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EXTERNAL ULTRAPOWERS OF HOD BY

We assume $V = L(\mathbb{R}) + AD$

Def $M \subseteq L(\mathbb{R})$ is an inner model and $\mu \in L(\mathbb{R})$ a measure on some ordinal.

Define external ultrapower of M, $Ext(M, \mu)$ using all functions $f \in V$.

$$j_\mu^M : M \longrightarrow Ext(M, \mu)$$

If $M \models AC$ then j_μ^M is fully elementary

Background Woodin used external ultrapower of HOD to show

$$\cancel{HOD}\ \delta_1^2 \text{ is strong up to } \Theta \text{ in HOD}$$

Def (Jackson) W_1^1 is the club measure on ω_1

Will be studying $j_{W_1^1} : HOD \longrightarrow Ext(HOD, W_1^1)$.

Outline

(1) (ω_1, ω_2) - extender from $j_{W_1^1}$

(2) codes for ordinals less than ω_2

(3) as much $j_{\delta_1^1}$ as I can

(4) Global results

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Review of Jackson-Ketchersid paper

(Comparison initial segment property + supercompactness quantification)

(1) HOD is minimal

Def M is minimal (→ $L(E)$-model) iff $\exists$ a sentence Θ s.t.

(1) $L[M|\delta^M] \not\models \Theta$

(2) $\forall \gamma < \delta^M \ L[M \restriction \gamma] \models \Theta$

(2) M, N are l.M. with minimal witnesses by Θ and if M, N have comparison

$$M \rightsquigarrow P$$
$$N \rightsquigarrow Q$$

} with no drops on either side

$J\omega_1$ below ω_2

Thm HOD and $Ext(HOD, \omega_1^1)$ have a successful comparison (Steel)

Notation $\mu = \omega_1^1 \cap HOD$

$Ult^\alpha(HOD, \mu)$ = the α-th u.p.

$i_\alpha : HOD \longrightarrow Ult^\alpha(HOD, \mu)$ the embedding

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Thm if HOD $\rightsquigarrow P$

Ext $\rightsquigarrow Q$

Then in the first ω_2 steps the Ext does not move while HOD iterates μ.

ω_2^V is the least measurable. Analogously HOD and ω_1. $\boxtimes$

Thm Let E be (ω_1, ω_2)-extender derived from j_{ω_1} and let $\{\xi_\alpha \mid \alpha < \gamma\}$ enumerates the generators. Let μ_α = the measure on ξ_α derived from

$$\mathrm{Ult}(\mathrm{HOD}, E \restriction \xi_\alpha) \to \mathrm{Ult}(\mathrm{HOD}, E)$$

then

(1) $\xi_\alpha = i_\alpha(\omega_1)$

(2) $\mu_\alpha = i_\alpha(\mu)$

Lemma (J-K) Let M be an iterate of HOD. Let $\kappa \in M$ be an M-cardinal s.t. there are no total extenders overlapping κ. Let $\Gamma \subseteq On$ be a proper class. Then

$$\mathcal{P}(\kappa)^M \subseteq \mathrm{Hull}^M(\kappa \cup \Gamma)$$

Pf: $M \overset{\mathcal{T}}{\rightsquigarrow} P$

Hull in Q $\qquad \mathrm{crit}(i_{\mathcal{T}}) \geq \kappa$. $\square$

Pf of Thm IH at $\alpha < \omega_2$. If

(i) $\beta \leq \alpha \longrightarrow \xi_\beta = i_\beta(\omega_1)$

(ii) $\mu_\beta = i_\beta(\mu)$

Verify for $\alpha+1$: Assume $\xi_{\alpha+1} < i_{\alpha+1}(\omega_1)$

Now $\xi_{\alpha+1} = i_{\alpha,\alpha+1}(f)(\xi_\alpha)$

where $f : \xi_\alpha \longrightarrow \xi_\alpha$

f is definable in $\mathrm{Ult}^\alpha(\mathrm{HOD}, \mu)$ from ordinals in Γ where $\Gamma \subseteq \mathrm{On}$ is fixed by i_σ, i_ψ and k. So $i_{\alpha,\alpha+1}(f)$ is definable in $\mathrm{Ult}^{\alpha+1}(\mathrm{HOD}, \mu)$ from $\xi_\alpha \cup \Gamma$ so $\xi_{\alpha+1}$ is definable in $\mathrm{Ult}^{\alpha+1}(\mathrm{HOD}, \mu)$ from $\xi_\alpha \cup \{\xi_\alpha\} \cup \Gamma$.

Point $\xi_{\alpha+1} = \tau(\bar\beta)$ where $\bar\beta$ is fixed by all maps. So

$$\xi_{\alpha+1} = \tau^P(\bar\beta) = \tau^Q(\bar\beta) = i_\psi(k(\xi_{\alpha+1})) > \xi_{\alpha+1}$$

$$\begin{tikzcd} P & Q \\ \mathrm{Ult}^{\alpha+1} \arrow[u, "\sigma"] \arrow[r, "k"] & \mathrm{Ext} \arrow[u, "\psi"'] \end{tikzcd}$$

Assume $A \in \mu_{\alpha+1} \setminus i_{\alpha+1}(\mu)$.

