COMBINATORIAL COMPUTATION OF MODULI DIMENSION OF NIELSEN CLASSES OF COVERS

Emphasis on the solvable cover case with historical comments from Zariski Vol. 3

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Abstract: Consider a rational function field $\mathcal{C}(x)$ in one variable. There have been quite a number of attempts to use Riemann's existence theorem to organize both the lattice of subfields and the algebraic extensions of it. This exposition describes a further attempt that includes exposition on ground (§2) covered sporadically by Zariski [Z]. A rough phrasing of the particular problem: For each nonnegative integer g describe explicitly all of the ways that the function field of the "generic curve" of genus g contains $\mathcal{C}(x)$ (§1).

Although a general program has been envisioned by John Thompson (c.f. §2.2), we narrow to the case where $g \geq 2$ and the containment of fields gives a solvable Galois closure. This alone illustrates that Zariski's most definitive conjecture touching on this is wrong (§2.3). Theorem 3.5 gives a presentation of the fundamental group $\pi_1(X)$ and 1st homology $H_1(X, \mathcal{Z})$ of a Riemann surface X appearing in a (not necessarily Galois) cover $X \to \mathcal{P}^1_x$ of the sphere in terms of branch cycles for the cover. In particular this offers an action of the Hurwitz monodromy group $H(r)$ on $H_1(X, \mathcal{Z})$ where r is the number of branch points of the cover. The remainder of §3 interprets the dimension of the image of the deformations of the cover in the moduli space of curves of genus $g = g(X)$ in terms of this group action.

§1. STATEMENT OF THE PROBLEM AND OUTLINE OF THE SECTIONS

§1.1. FAMILIES OF POLYNOMIALS WITH GIVEN MONODROMY GROUP

One way to give an (irreducible) algebraic curve is to give a polynomial (irreducible) in two variables $f(x, y) \in \mathcal{C}[x, y]$ where $\mathcal{C}$ denotes the complex numbers. Then the curve in question is

$$\{(x, y) \mid f(x, y) = 0\} \stackrel{\text{def}}{=} X.$$

This curve may, however, have singular points: points $(x_0, y_0) \in X$ for which $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ evaluated at (x_0, y_0) are both 0. Furthermore, we are missing the points at infinity obtained by taking the closure of X in the natural copy of projective 2-space $\mathcal{P}^2$ that contains the affine space $\mathcal{A}^2$ with variables x and y. (And these points, too, might be singular.)

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After this opening section we will assume that our algebraic curves X don't have these defects; they will be projective nonsingular curves, so we may not be able to regard them as given by a single polynomial in 2-space. But the essential ingredient of this presentation, represented by the x-coordinate will still be there.

That is, we have a *covering* map

$$\{(x,y) \mid f(x,y)=0\} \to \mathcal{P}^1_x \stackrel{\text{def}}{=} \mathcal{C} \cup \infty \quad \text{or} \quad X \to \mathcal{P}^1_x \tag{1.1}$$

given by projection of the point (x,y) onto its first coordinate. The *monodromy group* of this cover is defined to be the Galois group G of the Galois closure of the field extension $\mathcal{C}(X)/\mathcal{C}(x)$ where $\mathcal{C}(X)$ denotes the quotient field of the ring $\mathcal{C}[x,y]/(f(x,y))$. In the sequal we will denote this Galois closure by $\widehat{\mathcal{C}(X)}$ or by the geometric version $\hat{X}$, the smallest Galois cover of $\mathcal{P}^1_x$ that factors through $X \to \mathcal{P}^1_x$. Note that in this situation G automatically comes equipped with a transitive permutation representation $T: G \to S_n$. Denote the stabilizer in G of an integer (say, 1) by $G(T)$. Also, for later reference we point out that T is *primitive* (i.e., there are no proper groups between G and $G(T)$) if and only if there are no proper fields between $\mathcal{C}(X)$ and $\mathcal{C}(x)$ (equivalently, no proper covers fitting between $X \to \mathcal{P}^1_x$).

Actually, the problem of concern doesn't deal with one polynomial at a time, but rather with a parametrized family of them. We give the technical details for this in §1.2, but for a statement of the main problem it suffices to think of the coefficients of $f(x,y)$ lying in a field F, finitely generated over the rationals $\mathcal{Q}$. The problem comes when we simultaneously want to declare further properties of the (ramified) cover $X \to \mathcal{P}^1_x$ and for the field F. Here is the naive version of the constraints that we impose in terms of a priori given data, a group G and a nonnegative integer g:

(1.2) a) The monodromy group of the cover $X \to \mathcal{P}^1_x$ is equal to the group G; and

b) As we run over all specializations of the field F in the complex numbers, the field $\mathcal{C}(X)$ runs over "almost all" fields of functions of Riemann surfaces of genus g.

The phrase "almost all" means for all but a codimension 1 algebraic subset of the moduli space $\mathcal{M}_g$ of Riemann surfaces of genus g. The existence of such an (irreducible) algebraic variety and its properties ([M; Lecture II]—c.f. §2) is, of course, no triviality. Indeed, it is the abstractness of this object that causes all of our problems when we want to find out for which pairs (g,G) there exists such a polynomial $f(x,y)$ with coefficients in such a field F. The function fields of these polynomials depend only on the isomorphism class of the representing Riemann surface. Denote the points $\boldsymbol{m} \in \mathcal{M}_g$ whose representing Riemann surfaces have the same function field as such a polynomial f (as in (1.1)) by $\mathcal{M}_g(G)$. We may rephrase the conditions of (1.2) in the following form:

Question 1.1: *For which (g,G) is $\mathcal{M}_g(G)$ a Zariski open subset of $\mathcal{M}_g$?*

For fixed g denote the collection of groups for which the conclusion of the question is affirmative by $\mathcal{G}_g$ and denote the subset of solvable groups by $\mathcal{G}_g(\text{sol})$. The Main Theorem of [FrG] considers $\mathcal{G}_g(\text{sol})$ in the case that $g \geq 2$. Indeed, for a fixed g in order to show that $\mathcal{G}_g(\text{sol})$ is empty it suffices to show that the subset $\mathcal{G}_g(\text{prim}) \cap \mathcal{G}_g(\text{sol})$ consisting of primitive groups is empty. The Main Theorem of [FrG] gives the following.

Theorem 1.2: *For $g \geq 7$ $\mathcal{G}_g(sol)$ is empty; for $3 \leq g \leq 6$, $\mathcal{G}_g(prim) \cap \mathcal{G}_g(sol)$ consists of just S_3 and S_4; and for $g=2$, in addition to S_3 and S_4, $\mathcal{G}_g(prim) \cap \mathcal{G}_g(sol)$ is a subset of this list:*

$$G = D_{10}; \qquad (\mathcal{Z}/3)^2 \times^s D_8; \qquad (\mathcal{Z}/3)^2 \times^s GL(2,3); \quad \text{and} \tag{1.3}$$

$$(\mathcal{Z}/2)^2 \times (\mathcal{Z}/2)^2 \times^s ((S_3 \times S_3) \times^s (\mathcal{Z}/2)).$$

The most exciting *mathematical* considerations of [FrG] revolve around deciding which of the members of (1.3) are actually in $\mathcal{G}_g(\text{prim})$. This is a special case of the computation of the *moduli dimension* of a *Nielsen class* (§1.2). The *Hurwitz monodromy group*, a quotient of the Artin braid group plays the key role in reducing the problem to a computation in pure group theory. The computations, however, are difficult even if of general interest. Some of the list (1.3) has been eliminated by them, but there are two groups that are still in question at the time of this writing. Furthermore, the ideas can be applied to many problems, so it would be a shame if like minded calculations turned out to be unfeasible (§3).

In particular, Thompson's program (§2.2) conjectures that for fixed g, excluding A_n, $n = 5, 6, \ldots$, there are only finitely many simple groups that appear as composition factors of monodromy groups of covers by a Riemann surface of genus g. Therefore we illustrate further group theoretical computational difficulties on the problem of deciding for fixed g those n for which (g, A_n) gives an affirmative answer to Question 1.1.

The most interesting *historical* considerations of [FrG] are best summarized by noting that Zariski considered almost all aspects of the problem—unbeknownst to the authors of [FrG] at the appearance of the first draft of their paper—among a considerable subset of the papers in the 3rd volume of his collected works [Z]. In fact he knew everything in Theorem 1.2. except list (1.3). But, in the course of his formulation of a special case of the "moduli dimension problem" he conjectured results from which one would conclude that all of list (1.3) is in $\mathcal{G}_g(\text{prim}) \cap \mathcal{G}_g(\text{sol})$, contrary to our statement above. Also, he never explored the different ways that S_3 and S_4 belong in this list. More precisely, in the phraseology of §1.2: For which *Nielsen classes* $\mathbf{C}$ is $\text{Ni}(\mathbf{C})_T^{ab}$ of *full moduli dimension* where T is the standard representation of either S_3 or S_4. The minimal integer n for which (g, S_n) gives an affirmative answer to Question 1.1 is $n = [\frac{g+3}{2}]$ where [ ] denotes the "greatest integer" function [KL] (c.f. §2.1). The §1.2 formulations show that this is the obvious first case of the problem of computation of moduli dimension of a Nielsen class: the case when the group is S_n and the conjugacy classes $\mathbf{C} = (\mathrm{C}_1, \ldots, \mathrm{C}_r)$ in $G = S_n$ are each the conjugacy class of a 2-cycle. In §2 we explore the relation between [Z] and [FrG], with pointed remarks about [AM].

A paraphrase of Theorem 1.2 might start like this.

Statement 1.3: *The generic curve of genus $g > 6$ is not uniformized by radicals.*

Indeed, this was the first draft title of [FrG], which turned out to be essentially the English version of the Italian title of item [8] of [Z] (c.f. §2.1). I think that those who are comfortable with the classical treatment of algebraic geometry will have no difficulty with the limits of this result. But it is illuminating to point out that at this time it is not known for any g whether or not $\bigcup_{\text{solvable } G} \mathcal{M}_g(G)$ is dense in $\mathcal{M}_g$. In particular, it is (vaguely) possible for some $g > 6$ that each curve of genus g defined over $\bar{\mathcal{Q}}$, the algebraic closure of the rationals, has *some* map to $\mathcal{P}_x^1$ whose monodromy group is solvable.

Acknowledgements: It was John Ries who realized that a number of points of [FrG] are related to [Z], and who also, in looking back, had the first counterexamples to the conjecture of item [18] (see §2) of [Z]. A number of readers of a draft of this article have warned me that while it is historically conservative to be cavalier about the definition of "generic," the modern reader will not allow such liberties. Here, at least, I have tried to keep the reader's comfort in mind. It seems, however, inevitable that some intended readers might be unwilling to suspend concern that they haven't the background to visualize a hard core algebraic geometry object like $\mathcal{M}_g$ or $\mathcal{H}(\mathbf{C})_T$. For those willing to travel adventurously in the direction of a proferred arrow, I give a guide to their properties. Other than that I can only say that all pedantry is unintended.

