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# Bukovský and the long extender algebra

Def $W \subseteq V$. Let $\lambda$ be a cardinal. We say that $W$ uniformly $\lambda$-covers $V$ iff f.a. $f : \theta \to On$, $f \in V$ there is some $g \in W$ s.t.

- $dom(g) = \theta$
- $f(\xi) \in g(\xi)$ all $\xi < \theta$
- $card(g(\xi)) < \lambda$ for all $\xi < \theta$

Thm Let $W \subseteq V$ such that $W$ uniformly $\lambda$-covers $V$ and $\mathcal{P}(2^{<\lambda}) \subseteq W$. Then $W = V$.

Thm (Bukovský) Let $W \subseteq V$ uniformly $\lambda$-covers $V$. Then $V$ is a generic extension of $W$.

Proof Work in $W$. Let $\mathcal{L}$ be the following infinitary language: Fixing $\lambda \leq \mu$:

Atomic formulae: "$\xi \in \dot{a}$" for $\xi < \mu$

Close under $\neg$ and $<\lambda$-conjunctions.

For $A \subseteq \mu$: Define "$A \models \varphi$". We write $A \models \Gamma$ iff $A \models \varphi$ for all $\varphi \in \Gamma$. Note $card(\mathcal{L}) = \mu^{<\lambda}$

Write $\Gamma \vdash \varphi$ (when $\Gamma \in W$) iff for all $A \subseteq \mu$, $A \in W^{Coll(\omega, \mu^{<\lambda})}$ we have $A \models \Gamma \Rightarrow A \models \varphi$

(equivalently $A \models \Gamma \Rightarrow A \models \varphi$ for any $A$)

$\Gamma$ is consistent iff $\exists A \in W^{Coll(\omega, \mu^{<\lambda})}$ $A \models \Gamma$

From now on write $\mathcal{L}$ for a (~~consistent set~~) set of all consistent set of formulae

Let $g : [\mathcal{L}]^{\lambda}_{\cap W} \longrightarrow [\mathcal{L}]^{<\lambda} \cap W$, $g \in W$

s.t. $g(\Gamma) \subseteq \Gamma$ for all $\Gamma$

Call $\varphi$ illegal iff $\varphi \in \Gamma \setminus g(\Gamma)$

$T^g$ = the set of all statements of the form

$$\varphi \rightarrow \bigvee g(\Gamma)$$

where $\varphi$ is illegal $\varphi \in \Gamma \setminus g(\Gamma)$, $\Gamma \in [\mathcal{L}]^{\lambda}$.

$\mathbb{P}^g = \{\varphi \in \mathcal{L} \mid T^g \cup \{\varphi\}$ is consistent $\}$

$\varphi \leq_{\mathbb{P}^g} \psi$ iff $T^g \cup \{\varphi\} \vdash \psi$

$(\mathbb{P}^g, \leq_{\mathbb{P}^g}) \in W$

<u>claim</u> $\mathbb{P}^g$ has the $\lambda$-c.c. in $W$.

<u>Pf</u> If $\Gamma \subseteq [\mathcal{L}]^{\lambda} \cap W$ pick $\varphi \in \Gamma \setminus \varphi(\Gamma)$

$T^g \vdash \varphi \rightarrow \bigvee g(\Gamma)$. So $\Gamma$ cannot be an antichain. ☒

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$A \in M$ $\quad G_A = \{\varphi \in \mathbb{P}^g \mid A \models \varphi\}$

<u>Claim 2</u> $A \models T^g \Rightarrow G_A$ is $\mathbb{P}^g$-generic $/W$

and $A \in V[G_A]$

<u>Pf</u>: $A \models \varphi$, $\varphi \leq \psi \Rightarrow A \models \psi$ By absoluteness (*)

$A \models \varphi$, $A \models \psi \Rightarrow A \models \varphi \wedge \psi$ straightforward

Let $\Gamma$ be a MAC, $\Gamma \in W$. So $\mathrm{card}(\Gamma) < \lambda$ in $W$.

If $G_A \cap \Gamma = \emptyset$ $A \models \neg\bigvee\Gamma$ $A \models T^g \Rightarrow \neg\bigvee\Gamma \in \mathbb{P}^g$

So $\Gamma \cup \{\neg\bigvee\Gamma\}$ is an antichain ↯. ☒ Claim 2

Proof of Bukovsky' Pick $g : [\mathcal{L}]^{\lambda} \cap W \to [\mathcal{L}]^{<\lambda} \cap W$ s.t.

- $g(\Gamma) \subset \Gamma$ all $\Gamma$
- If $A \models \varphi$ for some $\varphi \in \Gamma$ then $A \models \bigvee g(\Gamma)$ for any $A \in V$

Here we set $\mu = 2^{<\lambda}$ and let $A \subseteq \mu$ be s.t. $\mathcal{O}(2^{<\lambda}) \subseteq L[A]$.

Have $A \models T^g$. This proves Bukovsky'.

