C}), N_{T_i}(G, \mathbf{C}), i = 1, 2.$$

The Galois closure condition then gives us

$$\mathrm{Ni}(G, \mathbf{C})^{\mathrm{in}} \stackrel{\mathrm{def}}{=} \mathrm{Ni}(G, \mathbf{C})/G \text{ maps to both Nielsen classes.} \tag{1.9}$$

Fibers of $\mathrm{Ni}(G, \mathbf{C})^{\mathrm{in}} \to \mathrm{Ni}(G, \mathbf{C})^{\mathrm{abs}_i}$ equate to $N_{T_i}(G, \mathbf{C})/G$, $i = 1, 2$. App. B.2 reminds that all Nielsen classes have attached analytic spaces, and what inner Hurwitz space points signify.

§2.1 discusses U_r, the space of r distinct unordered points on $\mathbb{P}^1_z$. Diagram (1.10) consists of covering maps between four nonsingular spaces.

$$\begin{array}{ccccc} & \stackrel{\Psi_{\mathrm{in,abs}_2}}{\nearrow} & \mathcal{H}(G, \mathbf{C})^{\mathrm{abs}_2} & \stackrel{\Psi_{\mathrm{abs}_2}}{\searrow} & \\ \mathcal{H}(G, \mathbf{C})^{\mathrm{in}} & & & & U_r \\ & \stackrel{\searrow}{\Psi_{\mathrm{in,abs}_1}} & \mathcal{H}(G, \mathbf{C})^{\mathrm{abs}_1} & \stackrel{\nearrow}{\Psi_{\mathrm{abs}_1}} & \end{array} \tag{1.10}$$

The function $(\Psi_{\mathrm{in,abs}_1}, \Psi_{\mathrm{in,abs}_2})$ maps

$$\mathcal{H}(G, \mathbf{C})^{\mathrm{in}} \to \mathcal{H}(G, \mathbf{C})^{\mathrm{abs}_1} \times_{U_r} \mathcal{H}(G, \mathbf{C})^{\mathrm{abs}_2}.$$

Denote the image of $\mathcal{H}(G, \mathbf{C})^{\mathrm{in}}$ by $\mathcal{GC}_{\mathrm{abs}_1,\mathrm{abs}_2}$.

Proposition 1.6. *The Nielsen class diagram of* (1.9) *on* (T_1, T_2) *guarantees there are pairs of covers* (f, g) *in the respective Nielsen classes* $\mathrm{Ni}(G, \mathbf{C})^{\mathrm{abs}_i}$, $i = 1, 2$, *with the same Galois closures.Then, points of* $\mathcal{GC}_{\mathrm{abs}_1,\mathrm{abs}_2}$ *– up to absolute equivalence – parametrize such pairs.*

For such pairs, the fiber products are reducible if and only if

(1.11) $G(T_1, 1)$ *has at least 2 orbits under the representation* T_2.

Then, the pairs (f, g) *above are Schinzel pairs if and only if in addition,*

(1.12) $G(T_1, 1)$ *has one orbit under the coset representation for every* $H \leq G$ *containing* $G(T_2, 1)$ *properly.*

Proof. If we verify that points of $\mathcal{GC}_{\text{abs}_1,\text{abs}_2}$ properly account for the 1st paragraph, then Cor. 1.4 gives the remainder of the proposition.

By definition, up to absolute equivalence, a $\boldsymbol{p} \in \text{Ni}(G, \mathbf{C})^{\text{in}}$ corresponds to a Galois cover, $\hat{f}_{\boldsymbol{p}} : \hat{X}_{\boldsymbol{p}} \to \mathbb{P}^1_z$, in the Nielsen class with branch points $\boldsymbol{z} = \Psi_{\text{abs}_i} \circ \Psi_{\text{in,abs}_i}(\boldsymbol{p})$, $i = 1, 2$. Then, $\Psi_{\text{in,abs}_i}(\boldsymbol{p})$ corresponds to an absolute class of covers in $\text{Ni}(G, \mathbf{C})^{\text{abs}_i}$ with the same branch points, for which the two covers have the equivalence class of $\hat{f}_{\boldsymbol{p}}$ as their common Galois closure. □

Definition 1.7. If (T_1, T_2) satisfy (1.11) (resp. (1.12)) we say the Nielsen classes $\text{Ni}(G, \mathbf{C})^{\text{abs}_i}$, $i = 1, 2$, are a *reducible* (resp. *Schinzel*) pair.

The groups $N_{T_i}(G, \mathbf{C})/G$ may be significantly different. Example: Their corresponding spaces $\mathcal{H}(G, \mathbf{C})^{\text{abs}_i}$ may even have a different number of components. That depends on whether those outer automorphisms are *braidable* (§B.3). In Cor. 1.11 a different kind of outer automorphism appears in Schinzel's problem.

1.3. **Möbius condition.** Recall, $r \geq 3$. Take $g = \alpha \circ f$, with $\alpha \in \text{PGL}_2(\mathbb{C})$: (f, g) is a Möbius pair as in (1.1a). §1.3.2 extends Prop. 1.6. This section starts our treatment of the major problem of the paper.

Problem 1.8. Give the sublocus of $\mathcal{GC}_{\text{abs}_1,\text{abs}_2}$ of Galois Möbius – even, extending Def. 1.7 reducible Möbius, or Schinzel Möbius – pairs.

§1.3.1 relates reducible Möbius to its special case, Schinzel Möbius.

1.3.1. *Unique imprimitivity.* Now assume the fiber product of f and g is reducible. Yet, (f, g) is not automatically a Schinzel pair. So, we cannot immediately conclude their Galois closure covers are identical.

Here we consider an hypothesis that replaces f by a maximal composition factor for which: (1.1a) and (1.1b) simultaneously hold; and components of the original fiber product map one-one to those of the new fiber product.

Consider a permutation representation T of G. Denote the set of groups H, with $G(T, 1) \leq H \leq G$ by I_T. For $H \in I_T$, denote $G/\cap_{g \in G} gHg^{-1}$ by G_H. Finally, denote the representation of G on cosets of H by T_H and the image of $\mathbf{C}$ in G_H by $\mathbf{C}_H$. Here $\mathbf{C}_H$ may have trivial classes, a la the statement on $\mathbf{C}$ at the top of §1.1.

Definition 1.9. Refer to $\text{Ni}(G, \mathbf{C})^{\text{abs}}$ as having *unique imprimitivity* if the map $H \in I_T \mapsto (G_H, \mathbf{C}_H)$ is one-one. The monodromy group and conjugacy classes determine uniquely any cover through which f factors.

Lemma 1.10. *Assume the Nielsen class* $\mathrm{Ni}(G,\mathbf{C})^{\mathrm{abs}}$ *of a cover* f *has unique imprimitivity. Then, we may assume* f *and* g *have the same Galois closure. Def. 1.9 applies if* f *has a totally ramified place (as when* $f \in \mathbb{C}[x]$*).*

Proof. Suppose $f^*: X^* \to \mathbb{P}^1_z$ is any cover in the Nielsen class $\mathrm{Ni}(G,\mathbf{C})^{\mathrm{abs}}$.

Prop. 1.3 then allows replacing f by a composition factor f', and g by a composition factor g'. Still, it doesn't conclude that $g' = \alpha \circ f'$. This requires knowing that the composition factor $\alpha \circ f'$ is the only possible one of g (up to affine equivalence) that could have the same Galois closure and conjugacy classes. This is what Def. 1.9 says.

If some conjugacy class, say C_r, is totally ramified, then the order of the elements in its image among $\mathbf{C}_H$ tracks the degree $(G:H)$. An elementary argument gives a degree preserving embedding of I_T to the cyclic quotient groups of $\mathbb{Z}/n$ [FrM69, Prop. 3.4]. That is, any composition factor (up to affine equivalence) has a unique degree. □

1.3.2. *Möbius fixed branch points.* Assume the Lem. 1.10 conclusion:

(1.13) f and $\alpha \circ f = g$ have identical Galois closures, and write f is in $\mathrm{Ni}(G,\mathbf{C})^{\mathrm{abs}_1}$ and g is in $\mathrm{Ni}(G,\mathbf{C})^{\mathrm{abs}_2}$.

We don't immediately assume the fiber product of f and g is reducible. If $T_1 = T_2$, then (1.13) says composing f with α gives an equivalent cover.

We need notation for partitionings of $\boldsymbol{z}$ based on an integer $v \ge 2$:

$$\begin{aligned}\{z_1, z_{t+1}\dots, z_{(v-1)t+1}\}, \{z_2, z_{t+2}, \dots, z_{(v-1)t+2}\},\\ \dots, \{z_{(t-1)v+1}, \dots, z_{tv}\}, \{z_t, z_{t+t}, \dots, z_{(v-1)t+t}\},\end{aligned} \tag{1.14}$$

where (ϵ_1,ϵ_2) is one of $(0,0)$, $(0,1)$ or $(1,2)$, and if $\epsilon_i = 0$, leave $z_{vt+\epsilon_i}$ out of the partition. Notice then:

(1.15) if $(\epsilon_1,\epsilon_2) = (0,0)$ (resp. $(0,1)$, $(1,2)$) then $vt = r$ (resp. $r-1$, $r-2$).