§1.2. NIELSEN CLASSES AND THE HURWITZ MONODROMY GROUP

Suppose that we are given a finite set $\{x_1, \dots, x_r\}$ of distinct points of $\mathcal{P}^1_x$. For any element $\boldsymbol{\sigma} \in S_n^r$ denote the group generated by its coordinate entries by $G(\boldsymbol{\sigma})$. We recall the classical classification data for the connected (ramified) degree n covers of the x-sphere. Consider $\varphi : X \to \mathcal{P}^1_x$, ramified only over $\boldsymbol{x}$ up to the relation that regards $\varphi : X \to \mathcal{P}^1_x$ and $\varphi' : X' \to \mathcal{P}^1_x$ as equivalent if there exists a homeomorphism $\lambda : X \to X'$ such that $\varphi' \circ \lambda = \varphi$. These equivalence classes are in one-one correspondence with

$$\{\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_r) \in S_n^r \mid \sigma_1 \cdots \sigma_r = 1,\ G(\boldsymbol{\sigma}) \text{ is a transitive subgroup of } S_n\} \tag{1.4}$$

modulo the relation that regards $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ as equivalent if there exists $\gamma \in S_n$ with $\gamma\boldsymbol{\sigma}\gamma^{-1} = \boldsymbol{\sigma}'$. This correspondence goes under the heading of *Riemann's existence theorem*. The collection of ramified points $\boldsymbol{x}$ will be called the branch points of the cover $\varphi : X \to \mathcal{P}^1_x$. (In most practical situations we shall mean that there truly is ramification over *each* of the points x_i, $i = 1, \dots, r$.)

Our next step is to generalize Riemann's existence theorem to a combinatorial group situation that allows us to consider the covers above, not one at a time, but as topologized collections of families: the branch points $\boldsymbol{x}$ run over the set $(\mathcal{P}^1_x)^r \setminus \Delta^r$ with Δ_r the r-tuples with two or more coordinates equal. The key definition is of a *Nielsen class*.

Suppose that $T : G \to S_n$ is any faithful transitive permutation representation of a group G. Let $\mathbf{C} = (\mathrm{C}_1, \dots, \mathrm{C}_r)$ be an r-tuple of conjugacy classes from G. It is understood in our next definition that we have fixed the group G before introducing conjugacy classes from it.

Definition 1.4: The Nielsen class of $\mathbf{C}$ is

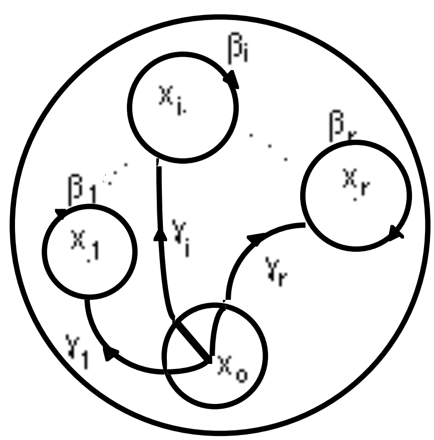
$$\mathrm{Ni}(\mathbf{C}) \stackrel{\text{def}}{=} \{\boldsymbol{\tau} \in G^r \mid G(\boldsymbol{\tau}) = G \text{ and there exists } \beta \in S_r \quad \text{such that } \tau_{\beta(i)} \in \mathrm{C}_i,\ i = 1, \dots, r\}.$$

We always assume that any given Nielsen class under consideration is nonempty—but, of course, this must be checked in each case. Also, for simplicity we assume that C_i is not the conjugacy class of the identity, $i = 1, \dots, r$.

Relative to *canonical* generators $\bar\sigma_1, \dots, \bar\sigma_r$ (see Figure 1) of the fundamental group $\pi_1(\mathcal{P}^1_x - \boldsymbol{x}, x_0)$, we say that a cover ramified only over $\boldsymbol{x}$ is in $\mathrm{Ni}(\mathbf{C})$ if the classical representation of the fundamental group sends the respective canonical generators to an r-tuple $\boldsymbol{\sigma} \in \mathrm{Ni}(\mathbf{C})$. What we would like to have is a total family of covers of $\mathcal{P}^1_x$ representing these equivalence classes. There are subtleties to forming this—even talking about it. Our next simplifying assumption on G holds for all examples of this paper.

Figure 1: Sample Bouquet

$\gamma_i \beta_i \gamma_i^{-1}$ represents one of the oriented generators $\bar\sigma_i$ on r-punctured sphere.



Assume that the permutation representation $T : G \to S_n$ has the property that

$$\text{the centralizer} \quad \mathrm{Cen}_{S_n}(G) \quad \text{of } G \text{ in } S_n \text{ is trivial.} \tag{1.5}$$

For example, any primitive subgroup of S_n satisfies (1.5). As all of §4 of [Fr,1] makes clear, the practical use of families without condition (1.5) is difficult, but not impossible.

Each permutation representation $T : G \to S_n$ provides us with an important equivalence relation on $\mathrm{Ni}(\mathbf{C})$. Consider the normalizer $N_{S_n}(G)$ (or $N_T(G)$) of G in S_n. The subgroup of the normalizer that consists of elements that permute the conjugacy classes C_i, $i = 1, \ldots, r$ (under conjugation) is denoted $N_T(\mathbf{C})$. The quotient of $\mathrm{Ni}(\mathbf{C})$ by this group called the *absolute Nielsen classes* (relative to T) and it is denoted by $\mathrm{Ni}(\mathbf{C})^{ab}_T$.

We now define the Hurwitz monodromy group $H(r)$—a quotient of the Artin braid group (c.f. §2.4 for Zariski's research into this group). The generators $Q_1, \ldots, Q_{r-1}$ of $H(r)$ satisfy the following relations:

(1.6) a) $Q_iQ_{i+1}Q_i = Q_{i+1}Q_iQ_{i+1}$, $i = 1, \ldots, r-2$;

b) $Q_iQ_j = Q_jQ_i$ for $1 \le i < j-1 \le r-1$; and

c) $Q_1Q_2 \cdots Q_{r-1}Q_{r-1} \cdots Q_1 = 1$.

Relations (1.6) a) and b) alone give the braid group. Their "strings" are not directly a part of our setup. It is relation (1.6) c) that truly indicates our involvement with projective algebraic geometry; the Artin braid group is the fundamental group of $\mathcal{A}^r - D_r$ while the Hurwitz monodromy group is the fundamental group of $\mathcal{P}^r - D_r$. Here D_r is the classical discriminant locus in the respective spaces. The natural embedding of $\mathcal{A}^r$ in $\mathcal{P}^r$ gives the natural surjective homomorphism from the braid group to the monodromy group.

From the relations we compute that $H(r)$ acts on the absolute Nielsen classes by extension of the following formula:

$$(\tau_1, \ldots, \tau_r)Q_i = (\tau_1, \ldots, \tau_{i-1}, \tau_i\tau_{i+1}\tau_i^{-1}, \tau_i, \tau_{i+2}, \ldots, \tau_r). \tag{1.7}$$

In the notation above we say that $\varphi_T : X_T \to \mathcal{P}^1_x$ is in the absolute Nielsen class $\mathrm{Ni}(\mathbf{C})^{ab}_T$. In many contexts it would be impossible to drop the subscript T without confusion. But such is unlikely to occur in this paper. Therefore we drop the subscript T quite often.

Each absolute Nielsen class $\mathrm{Ni}(\mathbf{C})^{ab}_T$ defines a *moduli space* $\mathcal{H}(\mathbf{C})_T$ of covers $\varphi_T : X_T \to \mathcal{P}^1_x$ of degree equal to $\deg(T)$ in that Nielsen class [Fr,1; §4]. Each point of $\mathcal{H}(\mathbf{C})_T$ corresponds to exactly one equivalence class of covers of $\mathrm{Ni}(\mathbf{C})^{ab}_T$. To explain the meaning of this we need a little notation to explain *families* of covers in $\mathrm{Ni}(\mathbf{C})^{ab}_T$.

Indeed, such a family $\mathcal{F}$ consists of a *parameter space* $\mathcal{H}$, a *total space* $\mathcal{T}$ and a map $\Phi : \mathcal{T} \to \mathcal{H} \times \mathcal{P}^1_x$, all complex manifolds, with this property: For each $\boldsymbol{m} \in \mathcal{H}$ the restriction of Φ to the fiber

$$\mathcal{T}_{\boldsymbol{m}} \stackrel{\text{def}}{=} \{\, t \in \mathcal{T} \mid \Phi(t) \in \boldsymbol{m} \times \mathcal{P}^1_x \}$$

presents it as a cover in the absolute Nielsen class $\mathrm{Ni}(\mathbf{C})^{ab}_T$. For brevity denote this cover by $\mathcal{F}_{\boldsymbol{m}}$, and its equivalence class (representing a point of $\mathcal{M}_g$) by $[\mathcal{F}_{\boldsymbol{m}}]$. The main moduli space property is that the natural map

$$\Psi(\mathcal{H}, \mathcal{M}_g) : \mathcal{H} \to \mathcal{M}_g \quad \text{by} \quad \boldsymbol{m} \to [\mathcal{F}_{\boldsymbol{m}}] \tag{1.8}$$

is complex analytic (actually algebraic as both spaces are quasi-projective varieties).

A final point: If (1.5) holds, then there is a unique family (up to the obvious equivalence) $\mathcal{F}(\mathbf{C})_T$,

$$\Phi(\mathbf{C})_T : \mathcal{T}(\mathbf{C})_T \to \mathcal{H}(\mathbf{C})_T \times \mathcal{P}^1_x\ , \tag{1.9}$$

such that for each $\boldsymbol{m} \in \mathcal{H}(\mathbf{C})_T$ the restriction of $\Phi(\mathbf{C})_T$ to the fiber $\mathcal{T}(\mathbf{C})_{T,\boldsymbol{m}}$ presents it as a cover in the absolute Nielsen class $\mathrm{Ni}(\mathbf{C})^{ab}_T$ (e.g., [Fr,1; §4]). The parameter variety $\mathcal{H}(\mathbf{C})_T$ ties us back to the elementary phrasing of §1.1. We are forced to consider the function field $\mathcal{C}(\mathcal{H}(\mathbf{C})_T)$ of $\mathcal{H}(\mathbf{C})_T$ for suitable $\mathbf{C}$ as a candidate for the field F in (1.2). And if for no suitable Nielsen class does this choice of F work, then Question 1.1 has a negative answer for the given (g, G).

Definition 1.5: The *moduli dimension* of the Nielsen class $\mathrm{Ni}(\mathbf{C})^{ab}_T$ is the dimension of the image of the morphism $\Psi(\mathcal{H}(\mathbf{C})_T, \mathcal{M}_g)$. We say that $\mathrm{Ni}(\mathbf{C})^{ab}_T$ is of *full* moduli dimension if this map is dominant (i.e., generically surjective). In other words, if the range has dimension $3g-3$ (resp., g) if $g \ge 2$ (resp., g=0 or 1).

