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Weak DJ property for (M, Σ): $M = \{e_i : i \in \omega\}$. Get in the DJ situation: N-to-N does not drop, $i \leq_{lex} \pi$, i.e. if k is s.t. $i(e_k) \neq \pi(e_k)$ and is the least such then $i(e_k) < \pi(e_k)$.

Thm (Existence of cores). Let M be a countable lpm, Σ a complete strategy for M defined on all countable stacks $\vec{s}$ Σ, last model N, $(N, \Sigma_{s,N})$ is a lbr hod pair. Suppose Σ is coded by a set of reals that is Suslin co-Suslin in some $L(\Gamma, \mathbb{R}) \models AD^+$. Then

core((M, Σ)) exists:

(1) $p(M)$ is solid

(2) $M | \rho^{+M} \trianglelefteq \mathrm{Hull}^M_{k(m)+1}(\rho \cup p(M))$ for $\rho = \rho(M)$.

Pf By the usual weak DJ strategy proof: WMA $M = \{e_i : i \in \omega\}$ and Σ has the weak DJ property relative to $\vec{e}$. Let $r = p(M)$.

Assume $r = \langle e_0, \dots, e_k \rangle$ some k, $e_0 > e_1 > \dots > e_k$. Let $r = q \cup s$ where q is the longest solid initial segment of r. $\max(s) < \min(q)$. Assume $q \neq r$ and get a contradiction. Similarly for universality. Let

$\alpha_0 =$ least β s.t. $\mathrm{Th}^M_{k+1}(\beta \cup q) \in M$.

WMA $\alpha_0 \in M$. Let $K =$ the collapse of $\mathrm{Hull}^M(\alpha_0 \cup q)$

$\pi : K \to M$, WMA $\alpha_0 \in K$, OW. $K \triangleleft M$.

Claim 0 (a) if $q = n$ then $\rho = \alpha_0$
(b) if $q \neq n$ then $\rho < \alpha_0 \leq \max(c)$
(c) $K \models \alpha_0$ is a cardinal

Prf strategy: compare (M, K, α_0) vs M.

Use ~~$\Phi(\ldots)$~~ $\Psi^{(id,\pi)}$ to iterate (M, K, α_0), lifts to T on M. Use Ψ to get U on M, on the other side, ~~of~~, i.e. M-side of the comparison. Now use weak DJ to conclude that the last model P on the phalanx side is above K, and $P \trianglelefteq Q = M^U_{\infty}$. This is in the $L[\vec{E}]$ case.

In the current case we are comparing against the background.

Problem: when $P = Q$, P above M,

$i^T : M \to P$ $\quad$ $i^U : M \to Q$ $\quad$ and $i^T = i^U$ we get 1st extenders used and i^U, i^T are compatible. This gives contradiction in the $L[\vec{E}]$-case but not our case, as they may have been hit because of disagreement with an $M^F_{\nu,k}$ and not 2 one between themselves.

Let N^* be a coarse Γ Woodin model, in ~~i.e. a model~~ $L(\Gamma, \mathbb{R})$ s.t. it captures Ψ. M is ctbl in N^* and N^* has a UB code for Ψ. The IS for N^* move the UB ~~code~~ codes of Ψ to UB codes of Ψ.

Let $\mathbb{C}$ be a maximal lpm construction of N^*

Let ν_0, k_0 be s.t. (M, Ψ) iterates to $(M^{\mathbb{C}}_{\nu_0,k_0}, \Omega^{\mathbb{C}}_{\nu_0,k_0})$

iterates strictly past each

$(M_{\eta,e}, \Omega_{\eta,e})$ whenever $\langle \eta, e\rangle <_{lex} \langle \nu_0, k_0\rangle$

Have $\mathcal{U}_{\eta,e}$ for $\langle \eta, e\rangle \leq_{lex} \langle \nu_0, k_0\rangle$ on (M, Ψ) witnessing this.