Corollary 1.11. *From* (1.13)*,* α *leaves* $\boldsymbol{z}$ *invariant (Prop. 1.3). Assume that* T_1 *and* T_2 *are inequivalent permutations representations.*

For some numbering of $\boldsymbol{z}$*, the* ϵ_i *s that are not 0 in* (1.14) *represent the fixed points of* α *in* $\boldsymbol{z}$*, and for each* $j \in \{0,\dots,t-1\}$*,* $s \bmod v$*,*

$$\alpha : z_{jv+s+1} \mapsto z_{jv+s+2}.$$

Denote by $\mathbb{P}^1_u$ *the quotient of* $\mathbb{P}^1_z$ *by* α*, and* $\mu: \mathbb{P}^1_z \to \mathbb{P}^1_u$ *the corresponding cyclic cover with group* $\mathbb{Z}/v = \langle a\rangle$*. Then,* $\mu \circ \hat f$ *and* $\mu \circ \hat g$ *are equivalent Galois covers with group* G^**. Any lift* a^* *of* a *to* G^* *has these properties.*

(1.16a) a^* *conjugates* $G(T_1,1)$ *to* $G(T_2,1)$ *up to conjugation by* G*.*

(1.16b) *No element of* $N_{S_n}(G,\mathbf{C})$ *represents conjugation by* a^**.*

Proof. [Ahl79, p. 78–80] reminds of the sharp triple transitivity of $\mathrm{PGL}_2(\mathbb{C})$ on points of $\mathbb{P}^1_z$. Here is one form that takes.

(1.17) $\frac{(z-z_3)(z_2-z_4)}{(z-z_4)(z_2-z_3)}$ is the unique Möbius transformation in z that takes (z_2, z_3, z_4) to $(1, 0, \infty)$.

Also, elements of $\mathrm{PGL}_2(\mathbb{C})$ take clockwise oriented circles to clockwise oriented circles [Ahl79, §3.2, Thm. 14].

Since $r \geq 3$, and α permutes the elements of $\boldsymbol{z}$, some 'power' (iteration) of α fixes all the elements of $\boldsymbol{z}$. So, α has finite order. Elements of PGL_2 of finite order have two distinct fixed points. By conjugation in $\mathrm{PGL}_2(\mathbb{C})$ we may assume they are 0 and ∞, and so $\alpha : z \to az$ for some $a \in \mathbb{C}^*$.

Finite order implies $a = e^{2\pi i j/v} \stackrel{\text{def}}{=} \zeta_v$, a primitive vth root of 1 for some v. Denote by $a^\dagger$ the effect of a on the collection $\boldsymbol{z}$. If none of the fixed points of α are branch points, then all orbits of $a^\dagger$ have length v. By renaming the elements of $\boldsymbol{z}$, the orbits partition according to (1.14), depending on how many fixed points of α are in $\boldsymbol{z}$.

Excluding the fixed points of α, all orbits of $a^\dagger$ consist of the vertices of a regular v-gon. Finally, consider (1.16). Extend α to the Galois closure cover. It takes the cover $f : X \to \mathbb{P}^1_z$ to $g : Y \to \mathbb{P}^1_z$. Denote this extension by a^*. If $\sigma \in G(\hat{X}/X)$, then $a^*\sigma(a^*)^{-1}$ fixes Y. This is equivalent to (1.16a). If, however, an element of $N_{T_1}(G, \mathbf{C})$ represents a^*, then T_1 and T_2 would be equivalent representations. This concludes the corollary. □

Definition 1.12. Refer to a $\boldsymbol{z} \in U_r$ fixed by some nontrivial element of $\mathrm{PGL}_2(\mathbb{C})$ as *Möbius fixed.*

A set on U_r contained in a real analytic proper subset of U_r is $\mathbb{R}$-*special*. Lem. 1.13, where $r > 4$, contrasts with Rem. 1.14, where $r = 4$.

Lemma 1.13. *If $r > 4$, the set of $\boldsymbol{z} \in U_r$ that are Möbius fixed is $\mathbb{R}$-special.*

Proof. The v-length orbits of an $\alpha \in \mathrm{PGL}_2(\mathbb{C})$ are on a circle under the Möbius image of the vertices of a regular polygon centered at the origin (as in the proof of Cor. 1.11). We want to show that such $\boldsymbol{z}$ must be $\mathbb{R}$-special. First assume $\boldsymbol{z}$ contains no fixed points of α.

Suppose $v \geq 3$. If $v = 3$ or 4, then $t \geq 2$. Fix any v-gon, $P_{v,0}$, centered at the origin. Then, $\boldsymbol{z}$ is the image of some Möbius transformation, β, that takes the vertices of $P_{v,0}$ to one of the orbits of α. Further, the remaining α orbits on $\boldsymbol{z}$ are the images under β of the vertices of another sequence of vertices of v-gons $P_{v,j}$, $j = 1, \dots, t-1$, centered around the origin.

To account for the real analytic dimension of the Möbius fixed $\boldsymbol{z}$ we have only to consider the real analytic dimension of the image of the vertices of $P_{v,0}$ under $\mathrm{PGL}_2(\mathbb{C})$; then add to that the real analytic dimension from varying the collection $P_{v,j}$, $j = 1, \dots, t-1$. There are two real dimensions to each of the latter: One each for rotation around the origin, and for radius of the circle containing the v-gon. The total real analytic dimension of Möbius fixed points is thus, $2 \cdot 3 + 2 \cdot (t-1) = 4+2t$ if $t > 1$, and 6 if $t = 1$. In each case this is less that $2vt$, the real analytic dimension of U_r.

Finally, if $v = 2$, then $t \geq 3$. Now, in the estimate of the $\mathrm{PGL}_2(\mathbb{C})$ range on a 2-point set, reduce its dimension by 2. So, the result is true here because $2+2t < 4t$. The case where $\boldsymbol{z}$ contains α fixed points is even easier, because this cuts down the effect of $\mathrm{PGL}_2(\mathbb{C})$ translation. □

Remark 1.14 (When $r = 4$). When $r = 4$, the Möbius transformation $z \mapsto a/z$, $a \in \mathbb{C}$, switches 0 and ∞, and also 1 and a. Similarly, there is a transformation that switches the elements in any two doublets from $\{0, 1, \infty, a\}$. That giving a subgroup, $K_{\boldsymbol{z}}$, of $\mathrm{PGL}_2(\mathbb{C})$ that acts as a Klein 4-group on the permutations of $\{0, 1, \infty, a\}$. As a is arbitrary the same is true for any 4-tuple of distinct elements of $\mathbb{P}^1_z$.

2. Möbius effect on Branch Cycles

Suppose we have one cover $f : X \to \mathbb{P}^1_z$, with a particular property called P. If composing f with Möbius transformations preserves P, then the collection $\{\alpha \circ f\}_{\alpha \in \mathrm{PGL}_2(\mathbb{C})}$ is a 3-dimensional family of covers with property P. Yet, this somewhat trivial family can disguise more meaningful appearances of property P. So, we commonly mod out by by $\mathrm{PGL}_2(\mathbb{C})$, as in §2.1, by dealing with reduced Hurwitz spaces. We want to distinguish the significant presence of PGL_2 fixed points from the modest translation by PGL_2.

The program of the rest of the paper is to characterize the conditions (1.1), singly or in combination, by a Nielsen class statement. The technical tool is §2.2 which also characterizes singularities on reduced Hurwitz spaces. This includes characterizing how to generalize the AGZ version of Schinzel's problem by adding the $g = \alpha \circ f$ condition (1.1a). We call these *Schinzel Möbius pairs* in Prob. 1.8.

§2.3 does the many-application case $r = 4$. Here we get the elements of Nielsen classes corresponding to points on the reduced version of $\mathcal{GC}_{\mathrm{abs}_1,\mathrm{abs}_2}$ (Prop. 1.6) corresponding to cover Galois Möbius pairs (f, g) (as in (1.8)): a Galois pair with $g = \alpha \circ f$ for some $\alpha \in \mathrm{PGL}_2$. Then, §2.4 finishes the formulas, and the conclusion on $\mathcal{GC}_{\mathrm{abs}_1,\mathrm{abs}_2}$, for $r > 4$.

2.1. **Reduced Hurwitz spaces.** There is a simple reason the spaces, like $\mathcal{H}(G,H)^{\mathrm{in}}$ and $\mathcal{GC}_{\mathrm{abs}_1,\mathrm{abs}_2}$ in Prop. 1.6 are complex manifolds – without singularity. They are each defined by a subgroup of a fundamental group, $\pi_1(U_r, \boldsymbol{z}_0)$ of the space, U_r, of r distinct points on $\mathbb{P}^1_z$. The Hurwitz monodromy group H_r, a quotient of the braid group, identifies with $\pi_1(U_r, \boldsymbol{z}_0)$, and it acts on any Nielsen class that is a quotient of $\mathrm{Ni}(G, \mathbf{C})^{\mathrm{in}}$. This is in detail in [Fr77, §4]. It has expositions in several places, including [Vo96, §10.1]. Its applications have appeared in many papers, with gently handled examples like those of this paper in [Fr12a, §6.4].

