INVERSE GALOIS THEORY
(Springer Monographs in Mathematics)
By Gunter Malle and B. Heinrich Matzat: 436 pp., £37.50, ISBN 3-540-62890-8 (Springer, Berlin, 1999)
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In this review *simple group* always means a non-abelian finite simple group. For K a field, G_K is the absolute Galois group of K. We denote the field generated by all roots of 1 over the rationals $\mathbf{Q}$ by $\mathbf{Q}_{\rm ab}$.

Unsolved problems about equations motivated the late 1700s and early 1800s device of attaching a (Galois) group to equations. These applications produced key-words: Abelian, solvable and simple groups; composition factors of a finite group; and transitive and primitive permutation representations. This *Group Classification Domain* had a parallel *Equation Classification Domain* with its key- words: elliptic functions, moduli of equations and uniformization. Galois applied *solvability* in algebraic equations to measure how varying equation coefficients changes their solutions. Today a great divide still separates abelian equations from those with simple groups. While equations with nilpotent groups are a natural (though difficult) extension of abelian equations, general solvable equations are another matter. Solvable equations still contribute to graduate algebra, through Galois' famous equation: solvable group = solvable algebraic relation.

A Galois topic (handled by Galois) considered modular curve covers of the j-line. The curves are upper half-plane quotients by the subgroups $\Gamma_0(p^{k+1})$ of $SL_2(\mathbf{Z})$. Modular functions give coordinates for such curves. It is their relation to the variable $j = j(\tau)$ from complex variables that he was testing for solvability. He found that the groups of these covers were rarely solvable, — noting the exceptions. These groups have $PSL_2(p)$ (p a prime) as a quotient. This usually is simple. Yet, as k increases, the $PSL_2(p^{k+1})$ quotients accrue more p-group behavior.

Extensions of simple groups by nonsplit p-group tails were a big theme in Galois' short life. Galois and modular curves: could that be? Yes! Documented on the last pages of [12] is a story corroborating Galois' problems with Cauchy and his essential *suicide* the morning of May 30, 1832. It is a story far sadder, yet more significant to credit in mathematics, than any legend preceding it. There are examples where modern mathematics honours this tradition. Galois recognized the significance of the two special embeddings of $\mathrm{PSL}_2(5) = A_5$ in $\mathrm{PSL}_2(11)$. In their relation to the *buckyball*, [10] suggests why Nature picks one embedding over the other. That's the simple group representation theory part of Galois' work. The book [13] was a long investigation into that p-projective tail that Galois discovered. Its topic is the dynamics of G_K action on projective systems of points (over a value of j in K) on modular curve towers: Serre's renowned *open image theorem*.

You won't, however, find the relation between the Inverse Galois Problem and the works of Abel, Galois and Riemann in the book under review. Finding which groups are Galois groups of regular extensions of arithmetic fields dominates any secondary themes. Regular extensions over $\mathbf{Q}$ are synonymous with geometric curve covers whose automorphisms have definition field $\mathbf{Q}$. This book assumes that the Inverse Galois Problem is significant without question. Few special problems inform us beyond their computational consequences.

The book under review has the feel of group theory emphasizing computation over inspiration. Yet, with all it tackles, it can't avoid resonant problems resisting manifold technique. For example, on page 245 the authors apply their version of [3]. This is a (braid group) criterion for checking, in a family of genus 0 curves, if some member has a $\mathbf{Q}$ rational point. Their example starts with a regular realization of the Mathieu group M_{24}. It comes from the Galois closure of members of a family of genus 0 covers. The goal is a regular realization of M_{23}. Such would arise if one of those genus 0 curves in the family has a rational point. The style is reminiscent of examples of Hilbert. Mestre ([11] or [14, §9.3]) recently applied it to go from realization of spin cover representations of A_n with n odd to n even. It brings an echo of a Thompson phrase– "In a lecture of an hour, I'd have more success explaining the *Monster* than the Mathieu groups."

Chapter I explains and applies the rigidity method, a special case of the so- called *braid-rigidity* method from [4] (see below). While it applies only to groups with very special conjugacy classes, it opened a territory of Galois group realizations in the late 1980s.

In Chapter II, on applications of rigidity, the authors apply Chevalley simple groups and the rigidity technique alone. Satisfaction of simple linear algebra conditions allows realizing such groups over $\mathbf{Q}_{\text{ab}}$. It starts with generators of the classical groups that satisfy Belyi's criterion [1], having one common large eigenspace. This chapter is the book's attempt to prove Shafarevich's conjecture: $G_{\mathbf{Q}_{\text{ab}}}$ is pro-free. Since $G_{\mathbf{Q}_{\text{ab}}}$ is projective (in the category of profinite groups), a technical result reduces this conjecture to proving that every single finite simple group has a special regular realization over $\mathbf{Q}_{\text{ab}}$. You can't leave out even one simple group. Thus, the chapter runs parallel to aspects of the classification of finite simple groups. The authors manage to get all sporadic simple groups. As expected, exceptional Lie-type groups are a big problem. The simple groups they get and those they did not appear in a list in §10.