The case when $G = S_n$ and $\mathbf{C}$ consists of just the conjugacy class of 2-cycles should be regarded as the classical case of this problem. Covers in this Nielsen class are said to be *simple branched covers*. The result in this case is that the n for which $\mathrm{Ni}(\mathbf{C})^{ab}_T$ has full moduli dimension are exactly the $n \ge [\frac{g+3}{2}]$. This is hardly trivial (c.f. §1.1). Indeed, while Zariski clearly "knew" this during the writing of his papers, [AM] regards it as still open until [KL] (§2.1). Nevertheless, there is a principle— known to the ancients—that applies to this situation.

Principle 1.6: *If any Riemann surface X of genus g has a covering $X \to \mathcal{P}^1_x$ of degree n, then some Riemann surface X' of genus g appears as a simple branched cover of $\mathcal{P}^1_x$ of degree n.*

Since nothing like this holds for Nielsen classes in general (c.f. Statement 2.16). we feel that some additions to the classical geometry ideas of, say [KL] and [ArC], would be required to decide the moduli dimension of $\mathrm{Ni}(\mathbf{C})^{ab}_T$ in general. The goal of [FrG] is to return this problem to a computation—if possible, practical—in group theory involving just the Nielsen classes and the action of the Hurwitz monodromy group.

In §3.2 we give the computational approach to the action of a subgroup $H_{\boldsymbol{\sigma}}$ of $H(r)$ on the fundamental group $\pi_1(X)$ (or on $H_1(X, \mathcal{Z})$) of a cover $X \to \mathcal{P}^1_x$ in a given absolute Nielsen class $\mathrm{Ni}(\mathbf{C})^{ab}_T$. The subgroup $H_{\boldsymbol{\sigma}}$ is the stabilizer in $H(r)$ of an element $\boldsymbol{\sigma} \in \mathrm{Ni}(\mathbf{C})^{ab}_T$. Since this note is intended to be expositional we comment on just two points: for $g = 1$ or 2, the action of $H_{\boldsymbol{\sigma}}$ on $H_1(X, \mathcal{Z})$ is through a finite group if and only if the moduli dimension of the Nielsen class is 0; and the general computation of whether the action is through a finite group must be difficult (albeit, primitive recursive). Finally, we illustrate the "endomorphism computations" discussed in [FrG; §5] and §2.3 by mentioning the problems in computing those endomorphisms of $H_1(X, \mathcal{Q})$ that arise from the group ring $\mathcal{Z}[G]$ in this special case: $G = A_n$; and the cover is in the Nielsen class $\mathrm{Ni}(\mathbf{C})$ with each of the conjugacy classes in $\mathbf{C}$ equal to the conjugacy class of a 3-cycle. The main tool here is just the Lefschetz trace formula.

§2. VOL. 3 OF ZARISKI'S COLLECTED WORKS APPLIED TO [FrG]

The papers of [Z] that apply to this note and to [FrG] are as follows:

- [8] Sull'impossibilitá di risolvere parametricamente per radicali un'equazione algebrica $f(x, y)$ =0 di genere $p > 6$ a moduli generali, 1926, 43–49.
- [12] Sopra una classe di equazioni algebriche contenenti linearmente un parametro e risolubili per radicali, 1926, 58–80.
- [13] On a theorem of Severi, 1928, 81–86.
- [18] On the moduli of algebraic functions possessing given monodromie group, 1930, 155–175.
- [28] On the Poincaré group of rational plane curves, 1936, 266–278.
- [29] A theorem on the Poincaré group of an algebraic hypersurface, 1937, 279–289.
- [31] The topological discriminant group of a Riemann surface of genus p, 1937, 307–330.

There are comments by M. Artin and B. Mazur [AM] on [8] and [12] on p.2, [13] on p.3–4, and on [28], [29] and [31] on p.9–10. There are no comments on [18], which does happen to be at the center of our discussion. What we call the Hurwitz monodromy group, $H(r)$ in §1.2 appears in [28] where Zariski says (p.266) that it "practically coincides with the braid group." Also, [AM; p.9] say that this is essentially the rth braid group defined by Artin. There is, however, a practical distinction between the groups that requires more than a "Hey, you!" when it is the turn of $H(r)$. No one has yet personally objected to the author's naming of $H(r)$. So it shall stand with the author until good objection comes forth. It was M. Artin (in 1972 during the writing of [Fr,1]) who pointed out the prior application that Hurwitz [Hu] had made of it to simple branched covers. An aside: It is traditional in classical Riemann surface theory to use p, instead of g, for the genus.

§2.1. ON [8] AND THE FOUNDATIONAL THEOREM OF [FrG].

Zariski's papers are "talky" in comparison to modern papers in algebraic geometry. Some caution is advisable since several of them are in Italian. Curiously, the preprint title of [FrG], prior to our awareness of Zariski's work was essentially the English version of the title of [8].

Statement 2.1: *[8] contains the proof that the generic curve of genus $g > 6$ has no map to $\mathcal{P}^1_x$ with solvable monodromy group.*

The old Italian understanding of the word generic was quite loose, but it definitely is used here as in [FrG] and as in the notation of §1.1. Consider a group G with a faithful permutation representation $T : G \to S_n$. Let $\mathcal{M}_g(G)$ be the collection of points $\boldsymbol{m} \in \mathcal{M}_g$ such that $\boldsymbol{m}$ is represented by a curve $X_{\boldsymbol{m}}$ that has a map $\varphi : X \to \mathcal{P}^1_x$ with monodromy group G (and permutation representation—via the cover—equal to T). If G runs over solvable groups and $g > 6$, then $\mathcal{M}_g(\mathrm{sol}) = \bigcup_G \mathcal{M}_g(G)$ contains no nonempty Zariski open subset of $\mathcal{M}_g$. This is much the same as the main part of the statement of [FrG; Prop. 3.1].

Statement 2.2: *Historical background.*

Zariski attributes the main lemma for the reduction to the primitive case to the 1897 International Congress talk of Enriques. Enriques also states, as unsolved, the problem of showing that the generic curve of genus $g > 6$ is not uniformized by radicals. The reduction to the primitive case in [FrG] comes from a throw-away paragraph in [Fr,1; p.26], but it was for exposition purposes there—no claim was made of originality. The motivation for consideration of the problem by [FrG] is manifold, and it has been pushed forth at this time as a part Thompson's program (Statement 2.6). More modest motivation comes from [FrJ; p.137] which has a near outline of the proof of Prop. 3.1 of [FrG]; and it notes that if this were false, then most likely the solvable closure of the rationals would be a PAC (pseudo- algebraically closed) field. This latter question is still undecided. Actually (and Mumford noted this independently), there is an apparently much more difficult question: Given g, is the function field of the general curve of genus g a subfield of the function field of some X where $X \to \mathcal{P}^1_x$ has solvable monodromy group. If someone doesn't show this is impossible for large g soon, [FrG] will comment on why we believe this is difficult.

Statement 2.3: *Zariski on the exceptional values of g.*

Here we take g to be exceptional if $\mathcal{G}_g(\mathrm{sol})$ is nonempty. Zariski is satisfied to comment on the exceptional values of g by noting that $\mathcal{G}_g(\mathrm{prim})$ contains $S_{[g+3/2]}$ which is a solvable group in each of the cases g=1,. . .,6. Artin-Mazur comments [AM;p.2], using the language from classical Riemann surface theory, that the existence of a g^1_4 in the case of each curve of genus 6 was still unproved at the time of Zariski's paper because of gaps in papers of the Italian school. The use of [KL] in [FrG] is exactly what Artin-Mazur recommend, and therefore [FrG; Principle 2.5] is right on target in declaring that $S_n \in \mathcal{G}_g(\mathrm{prim})$ if and only if $n \geq [g + 3/2]$. This, of course, is foundational for the interesting calculations that have arisen in [FrG; §5.3] for deciding, given g, those n for which $A_n \in \mathcal{G}_g(\mathrm{prim})$.

§2.2. [12]; EXCEPTIONAL SOLVABLE GROUPS; THOMPSON'S PROGRAM

Statement 2.4: *Zariski's use of group theory and Ritt's results.*

The argument used by Zariski [8; p.46-49] to conclude his equivalent to [FrG; Prop. 3.1] is slightly longer than the one page of [FrG], and it is less valuable for listing possible exceptions. It rests, however, on exactly the same group theoretical principles—going back to Galois. This is a minuscule portion of [FrG]. Justification for the the work of [FrG; §3 and §4] comes from the desire to display all "branch cycles" $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_r)$ for all of the solvable groups G $(= G(\boldsymbol{\sigma}))$ with these properties:

$$\sum_{i=1}^{r} \text{ind}(\sigma_i) = 2(n+g-1) \quad \text{and} \quad r \geq 3g \ , \tag{2.1}$$

where g is the genus of a cover $X \to \mathcal{P}^1_x$ of degree n with $\boldsymbol{\sigma}$ as a description of its branch cycles.

Zariski's own papers provide two motivations for this. The most important of these is in [18], which is also the topic of [FrG; §5]. Statement 2.8 reports that his conjectures in [18] are wrong—by examples that already appear in [FrG; §5.2] (e.g., following Ries [R]). An explanation of the condition $r \geq 3g$ appears in §2.3.

Here, however, he was already ahead of his time. More naive motivation appears in [12], based on work of Chisini and Ritt [Ri].

Statement 2.5: *How Zariski continues a long tradition.*

In his introduction Zariski mentions papers of Klein (1874), Bianchi and Chisini (1900) and Chisini (1915) as precursers to Ritt's classification of the covers $\varphi : X \to \mathcal{P}^1_x$ with the genus $g(X)$ of X equal to 0, $\deg(\varphi)$ a prime and with solvable monodromy group. Ritt observes that the Galois closures $\hat{X}$ of each of his 5 cases are of genus 0 or 1. This puts an interesting structure into this problem. Although the Schur problem for rational functions [Fr,2; p.148] seems to have nothing to do with "uniformization by radicals," after a quote of a theorem of Burnside the geometric territory for the Schur problem of prime degree is the same as given by Ritt's list [12; p.59]. It was precisely the availability of the *arithmetic* of elliptic curves that solved the Schur problem for rational functions of prime degree. We quote [AM; p.2]:

> " . . . Zariski generalizes their [Chisini and Ritt] elegant result to arbitrary degree. He classifies solvable $[\varphi : \mathcal{P}^1_y \to \mathcal{P}^1_x]$ having the familiar property that all points of $[\varphi^{-1}(x)]$ can be expressed rationally in terms of any pair. The key point is that the [Galois closure of $\mathcal{P}^1_y \to \mathcal{P}^1_x$] is either rational or an elliptic curve."

Despite our own personal motivations just mentioned, and in light of the rather obvious group theory, it is hard for us to understand why Artin and Mazur make much of the Chisini-Ritt result considering what they ignore. John Thompson suggests that an error in Galois [B; p.162-165] persisted as the motivation for the problem. Galois incorrectly asserted that it was usual for solvable covers to have the property of the quote above. (He knew of couterexamples, but considered them, contrary to modern understanding, to be rare.) Zariski points out that his classification yields the expected generalizations excluding only one new type—"scarcely of interest" (we agree)—from a cover of degree 4. The proof is long and we know of no unusual applications or reasonable generalizations.