2.1.1. *Modding out by* $\mathrm{PGL}_2(\mathbb{C})$. The reduction of Hurwitz spaces (of covers) is based on this idea. Given $f : X \to \mathbb{P}^1_z$, branched at $\boldsymbol{z}$, we can calculate its Nielsen class precisely using *classical generators* $\mathcal{P} = \{P_1, \dots, P_r\}$ of the fundamental group of $\mathbb{P}^1_z \setminus \{\boldsymbol{z}\} \stackrel{\text{def}}{=} U_{\boldsymbol{z}}$ based at $z_0 \in U_{\boldsymbol{z}}$. The example of App. C displays the essential properties – especially, (C.1) – of $\mathcal{P}$. The special symmetrical paths here would not be generally appropriate; general deformations of the branch points won't preserve the symmetry.

Suppose we apply $\alpha \in \mathrm{PGL}_2(\mathbb{C})$ to these paths to get $\alpha(\mathcal{P})$. Then, as noted below (1.17) the paths ending in (clockwise oriented) circles around $\boldsymbol{z}$ will go to paths ending in (clockwise oriented) circles around $\alpha(\boldsymbol{z})$. So, α takes classical generators to classical generators, except for one subtlety.

(2.1) Unless z_0 is a fixed by α, $\alpha(z_0) \neq z_0$; we moved the base point.

Therefore, in §2.4 we consider first the case where

(2.2) z_0 is a fixed point of α.

2.1.2. *When* α *fixes* $\boldsymbol{z}$. In considering Galois Möbius (pairs of) covers, we must assume $\alpha(\boldsymbol{z}) = \boldsymbol{z}$. Then, as in [Fr77, Lem. 1.1], the Nielsen class of $\alpha \circ f$ is the same as that of f. The $*$ class ($*$ = abs or in) branch cycles for f relative to $\mathcal{P}$, $\boldsymbol{\sigma}_{f,\mathcal{P}}$, are the same as branch cycles for $\alpha \circ f$ relative to $\alpha(\mathcal{P})$. The equivalence must include inner classes to make this unambiguous; to account for possibly moving the base point, as in (2.1).

On a reduced Hurwitz space $\mathcal{H}(G, \mathbf{C})^{*,\mathrm{rd}}$ we identify $\boldsymbol{p}, \boldsymbol{p}' \in \mathcal{H}(G, \mathbf{C})^*$ if $f_{\boldsymbol{p}'}$ is a $*$-equivalent cover to $\alpha \circ f_{\boldsymbol{p}}$. This reduces the dimension of $\mathcal{H}(G, \mathbf{C})^*$ by 3. The resulting space is still an affine variety [Fr10, Prop. A.8]. It covers the natural space $U_r/\mathrm{PGL}_2(\mathbb{C}) \stackrel{\text{def}}{=} J_r$, which for $r = 4$ (resp. $r = 3$), is the classical j-line (resp. a point). Denote the image in J_r of $\boldsymbol{z} \in U_r$ by $[\boldsymbol{z}]$.

§2.2 gives the branch cycles of $\alpha \circ f$ relative to $\mathcal{P}$: the $*$ class of $\boldsymbol{\sigma}_{\alpha \circ f,\mathcal{P}}$. If the $*$ classes of $\boldsymbol{\sigma}_{f,\mathcal{P}}$ and $\boldsymbol{\sigma}_{\alpha \circ f,\mathcal{P}}$ are the same we say α fixes f. Otherwise, α *moves* f. When some nontrivial α fixes f in the Nielsen class, we say f is Möbius fixed. Prob. 2.1 refines part of the goal of Prob. 1.8.

Problem 2.1. Describe the $f \in \mathrm{Ni}(G, \mathbf{C})^{\mathrm{abs}_1}$ whose Galois closure (resp. f itself) is Möbius fixed.

There is further refinement needed when $r = 4$ because all $\boldsymbol{z}$ are fixed by a Klein 4-group, $K_{\boldsymbol{z}}$ (Rem. 1.14).

(2.3a) For $r = 4$ we need to consider if a subgroup of $K_{\boldsymbol{z}}$ fixes f.

(2.3b) Also, for $r = 4$ and the special $\boldsymbol{z}$ fixed by some α outside $K_{\boldsymbol{z}}$, we need to decide which f are moved by the extra α.

(2.3c) For $r > 4$, we need only decide which f are moved by α.

When $r = 4$, $\mathcal{H}(G, \mathbf{C})^{*,\mathrm{rd}}$ is still nonsingular. A point $\boldsymbol{p}$ corresponding to fixed f ramifies over $[\boldsymbol{z}]$, but is not singular. For (2.3c), that $\boldsymbol{p}$ is a singular point in the fiber of the reduced Hurwitz space over $[\boldsymbol{z}]$.

Remark 2.2. There are two extra cases of (2.3b) respectively represented by $\boldsymbol{z} = \{0, e^{2\pi \mathrm{i} j/3}, j = 1, 2, 3\}$ and $\boldsymbol{z} = \{e^{2\pi \mathrm{i} j/4}, j = 1, 2, 3, 4\}$. In a standard normalization of the $j = J_4$-line, the 1st represents $j = 0$, the 2nd $j = 1$.

2.1.3. *Compactification of* $\mathcal{H}^{*,\mathrm{abs}}$. Every (projective) algebraic variety V has a unique projective normalization $\bar{V}$. Inside $\bar{V}$ is a subvariety V' for which the following two properties hold:

(2.4a) $\bar{V} \setminus V'$ has codimension at least 1; and

(2.4b) a surjective, birational, morphism $V' \to V$ – from normalization – is one-one off a locus of V of codimension at least 1.

When V is irreducible and has dimension 1, the result is the unique nonsingular projective model of V.

This applies to the spaces J_r and $\mathcal{H}^{*,\mathrm{abs}}$ for which the results are respectively $\bar{J}_r$ and $\bar{\mathcal{H}}^{*,\mathrm{abs}}$, with the latter naturally mapping to the former. [Fr10, §A.8, esp. Prop. A.8] has details. When $r = 4$, $\bar{J}_r$ identifies with the classical j-line $\mathbb{P}^1_j$. Then, $\bar{\mathcal{H}}^{*,\mathrm{abs}} \to \mathbb{P}^1_j$ is a nonsingular curve covering to which we can apply Riemann-Hurwitz (as in Prop. 2.12) to compute the genus of $\bar{\mathcal{H}}^{*,\mathrm{abs}}$.

For $r > 4$ it has become standard among many – as a variant on the first author's approach in [Fr95, proof of Thm. 3.21] – to desingularize $\bar{\mathcal{H}}^{*,\mathrm{abs}}$ using a Deligne-Mumford type compactification for Hurwitz spaces introduced by Wewers [We98].

When $r > 4$, the resulting nonsingular spaces feature above the singularities of $\bar{\mathcal{H}}^{*,\mathrm{rd}}$ divisors with normal crossings. We here are doing something preliminary to considering that – explicitly identifying the singular points – and we will only use the compactification when $r = 4$.

2.2. **Identifying Galois Möbius pairs.** [FrGu12, App. A] arranges branch points $z_1, \dots, z_{r-1}$ along the vertices of a regular polygon. It has a set of classical generators $P_{\mathrm{fg},1}, \dots, P_{\mathrm{fg},r}$ based at the origin, arranged in clockwise order. The loop around ∞ goes between the loops around z_{r-1} and z_1. This is where $\infty = z_r$ is a fixed point of α and all finite branch points fall in one α orbit. Applying it to polynomials, we took $\sigma_r = \sigma_\infty = (1\,2\,\dots\,n)^{-1}$.

Use the Nielsen class of the cover f, denoting it $\mathrm{Ni}(G, \mathbf{C})$. Suppose, relative to $P_{\mathrm{fg},1}, \dots, P_{\mathrm{fg},r}$, $\boldsymbol{\sigma}_f = (\sigma_1, \dots, \sigma_r) \in \mathrm{Ni}(G, \mathbf{C})$ represents f. Then, with $\zeta_v = e^{2\pi \mathrm{i}/v}$, $g = \zeta_v f$ is represented by

$$\boldsymbol{\sigma}_g = (\sigma_2, \dots, \sigma_{r-1}, \sigma_1, \sigma_1^{-1}\sigma_r\sigma_1)(\text{with } \sigma_r = \sigma_\infty) \tag{2.5}$$

relative to the same classical generators.

Let $\boldsymbol{\sigma} \in \mathrm{Ni}(G, \mathbf{C})$, with T_i, $i = 1, 2$, *distinct* faithful transitive permutation representations of G. Assume $\boldsymbol{z} \in U_r$ is fixed by some nontrivial $\alpha \in \mathrm{PGL}_2$. The hypothesis of [FrGu12, §2] is the following:

(2.6) $\boldsymbol{z}$ consists of one fixed point of α, and one other α orbit.

The expressions in (2.14) give three formulas – referred to from here as the *BC Formulas*. They correspond to the cases on (ϵ_1, ϵ_2) in (1.15). Then, (2.6) is a special case of (2.14b) where $(\epsilon_1, \epsilon_2) = (0, 1)$.

