REM There is an even stronger representation called weakly homogeneously j-Suslin. This one combines a j-Suslin with a $U(j)$-representation. These representations are easier to propagate through $L(V_{\lambda+1})$

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PROOF SKETCH OF GENERIC ABSOLUTENESS RESULT

Def. T is a uniform (j,κ)-Suslin representation for A if for ~~for~~ any $\vec{x} \nearrow \lambda$: $T^{*}(\vec{x})$ is a (j,κ)-Suslin representation for A, given continuously

Main Lemma Suppose $T \in L_\delta(V_{\lambda+1})$ is a uniform (j,κ)-Suslin representation for $A \subseteq V_{\lambda+1}$ s.t. $j(T) = T$ and $\vec{\kappa}$ is the critical sequence of j. Then

$$M_\omega[\vec{\kappa}] \cap A \neq \emptyset$$

Pf Assume $j(\delta) = \delta$ and call

- $j_{0,\omega}(T) = T^\omega$
- $T^* = T^\omega(\vec{\kappa})$

<u>Claim 1</u> T^* is illfounded

<u>Claim 2</u> This implies $M_\omega[\vec{\kappa}] \cap A \neq \emptyset$

<u>Proof of Claim 1</u> Since $A \neq \emptyset$, $T(\vec{\kappa})$ is illfounded. Let b be a branch through $T(\vec{\kappa})$.

$$\begin{tikzcd}
b \in [T(\vec{\kappa})] \arrow[d] & & b^* \in [T^\omega(\vec{\kappa})] = [T^*] \\
b \in [T(\vec{\delta})] \arrow[rr, "j_{0,\omega}"] & & j_{0\omega}(\vec{b}) \in [T^\omega(\vec{\delta})] \arrow[u]
\end{tikzcd}$$

<u>Def</u> Let j be an I_0-embedding. Fix δ with $j(\delta) = \delta$. An ~~[illegible]~~ inverse limit root of j is a sequence $\langle k_n \mid n \in \omega \rangle$ s.t.

(1) $\forall i \in \omega \quad k_i : L_{\delta+1}(V_{\lambda+1}) \to L_{\delta+1}(V_{\lambda+1})$

(2) $\forall i \in \omega \quad k_i(k_i \restriction V_\lambda) = j \restriction V_\lambda$

(3) $\forall i \in \omega \quad k_0 \restriction V_\lambda, \ldots, k_i \restriction V_\lambda \in \operatorname{rng}(k_{i+1})$

<u>Definition</u> For $\langle k_i \mid i \in \omega \rangle$ an inverse limit root of j define the partial functions

$K^e, K^v : L_\delta(V_{\lambda+1}) \to L_\delta(V_{\lambda+1})$

as follows:

$$k^e(a) = \lim_{\substack{n \to \infty \\ \omega}} (k_0 \circ \cdots \circ k_n)(a)$$

if this limit exists

$$k^e(\vec{a}) = (k^e(a_0), k^e(a_1), \ldots)$$

$$k^*(a) = \lim_{n \to \omega} (k_0 \circ \cdots \circ k_n)^{-1}(a)$$

if this limit exists

Example Suppose $\langle \kappa_i \mid i<\omega \rangle$ is increasing cofinal in λ and $\forall n$ $\langle \kappa_i \mid i<\omega \rangle \in \operatorname{rng}(k_n)$

wt $k^*(\langle \kappa_i \mid i<\omega \rangle) = \langle \delta_i \mid i\in\omega \rangle$

such that $\langle \delta_i \mid i\in\omega \rangle$ is increasing cofinal in $\bar\lambda \overset{\text{def}}{=} \sup_{n\in\omega} \operatorname{crit}(k_n)$. (the assumptions imply that $\operatorname{crit}(k_i)$ are increasing:

Fact 1 $k(k \restriction V_\lambda) = j \restriction V_\lambda$ so $k(\operatorname{crit}(k)) = \operatorname{crit}(j)$,

and Fact 2 $k_i \in \operatorname{rng} \cancel{\ }(k_{i+1}) \Rightarrow \operatorname{crit}(k_i) < \operatorname{crit}(k_{i+1})$)

So $k^e(\langle \delta_i \mid i<\omega \rangle) = \langle \kappa_i \mid i<\omega \rangle$.