Statement 2.6: *The geometric portion of Thompson's program and moduli dimension.*

There are two main conjectures about the monodromy group of covers in Thompson's program:

Solvable group conjecture: *Excluding the Ritt-Zariski list there are only finitely many primitive solvable groups that occur as monodromy groups of covers $X \to \mathcal{P}^1_x$; and for each $g \geq 1$ there are only finitely many solvable primitive groups that occur as the monodromy groups of covers $X \to \mathcal{P}^1_x$ with $g(X) = g$.*

Actually the case $g = 0$ is quite significant, but it is already a theorem [GTh]. Clearly $\mathcal{G}_0(\text{sol})$ is entirely composed of subgroups of sequences of wreath products from $\mathcal{G}_0(\text{prim}) \cap \mathcal{G}_0(\text{sol})$ (§1.1) and a major portion of the groups of $\mathcal{G}_g(\text{sol})$ are comprised from subgroups of wreath products of elements of $\mathcal{G}_g(\text{prim}) \cap \mathcal{G}_g(\text{sol})$ and $\mathcal{G}_0$. A brave venture might be that in the "Solvable group conjecture," excluding the Ritt-Zariski list, A_n and S_n, $n = 5, 6, \ldots$—these both occur for several distinct types of Nielsen classes even in genus 0—there are only finitely many primitive groups that occur as monodromy groups of covers $X \to \mathcal{P}^1_x$ with $g(X) = g$. The following, however, seems more certain.

Composition factor conjecture: *Excluding A_n and cyclic groups, for each $g \geq 0$ there are only finitely many simple groups that occur as composition factors of monodromy groups of covers $X \to \mathcal{P}^1_x$ with $g(X) = g$.*

Ultimately the program is concerned with the arithmetic of these covers, but it is easier to state the results of [FrG] and [GTh] if we bring up just one geometric quantity, *moduli dimension*: the moduli dimension of (g, G) is the dimension of $\mathcal{M}_g(G)$ (Definition 1.6). For $g > 1$ the moduli dimension of (g, G) is at most $3g - 3$. Note that we assume that G is a transitive permutation group, and that the permutation representation of the monodromy group of the cover is the same as that attached to the group. When the the moduli dimension is maximal possible, Zariski says that G is *nonspecial*. Even though Zariski got some motivation from the appearance of *special* divisors in [18] (c.f. Statement 2.7) the word seems bland. Therefore we say that (g, G) has *full moduli dimension*. In the context of [FrG], when there can be no confusion we might say that (g, G) (or when g is understood, just G) is *exceptional*.

§2.3. ON [18]:MODULI DIMENSION FOR THE EXCEPTIONAL g

Statement 2.7: *Groups with sufficient branch point parameters to have a chance to have full moduli dimension.*

Since the moduli space of curves of genus g is of dimension $3g - 3$ it is an ancient observation, based on Principle 1.6, that in order that (g, G) have full moduli dimension there must be a cover $\varphi : X \to \mathcal{P}^1_x$ with monodromy group G, $g(X) = g$ and at least $r = 3g$ branch points. It is obvious that this is not (usually) sufficient unless the group is primitive—equivalently, no curves are properly contained between X and $\mathcal{P}^1_x$. Zariski notes this by example. Then he repeats several times [18; p.156,157 and generally along into the paper] that he believes that if $r \geq 3g$ and G is primitive, G has full moduli dimension. He phrases the remainder of his paper as generalization of the result of [KL] and he develops a complicated formula for the moduli dimension.

Well, actually, it isn't really so complicated from a modern viewpoint, for he has actually rephrased the problem in terms of the dimensions of fibers of the Picard bundle

$$\underbrace{X \times X \times \cdots \times X}_{n \text{ times}} / S_n \to \mathrm{Pic}^n(X), \tag{2.2}$$

where $X \times X \times \cdots X / S_n \stackrel{\text{def}}{=} X^{(n)}$ denotes the symmetric product of X, n times, and $\mathrm{Pic}^n(X)$ denotes the divisor classes of degree n on X, as X varies over representatives of points of a Zariski open subset of $\mathcal{M}_g$.

In particular [FrG] provides a list of test examples for the solvable group version of the problem of computing the moduli dimension of (g, G). If we exclude S_n's then 0,1 and 2 are the only values of g for which (g, G) has full moduli dimension with G a (primitive) solvable group. But, in the case $g = 2$ the four groups that are monodromy groups of covers with at least 6 branch points [FrG; Theorem 3.3] appear in the statement of Theorem 1.2.

Indeed, [FrG; §4] lists the complete set of *Nielsen classes* that are associated to covers with these groups as monodromy groups, and thus through Hurwitz monodromy action collects the covers into algebraic subsets whose images in $\mathcal{M}_g$ will be irreducible varieties. Much of [FrG; §5] brings together tools that bear on which of these examples do have full moduli dimension. Contrary to Zariski's conjecture: Not all (Statement 2.8)!

The list of the cases where $g = 1$, G is primitive solvable G and the Nielsen class has full moduli dimension (=1) is not complete. All have degree p^r with $p \leq 7$ [FrG; Theorem 3.2] and it is likely that the exceptions include the list with g=2. But this hasn't yet been checked. The list for g=0 (also not yet complete) is part of [GTh], and it includes many groups that don't appear in the list for $g = 2$.

Statement 2.8: *The endomorphism test for complete moduli dimension in the case $g = 2$ gives counterexamples to the main conjecture of [18].*

Suppose we are given a cover $\varphi : X \to \mathcal{P}^1_x$. Let $\hat{\varphi} : \hat{X} \to \mathcal{P}^1_z$ be the Galois closure of this cover. There is a fairly explicit algorithm for computing those endomorphisms of the Jacobian, $J(X)$, of X (identified with $\mathrm{Pic}^0(X)$) that arise from the group ring $\mathcal{Z}[G]$ with $G = G(\hat{X}/\mathcal{P}^1_x)$ acting on $J(\hat{X})$ (leaving $J(X)$ stable). Indeed, in the examples here, and in the other test case (g, A_n) of [FrG; §5.3], it suffices to carry out all computations on $H_1(X, \mathcal{Z})$ using just the Lefschetz trace formula ([FrG; Theorem 5.5] and §3.2).

Identify $H_1(X, \mathcal{Q})$ as the image of $H_1(\hat{X}, \mathcal{Q})$ under $pr_X = \sum_{\sigma \in G(\hat{X}/X)} \sigma$ and consider those elements of $\mathcal{Q}[G]$ that commute with pr_X. Denote the action of this subring on $H_1(X, \mathcal{Q})$ by $\mathrm{End}_{\hat{X}}(X)$. If $\mathrm{End}_{\hat{X}}(X)$ properly contains $\mathcal{Q}$, then (g, G) doesn't have full moduli dimension (c.f. Comments on [13]). These computations aren't yet complete, but two at least of the exceptional groups in the case of g=2 don't have full moduli dimension by this criteria, and none of the list of (1.3) yet has been shown to have full moduli dimension. Ries was the first to give an example of the failure of Zariski's conjecture [R] (for D_{10}—albeit in a somewhat intricate format).

Statement 2.9: *Continuation of $g = 2$, a full moduli formula and the hyperelliptic involution.*

The spaces $\mathcal{M}_g$ are special for the values $g = 1$ and 2. They are *affine* open subsets of a natural (Igusa) projective compactification. There is a general test for moduli dimension being 0 that is "if and only if" in the case $g = 1$ or 2: All of the Picard-Lefschetz transformations around "branches at ∞" act on $H_1(X, \mathcal{Q})$ as elements of finite order. By identifying the "braid group" generators of the Hurwitz monodromy group with P-L transformations and by expressing $H_1(X, \mathcal{Z})$ in terms of "branch cycles," ([FrG; Theorem 5.4] and §3, Theorem 3.6) provides lower bounds for the moduli dimension. It would be a considerable refinement of the ideas attributed to Mayer-Mumford in [Gr; §13] to rephrase the whole problem of computing the moduli dimension in terms of P-L transformations even in Hurwitz family situations where, as we have just noted, things can be computed explicitly. Also, "coalescing" of branch cycles provides much information on boundary behaviour of the Hurwitz family, and indirectly on the moduli dimension.

In the case $g = 2$, when it is possible to reconstruct the canonical involution from the branch cycles in the Nielsen class giving the collection of covers, then we have a precise handle (via Igusa) on the relation of the family of covers to $\mathcal{M}_g$. We could do this from pure group theory if we are in a situation where our cover $\varphi : X \to \mathcal{P}^1_x$ is part of a commutative diagram

$$(2.4) \qquad \begin{array}{ccccccc} \hat{X}_u & \longrightarrow & \hat{X}_x & \longrightarrow & X & \xrightarrow{\lambda} & \mathcal{P}^1_t \\ & & & & \downarrow \varphi & & \downarrow \varphi(h) \\ & & & & \mathcal{P}^1_x & \xrightarrow{\varphi(f)} & \mathcal{P}^1_u \end{array}$$

where we have renamed $\hat{X}$ to be $\hat{X}_x$, $\lambda : X \to \mathcal{P}^1_t$ is the canonical hyperelliptic involution and $\hat{X}_u$ is the Galois closure of the cover $X \to \mathcal{P}^1_u$. The maps $\varphi(f)$ and $\varphi(h)$ derive from rational functions f and h.

Of course, the most important point of this diagram is that there are such rational functions. This is not to be expected in general. But we don't know if this is precluded for Nielsen classes of full moduli dimension. It is an example of the "finite correspondence situation" that was featured in [Fr,3]: given everything in the diagram, except the lower right corner, we ask when we can fill it in using rational function maps f and h.

The Basic Problem: *Which covers $\varphi : X \to \mathcal{P}^1_x$, expressed in terms of branch cycles are part of a diagram like (2.4).*

Comments: Since the cover $\varphi : X \to \mathcal{P}^1_x$ is primitive, we may assume that X is a component of the fiber product $\mathcal{P}^1_x \times_{\mathcal{P}^1_u} \mathcal{P}^1_t$, and the maps φ and λ arise from projection on the two factors. The Galois closures of each of these covers give groups, G_f and G_h, that are homomorphic images of the group $G_u = G$. This can be rephrased entirely in terms of group theory with "branch cycle" generators $\boldsymbol{\tau}$.

Assume given a description $\boldsymbol{\sigma}$ of the branch cycles of $\varphi : X \to \mathcal{P}^1_x$. We seek a group G_u with "branch cycle" generators $\boldsymbol{\tau}$ and with three transitive permutation representations (named for the above situation) T_X, T_f and T_h with these properties:

(2.5) a) T_X arises from an orbit of $T_f \otimes T_h$;

b) $\boldsymbol{\sigma}$ is in the Nielsen class of a Schreier construction arising from $T_f(\boldsymbol{\tau})$ applied to $G(T_X)$, the stabilizer of an integer in the representation T_X; and

c) branch cycles for a degree 2 hyperelliptic cover are the result of a Schreier construction arising from $T_h(\boldsymbol{\tau})$ applied to $G(T_X)$.