We now make a series of formal conclusions involving The BC Formulas; §2.4.3 fills in their details. In each of the three cases we must adjust the classical generators $P_{\mathrm{fg},1}, \dots, P_{\mathrm{fg},r}$ slightly to get appropriate classical generators, which we name respectively $P_{\mathrm{fg,a},1}, \dots, P_{\mathrm{fg,a},r}$, $P_{\mathrm{fg,b},1}, \dots, P_{\mathrm{fg,b},r}$ and $P_{\mathrm{fg,c},1}, \dots, P_{\mathrm{fg,c},r}$. Their pictures are in §C.

Proposition 2.3. *Assume $\boldsymbol{z}$ is Möbius fixed and* (2.14a) *(resp.* (2.14b) *or* (2.14c)*) applies. Then, there is a $\beta \in \mathrm{PGL}_2(\mathbb{C})$ such that α is conjugation of $z \mapsto \zeta_v z$ by β with $v \cdot t = r$ (resp. $v \cdot t = r-1$, or $v \cdot t = r-2$) and the image of $P_{fg,\ a,1}, \dots, P_{fg,a,r}$ (resp. $P_{fg,\ b,1}, \dots, P_{fg,b,r}$ or $P_{fg,\ c,1}, \dots, P_{fg,c,r}$) are classical generators under β relative to $\boldsymbol{z}$ based at $\beta(0)$ in the first two cases, but based at $\beta(0 + \epsilon)$ with ϵ arbitrarily small in the 3rd case.*

For each $\boldsymbol{\sigma} \in \mathrm{Ni}(G, \mathbf{C})^{\mathrm{abs}_1}$, there is a corresponding cover f branched at $\boldsymbol{z}$. Then, α fixes f (resp. its Galois closure) if and only if some $h \in N_{T_1}(G, \mathbf{C})$ (resp. $h \in \mathrm{Aut}(G)$) conjugates $\boldsymbol{\sigma}$ to the right side of the equation in (2.14) *corresponding to the case – as stated above – a, b or c.*

For $\alpha \circ f = g$ to be in $\mathrm{Ni}(G, \mathbf{C})^{\mathrm{abs}_2}$ is the same as above, except, apply T_2 to the right side of the corresponding equation in (2.14) *and $h \in N_{T_2}(G, \mathbf{C})$.*

The right side of (2.5) comes from applying the braid $\mathbf{sh} \circ q_{r-1}^{-1}$ (§B.1) to the left side. For branch cycles for $e^{2\pi \mathrm{i} j/v} f$, apply $(\mathbf{sh} \circ q_{r-1}^{-1})^j$.

Corollary 2.4 (Braiding $\mathrm{PGL}_2(\mathbb{C})$)**.** *There is an analogous braid in each case listed in Prop. 2.3. In particular, not only is the Nielsen class of a cover preserved by* $\mathrm{PGL}_2(\mathbb{C})$ *(§2.1.2), but so is its braid orbit.*

(2.7a) *When $r = 4$ and $\boldsymbol{z}$ does not correspond to $j = 0$ or 1 the braids consist of the elements of the moduli group, $\mathcal{Q}''$ (§B.1).*

(2.7b) *For the $r = 4$ special cases (§2.3.3), the braids are respectively iterates of $\mathbf{sh} \circ q_3^{-1}$ ($j = 0$) and the iterates of $\mathbf{sh}$ ($j = 1$).*

(2.7c) *For $r > 4$, Rem. 2.16 and Rem. 2.19 give the generalizations.*

Corollary 2.5. *Assume the hypotheses of Prop. 2.3 (1st sentence), and that α fixes f in the Nielsen class (as above, T_1 and T_2 are distinct). Then, there is an automorphism μ of G that conjugates $(G(T_1, 1)$ to $(G(T_2, 1)$ taking the branch cycles for f to the branch cycles for g. Conversely, if there is such an automorphism, then g arises from f by an $\alpha \in \mathrm{PGL}_2(\mathbb{C})$.*

Remark 2.6 (Polynomial case). [FrGu12, Prop. 1.2] considers the case α is multiplication by ζ_v, and f is a polynomial cover, with a branch cycle $\sigma_\infty = (n \cdots 1)$ over ∞, the case of the original case in [AZ03] and [Gu10]. It denotes the conjugation by a choice of μ by c_{AZ}. Then, μ has trivial action on σ_∞ and there is no element of S_n that represents it.

2.3. **$\mathcal{Q}''$ invariant orbits; 4 branch points.** Using **H**(arbater)**M**(umford) braid orbits originated in [Fr95, Part III]. In applications when $r = 4$, these braid orbit types are most common.

2.3.1. *Klein-dihedral groups.* We give two definitions based on the dihedral group, D_d, of order $2d$ (App. A).

Definition 2.7. Suppose G^* is generated by three involutions $\{\alpha_1, \alpha_2, \alpha_3\}$. We say they form a *2-dihedral* group if $G^* = \langle \alpha_1\alpha_2, \alpha_1\alpha_3\rangle$.

The essential is that pairwise products of all the α_i s generate $\langle \alpha_1, \alpha_2, \alpha_3\rangle$. Two dihedral subgroups come together at α_1 to generate G^*. Another group generated by three involutions contains a 2-dihedral.

Definition 2.8. Suppose three involutions $\{\alpha'_0, \alpha'_1, \alpha'_2\}$ generate $G^\dagger$ where $\langle \alpha'_0, \alpha'_1\rangle$ is a Klein 4-group and $\langle \alpha'_1, \alpha'_2, \alpha'_0\alpha'_2\alpha'_0\rangle$ is 2-dihedral. We call $G^\dagger$ a *Klein-dihedral* group if in addition $G^\dagger = \langle \alpha'_1\alpha'_2, \alpha'_1\alpha'_0\alpha'_2\alpha'_0\rangle$.

For $r = 4$, a Klein 4-group $K_{\boldsymbol{z}} \le \mathrm{PGL}_2(\mathbb{C})$ fixes each $\boldsymbol{z} \in U_4$ (Rem. 1.14). If $\boldsymbol{z}$ is not a special $\mathrm{PGL}_2(\mathbb{C})$ orbits (corresponding to $j = 0$ or 1 in Rem. 2.2), no other elements fix $\boldsymbol{z}$. Let O be a braid orbit of $\mathrm{Ni}(G, \mathbf{C})^*$ ($* =$ in or abs equivalence).

§B.1 says we can decide if elements in $K_{\boldsymbol{z}}$ fix a point of the Hurwitz space representing $\boldsymbol{\sigma} \in O$ over $\boldsymbol{z}$ by applying $\mathcal{Q}''$ to $\boldsymbol{\sigma}$. §2.3.3 notes a corresponding test over just the special points $j = 0$ and 1.

When $\mathcal{Q}''$ fixes all elements in a braid orbit, then Nielsen classes and reduced Nielsen classes in that orbit are the same. Thm. 2.9 (1st sentence) notes that invariance by $\mathcal{Q}''$ is a braid invariant. The remainder characterizes $\mathcal{Q}''$ invariance of H-M braid orbits.

For $\boldsymbol{\sigma} \in \mathrm{Ni}(G, \mathbf{C})$, denote its (inner) braid orbit by $O_{\boldsymbol{\sigma}}$. Write an H-M representative, $\boldsymbol{\sigma}_{\text{H-M}}$, as $(\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1})$. If $\boldsymbol{\sigma}_{\text{H-M}}$ is $\mathcal{Q}''$ invariant, then the collection $\{\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1}\}$ automatically consists of conjugate elements. Denote their common orders by d. We *always* assume G is not cyclic.

Theorem 2.9. *If $q' \in \mathcal{Q}''\setminus\{1\}$ fixes $\boldsymbol{\sigma} \in \mathrm{Ni}(G, \mathbf{C})^{\mathrm{in}}$, then $q^{-1}q'q$ is in $\mathcal{Q}''\setminus\{1\}$, and it fixes $(\boldsymbol{\sigma})q$. Therefore, invariance by $\mathcal{Q}''$ is a braid invariant. Condition* (2.8) *characterizes invariance of $\boldsymbol{\sigma}_{\text{H-M}}$ under two particular elements of $\mathcal{Q}''$.*

(2.8a) $\mathbf{sh}^2$ *invariance: For some involution σ', $\sigma_2 = \sigma'\sigma_1(\sigma')^{-1}$.*

(2.8b) $q_1q_3^{-1}$ *invariance: For some involution $\sigma'' \in G$, $\langle\sigma'', \sigma_i\rangle$, $i = 1, 2$, are dihedral groups; or (degenerate case) $\sigma_i = \sigma_i^{-1}$, $i = 1, 2$.*

Then, $\mathcal{Q}''$ invariance of $O_{\boldsymbol{\sigma}_{\text{H-M}}}$ is equivalent to both (2.8a) *and* (2.8b) *holding. In turn, that is equivalent to the following:*

(2.9) *$\langle\sigma'', \sigma', \beta\rangle$ is a Klein-dihedral with $\sigma_1 = \sigma'\beta$ and $\sigma_2 = \sigma'\sigma''\beta\sigma''$; or (degenerate case) σ_1 and σ_2 are conjugate involutions.*

Proof. For $q' \in \mathcal{Q}''$, as $\mathcal{Q}''$ is a normal in H_4 [BaFr02, (2.11b)], $qq'q^{-1} \in \mathcal{Q}''$. The 1st sentence is now clear from the hypothesis $(\boldsymbol{\sigma})q' = \boldsymbol{\sigma}$. The 2nd sentence starts with the hypothesis that every $a' \in \mathcal{Q}''$ fixes $\boldsymbol{\sigma}$. From the 1st sentence the same is true of $(\boldsymbol{\sigma})q$.