Dedicated experts might already know much of the material from Chapter I. That also holds in Chapter II, except the relevant experts change. Exceptional groups of Lie Type don't have standard matrix representations. Rather, one uses pure character theory to apply basic rigidity. The prestige of Deligne-Lustig results will recommend §5 especially. Even near experts may not have met this compendium of facts.

The book [15, Chap. 8] uses a Matzat idea, *GAR* realizations, that appears here and should attract many. It continues to appear in the continuation to §7 for the sporadic groups.

Serre's book [14] constructs a story around the *Monster*. That simple group interests mathematicians. Serre and others worked it into questions. Where does the classification stand? Can the general mathematician use it efficiently?

By contrast, the end of Chapter II feels like the end of the trail. Applying rigidity to the sporadic simple groups is a finite task. Its finality depends on a belief there are no unknown sporadic simple groups.

Chapter III applies braid action, necessary to understand moduli of algebraic equations. The Inverse Galois technique started with monodromy action through braids (what [15] calls braid-rigidity), in [4]. Though technically more difficult that its special rigidity technique, it also allows including connections to classical spaces, like modular curves. The group for such moduli questions is the *Hurwitz* (*monodromy* or braid) group H_r. It corresponds to the monodromy from deformation of Riemann surface covers of the sphere branched at r points. The Artin braid group on r strings has H_r as a quotient.

The chapter starts with pleasant presentations of group results: Theorem 1.13 shows that H_r is residually finite. It misses an opportunity when it alludes, without precise quotation, to a result of Lyndon and Schupp. The consequence is that H_r has solvable word problem. In practice, though H_r has many relations, it is a group you can understand. So you can often compute nicely with the moduli spaces produced from it.

The authors use early results (including [2] and [5]) that turned the Inverse Galois Problem into an existential diophantine problem about rational points on Hurwitz spaces. This led in two directions. One assumed conditions ensuring that the Hurwitz space cover of projective r-space is somewhat trivial. Völklein has been single-minded and successful in this direction. The book under review documents this from six years ago, with [16] giving an extensive update. In their quest for the Inverse Galois Problem the authors add something to the potential of their method in §4.2 on the braid orbit genera. So far this has worked only for covers with four branch points. Still, it is now in their book for the next generation.

The other direction has been to prove versions of the Inverse Galois Problem over large fields. Again, one can go either for classical connections or entertaining results. For the former: [9] shows that the field of totally real numbers, from the theory of complex multiplication, has absolute Galois group profreely generated by involutions. For the latter: [8] shows that there is an exact sequence

$$1 \to \tilde{F}_\omega \to G_{\mathbf{Q}} \to \prod_{n=2}^{\infty} S_n \to 1.$$

The group on the left is the profree group on a countable number of generators. The group on the right is the direct product of the symmetric groups, one copy for each integer. This catches $G_{\mathbf{Q}}$ between two known groups.

In Chapter IV, the book finally lightens. An interesting example comes from the definition of semi-abelian groups. These are groups generated by a finite set of abelian subgroups, $A_1, \ldots, A_n$ with A_i in the normalizer in G of A_j for $j \geq i$. Theorem 2.7: G semi-abelian is equivalent to G is a homomorphic image of $A \times^s U$, with A abelian and U a proper semi-abelian subgroup of G. It uses wreath products to conclude that every semi-abelian group is a Galois group over a Hilbertian field.

Here, attentiveness to useful details leads to Wreath products, barely mentioned in [14]. This is one of the few general constructive tools in the area. Further topics include *GAR* realizations (close in spirit to the way Hilbert realized A_n). This commands a fine observation: if a group has composition factors with GAR realizations, then the group has a realization over the field. Shafarevich's conjecture would follow from showing each simple group has a GAR realization over $\mathbf{Q}_{\rm ab}$.

A favorite topic of theirs, central Frattini extensions, deserves high marks and a favorable comparison with [14].

The book is much about examples illustrating technique. Most readers would require applications from the literature calling for these techniques. These might inspire new researchers to say: This work extends, say, that of Klein for producing Riemann surface covers of significance. Other writers might mine the book's examples for producing results on the significance the moduli of algebraic equations.

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University of California, Irvine — Michael D. Fried