We don't explain the phrase "Schreier construction", but it is essentially the construction of free generators of a subgroup of finite index in a free group. Without experimenting it is difficult to say whether, given $\boldsymbol{\sigma}$, the existence of G_u, $\boldsymbol{\tau}$, etc. is a reasonably effective calculation, but it doesn't look easy. □

§2.4. ON [13], [28]–[31]: ENDOMORPHISMS ARE SPECIAL AND MORE ON $H(r)$

Statement 2.10: *Comments of [AM; p.3–4] on Endomorphisms of generic curves.*

Let C be a curve and T a curve on $C \times C$. Then T is called a singular correspondence if its cohomology class in $H^2(C \times C, \mathcal{Q})$ isn't a linear combination of the classes of the diagonal, $\boldsymbol{p} \times C$ and $C \times \boldsymbol{p}$ with $\boldsymbol{p} \in C$. Hurwitz conjectured and Severi tried to prove that a sufficiently general linear system $|C|$ on a surface F contains no singular correspondence. He based his argument on families of plane curves with only nodes as singularities. As a consequence the result of Statement 2.8 follows.

Zariski points out that there are difficulties with Severi's argument, but concludes the result in the case that the rational map from F to $\mathcal{P}^1_x$ induced from $|C|$ is birational. In the proof he uses a result of Severi that has been proved only for Lefschetz pencils. Here [AM; p.4] claim that it is not difficult to verify this extra condition in the case of plane curves with nodes—thus the result: "So the proof that the general curve of genus [g] has no singular correspondence can be distilled from the two papers of Severi and Zariski."

Also [AM] includes an allusion to a *preprint* of Mori, but there is no description of contents, so its relation to the topic of correspondences is obscure. What was left out was any reference to Lefschetz's paper [L] which appeared in the same volume as Zariski's, even though Zariski himself includes it as a footnote. Lefschetz is quite clear: He shows that if the "abelian functions of every [curve of genus g] have a complex multiplication, then there exists a fixed complex multiplication common to them all;" and then he shows, by an explicit induction on g—using explicit computation of periods—that the general hyperelliptic curve has no complex multiplication.

Statement 2.12: *Zariski seems to be the first to write out the relation between the Artin braid group and the Hurwitz monodromy group.*

Zariski's interest in the Hurwitz monodromy group seems to have nothing to do with moduli families. Indeed, the only evidence in [Z] of any motivation coming from classical moduli space thinking related to families of curves is his considerable work on curves in $\mathcal{P}^2$ with, say, only nodes (or nodes and cusps) as singularities. But even here he concentrates on the fundamental group of the complement of such a curve. Here is the progression of his papers on this.

In [29] he argues for $r > 2$ that if V^{r-1} is a hypersurface in $\mathcal{P}^r$, then $\pi_1(\mathcal{P}^r - V^{r-1})$ is isomorphic to $\pi_1(L - L \cap V^{r-1})$ where L is a generic hyperplane section of $\mathcal{P}^r$. There is a claim in [AM; p.7] that his proof requires amplification on several points. They mention [AM; p.15] the Morse theory proof of D. Cheniot.

In [28] Zariski notices that "maximal cuspidal curves" of even order $2r-2$ are generic sections of the discriminant locus, and that among rational curves C in $\mathcal{P}^2$ with only nodes and cusps, other than the maximal cuspidal curves (recall the connection with the Hurwitz monodromy group), $\pi_1(\mathcal{P}^2 - C)$ is cyclic.

Finally, in [31] there is something that the author hasn't seen used before. Use the notation of (2.2) with D_r denoting the natural discriminant locus in $X^{(r)}$. For a nonsingular projective curve X of genus g, not only does Zariski compute $\pi_1(X^{(r)} - D_r)$, denoted $G_{r,g}$, in terms of generators and relations, but he considers a fascinating normal subgroup of it. Assume that $r > g$ and apply the map (2.2) from $X^{(r)}$ to $\mathrm{Pic}^r(X)$. The general fibers of this are well known to be copies of $\mathcal{P}^{r-g}$. Thus, in Zariski's language, $X^{(r)}$ contains a system of ∞^g , $\mathcal{P}^{r-g}$'s. He then computes $\pi_1(\mathcal{P}^{r-g} - D')$ where D' is the intersection of the discriminant locus with a general one of these $\mathcal{P}^{r-g}$'s.

§2.5. THE SOLVABLE AND GENUS 0 HULLS

Statement 2.13: *The genus zero hull of an element of a function field.*

The title contains a neologism which we will define as follows. Let x be a nonconstant function on a curve X (over $\mathcal{C}$). Denote an algebraic closure of $\mathcal{C}(X)$ by $\bar{K}$. Starting from x consider the smallest field $\bar{K}_o$ containing $\mathcal{C}(x)$ that includes the Galois closure of each field extension $\mathcal{C}(u)/\mathcal{C}(v)$ with nonconstant u and $v \in \bar{K}_o$. Part of Thompson's program is to describe $\bar{K}_o$ (and the similarly defined $\bar{K}_g$ where $\mathcal{C}(u)$ is replaced by any function field of genus g). Note that this includes data on the lattice of subfields of $\mathcal{C}(x)$ compatible with the discussion of Statement 2.9 and of many of Ritt's papers (cf. [Fr,3]).

Statement 2.14: *For which g is $\mathcal{M}_g(\mathrm{sol})$ ($= \cup_{G \text{ solvable}} \mathcal{M}_g(G)$ (conclusion of §1.1 and Statement 2.1) dense in $\mathcal{M}_g$?*

The result of Zariski does not preclude that the g's that satisfy this statement exceed 6. But in this direction it seems possible that for fixed $g > 0$ there exists $N = N(g)$ such that for $n > N$ there are no primitive solvable groups of genus g (i.e., appearing as monodromy groups of covers of $\mathcal{P}^1_x$ by some genus g curve— c.f. Statement 2.6). Recall that n denotes the degree of the permutation representation that goes with G. Some understanding of the solvable closure of $\mathcal{C}(x)$ would follow from this if we also understood, for fixed g, how to bound the degrees of *all* covers $X \to Y$ with solvable monodromy group where $g(X) = g$ and $g(Y) > 0$.

Statement 2.15: *Nielsen classes consisting entirely of hyperelliptic curves, and generalities relating one Nielsen class to another.*

Suppose given a Nielsen class $\mathrm{Ni}(\mathbf{C})$ where $\mathbf{C}$ represents an r-tuple of conjugacy classes in a group G. Statement 2.9 can be generalized beyond the case $g = 2$. We can inspect whether the set of covers $\varphi : X \to \mathcal{P}^1_x$ in this Nielsen class each fit in a commutative diagram like (2.4) that displays a hyperelliptic involution for X. This implies the existence of (complex analytic) $\Psi(\mathbf{C}, \mathbf{C}') : \mathcal{H}(\mathbf{C}) \to \mathcal{H}(\mathbf{C}')$ where $\mathbf{C}'$ denotes the Nielsen class for hyperelliptic covers of the genus of the family. But such a morphism exists if each X that appears in the Nielsen class is hyperelliptic. If this situation occurs, we would have all of the apparatus for computing the moduli dimension of the Nielsen that arises from the special "Igusa-like" compactification of the hyperelliptic curves of genus g.

More generally, for a given Nielsen class $\mathrm{Ni}(\mathbf{C})$ we would ask how one might effectively compute the possibility for a natural map $\Psi(\mathbf{C}, \mathbf{C}') : \mathcal{H}(\mathbf{C}) \to \mathcal{H}(\mathbf{C}')$ where $\mathbf{C}'$ denotes some other Nielsen class. We shall say that $\mathbf{C}$ and $\mathbf{C}'$ are *concatenated* if the obvious analogue of diagram (2.4) exists. It must be a difficult problem to decide if $\mathbf{C}$ and $\mathbf{C}'$ are concatenated. The main theorem of [ArC] considers the case that the conjugacy classes of the Nielsen class $\mathbf{C}$ are all 2-cycles and the degree of the representation is smaller than $g/2 + 1$. If the degree of the group associated to C$'$ also does not exceed $g/2 + 1$ then any concatenation must be particularly simple: the equivalence classes of the covers represented by points of $\mathcal{H}(\mathbf{C}')$ are of the form $X \to \mathcal{P}^1_x \to \mathcal{P}^1_z$ where the covers $X \to \mathcal{P}^1_x$ are in $\mathrm{Ni}(\mathbf{C})$.

Statement 2.16: *The relation between Nielsen classes for a subgroup H of G and Nielsen classes for G.*

Suppose that $\mathrm{Ni}(\mathbf{C})$ is a Nielsen class for the group $H \subset G$, and assume that $\mathrm{Ni}(\mathbf{C})$ has full moduli dimension. In the light of Principle 1.6 it is tempting to think that there must be a Nielsen class $\mathrm{Ni}(\mathbf{C}')$ for G for which $\mathrm{Ni}(\mathbf{C}')$ is also of full moduli dimension. But the principle of "coalescing of branch cycles" would say that there is probably little chance that such a statement holds generally without adding the the following condition; in which case the statement is true by the same ideas that give Principle 1.6. Suppose that $\boldsymbol{\sigma} \in \mathrm{Ni}(\mathbf{C})$. There should exist generators of G,

$$\sigma_{1,1}, \ldots, \sigma_{1,s_1}, \sigma_{2,1}, \ldots, \sigma_{2,s_2}, \ldots, \sigma_{r,1}, \ldots, \sigma_{r,s_r}, \; i = 1, \ldots, r,$$

whose sum of indices gives the same genus g as does $\boldsymbol{\sigma}$ and with $\sigma_{i,1} \cdots \sigma_{i,s_i} = \sigma_i$, $i = 1, \ldots, r$. In practical situations checking for this situation would be a nontrivial computation.

§3. HURWITZ MONODROMY ACTION ON $\pi_1(X)$

Use the notation of the previous sections for an absolute Nielsen class $\mathrm{Ni}(\mathbf{C})^{ab}_T$. Consider the Hurwitz space $\mathcal{H}(\mathbf{C})_T$ parametrizing equivalence classes of covers $X \to \mathcal{P}^1_x$ in this Nielsen class. We assume that this representative cover corresponds to a point $\boldsymbol{m}_0 \in \mathcal{H}(\mathbf{C})_T$. The goal of this section is to discuss the combinatorial computation of the natural action of the fundamental group of $\mathcal{H}(\mathbf{C})_T$ on the fundamental group $\pi_1(X)$. This starts with the computation of $\pi_1(X)$ in terms of branch cycles for the cover $X \to \mathcal{P}^1_x$ (§3.1, Theorem 3.5). As usual $g = g(X)$ is the genus of X.