By assumption $\mathbf{sh}^2$ and $q_1q_3^{-1}$ take $\boldsymbol{\sigma}_{\text{H-M}}$ to $\boldsymbol{\sigma}' = (\sigma_2, \sigma_2^{-1}, \sigma_1, \sigma_1^{-1})$ and $\boldsymbol{\sigma}'' = (\sigma_1^{-1}, \sigma_1, \sigma_2^{-1}, \sigma_2)$, respectively. That Q'' is trivial on $\sigma_{\text{H-M}}$ means there are respective $\sigma', \sigma'' \in G$ with these properties.

(2.10a) σ' conjugates σ_1 to σ_2 and σ_2 to σ_1.

(2.10b) σ'' conjugates σ_1 to σ_1^{-1} and σ_2 to σ_2^{-1}.

Rem. 2.10 handles the degenerate case where $\sigma'' = 1$ and both σ_is are involutions. Otherwise, $\mathcal{Q}''$ induces a regular Klein 4-group action on the following 4 pairs $\{(\sigma_1, \sigma_2), (\sigma_1^{-1}, \sigma_2^{-1}), (\sigma_2, \sigma_1), (\sigma_2^{-1}, \sigma_1^{-1})\}$ through a homomorphism into G. Then, σ' and σ'' play the roles of the elements in (2.8). □

Remark 2.10 (Degenerate Klein-dihedrals). The degenerate case of Thm. 2.9 is where $\sigma'' = 1$. Then, the H-M rep. consists of $(\sigma_1, \sigma_1, \sigma_2, \sigma_2)$ with σ_1 and σ_2 conjugate involutions. That is, G is the dihedral group D_n for some odd integer n. The corresponding reduced Hurwitz space then identifies with a modular curve as in [Fr78, §2]. §2.3.2 does the example where $n = 4$.

2.3.2. *Example for covers invariant under elements of $\mathcal{Q}''$.* This example is relevant to all aspects of this paper. The subsection concludes with comments on variant examples.

Take G to be D_4, the dihedral group of order 8. If you write D_4 as

$$\{\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) \mid a \in \{\pm 1\}, b \in \mathbb{Z}/4\},$$

2×2 matrices under multiplication, then conjugacy classes for the Nielsen class $\mathrm{Ni}(D_4, \mathbf{C}_{a^2b^2})$ are two repetitions of the involution classes C_a and C_b given, respectively, by $A = \left(\begin{smallmatrix} -1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. The given matrices generate D_4. If you write D_4 in its action on the integers $\mathbb{Z}/4$ as a subgroup of S_4, then we can take C_a as the class of $(1\,4)(2\,3)$ and C_b as the class of $(1\,3)$ in the group generated by these two elements.

The second representation suits the absolute Nielsen class. Consider the H-M rep. $\boldsymbol{\sigma}_{\text{H-M}} = ((1\,4)(2\,3), (2\,4), (2\,4), (1\,4)(2\,3))$. Recall we need classical generators around a specific collection of branch points $\boldsymbol{z}$ to produce a cover $f_{\text{H-M}} : X_{\text{H-M}} \to \mathbb{P}^1_z$. There is nothing intrinsic about the cover to suggest that it should have such branch cycles. Since both $\mathbf{sh}^2$ and $q_1 q_3^{-1}$ interchange the conjugacy classes, they don't fix the absolute class of $\boldsymbol{\sigma}_{\text{H-M}}$. Yet, their product does, and the result is again the inner class of $\boldsymbol{\sigma}_{\text{H-M}}$.

Each of the classes C_a and C_b has two elements. Each 4-tuple in the Nielsen class has two entries in C_a; the remaining two in C_b. All allocations are achieved as 4-tuples by applying braids to any one of them. Example: $\boldsymbol{\sigma}_{\text{H-M}}$ has the entries, in order, in $\mathrm{C}_a, \mathrm{C}_b, \mathrm{C}_b, \mathrm{C}_a$. Then, $(\boldsymbol{\sigma}_{\text{H-M}})q_1$ reallocates these as $\mathrm{C}_b, \mathrm{C}_a, \mathrm{C}_b, \mathrm{C}_a$. By conjugation in G you can choose the first appearance of C_a to be $(1\,4)(2\,3)$, and the 1st appearance of C_b to be $(2\,4)$.

With the conjugations above, we have the leeway to choose the 2nd appearance of both C_a and C_b in each 4-tuple, so that both these second choices occur simultaneously. Here is a typical case:

$$(\boldsymbol{\sigma}_{\text{H-M}})q_3^2 = ((1\,4)(2\,3), (2\,4), (1\,3), (1\,2)(4\,3)).$$

There are therefore 12 absolute or inner Nielsen classes with one braid orbit among them. (The one braid orbit conclusion is common, though rarely so easy to prove.) The 1st sentence of Thm. 2.9 says just one element of $\mathcal{Q}''$ fixes any given element of the Nielsen class. We need a notation for forming a kind of fiber product from two r-tuples $(\sigma_1, \dots, \sigma_r)$ and $(\sigma'_1, \dots, \sigma'_r)$:

$$\boldsymbol{\sigma} \cdot \boldsymbol{\sigma}' \stackrel{\text{def}}{=} ((\sigma_1, \sigma'_1), \dots, (\sigma_r, \sigma'_r)). \tag{2.11}$$

This arises from the following principle.

Principle 2.11. *Suppose $f : X \to \mathbb{P}^1_z$ and $g : X' \to \mathbb{P}^1_z$ have the same branch points $\boldsymbol{z}$, and respective branch cycles $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ with respective to given classical generators. Then,* (2.11) *gives the branch cycles for the monodromy of their fiber product.*

The detail in Prop. 2.12 is to show those unaccustomed to branch cycles how to handle them. It produces all the Schinzel pairs in $\mathcal{GC}_{\text{abs}_1,\text{abs}_2}$ of Prop. 1.6 where the representations T_1 and T_2 are respectively on cosets of $\langle(1\,4)(2\,3)\rangle$ and $\langle(2\,4)\rangle$. According to Cor. 2.4 we expect Möbius pairs to be among fiber products whose branch cycles have the form $\boldsymbol{\sigma}\cdot(\boldsymbol{\sigma})q$, $q \in H_r$.

Proposition 2.12. *Let $f_{\text{H-M}}$ be the cover above. Then, $X_{\text{H-M}}$ is isomorphic to $\mathbb{P}^1$ (over $\mathbb{C}$); a rational function in one variable represents $f_{\text{H-M}}$.*

The fiber product of $X_{\text{H-M}}$ and $Y_{(\text{H-M})\mathbf{sh}^2}$ over $\mathbb{P}^1_z$ is a Schinzel Möbius pair. Any path along which you drag $\boldsymbol{z}$ on U_r gives a similar Schinzel Möbius pair. For each fixed $\boldsymbol{z}$ there are 12 such pairs with those branch points.

Proof. Apply Riemann-Hurwitz ((B.3)) to $f_{\text{H-M}}$ to conclude the genus of $X_{\text{H-M}}$ is $\mathbf{g}_{f_{\text{H-M}}} = 0$ from $2(4 + \mathbf{g}_{f_{\text{H-M}}} - 1) = 2(2+1) = 6$. Therefore, $X_{\text{H-M}}$ is isomorphic to the Riemann sphere. Even if $X_{\text{H-M}}$ has definition field $\mathbb{Q}$, it may not be isomorphic to $\mathbb{P}^1$ over $\mathbb{Q}$ (Rem. 2.14).

From Princ. 2.11, $\boldsymbol{\sigma}_{\text{H-M}}\cdot(\boldsymbol{\sigma}_{\text{H-M}})\mathbf{sh}^2$ gives branch cycles for the fiber product $X_{\text{H-M}} \times_{\mathbb{P}^1_z} Y_{(\text{H-M})\mathbf{sh}^2} = W_f$ as a cover $\mathbb{P}^1_z$. That is, the 1st branch cycle would be $((1_1\,4_1)(2_1\,3_1),(2_2\,4_2))$, the 2nd $((2_1\,4_1),(1_2\,4_2)(2_2\,3_2))$, etc. where the 1st entry is acting on the integers – designated by the subscript 1 – of the representation for $G_{f_{\text{H-M}}}$; the second entry on the integers – designated by the subscript 2 – of the representation of $G_{g_{(\text{H-M})\mathbf{sh}^2}}$.

The group generated by the four 2-tuples is the monodromy, $G_{f_{\text{H-M}},g_{(\text{H-M})\mathbf{sh}^2}}$ of the fiber product. Components correspond to orbits on $\{1_2,2_2,3_2,4_2\}$ of the subgroup of $G_{f_{\text{H-M}},g_{(\text{H-M})\mathbf{sh}^2}}$ that fixes 1_1. That subgroup is

$$\langle((2_1,4_1),(1_2,4_2)(2_2\,3_2))\rangle.$$

The 2 orbits on $\{1_2,2_2,3_2,4_2\}$ gives two components of degree 8 over $\mathbb{P}^1_z$.