We now define by induction on pairs $\langle \nu, e\rangle \leq_{lex} \langle \nu_0, k_0\rangle$ normal pseudo trees $\mathcal{Y}_{\nu,e}$ on (M, K, α_0). lifted to $\mathcal{T}_{\nu,e}$ on M via (id, π). As we go in defining $\mathcal{Y}_{\nu,e}$, we look at $\mathcal{U}_{\nu,e}$ and based on that declare certain modes unstable. All exit extenders from $\mathcal{Y}$ come from stable modes. $\mathcal{Y}^{\eta}_{\nu,e}$ has a last model, and it's stable $\mathcal{Y}^{\eta+1}_{\nu,e}$ has ≤ 2 more models than $\mathcal{Y}^{e}_{\nu,e}$

Also define $\lambda^{\mathcal{Y}}_{\theta}$ for $\theta < lh(\mathcal{Y})$. Also define $\pi_\theta : M^{\mathcal{Y}}_\theta \to N^{\mathcal{T}}_\theta$

$\theta \leq \gamma \Rightarrow \lambda^{\mathcal{Y}}_\theta \leq \lambda^{\mathcal{Y}}_\gamma$

Fix ν, e. Let $\mathcal{U} = \mathcal{U}_{\nu,e}$.

$\mathcal{Y} = \mathcal{Y}_{\nu,e}$ is defined by:

- $M^{\mathcal{Y}}_0 = M$ $\quad M^{\mathcal{Y}}_1 = K$ $\quad \lambda^{\mathcal{Y}}_0 = \alpha_0$ $\quad M^{\mathcal{T}}_0 = M^{\mathcal{T}}_1 = M$

  $\pi_0 = id$ $\quad \pi_1 = \pi$ $\quad$ 0 is unstable, 1 is stable.

<u>We will have induction hypos</u>: if θ is unstable:

(i) $0 \leq_{\mathcal{Y}} \theta$ and $[0, \theta]_{\mathcal{Y}}$ does not drop

<u>Def</u> For θ unstable: $\alpha_\theta = \sup i^{\mathcal{Y}}_{0\theta}[\alpha_0]$

Ind hypos again If Θ is unstable then

(i)

(ii) $\lambda^y_\theta \le \alpha_\theta \le \rho_k(M^y_\theta)$

(iii) every $\sigma \le_S \Theta$ is unstable

(iv) $\exists \xi\ \ M^y_\theta = M^u_\xi$

(v) $\rho(M^y_\theta) = \sup i^{y}_{0,\theta}[$

(vi) α_θ = least β s.t. $Th^{M^y_\theta}_{k+1}(\beta \cup i^y_{0,\theta}(q)) \notin M^y_\theta$

At any stage in the definition of $\mathcal{Y}_{r,e}$ have some last model M^y_γ, γ stable.

The construction of $\mathcal{Y}_{r,e}$ can terminate in one of the two ways:

(1) We reach a stable Θ s.t. either

(a) $M_{r,e} \lhd M^y_\theta$ (and $\Omega_{r,e} = \Sigma_{\Theta, M_{r,e}}$)

(b) $M^y_\theta \unlhd M_{r,e}$ and $[root(\theta), \theta]_S$ does not drop in model or degree

This is a successful comparison of (M, k, α_0) with M.

(2) We reach a stable Θ s.t. for some ξ $M^y_\theta = M^u_\xi$ and neither ~~~~ $[root(\theta), \theta]_S$ nor $[0,\xi]_u$ has dropped. Moreover, $\varphi = M^y_\theta | \langle \partial(M^y_\theta), -1\rangle$ we have: $\varphi \unlhd M_{r,e}$.

(2) is considered a successful comparison.

If $\gamma = \theta$ does not work as in (1) or (2) then we define $M^y_{\gamma+1}$ and maybe $M^y_{\gamma+2}$.

E^y_γ = the first extender on the sequence of M^y_γ s.t. is not on the $M^\Phi_{\nu,\ell}$ - sequence

Need to prove : $M_{\nu,\ell}|\xi$ is passive

$$(\Sigma_\gamma)_{\langle - \rangle} = (\Omega_{(u,\ell)})_{\langle - \rangle}$$