~~Lemma~~ Def $\operatorname{Fix}^\delta(j) = \{ a \in L_\delta(V_{\lambda+1}) \mid j(a) = a \}$

$$\operatorname{Fix}^\delta_\omega(j) = \bigcup_{n<\omega} \operatorname{Fix}^\delta(j_n).$$

Lemma Suppose $a \in Fix^\delta_\omega(j)$ and $\forall n \in \omega$ $a \in rng(k_n)$

Then $k^*(a)$ exists and

$$k^e(k^*(a)) = a$$

Proof Let m be s.t. $a \in Fix^\delta(j_m)$

Prove by induction that

$$k_n((k_0 \circ \cdots \circ k_{n-1})^{-1}(a)) = (k_0 \circ \cdots \circ k_{n-1})^{-1}(a)$$

$m=0$ $a \in Fix^\delta(j)$, so $j(a)=a$. Since $a \in rng(k_0)$:

$k_0(a) = j(a) = a$

$m>0$ $a \in Fix^\delta(j_m)$

then $k_0^{-1}(a) \in Fix^\delta(j_{m-1})$.

Now apply the lemma to $\langle k_i \mid 1 \le i < \omega \rangle$ and $k_0^{-1}(a)$. $\boxtimes$(lem)

Now $b = (\vec{x}, \vec{c}) \in [T(\vec{\kappa})]$ where $\vec{\kappa}$ is fixed in advance, with everything in the range

$$\begin{array}{c} b = (\vec{x}, \vec{c}) \in [T(\vec{\kappa})] \\ \downarrow k^* \\ \bar{b} = (\vec{y}, \vec{d}) \in [T(\vec{\delta})] \end{array}$$

$k^*(\vec{\kappa}) = \vec{\delta}$

$k^*(\vec{x}) = \vec{y} \in V^\omega_{\bar{\lambda}}$

$k^*(\vec{c})$ exists

To see that $\bar{b} \in [T(\vec{\delta})]$ follows as $j(T) = T$ and then by elementarity and continuity.

$$\begin{array}{ccc} \bar{b} = (\vec{y}, \vec{d}) \in [T(\vec{\delta})] & \xrightarrow{j_{0,\omega}} & j_{0,\omega}(\bar{b}) = (\vec{y}, j_{0,\omega}(\vec{d})) \in T^{\omega}(\vec{\delta}) \\ & & \Big\downarrow \pi^{e} \\ & & [T^{\omega}(\vec{n})] \ni b^{*} = (\vec{x}, \pi^{e}(j_{0,\omega}(\vec{d}))) \end{array}$$

Need to check the existence (*)

(*) <u>Check</u>: $\pi^{e}(j_{0,\omega}(\vec{d}))$ exists.

$\pi^{*}(\vec{c}) = \vec{d}$. Let n be arbitrary.

$\pi^{*}(c_n) = d_n = \lim\limits_{m \to \omega} (k_0 \circ \dots \circ k_m)^{-1}(c_n)$

For large enough m: $k_m(d_n) = d_n$

$j(j_{0,\omega}(d_n)) = j(j)_{0,\omega}(j(d_n)) = j_{1,\omega}(j(d_n)) = j_{0,\omega}(d_n)$

$k_m(j_{0,\omega}(d_n)) = j_{0,\omega}(d_n)$.

So T^{*} is illfounded $\boxtimes$ (Claim 1).

So by absoluteness, there is a branch $b_1^{*} \in [T^{*}]$ in $M_\omega[\vec{n}]$.

$b^{1,*} = (\vec{x}', \vec{c}')$

We need to show $\vec{x}' \in p[T(\vec{n}')]$.

This is essentially the same argument as above, but we work in reverse. $\boxtimes$ Claim 2, $\boxtimes$ Main Lemma