The monodromy group of the fiber product remains constant as we drag along a movement of $\boldsymbol{z}$. Therefore, the number of fiber product components (and degrees) will remain constant. For $\boldsymbol{z}$ fixed, we now compute the complete orbit of such dragged pairs that have $\boldsymbol{z}$ as branch cycles starting from W_f. This comes by applying H_4 to the branch cycles for W_f:

$$\{(\boldsymbol{\sigma}_{\text{H-M}})q\cdot((\boldsymbol{\sigma}_{\text{H-M}})\mathbf{sh}^2)q \mid q \in H_4\}.$$

There are at least 12 such pairs from transitivity of H_4 on the Nielsen class. Suppose $\boldsymbol{\sigma},\boldsymbol{\sigma}' \in \mathrm{Ni}(D_4,\mathbf{C}_{\mathrm{C}_{a^2b^2}})^{\mathrm{abs}}$ give a Schinzel Mobius pair. We show that there can be no more than 12 such pairs by dividing into cases according to how the allocation of C_a s and C_b s match between them.

(2.12a) The C_a s and C_b s are in the same positions in $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$.
(2.12b) The C_a s and C_b s are in totally complementary positions.
(2.12c) Neither of the above.

Eliminate (2.12a) because the resulting fiber product – while reducible – would not be newly reducible: Both $f_{\boldsymbol{\sigma}}$ and $g_{\boldsymbol{\sigma}'}$ would factor through degree 2 covers branched at the same two points (corresponding to the C_a s). Eliminate (2.12c) because this would give both

$$((2_1,4_1),\sigma_a)) \text{ and } ((2_1,4_1),\sigma_b))$$

in the stabilizer of 1_1 with $\sigma_a \in \mathrm{C}_a$ and $\sigma_b \in \mathrm{C}_b$ (acting on $\{1_2,2_2,3_2,4_2\}$). But the action of σ_a and σ_b is transitive. So the fiber product would not be reducible. Finally, for a given $\boldsymbol{\sigma}$ in (2.12b), we must eliminate the other possibility for $\boldsymbol{\sigma}'$ not accounted for above. That reverts immediately, to showing that $\boldsymbol{\sigma}_{\text{H-M}}\cdot\boldsymbol{\sigma}'$ doesn't work where $\boldsymbol{\sigma}'$ is the complementary 4-tuple to $(\boldsymbol{\sigma}_{\text{H-M}})\mathbf{sh}^2$ replacing its 3rd entry by $(1\,2)(3\,4)$. This would give both

$$((2_1,4_1),\sigma_a)) \text{ and } ((2_1,4_1),\sigma_a'))$$

in the stabilizer of 1_1 with $\sigma_a=(1_2\,4_2)(2_2\,3_2)$ and $\sigma_a'=(1_2\,2_2)(3_2\,4_2)$. Since this is transitive on $\{1_2,2_2,3_2,4_2\}$, the fiber product is irreducible. □

Example 2.13 (A_5 and 3-cycles)**.** In place of $\boldsymbol{\sigma}_{\text{H-M}}$ in §2.3.2, consider the H-M rep. $\boldsymbol{\sigma}_{\text{H-M},A_5}=(\sigma_1,\sigma_1^{-1},\sigma_2,\sigma_2^{-1})$ with $\sigma_1=(1\,2\,3)$, and $\sigma_2=(1\,4\,5)$ in $G=A_5$. Then, we can take $\sigma'=(2\,4)(3\,5)$ and $\sigma''=(2\,3)(4\,5)$ in (2.8) to get one Nielsen class that is $\mathcal{Q}''$ invariant. But from this one example, there comes a large collection of several types of examples as described in [Fr12b, §2.3.2] that are archetypes for generalizing modular curves. For example, identify A_5 with $\mathrm{PSL}_2(\mathbb{Z}/5)$. Then, $\mathcal{Q}''$ acts trivially on the Nielsen class with 4-tuples of order 3 elements in $\mathrm{PSL}_2(\mathbb{Z}/5^{k+1})$, $k\ge 0$.

Remark 2.14. Assume $(f_{\text{H-M}},X_{\text{H-M}})$ (Prop. 2.12) has definition field $K\le\mathbb{C}$. We need only one odd degree divisor over K on $X_{\text{H-M}}$ to conclude (from Riemann-Roch) it is isomorphic over K to $\mathbb{P}^1$. For example, it would suffice if a point in $\boldsymbol{z}$ attached to the conjugacy class C_b is in K. Sometimes that happens, and sometimes it doesn't.

2.3.3. *$j=0$ or 1 Mobius fixed points.* Use the notation for reduced Hurwitz spaces in the case $r=4$ from §2.1.2. When $j=0$, $\boldsymbol{z}=(0,e^{2\pi i/3},e^{4\pi i/3},1)$ is above it. This is the case (2.14b) with one non-trivial orbit, where the braid is already covered by (2.5) as $\mathbf{sh}q_3^{-1}$.

When $j=1$, $\boldsymbol{z}=(e^{2\pi i/4},e^{4\pi i/4},e^{6\pi i/4},1)$ is above it. So, we are in the case (2.14a) with one orbit where the braid is $\mathbf{sh}$.

The formula for the genus of the (compactification) of the reduced Hurwitz space $\mathcal{H}(D_4,\mathbf{C}_{a^2b^2})^{\mathrm{abs,rd}}$ requires knowing:

(2.13a) which reduced Hurwitz classes are fixed by the braids $\mathbf{sh}q_3^{-1}$ and $\mathbf{sh}$ attached to the elements $\alpha\in\mathrm{PGL}_2(\mathbb{C})$ as above; and

(2.13b) the action of q_2 on reduced classes to produce the cusp orbits.

Proposition 2.15. *There are six reduced (absolute or inner) Nielsen classes corresponding to the j-line cover $\bar{\mathcal{H}}(D_4,\mathbf{C}_{a^2b^2})^{\mathrm{abs,rd}}\to\mathbb{P}^1_j$. There are two cusps (resp. 1) ramified of order 1 (resp. 4) over ∞. Each point over $j=0$ (resp. $j=1$) is ramified of order 3 (resp. 2).*

Proof. Thm. 2.9 says that exactly one nontrivial element of $\mathcal{Q}''$ fixes each element of $\bar{\mathcal{H}}(D_4,\mathbf{C}_{a^2b^2})^{\mathrm{abs,rd}}\to\mathbb{P}^1_j$. Reduced Nielsen classes are given by $\mathrm{Ni}(G,\mathbf{C})^{*,\mathrm{rd}}/\mathcal{Q}''$ whatever the equivalence $*$.

The genus of the corresponding cover to a braid orbit O. In our case there is one orbit, and the action of $\mathcal{Q}''$ equivalences the elements in pairs, giving a total of six reduced Nielsen classes.

The genus of $\bar{\mathcal{H}}(D_4,\mathbf{C}_{a^2b^2})^{\mathrm{abs,rd}}$ comes from (B.4). We must compute the respective indices of $\gamma'_0,\gamma'_1,\gamma'_\infty$ coming from the action on O of the three braids $q_1q_2=\mathbf{sh}q_3^{-1},\mathbf{sh},q_2$. These braids correspond to the local monodromy over the three ramified points $j=0,1,\infty$.

Below we only need compute the ramification over $j=\infty$. To do so, compute the length of q_2 orbits on the 12 Nielsen classes from Prop. 2.12. There are 4 Nielsen classes in the q_2 orbit of $\boldsymbol{\sigma}=((1\,4)(2\,3),((1\,4)(2\,3),(2\,4),(2\,4))$. Similarly, the q_2 orbit of $\boldsymbol{\sigma}'=((2\,4),(1\,4)(2\,3),(2\,4),(1\,2)(3\,4))$ has 4 elements. But $(\boldsymbol{\sigma})q_1q_3^{-1}=\boldsymbol{\sigma}'$, so those two orbits are reduced equivalent.

Then there are four q_2 orbits of length 1, 2 each represented by allocations with C_a (resp. C_b) in the 2nd and 3rd positions. But $\mathbf{sh}^2$ takes the 1st two to the 2nd 2, showing the former are reduced equivalent to the latter. That gives γ'_∞ an index of 3.

The genus, $\mathbf{g}_{a^2b^2}$ of $\bar{\mathcal{H}}(D_4,\mathbf{C}_{a^2b^2})^{\mathrm{abs,rd}}$ appears in

$$2(6+\mathbf{g}_{a^2b^2}-1)=\mathrm{ind}(\gamma'_0)+\mathrm{ind}(\gamma'_1)+\mathrm{ind}(\gamma'_\infty)$$

where the last term is 3, and $\text{ind}(\gamma'_0) \le 4$ and $\text{ind}(\gamma'_1) \le 3$. Since $\mathbf{g}_{a^2b^2} \ge 0$, the only possibility is that it is 0, $\text{ind}(\gamma'_0) = 4$ and $\text{ind}(\gamma'_1) = 3$. In particular, the only covers in the Nielsen classes that lie over $j = 0$ or $j = 1$ that are Möbius fixed, are those already indicated in Prop. 2.12. □

2.4. **Branch cycle formulas.** As previously our formulas in §2.4.1 assume $\zeta_v = e^{2\pi \mathrm{i}/v}$, and subdivide according to whether $\boldsymbol{z}$ contains 0, 1 or 2 fixed points of $z \mapsto \zeta_v z$. §2.4.2 gives the details of the one case that is substantially more difficult than what was done in [FrGu12].