Similarly $A = \tau^{\mathrm{Ult}^{\alpha+1}}(\bar{\beta})$ where $\bar{\beta}$ is fixed by k, i_{γ}, i_{ψ}. Then

$i_{\alpha}(\omega_1) \in k(A) = \tau^{\mathrm{Ext}}(\bar{\beta})$

$i_{\alpha}(\omega_1) \notin i_{\gamma}(A) = \tau^{P}(\bar{\beta})$

$\mathrm{crit}(i_{\psi}) > i_{\alpha+1}(\omega_1)$

$i_{\alpha+1}(\omega_1) \in i_{\psi}(k(A)) = \tau^{Q}(\bar{\beta}) = \tau^{P}(\bar{\beta})$ ⊠

<u>Codes for ordinals ω_ω</u>

<u>Fact</u> (Martin): $j_{W_1^n}(\omega_1) = \omega_{n+1}$.

(essentially $\delta^1_{3} = \omega_{\omega+1}$)

So $f: \omega_1^n \longrightarrow \omega_1$, view f as coding $[f]_{W_1^n} < \omega_\omega$

<u>Question</u> (J-K) Which ordinals are coded by $f \in \mathrm{HOD}$

<u>Def</u> $f: \omega_1^n \longrightarrow \omega_1$ "codes a gap" iff
$f \in \mathrm{HOD}$

$\sup\{[g] \mid g \in \mathrm{HOD} \wedge [g] < [f]\} < [f]$.

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Thm (J-K) For $f : \omega_1 \to \omega_1$, $f \in \mathrm{HOD}$, f does not begin a gap.

Thm $f : \omega_1^n \to \omega$, $f \in \mathrm{HOD}$ then f begins a gap iff
$$\mathrm{cof}^{\mathrm{Ult}(\mathrm{HOD},\mu^n)}([f]_{\mu^n}) \in \{ i_1(\omega_1), \dots, i_{n-1}(\omega_1) \}$$

Pf for $n = 2$.

Key idea:

$$\begin{array}{ccc} \mathrm{HOD} & \xrightarrow{i_{\mu^2}} & \mathrm{Ult}(\mathrm{HOD}, \mu^2) \\ \downarrow j_{\omega_1^2} & \swarrow k & \\ \mathrm{Ext}(\mathrm{HOD}, \omega_1^2) & & \end{array}$$

Analyze k. Let F be extender that comes from iterating $i_1(\mu)$ ω_2 times

Fact 1 $\mathrm{Ult}(\mathrm{Ult}(\mathrm{HOD},\mu^2), F) = \mathrm{Ult}^{\omega_2+1}(\mathrm{HOD},\mu)$

Pf Let $M \lhd \mathrm{Ult}(\mathrm{HOD}, \mu \times \mu)$

WTS: $i_F(M) \lhd \mathrm{Ult}^{\omega_2+1}(\mathrm{HOD},\mu)$.

$\mathrm{Ult}(\mathrm{HOD},\mu) \models M$ is an I.S. of ~~...~~ any u.p. by $i_1(\mu)$

Now apply F. $\boxtimes$

Fact F is the $(i(\omega_1), \omega_2)$-extender derived from k.

Pf

$$\begin{tikzcd}
P \arrow[rr, phantom, "="] & & Q \\
\mathrm{Ult}^{\omega^2+1}(\mathrm{HOD}, \mu) \arrow[u, "i_\sigma"] & & \mathrm{Ext}(\mathrm{HOD}, \omega_1^2) \arrow[u, squiggly, "i_y"'] \\
\mathrm{Ult}(\mathrm{HOD}, \mu^2) \arrow[u] \arrow[urr] & &
\end{tikzcd}$$

Now run the argument as above using definability from a class of ordinals that are fixed. $\boxtimes$

$$\begin{tikzcd}
\mathrm{HOD} \arrow[rr, "i_{\mu^2}"] \arrow[d, "j_{\omega_1^2}"'] & & \mathrm{Ult}(\mathrm{HOD}, \mu^2) \arrow[d, "i_F"] \arrow[dll, "k"] \\
\mathrm{Ext}(\mathrm{HOD}, \omega_1^2) & & \mathrm{Ult}^{\omega_2+1}(\mathrm{HOD}, \mu) \arrow[ll, "k'"]
\end{tikzcd}$$

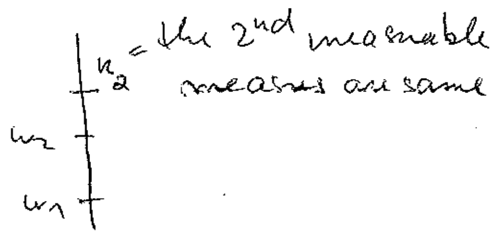
f begins a gap iff k is discontinuous at $[f]_{\mu^2}$.

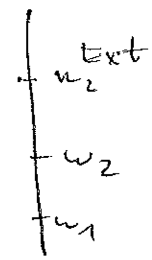
$\mathrm{crit}(k') = i_{\omega_2+1}(\omega_1)$ and $i_{\omega_2+1}(\omega_1) > i_F([f]_{\mu\times\mu})$

so k is discontinuous at $[f]_{\mu\times\mu}$ iff i_F is discontinuous at $[f]_{\mu\times\mu}$ iff $\mathrm{cf}^{\mathrm{Ult}(\mathrm{HOD}, \mu\times\mu)}([f]_{\mu\times\mu}) = i_1(\omega_1)$ $\boxtimes$

J_{ω_1} and the order 0 measures

Let κ be the least measurable of Mitchell order 1 in HOD and let $\gamma = \sup J_{\omega_1}[\kappa]$. We can analyze the (ω_1, γ)-extender from J_{ω_1}.





Ext

$$\begin{array}{c} \mathrm{Ult}^{\omega_\alpha}(\mathrm{HOD}, \mu) \\ \Big\uparrow {\scriptstyle i_{\omega_2}(\bar\kappa_2)} \\ \mathrm{HOD} \end{array}$$

i_{ω_2} is continuous at $\bar\kappa_2$ while J_{ω_1} is not.