The action of $\pi_1(\mathcal{H}(\mathbf{C})_T)$ on this (§3.2) can be regarded as giving data for a global version of a local problem, the computation of the variation of the complex structure of X as $\boldsymbol{m}$ varies in a suitably small neighborhood U of $\boldsymbol{m}_0$ on $\mathcal{H}(\mathbf{C})_T$. This local problem can be made quite explicit by considering an analytic basis $(\omega(\boldsymbol{m})_1, \dots, \omega(\boldsymbol{m})_g)$ for the space of holomorphic 1-forms $\Gamma(\Omega^1(X_{\boldsymbol{m}}))$ on $X_{\boldsymbol{m}}$ as $\varphi_{\boldsymbol{m}} : X_{\boldsymbol{m}} \to \mathcal{P}^1_x$ runs over representatives for the equivalence class of covers $\boldsymbol{m} \in U$. Denote the tangent space to $\mathcal{H}(\mathbf{C})_T$ at $\boldsymbol{m}_0$ by $\mathbf{T}_{\boldsymbol{m}_0}$. The directional derivatives $D_{\boldsymbol{t}}$ applied to $(\omega(\boldsymbol{m})_1, \dots, \omega(\boldsymbol{m})_g)$, $\boldsymbol{t} \in \mathbf{T}_{\boldsymbol{m}_0}$—the result is a vector of differentials of second kind at worst—gives sufficient information to determine the moduli dimension of the Nielsen class. This is complicated, but it is outlined in the opening section of [Gr] under the heading of the Gauss-Manin connection. A simple case is illustrative.

Since we may choose U so that the family of covers over U is locally constant we may regard $H^1(X_{\boldsymbol{m}}, \mathcal{C})$ as constant in $\boldsymbol{m}$. Suppose that the application of $D_{\boldsymbol{t}}$ to $(\omega(\boldsymbol{m})_1, \dots, \omega(\boldsymbol{m})_g)$ gives an analytic vector of *holomorphic* differentials on $X_{\boldsymbol{m}_0}$ for each $\boldsymbol{t} \in \mathbf{T}_{\boldsymbol{m}_0}$. Then the vector subspace $\Gamma(\Omega^1(X_{\boldsymbol{m}}))$ is a constant subspace of $H^1(X_{\boldsymbol{m}}, \mathcal{C})$ as $\boldsymbol{m}$ varies. From the local Torelli theorem [Gr; p.247] the moduli dimension of the Nielsen class is 0.

Figure 2:
Gradient of Differentials
Gradient with respect to parameter space variables gives variation of the complex structure of the family.

The abstract construction of the Hurwitz family would seem to mitigate against the computation of the holomorphic differentials $(\omega(\boldsymbol{m})_1, \dots, \omega(\boldsymbol{m})_g)$. Data from Nielsen classes—of an essentially global nature—gives the appearance of a more tractible computation. Since, however, the theory for determining the moduli dimension from such data hasn't been completed, we make a local–global comparison here only for when the moduli dimension is nonzero.

§3.1. FUNDAMENTAL GROUP COMPUTATION FROM BRANCH CYCLES

Given a cover $\varphi : X \to \mathcal{P}^1_x$ we compute $\pi_1(X)$ (and $H_1(X, \mathcal{Z})$) in terms of a description of the branch cycles for the cover (Theorem 3.5). Indeed, if the cover is Galois it is back to the well known Schreier construction for generators of a subgroup of a free group in an explicit way.

The Galois case: Denote by $S = \{\bar\sigma_1, \dots, \bar\sigma_r\}$ generators of the free group F_r on r generators. With $\boldsymbol{\sigma}$ a description of the branch cycles of the cover, let G be $G(\boldsymbol{\sigma})$ and $G(1) = \{\gamma \in G(\boldsymbol{\sigma}) \mid \gamma(1) = 1\}$. Consider the homomorphism $\delta : F_r \to G(\boldsymbol{\sigma})$ induced by $\bar\sigma_i \to \sigma_i$, $i = 1, \dots, r$. Finally, let $H(1)$ be $\delta^{-1}(G(1))$.

In order to get free generators of $H(1)$ we need a function $\rho : F_r \to F_r$ representing right cosets of $H(1)$, with the following properties : $\rho(1) = 1$; $\rho(\alpha) \in H(1)\alpha$; and $\rho(h\alpha) = \rho(\alpha)$ for each $h \in H(1)$ and $\alpha \in F_r$. Furthermore ρ may be selected to have the following property:

$$\text{length}_{\bar\sigma}(\rho(\alpha)) = \min_{h\in H(1)} \text{length}_{\bar\sigma}(h\alpha) \quad \text{for} \quad \alpha \in F_r \tag{3.1}$$

where $\text{length}_{\bar\sigma}$ denotes the length of a word in the $\bar\sigma$'s.

Automatic from this is the following property: if $\rho(\alpha) = s_1^{\epsilon_1} \cdots s_n^{\epsilon_n}$ is a reduced representation of $\rho(\alpha)$, $s_i \in S$, $\epsilon_i \in \{\pm 1\}$, $i = 1, \ldots, n$, then

$$s_1^{\epsilon_1} \cdots s_i^{\epsilon_i} \in \rho(F_r) \quad \text{for each } i = 1, \ldots, n. \tag{3.2}$$

With these conditions, the collection

$$M = \{rs\rho(rs)^{-1} \mid r \in \rho(F_r),\ s \in S \text{ and } rs \notin \rho(F_r)\}$$

generates H(1) freely (e.g., [FrJ; Lemma 15.23]).

Lemma 3.1: *If $\varphi : X \to \mathcal{P}^1_x$ is a Galois cover, then the fundamental group of X is isomorphic to the image of $H(1)$ in F_r/N where N is the smallest normal subgroup of F_r containing $\bar\sigma_1 \cdots \bar\sigma_r$ and $\bar\sigma_i^{\text{ord}(\sigma_i)}$, $i = 1, \ldots, r$.*

This definitely doesn't hold if $\varphi : X \to \mathcal{P}^1_x$ isn't Galois . Here is the easiest example.

Example 3.2: *S_3 in its regular and nonregular representations.* Let G be S_3, let $r > 3$ be an even integer and let C_i be the conjugacy class of a 2-cycle in G, $i = 1, \ldots, r$. Let $\varphi : X \to \mathcal{P}^1_x$ be a cover in the Nielsen class $\mathrm{Ni}(\mathbf{C})$ where G is in its representation of degree 3. As usual let $\hat\varphi : \hat X \to \mathcal{P}^1_x$ be the Galois closure of $\varphi : X \to \mathcal{P}^1_x$. Then we may compute $\pi_1(\hat X)$ from Lemma 3.1. For simplicity, and with no loss, assume that $\sigma_1 = \sigma_2 = (1\,3)$, and that $\sigma_3 = \cdots = \sigma_r = (1\,2)$. Then $\{1, \bar\sigma_2, \bar\sigma_3\} = \Lambda$ consists of coset representatives for $H(1)$, and

$$M = \{\bar\sigma_1\bar\sigma_i,\ \bar\sigma_3\bar\sigma_i\bar\sigma_3^{-1},\ i = 1, 2,\ \bar\sigma_3\bar\sigma_j,\ \bar\sigma_1\bar\sigma_j\bar\sigma_1^{-1},\ j = 3, \ldots, r,\ \bar\sigma_2\bar\sigma_1^{-1},\ \bar\sigma_j\bar\sigma_3^{-1},\ j = 4, \ldots, r\}$$

generates $H(1)$ freely.

Similarly , appropriate coset representatives for $\widehat{H(1)}$ defined by the regular representation of G are given by $\hat\Lambda = \Lambda \cup \{\bar\sigma_1\bar\sigma_3,\ \bar\sigma_3\bar\sigma_1,\ \bar\sigma_1\bar\sigma_3\bar\sigma_1\}$. Thus,

$$\begin{aligned}\hat M =&\{\bar\sigma_2\bar\sigma_1^{-1},\ \bar\sigma_j\bar\sigma_3^{-1},\ \bar\sigma_1\bar\sigma_j(\bar\sigma_1\bar\sigma_3)^{-1},\ j = 4, \ldots, r,\ i = 1, 2,\\ &\bar\sigma_3\bar\sigma_j,\ j = 3, \ldots, r,\ \bar\sigma_3\bar\sigma_2(\bar\sigma_3\bar\sigma_1)^{-1},\ \bar\sigma_1\bar\sigma_3\bar\sigma_2(\bar\sigma_1\bar\sigma_3\bar\sigma_1)^{-1},\ \bar\sigma_1\bar\sigma_3\bar\sigma_j(\bar\sigma_1)^{-1},\\ &j = 3, \ldots, r,\ \bar\sigma_3\bar\sigma_1\bar\sigma_i(\bar\sigma_3)^{-1},\ i = 1, 2,\ \bar\sigma_3\bar\sigma_1\bar\sigma_j(\bar\sigma_1\bar\sigma_3\bar\sigma_1)^{-1},\ j = 3, \ldots, r,\\ &\bar\sigma_1\bar\sigma_3\bar\sigma_1\bar\sigma_i(\bar\sigma_1\bar\sigma_3)^{-1},\ i = 1, 2, \text{ and } \bar\sigma_1\bar\sigma_3\bar\sigma_1\bar\sigma_j(\bar\sigma_3\bar\sigma_1)^{-1},\ j = 3, \ldots, r\,\}\end{aligned}$$

generates $\widehat{H(1)}$.

This all simplifies once we go to the quotient by the relations defining N in Lemma 3.1. For simplicity we do just the case $r = 4$. It is easy to see that the following subsets of M and $\hat M$, respectively, generate the same quotient groups modulo N as do M and $\hat M$:

$$M' = \{\bar\sigma_1\bar\sigma_2,\ \bar\sigma_3\bar\sigma_1\bar\sigma_3,\ \bar\sigma_1\bar\sigma_3\bar\sigma_1,\ \bar\sigma_3\bar\sigma_2\bar\sigma_3,\ \bar\sigma_1\bar\sigma_4\bar\sigma_1\};\ \text{and}$$

$$\begin{aligned}\hat M' = \{&\bar\sigma_1\bar\sigma_2,\ \bar\sigma_1\bar\sigma_4\bar\sigma_3\bar\sigma_1, \bar\sigma_3\bar\sigma_1\bar\sigma_2\bar\sigma_3,\ \bar\sigma_3\bar\sigma_1\bar\sigma_4\bar\sigma_1\bar\sigma_3\bar\sigma_1,\\ &\bar\sigma_1\bar\sigma_3\bar\sigma_1\bar\sigma_3\bar\sigma_1\bar\sigma_3,\ \bar\sigma_1\bar\sigma_3\bar\sigma_1\bar\sigma_4\bar\sigma_1\bar\sigma_3\}.\end{aligned}$$

Indeed , only the 1st and 5th elements of $\hat M'$ are really needed. For example

$$\bar\sigma_1\bar\sigma_4\bar\sigma_3\bar\sigma_1 = \bar\sigma_1\bar\sigma_1\bar\sigma_2\bar\sigma_1 = \bar\sigma_2\bar\sigma_1 \bmod N.$$

This is the 1st element's inverse. And the inverse of the 5th times the 6th is

$$\bar\sigma_3\bar\sigma_1\bar\sigma_3\bar\sigma_4\bar\sigma_1\bar\sigma_3 = \bar\sigma_3\bar\sigma_1\bar\sigma_2\bar\sigma_1\bar\sigma_1\bar\sigma_3 = \bar\sigma_3\bar\sigma_4 \bmod N.$$

Thus we may regard $\pi_1(\hat X)$ as the quotient $\bmod N$ of the group generated by

$$\hat M'' = \{\bar\sigma_1\bar\sigma_2 \quad \text{and} \quad \bar\sigma_1\bar\sigma_3\bar\sigma_1\bar\sigma_3\bar\sigma_1\bar\sigma_3\}.$$

It is reassuring to see that these two commute :

$$(\bar\sigma_1\bar\sigma_2)\bar\sigma_1\bar\sigma_3\bar\sigma_1\bar\sigma_3\bar\sigma_1\bar\sigma_3 = \bar\sigma_1\bar\sigma_3\bar\sigma_4\bar\sigma_3\bar\sigma_1\bar\sigma_3\bar\sigma_1\bar\sigma_3$$

$= \bar\sigma_1\bar\sigma_3\bar\sigma_1\bar\sigma_2\bar\sigma_1\bar\sigma_3\bar\sigma_1\bar\sigma_3 \bmod N$, etc. Thus everything is in agreement with the computation that $g(\hat X) = 1$. Now consider M'.