2.4.1. *Listing of cases.* We list cases according to the (ϵ_1, ϵ_2) values in (1.15):

(2.14a) $\boldsymbol{z}$ contains neither 0 nor ∞ ($(\epsilon_1, \epsilon_2) = (0, 0)$);
(2.14b) $\boldsymbol{z}$ contains ∞, but not 0 ($(\epsilon_1, \epsilon_2) = (0, 1)$); or
(2.14c) $\boldsymbol{z}$ contains both 0 and ∞ ($(\epsilon_1, \epsilon_2) = (1, 1)$).

The work that went into (2.5) works as well when α has many orbits on $\boldsymbol{z}$ with the exception of when the origin and ∞ are both fixed points of α. So we merely list the formulas for the first two cases. Then, we are explicit about the 3rd case, where $z_{r-1} = 0$, and $z_r = \infty$. The complication is that we can't use the origin as a base point as was done in [FrGu12]. Our graphic (Fig. 1) for the 3rd case – with its lollypop loops – simplifies to give graphics for the first two cases merely by taking $\rho = 0$ instead of $\rho\mathbf{i}$ as below.

To get the BC formulas in case (2.14b), use the $\boldsymbol{z}$ partition in (1.14):

(2.15) $r = tv+1$, there are t orbits of length v (and different radia generally), and $\alpha = \zeta_v$ fixes the branch point set and also ∞.

Ordering the finite branch points by their clockwise angles we have

$$\begin{array}{rl} \text{1st orbit :} & z_1, z_{t+1}, z_{2t+1}, \dots, z_{(v-1)t+1}; \\ \text{2nd orbit:} & z_2, z_{t+2}, z_{2t+2}, \dots, z_{(v-1)t+2}; \\ & \dots \\ t\text{-th orbit:} & z_t, z_{t+t}, z_{2t+t}, \dots, z_{(v-1)t+t} = z_{tv}. \end{array} \tag{2.16}$$

We don't permit loops – here emanating from the origin – to go through a branch point. So, if some branch points lie on the same radius from the origin, we need a slight adjustment: whenever we come upon such a branch point, jog a small half circle *to the left* around that branch point. [FrGu12, Ex. 3.1] is a 4-branch point example of exactly this.

Apply (almost directly) the argument of [FrGu12, Lem. 2.1] (as well as the corresponding figure), using analogous symmetrical classical generators with zero as the base point. Then, if $(g_1, \dots, g_{tv}, g_\infty)$ are branch cycles for f (relative to the set of classical generators above), then

$$(g_{t+1}, g_{t+2}, \dots, g_{tv}, g_1, g_2, \dots, g_t, (g_1 g_2 \cdots g_t))^{-1} g_\infty (g_1 g_2 \cdots g_t)), \tag{2.17}$$

are branch cycles for α_f (relative to the same set of classical generators).

The form of branch cycle at ∞ can be explained by the figure, but also follows from the product one condition and the obvious form of the leading part of the branch cycles (which, according to the rotation, has to be $(g_{t+1}, g_{t+2}, \dots, g_{r-1}, g_1, g_2, \dots, g_t, \dots)$.

Case (2.14a) is even easier, since there is no fixed point, we have only the rotation. We get the result – essentially – by putting $g_\infty = 1$ in (2.17).

Remark 2.16 (Braids giving (2.14a) and (2.14b)).

2.4.2. *Details of* (2.14c). In reminding about the conditions for classical generators, we usually say in (C.1a) that the paths $\delta_1, \dots, \delta_r$ from the base point z_0 to a neighborhood of their respective z_i s satisfy the following.

(2.18) They are pairwise nonintersecting, except at z_0.

It simplifies the look of our paths, and so the argument we now make, if we relax the phrase "except at z_0" to this condition:

(2.19) except along a segment of δ_i s starting from z_0.

What matters is that it is possible to isotopically deform a set of classical generators to those satisfying (2.18) (in addition to the other conditions). Given (2.19), we can always deform those paths (slightly) to separate the segments in an isotopy so that the original (2.19) holds. As in our example, however, doing so complicates the description.

Example: Consider, as in the left half of Fig. 1, $\delta_1, \delta_2, \delta_3$ all starting along the x-axis from 0, going to the right, respectively 1, 2 and 3 units and then in each case veering to the north. Here, then, each ends, respectively, on a small disc about z_1, z_2, z_3. By contrast, we indicate the separating in the right half of Fig. 1: $\delta_1, \delta_2, \delta_3$, respectively come out of 0 at the counterclockwise angles θ, $\frac{2}{3}\theta$, $\frac{1}{3}\theta$, where we can take θ suitably small. Still, this represents an awkward picture when there are several clusters of this phenomenon.

In our Fig. 2 example, you see we adapt this going on a circle, δ_0^*, around the origin of radius ρ instead of along the x-axis. On that circle is our base point at ρbf i $= b$, with $\rho > 0$ and also its result after rotation, $\rho\zeta_v\mathbf{i} = b'$. We have selected ρ so that b is closer to the origin than is any of $z_1, \dots, z_{tv}$.

Consult now with the description of the δ_i s for Fig. 2, in §D, to see their (almost) canonical description by the labeling of (2.16), except here $r = tv+2$, and $z_{tv+1} = \infty$ and $z_{tv+2} = 0$. Then, say, δ_i – after going clockwise around the circle around the origin from b to the angle $2\pi/v$ (in radians) of the point z_i – leaves that circle along the radius from the origin to z_i, except if it must zig-zag in a little circle around some other branch point along the radius from 0 to z_i.

Now, when we rotate the figure through a (clockwise) angle of $2\pi/v$, we have rotated the point ρi along δ_0^*, too, to b'. We can assert, as previously, that branch cycles at the end relative to the rotated classical generators (and basepoint b') are exactly the same as the original.

We want is to compare the new classical generators to the original classical generators of Fig. 2. To do so, modify the rotated $\delta_i\, s$ (notate these as δ_i's) by adding an arc (clockwise through $2\pi/v$) along their beginnings, and call these δ_i''s. The branch cycles of the cover relative to these won't be any different. Then, we can compare the new classical generators $\gamma_1'', \dots, \gamma_r''$ with the δ_i''s replacing the δ_i's with the old.

— Mike's statement: I await getting some figures to refer to. This will respond to Ivica's statement: In Case (2.14c), I think (although I don't see a clever proof) that the role of fixed points 0 and ∞ are symmetrical. Suppose we arrange finite non-zero branch points as above and starts with branch cycles, $(g_0, g_1, \dots, g_{tv}, g_\infty)$ for f. Then, the branch cycles for α_f are:

$$\begin{aligned}((g_1g_2\cdots g_t)^{-1}g_0(g_1g_2\cdots g_t), g_{t+1}, g_{t+2}, \dots, g_{tv},\\ g_1, g_2, \dots, g_t,\ (g_1g_2\cdot\cdots g_t)^{-1}g_\infty(g_1g_2\cdots g_t)).\end{aligned} \tag{2.20}$$

Theorem 2.17.

Remark 2.18 (Comments on $r = 4$). We have already noted in the proof of Cor. 1.11, for $r = 4$, what happens if the two fixed points are 0 and ∞. Then, the other two branch points under the assumption of this section are $\pm z'$ for some $z' \neq 0$ or ∞. §2.3.3 notes that this covers the case when the Möbius transformation corresponds to a cover contributing to ramification over $j = 1$. No element of $\mathrm{PGL}_2(\mathbb{C})$ representing $\mathcal{Q}''$ is included in this case, since these elements switch pairs of branch points in pairs.

Remark 2.19 (Braids giving (2.14c)).

2.4.3. *Comments on the use of the BC Formulas.*

3. The Main Conjecture

[FrGu12, Prop. 1.2] says that by change of variable, we may assume in Prob. 3.2, that α is multiplication by a vth root of 1, ζ_v. That is, $g = \zeta_v f$ with $\zeta_v = e^{2\pi \mathrm{i}/v}$ and multiplication by ζ_v permutes the branch points of f. We denote the permutation action of that multiplication by m_v.

3.1. **Three finite branch points.** In the case $r-1 = 3$, (2.5) suffices, unless one of the branch points is ∞.

Further, if you composite the Galois closure covers of f and g with the cyclic cover $\mathbb{P}^1_z \to \mathbb{P}^1_u$ by $z \mapsto z^v$, then the branch cycle σ_∞^* for this composite plays a special role. 1st $(\sigma_\infty^*)^v = \sigma_\infty = (1\,2\,\ldots\,n)$ can be taken as the branch cycle at ∞ in either of the monodromy representations T_f or T_g. Also, σ_∞^* conjugates the coset representation T_f to the coset representation T_g. If we denote this conjugation by c_{AZ}, there is no element of S_n that represents it.