Here $g$ is obtained by applying uniform $\lambda$-covering to the function $f : \Gamma \mapsto \varphi$ where

$$f(\Gamma) = \begin{cases} \varphi & \text{if } \varphi \in A \text{ s.t. } A \models \varphi \text{ exists (+ pick some such } \varphi) \\ \varphi_0 & \text{o.w.} \end{cases}$$

This $f$ is in $V$.

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The long extender algebra

Fix $W \subseteq V$, $\mu \geq \lambda$ regular in $V$, $\mu$ a $V$-cardinal.

Work in $W$. Fix $\mathcal{E}$ a collection of extenders s.t. $\mathrm{crit}(E) < \sigma(E) < \lambda$ for all $E \in \mathcal{E}$. Let

$\sigma(E)$ = space of $E$

$T^{\mathcal{E}}$ = the collection of all formulae of the form

$$\varphi \longrightarrow \bigvee \pi_E[\Gamma]$$

where $E \in \mathcal{E}$, $\Gamma \in [\mathcal{L}]^{\mathrm{crit}(E)} \cap V_{\sigma(E)}$,

$\varphi \in \pi_E(\Gamma) \cap V_{\rho(E)}$

Here $\rho(E) =$ strength $(E)$.

Say $\mathcal{E}$ is rich iff f.a. $\Gamma \in [\mathcal{X}]^\lambda$ there is some $E \in \mathcal{E}$ s.t.

- $\pi_E(\mathrm{crit}(E)) = \lambda$
- $\Gamma \in \mathrm{rng}(\pi_E) \cap V_{\rho(E)}$
- $\pi_E^{-1}(\Gamma) \in V_{\sigma(E)}$

REM If $\lambda$ is $(\lambda^{+2})$ supercompact then ~~there~~ is rich $\mathcal{E}$

If $\mathcal{E}$ is rich define $g : [\mathcal{X}]^\lambda \longrightarrow [\mathcal{X}]^{<\lambda}$ as follows
Let $\Gamma \in [\mathcal{X}]^\lambda$. Pick $E \in \mathcal{E}$ s.t. $\Gamma \in \mathrm{rng}(\pi_E) \cap V_{\rho(E)}$. Let

$$g(\Gamma) = \Gamma \cap \mathrm{rng}(\pi_E) = \pi_E[\pi_E^{-1}(\Gamma)]$$

Recall

$T^g =$ the collection of all $\varphi \rightarrow \bigvee g(\Gamma)$ s.t. $\varphi \in \Gamma \setminus g(\Gamma)$

Claim 1 Suppose $\mathcal{E}$ is rich. Then

$$T^{\mathcal{E}} \vdash T^g$$

Proof Consider "$\varphi \rightarrow \bigvee g(\Gamma)$" $\in T^g$.
Assume $E$ was used to define $g(\Gamma)$.
Let $\bar{\Gamma} = \pi_E^{-1}(\Gamma)$

$\bigvee \pi_E[\bar{\Gamma}] = \bigvee(\mathrm{rng}(\pi_E) \cap \Gamma) = \bigvee g(\Gamma)$ $\boxtimes$

$\mathbb{P}^{\mathcal{E}} = \mathbb{P}^g$

Claim 2 $\mathbb{P}^{\mathcal{E}}$ has the $\lambda$-c.c.

Claim 3 Suppose $\mathcal{E} \in W$ is rich in $W$. Assume $A \subseteq \mu$, $A \in V$ is s.t. $A \models T^{\mathcal{E}}$

Then $G_A$ (as defined before) is $\mathbb{P}^{\mathcal{E}}$-generic / $W$ and $A \in W[G_A]$.

Lemma $W \subseteq V$, $\lambda \leq \mu$ as before. Let $\mathcal{E} \in W$ be rich in $W$. Let $A \subseteq \mu$, $A \in V$ be s.t. for every $E \in \mathcal{E}$ there is some $\tilde{\pi} \supseteq \pi_E^W$, $\tilde{\pi} : V \to M$ s.t. $A \in \operatorname{rng}(\tilde{\pi})$. Then $G_A$ is $\mathbb{P}^{\mathcal{E}}$-generic / $W$ + $A \in W[G_A]$.

To see $A \in T^{\mathcal{E}}$: Say $A \models \varphi^{\in \Gamma}$. Have $\tilde{\pi} : V \to M$ s.t. $A \in \operatorname{rng}(\tilde{\pi})$ s.t. $M \models$ "$(\exists \varphi' \in \Gamma) A \models \varphi'$". So

$V \models$ "$\exists \varphi' \in \underbrace{(\pi_E^W)^{-1}(\Gamma)}_{\tilde{\pi}^{-1}(\Gamma)}$ $\underbrace{\tilde{\pi}^{-1}(A)}_{\bar{A}} \models \varphi'$"

$V \models$ "$\bar{A} \models \varphi'$" $\qquad M \models$ "$A \models \tilde{\pi}(\varphi)$ $\Rightarrow$ $A \models \bigvee \pi_E^W[\Gamma]$

$\tilde{\pi}(\varphi) = \pi_E^W(\varphi)$