The group generated by M' mod N is the same as the group generated by $M'' = \{\bar\sigma_1\bar\sigma_2,\ \bar\sigma_1\bar\sigma_3\bar\sigma_1,\ \bar\sigma_3\bar\sigma_1\bar\sigma_3\}$. Denote this group by $H(1)'$. The group generated by it is clearly nontrivial $\bmod N$, and since $g(X) = 0$, the group $H(1)'$ cannot possibly be $\pi_1(X)$. Below, however, we see that if we form the quotient by the torsion elements in $H(1)'$ we get $\pi_1(X)$. □

The non-Galois case: Now return to the group $H(1)$, a possibly nonnormal subgroup of F_r as derived from the Schreier construction. It is easy to interpret its quotient, $H(1)'$, modulo the group N of Lemma 3.1 : Subgroups of $H(1)'$ are in one-one correspondence with covers $X' \to X$ with the property that the pullback over $\hat X$ (i.e., a connected component of the fiber product $\hat X \times_X X'$) is unramified over X'.

Since $g(X) = 0$ in Ex. 3.2, simple principle will help us to find the natural normal subgroup N' of $H(1)'$ whose quotient will be trivial. The image of each subgroup H of $H(1)'$ in $H(1)'/N'$ should be the same as the image of $H \cap (\widehat{H(1)}/N)$ and N' should be minimal with this property. In particular this applies to $H = H(1)'$. But in this case we use that fundamental groups of Riemann surfaces have no torsion. Therefore the fundamental group quotient will have no torsion. Thus the image of the elements $\bar\sigma_1\bar\sigma_3\bar\sigma_1 = \alpha$ and $\bar\sigma_3\bar\sigma_1\bar\sigma_3 = \beta$, both of order dividing 2, must be 1. Just for the record, $\alpha\bar\sigma_1\bar\sigma_2 = \bar\sigma_1\bar\sigma_4\bar\sigma_1 = \bar\sigma_1\bar\sigma_2 \bmod N'$. Thus $\bar\sigma_1\bar\sigma_2$ has image $\bmod N'$ also of order dividing 2, and it too must be 1. We have now shown that $N' = H(1)'$ in Ex. 3.2. Thus we have established on general principles how to recognize that $\pi_1(X)$ is trivial.

For the general cover $X \to \mathcal{P}^1_x$ we show that the same idea works if each nontrivial cover $Y \to X$, fitting in a diagram $\hat X \to Y \to X$, is ramified. That is, replace X by the maximal unramified cover X^{un} of X fitting between $\hat X$ and X.

Lemma 3.3: *The cover X^{un} corresponds to the minimal subgroup H^{un} of $G(1)$ with this property: the length of any orbit O of σ_i on the right cosets of $G(1)$ is the same as the lengths of each of the orbits of σ_i on the right cosets of H that comprise O, $i = 1, ..., r$. Finally, this condition is equivalent to the following :*

(3.3) *$\alpha \in G(1)$ if and only if $\alpha \in H$ as α runs over all elements of the form $g\sigma_i^k g^{-1}$ with $g \in G$, k a divisor of $\mathrm{ord}(\sigma_i)$, $i = 1, \ldots, r$.*

Proof: Denote the cover of $\mathcal{P}^1_x$ given by any subgroup H of $G(1)$ by $X_H \to \mathcal{P}^1_x$ so that $X_H \to X \to \mathcal{P}^1_x$ corresponds to the chain of subgroups $H \subseteq G(1) \subseteq G$. As in §1.2 let $\boldsymbol{x} = (x_1, \ldots, x_r)$ be the branch points of the cover $\hat X \to \mathcal{P}^1_x$. From [Fr,2; p.147 expression (1.6)], a point $\boldsymbol{p}_H$ of X_H above x_i corresponds to an orbit of σ_i on the right cosets of H in G, and the image $\boldsymbol{p}$ of $\boldsymbol{p}_H$ in X corresponds to the extension of this orbit by $G(1)$ (i.e., by multiplication of the cosets for H on the left by $G(1)$).

Let $e(\boldsymbol{p}_H/\boldsymbol{p})$ (resp., $e(\boldsymbol{p}_H/x_i)$ and $e(\boldsymbol{p}/x_i)$) be the ramification index of $\boldsymbol{p}_H$ over $\boldsymbol{p}$ (resp., etc.). Then, $e(\boldsymbol{p}_H/\boldsymbol{p}) \cdot e(\boldsymbol{p}/x_i) = e(\boldsymbol{p}_H/x_i)$ and this expresses that the orbit of σ_i on the cosets of H that corresponds to $\boldsymbol{p}_H$ has length $e(\boldsymbol{p}_H/\boldsymbol{p})$ times the length of the orbit of σ_i on the cosets given by extension by $G(1)$. Of course, $X_H \to X$ is unramified if and only if $e(\boldsymbol{p}_H/\boldsymbol{p})$=1 for all points $\boldsymbol{p}_H \in X_H$. Thus the condition in the lemma restates exactly that this cover is unramified. The restatement of the condition in (3.3) follows by noting these two points: α fixes an H-coset Hg if and only if $g\alpha g^{-1} \in H$; and α has an orbit of length k on H-cosets including H if and only if k is the minimal integer t such that $\alpha^t \in H$. It only remains to note that there is a minimal subgroup H with this property. □

Again recall the natural surjective map $\delta : H(1)' \to G(1)$ induced from $\delta : F_r \to G(\boldsymbol{\sigma})$. Whenever there can be no confusion we denote the normal subgroup of $H(1)$ generated by torsion elements by **tor**. As a presented group it is generated by the image in $H(1)'$ of the set

$$T = \{\tau\alpha\tau^{-1} \mid \tau \in F_r,\ \alpha = \bar\sigma_i^k,\ k \text{ divides } \mathrm{ord}(\sigma_i),\ i = 1, ..., r, \text{ and } \delta(\tau\alpha\tau^{-1}) \in G(1)\}.$$

Denote the subgroup $< \delta(\beta) \mid \beta \in T >$ of $G(1)$ by $G_{\mathbf{tor}}$. In Ex. 3.2 $G_{\mathbf{tor}} = G(1)$.

Example 3.4: $G_{\mathbf{tor}} \neq G(1)$. Let $G = S_n$, with n even and at least 4, in its standard representation and take $\boldsymbol{\sigma}$ in the case where $\sigma_1 = (1\,2 \dots n)$, $\sigma_2 = \sigma_1^{-1}$, and all of the remainder of the σ's are 3-cycles which generate A_n. Then $G(1) = S_{n-1}$ (identified with the stabilizer of 1), but $G_{\mathbf{tor}} = A_{n-1}$. Note, however, in the notation of Lemma 3.3 that $H^{\mathrm{un}} = A_{n-1}$. That is, suppose that H is a subgroup of $G(1)$ containing H^{un} and σ runs over 3-cycles in $G(1)$ that are conjugate to those in $\{\sigma_i,\ 1, \dots, r\}$. By hypothesis these generate A_{n-1}. Then, according to Lemma 3.3, $H\sigma = H$ for each of these. Thus H contains all of these σ's and so equals A_{n-1}. □

The conclusion of Ex. 3.4 holds in general.

Theorem 3.5: *Use the previous notation with $\widehat{H(1)}$ the maximal normal subgroup of F_r contained in $H(1)$. We may compute $\pi_1(X)$ as the quotient $H(1)'/N'$ where N' is the smallest normal subgroup of $H(1)'$ with the property that the induced map from $\widehat{H(1)}$ is surjective onto $H(1)'/N'$. In particular the quotient $H(1)'/$**tor** of $H(1)'$ by **tor** maps surjectively to $\pi_1(X)$. Furthermore, this is an isomorphism if and only if in the natural map $\delta : H(1)' \to G(1)$, the image of **tor** is surjective. This holds if and only if $H^{\mathrm{un}} = G(1)$.*

Proof: Everything, except the conclusion that $H^{\mathrm{un}} = G(1)$ if and only if **tor** maps surjectively onto $G(1)$ by δ, follows from the previous discussion. But Lemma 3.3 shows that the images under δ of the elements of T generate H^{un}, and thus $\delta(\mathbf{tor}) = H^{\mathrm{un}}$. □

§3.2. $H_{\boldsymbol{\sigma}}$ ACTION ON $\pi_1(X)$

In the notation of §1.2 the subgroup $H_{\boldsymbol{\sigma}}$ is the stabilizer in $H(r)$ of an element $\boldsymbol{\sigma} \in \mathrm{Ni}(\mathbf{C})^{ab}_T$. To see the $H_{\boldsymbol{\sigma}}$ action explicitly recall from Theorem 3.5 that $\pi_1(X)$ is identified with $H(1)N/N = H(1)$ modulo N', where N' is the minimal normal subgroup of $H(1)'$ such that the induced map from $\widehat{H(1)}$ is surjective.

Denote the normal subgroup of F_r generated by $\bar\sigma_1 \cdots \bar\sigma_r$ by N_0. Then we identify $\pi_1(X - \varphi^{-1}(\boldsymbol{x}))$ with $H(1)N_0/N_0$. Consider $Q \in H(r)$. It is in the group generated by $Q_1, \dots, Q_{r-1}$ as in §1.2. Let Q act on $\bar{\boldsymbol{\sigma}} = (\bar\sigma_1, \dots, \bar\sigma_r)$ through the same formula as in (1.7). Clearly Q maps N_0 into itself. Furthermore, consider the application of Q to one of the generators (notation as in §3.1) $rs\rho(rs)^{-1} \in M$ of $H(1)$. Assume now that $Q \in H_{\boldsymbol{\sigma}}$. For simplicity assume also that $(\boldsymbol{\sigma})Q = \boldsymbol{\sigma}$. (Modification for the case when $\gamma(\boldsymbol{\sigma})Q\gamma^{-1} = \boldsymbol{\sigma}$ for some γ in the normalizer of G is easy.) Then the image of $(rs\rho(rs)^{-1})Q$ in $G(\boldsymbol{\sigma})$ is the same as the image of $rs\rho(rs)^{-1}$. Since $H(1)$ consists of those elements whose image is in $G(1)$, clearly Q maps $H(1)$ into itself. Similarly, Q maps $\widehat{H(1)}$ into itself.