It is automatic from Cor. 1.4 that c_{AZ} permutes the collection of conjugacy classes $\mathbf{C} = \{\mathrm{C}_1, \ldots, \mathrm{C}_{r-1}\}$ attached to the common finite branch points of the two covers (and centralizes σ_∞). [FrGu12, Prop. 2.4], however, shows that it cannot fix all the conjugacy classes in G. We keep the notation of previous papers so that r is the cardinality of the branch points, and $r-1$ indicates the (finite) branch points in $\mathbb{C}$. Then, the rth branch point, and its associated conjugacy class C_r is at ∞.

Conjecture 3.1 (Main)**.** The only possible polynomial Schinzel pairs have degree 4, essentially given by $T_4(x) + T_4(y)$ with $\zeta_v = -1$ and T_4 the 4th classical Chebychev polynomial.

Further, it holds if and only if the subgroup $\langle \sigma_\infty \rangle \le G$ is normal. [FrGu12, §1.4] also shows that if there is a counter-example to the Main Conjecture, then f just have at least $r-1 \ge 3$ finite branch points.

[FrGu12, (2.4)] characterized the effect of (f, g) being a Möbius paper, (1.1a), completely in terms of branch cycles from formula (2.5), when there is only one orbit of finite branch points under the action of ζ_v. §2.19 finishes that characterization by allowing for several orbits.

Schinzel pairs played a significant role in the genus 0 problem whose solution for polynomials goes like this [Mu95]. All primitive groups come from a procedure based either on simple groups or they are affine groups ([Fr12a, §7.4 and §A.3] has an exposition). That means there are natural series of primitive groups, and of course a limited set of exceptional groups.

For all degrees there are indecomposable polynomials with *standard* monodromy:

(3.1) cyclic or dihedral of prime degree, alternating or symmetric in their standard representations.

Excluding, however, the monodromy in (3.1), the actual occurring primitive monodromy of polynomials includes only finitely many other groups and [Mu95] lists them all. The affine groups that occur in (1.3) represent a good proportion of the "exceptional " primitive polynomial monodromy.

[Fr12a] expands on many papers affected by the monodromy method: especially to circumvent using only covers with primitive monodromy. We here take the next step to consider the problem left by R. Avanzi and

U. Zannier [AZ03], and the 2nd author [Gu10]. Consider those f for which there is a $g = \alpha \circ f$, with $\alpha \in \mathrm{PGL}_2(\mathbb{C})$, for which (f, g) is a Schinzel pair.

We do not always assume X and Y are copies of the projective line, or even if they are that the rational functions f and g are polynomials. Here is our main problem.

Problem 3.2. Condition (1.1a) is necessary for a Schinzel pair. Given that condition, we consider how then to detect if there is (f, g), a Schinzel pair of polynomials assuming (1.1c).

Appendix A. Group notation

A Klein (or Klein 4-) group is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$. The dihedral group of order $2d$, denoted D_d, is characterized by being generated by two involutions (order 2 elements) α_i, $i = 1, 2$, with $\mathrm{ord}\alpha_1\alpha_2 = d$. Another characterization is that it has generators $\langle \alpha_1, \beta \rangle$ with $\alpha_1 \beta \alpha_1^{-1} = \beta^{-1}$. In the first formulation, $\beta = \alpha_1\alpha_2$.

Any subset in a group G is called p' if the order of the elements in it is prime to p. We can replace p by any integer for this definition. For example: We can speak of a p' conjugacy class.

Appendix B. Review of Nielsen classes

B.1. **Braids.** The Hurwitz monodromy group, H_r, a quotient of the Artin Braid group B_r acts on Nielsen classes by mapping $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_r)$ to an r-tuple of words in the entries of $\boldsymbol{\sigma}$. These two elements generate H_r:

(2.1a) $q_1 : \boldsymbol{g} \mapsto (g_1 g_2 g_1^{-1}, g_1, g_3, \dots, g_r)$ the *1st* (coordinate) *twist*, and

(2.1b) $\mathbf{sh} : \boldsymbol{g} \mapsto (g_2, g_3, \dots, g_r, g_1)$, the *left shift*.

Each preserves generation, product-one and the conjugacy class collection conditions of (1.2), Conjugating q_1 by $\mathbf{sh}$, gives q_2, the twist moved to the right. Repeating gives $q_3, \dots, q_{r-1}$. Three relations generate all such in H_r:

(2.2a) Sphere: $q_1 q_2 \cdots q_{r-1} q_{r-1} \cdots q_1$;

(2.2b) Commuting: $q_i q_j = q_j q_i$, for $|i - j| \ge 2$ (subscripts mod $r-1$); and

(2.2c) (Braid) Twisting: $q_i q_{i+1} q_i = q_{i+1} q_i q_{i+1}$.

The group H_r inherits (B.2b) and (B.2c) from B_r.

A special normal subgroup $\mathcal{Q}'' = \langle \mathbf{sh}^2, q_1 q_3^{-1} \rangle$ of H_4 – called the *moduli group* – is isomorphic to a Klein 4-group [BaFr02, §2.10]. In §B.2 we note the effect of quotienting by $\mathcal{Q}''$ represents $\mathrm{PGL}_2(\mathbb{C})$ equivalence on Hurwitz spaces by the Klein 4-group denoted $K_{\boldsymbol{z}}$ in Rem. 1.14.

The index, $\mathrm{ind}(\sigma)$, of a permutation $\sigma \in S_n$ is just n minus the number of disjoint cycles in the permutation. Example: an n-cycle in S_n has index

$n-1$, and an involution has index equal to the number of disjoint 2-cycles in it. The Riemann-Hurwitz formula says the *genus*, $\mathbf{g}_X$ of X satisfies

$$2(n+\mathbf{g}_X-1)=\sum_{i=1}^{r}\operatorname{ind}(\sigma_i). \tag{B.3}$$

B.2. **Inner and reduced Nielsen classes.** As in §B.1, identify the equivalence from reduced Nielsen classes. RETURN

(2.4) Put the RH for the j-line cover

B.3. **Braidable elements of $N_{S_n}(G,\mathbf{C})$.**

Appendix C. Classical Generators Used in Thm. 2.17

These are ordered closed paths $\delta_i\sigma_i^*\delta_i^{-1}=\bar\sigma_i$, $i=1,\dots,r$.

Here are their properties. There are discs, $i=1,\dots,r$: D_i with center z_i; all disjoint, each excludes z_0; b_i is on the boundary of D_i. Their clockwise orientation refers to the boundary of D_i. The path σ_i^* has initial and end point b_i; δ_i is a simple *simplicial* path with initial point z_0 and end point b_i. We also assume δ_i meets none of $\sigma_1^*,\dots,\sigma_{i-1}^*,\sigma_{i+1}^*,\dots,\sigma_r^*$, and it meets σ_i^* only at its endpoint.

There is a crucial condition on meeting the boundary of D_0. First: D_0, with center z_0, is disjoint from each $D_1,\dots,D_r$. Consider a_i, the first intersection of δ_i and boundary σ_0^* of D_0. Then, $\delta_1,\dots,\delta_r$ satisfy these conditions:

(3.1a) they are pairwise nonintersecting, except along a segment of the δ_i s starting from z_0; and

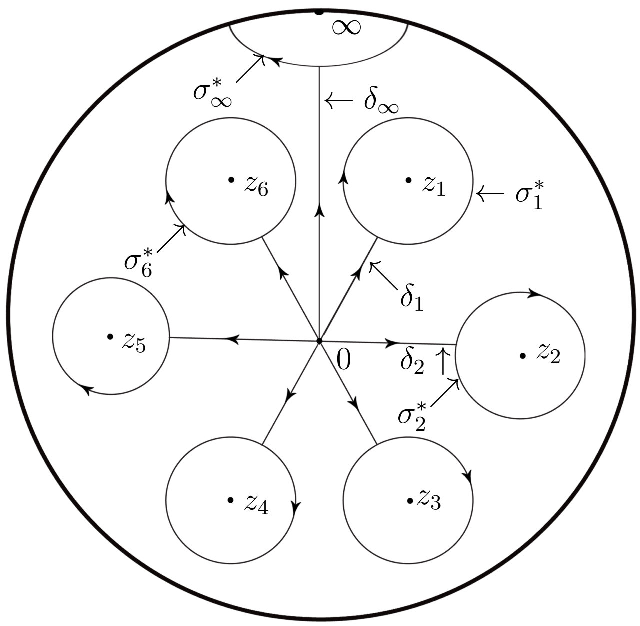
(3.1b) $a_1,\dots,a_r$ are in order clockwise around σ_0^*.

Appendix D. Regular polygon classical generators

The paths, $\delta_i\sigma_i^*\delta_i^{-1}$ (including the subscript $r=\infty$, going around ∞ in Fig. 1 satisfy all the conditions of *classical generators* based at $z_0=0$. Our notation is compatible with that of [Fr10, App. B.1], except we here use very regular paths, with punctures (except at ∞) arranged on a regular 6-gon.

FIGURE 1. Allowing initial overlapping segments of the δ_i s

FIGURE 2. $r = 7$, with 6 branch points on a regular polygon



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