Let N^* be the smallest normal subgroup of $H(1)$ containing N (discussion prior to Lemma 3.3) such that $\widehat{H(1)}/N^* = H(1)/N^*$. We have induced an action of Q on $\pi_1(X)$ if we show that Q maps N^* into itself. But $Q(N^*)$ clearly has the same properties as does N^*, once it has been established that $Q(N) = N$. This reduces to showing that $\bar\sigma_i^{\mathrm{ord}(\sigma_i)}$ is in N, $i = 1, \dots, r$. Going back to the generating elements of $H(r)$, $(\bar\sigma_i)Q = \alpha\bar\sigma_j\alpha^{-1}$ for some $\alpha \in F_r$ and some j. Since the image of $(\bar\sigma_i)Q$ and $\bar\sigma_i$ in $G(1)$ are the same, $\mathrm{ord}(\sigma_j) = \mathrm{ord}(\sigma_i)$. As N is a normal subgroup of F_r, $(\bar\sigma_i^{\mathrm{ord}(\sigma_i)})Q = \alpha\bar\sigma_j\alpha^{-1}$ is in N. The following includes a summary.

Theorem 3.6: *Suppose that $\mathcal{T}(\mathbf{C})_{\boldsymbol{m}} = X$ is one of the fibers of the family $\mathcal{F}(\mathbf{C})$, and that this fiber , as a cover of $\mathcal{P}^1_x$, has $\boldsymbol{\sigma}$ as a description of its branch cycles. The subgroup $H_{\boldsymbol{\sigma}}$ of $H(r)$ that leaves the image of $\boldsymbol{\sigma}$ in $\mathrm{Ni}(\mathbf{C})^{ab}_T$ fixed, induces an action on $\pi_1(X)$ through the action of $H(r)$ on $\bar{\boldsymbol{\sigma}}$. This action can be identified with the usual (Picard-Lefshetz) monodomy action of the fundamental group of a parameter space on the fibers of a smooth complex analytic family . Furthermore, in the case that $g = 1$ or 2, the image of $H_{\boldsymbol{\sigma}}$ in $\mathrm{Aut}(H_1(X, \mathcal{Z}))$ is a finite group if and only if $\Psi(\mathcal{H}(\mathbf{C}), \mathcal{M}_g) : \mathcal{H}(\mathbf{C}) \to \mathcal{M}_g$ is constant.*

Proof: The action is described above. The identification with the usual monodromy action follows from [Fr,1; §4] which shows the effect of the Q_i's on generating paths (e.g., Figure 1) of $\pi_1(\mathcal{P}^1_x - \boldsymbol{x})$. If these are represented by $\bar{\boldsymbol{\sigma}}$ the action is given by (1.7). The induced action on paths representing $\pi_1(X - \varphi_{-1}(\boldsymbol{x}))$ follows from the uniqueness up to homotopy of the natural fundamental group action.

The final statement comes by identifying generators of $H_{\boldsymbol{\sigma}}$ with the Picard-Lefshetz transformation around a branch at ∞, in the language of [Gr; §6, especially Theorem 6.4—The removable singularity theorem]. If each of these generators is of finite order on $H_1(X, \mathcal{Z})$, then $\Psi(\mathcal{H}(\mathbf{C}), \mathcal{M}_g) : \mathcal{H}(\mathbf{C}) \to \mathcal{M}_g$ extends to $\overline{\mathcal{H}(\mathbf{C})} \to \mathcal{M}_g$ where $\overline{\mathcal{H}(\mathbf{C})}$ is a nonsingular projective compactification of $\mathcal{H}(\mathbf{C})$. But, in the case that $g = 1$ or 2, $\mathcal{M}_g$ is an affine open subset of a projective variety $\overline{\mathcal{M}}_g$ [M; p.25]. By Chow's theorem the image of $\overline{\mathcal{H}(\mathbf{C})}$ is a projective subvariety of $\mathcal{M}_g$. Thus, unless it is just a point it must meet one of the divisors in $\overline{\mathcal{M}}_g \setminus \mathcal{M}_g$, contrary to our information. The converse of the last statement is much easier. □

Remark: For $g \geq 3$, $\mathcal{M}_g$ is not affine (e.g., it contains projective curves), but the "coalescing of branch points" argument (as in [FrG; §5.2] or Statement 2.16) often works to check for the possibility of extending $\mathcal{H}(\mathbf{C}) \to \mathcal{M}_g$ to a map into $\mathcal{M}_g$ along a specific branch at ∞ of $\mathcal{H}(\mathbf{C})$. □

§3.3. ENDOMORPHISMS AND THE LEFSCHETZ TRACE FORMULA

Now consider the group $\mathrm{Aut}(X/\mathcal{P}^1_x)$ of automorphisms of the cover $\varphi : X \to \mathcal{P}^1_x$. In terms of branch cycles, this is naturally identified with the centralizer, $\mathrm{Cen}_{S_n}(G(\boldsymbol{\sigma}))$, of $G(\boldsymbol{\sigma})$ in S_n [Fr,2; Lemma 2.1]. When we take X to be $\hat{X}$ (i.e., the cover is Galois) this is the regular representation of $G(\boldsymbol{\sigma})$ and $n = \hat{n} = |G(\boldsymbol{\sigma})|$ [Fr,2; Lemma 2.1]. This induces an action of $\mathrm{Aut}(\hat{X}/\mathcal{P}^1_x)$ on $H_1(\hat{X}, \mathcal{Z})$ (and $H_1(\hat{X}, \mathcal{Q})$) which is known to be faithful [FaK; p.253]. Thus the group ring $A = \mathcal{Q}[\mathrm{Aut}(\hat{X}/\mathcal{P}^1_x)]$ acts faithfully on $H_1(\hat{X}, \mathcal{Q})$.

For each $\alpha \in \mathrm{Aut}(\hat{X}/\mathcal{P}^1_x)$ choose a (homotopy class of) path $\bar{\sigma}_\alpha$ on $\hat{X}$ with initial point $\boldsymbol{p}_1$ and end point $\alpha(\boldsymbol{p}_1)$. (Note that such choices would have already been made in applying Schreier's construction to compute the fundamental group of $\hat{X}$ in terms of branch cycles.) The following uses a number of classical results, including the Lefschetz trace formula: The alternating sum of the traces of an automorphism of a Riemann surface on the integral homology spaces is the number of fixed points of the automorphism [FaK; p.265].

Principle 3.7: *In the notation above, α acts on $H_1(\hat{X}, \mathcal{Q})$ by conjugation by $\bar{\sigma}_\alpha$. Denote the number of disjoint cycles of σ_i (in the regular representation of G) that α centralizes by t_i. Then,*

(3.4) the trace of the action of α on $H_1(\hat{X}, \mathcal{Q})$ is $2 - \sum_{i=1}^r t_i$.

We may effectively do the following: identify $H_1(X, \mathcal{Q})$ with a subspace of $H_1(\hat{X}, \mathcal{Q})$; and given $\beta \in A$, decide if β maps $H_1(X, \mathcal{Q})$ into itself and in this case deduce whether the action of β is nontrivial. Denote the elements of $A \otimes \mathcal{Q}$ that leave $H_1(X, \mathcal{Q})$ stable by A_X. If $\mathrm{Ni}(\mathbf{C})$ has full moduli dimension $3g-3$, then $A_X = \mathcal{Q}$.

Denote the divisor classes (not necessarily positive) of degree k on X (resp., $\hat{X}$) by $\mathrm{Pic}^k(X)$ (resp., $\mathrm{Pic}^k(\hat{X})$). Consider an element

$$c = \sum_{\tau \in \mathrm{Aut}(\hat{X}/\mathbf{P}^1_x)} a_\tau \tau \in A.$$

Then c induces a map $c^* : \mathrm{Pic}^1(\hat{X}) \to \mathrm{Pic}^k(\hat{X})$ where $k = \sum_{\tau \in \mathrm{Aut}(\hat{X}/\mathbf{P}^1_x)} a_\tau$ by mapping $\hat{\boldsymbol{p}} \in \hat{X} \to \sum_{\tau \in \mathrm{Aut}(\hat{X}/\mathbf{P}^1_x)} a_\tau \tau(\hat{\boldsymbol{p}})$. Let $\psi : \hat{X} \to X$ be the natural map. Identify a point $\boldsymbol{p} \in X$ with $\sum_{\tau \in G(1)} \tau(\hat{\boldsymbol{p}})$ by choosing any point $\hat{\boldsymbol{p}} \in \hat{X}$ lying above $\boldsymbol{p}$. Then we recognize $\mathrm{Pic}^1(X)$ as the image of $\mathrm{Pic}^1(\hat{X})$ under the map $c(\psi)^* : \mathrm{Pic}^1(\hat{X}) \to \mathrm{Pic}^k(\hat{X})$ with $c(\psi) = \sum_{\tau \in G(1)} \tau$. Suppose $c \in A$ satisfies $c \cdot c(\psi) = c(\psi) \cdot c$. Then there exists a commutative diagram

$$\begin{array}{ccc} \mathrm{Pic}^1(\hat{X}) & \xrightarrow{c(\hat{\psi})} & \mathrm{Pic}^1(X) \\ \Big\downarrow c & & \Big\downarrow c_0 \\ \mathrm{Pic}^k(\hat{X}) & \xrightarrow{c(\psi)} & \mathrm{Pic}^k(X) \end{array} \qquad (3.5)$$

where c_0 is defined by application to $c(\psi)(\hat{\boldsymbol{p}})$ where it yields $c(\psi)(c(\hat{\boldsymbol{p}}))$. Note that c_0 induces an endomorphism on $\mathrm{Pic}^0(X)$.

At any time, up to isogeny, this allows us to identify $\mathrm{Pic}^0(X)$ with the image of $c(\hat{\psi})$ in $\mathrm{Pic}^0(\hat{X})$. In practical examples (e.g., [FrG; §5.3]) we check if there exists a c such that c_0 induces a nontrivial endomorphism (i.e., not in $\mathcal{Z}$) of $\mathrm{Pic}^0(X)$) by using the Lefschetz trace formula (Principal 3.7). That is, we decompose $H_1(\hat{X}, \mathcal{Q})$ in terms of representations of $\mathrm{Aut}(\hat{X}/\mathcal{P}^1_x)$ and then we apply $c(\psi)$ to these modules. If any of the resulting modules have an induced nontrivial endomorphism, then the Nielsen class is not of full moduli dimension (c.f. Statement 2.8).

Problem 3.8: *Apply the procedure above to the case when the Nielsen class $\mathrm{Ni}(\mathbf{C})^{ab}_T$ is given by $G = A_n$ and $\mathbf{C}$ has all conjugacy classes equal to that of a 3-cycle. What are the values of r for which this example has full moduli dimension?*

This particular example appears quite generally in Thompson's program (Statement 2.6).

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