DIAMOND, GCH AND WEAK SQUARE

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ABSTRACT. Shelah proved recently that if $\kappa > \omega$ and $S \subseteq \kappa^+$ is a stationary set of ordinals of cofinality different from $\mathrm{cf}(\kappa)$ then $2^\kappa = \kappa^+$ implies $\diamondsuit_\kappa(S)$. We show that for singular κ, an elaboration on his argument allows to derive $\diamondsuit_\kappa(T)$ from $2^\kappa = \kappa^+ + \Box^*_\kappa$ where $T = \{\delta < \kappa^+ \mid \mathrm{cf}(\delta) = \mathrm{cf}(\kappa)\}$. This gives a strong restriction on the existence of saturated ideals on κ^+.

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It is a well-known fact that $\diamondsuit_\lambda$ implies $2^{<\lambda} = \lambda$. In many situations the converse is also true. Jensen [3] proved that CH does not imply $\diamondsuit$, so when looking for the converse one has to focus on $\lambda > \omega_1$. Let

$$S^\lambda_\mu = \{\delta < \lambda \mid \mathrm{cf}(\delta) = \mu\}.$$

and

$$T_\kappa = S^{\kappa^+}_{\mathrm{cf}(\kappa)}.$$

Gregory observed that GCH below ω_2 implies $\diamondsuit_{\omega_2}(S^{\omega_2}_\omega)$. A sequence of improvements on his result, mainly by Gregory [4], Jensen (unpublished) and Shelah [7], resulted in the following theorem whose proof can be found in [2].

Theorem 0.1 (Gregory, Jensen, Shelah). *If $2^{<\kappa} = \kappa$ and $2^\kappa = \kappa^+$ then $\diamondsuit_{\kappa^+}(S)$ holds whenever $S \subseteq \kappa^+$ is a stationary set of ordinals of cofinality different from $\mathrm{cf}(\kappa)$. If κ is singular and additionally $\Box_\kappa$ holds then $\diamondsuit_{\kappa^+}(T_\kappa)$.*

Shelah also proved that for regular κ, the condition $2^\kappa = \kappa^+ + \Box_\kappa$ is not sufficient to guarantee $\diamondsuit_{\kappa^+}(T_\kappa)$, so the absolute ZFC result is possible only for singular κ.

The question remained whether the localized GCH, i.e. the equality $2^\kappa = \kappa^+$ alone implies $\diamondsuit_{\kappa^+}$. Shelah proved this to be true for sufficiently large κ, and recently [8] found an argument that proves it for every uncountable cardinal κ; see Komjáth's paper [6] for a simplified proof and an elaboration on Shelah's argument.

Theorem 0.2 (Shelah). *Let $\kappa > \omega$ and $2^\kappa = \kappa^+$. Then $\diamondsuit_{\kappa^+}(S)$ holds for every stationary $S \subseteq \kappa^+$ that is disjoint with T_κ.*

This note combines arguments from the proof of Theorem 0.1 with Shelah's argument for Theorem 0.2 to give a proof of $\diamondsuit_{\kappa^+}(S)$ for $S \subseteq T_\kappa$.

Theorem 0.3 (Main Theorem). *Assume κ is a singular cardinal and $T \subseteq T_\kappa$ is stationary with stationarily many reflection points. Then*

$$2^\kappa = \kappa^+ \; + \; \Box^*_\kappa \implies \diamondsuit_{\kappa^+}(T).$$

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It was proved by Cummings, Foreman and Magidor [1] that for singular κ, the principle $\square^*_\kappa$ is consistent with the requirement that every stationary $T \subseteq T_\kappa$ has stationarily many reflection points. Consequently, in their model we have $\diamondsuit_{\kappa^+}(S)$ for all stationary $S \subseteq \kappa^+$.

Corollary 0.4. *Assume κ is a singular cardinal. Then*

$$2^\kappa = \kappa^+ \ + \ \square^*_\kappa \implies \diamondsuit_\kappa(T_\kappa).$$

Both the above theorem and its corollary provides a very strong restriction on the existence of saturated ideals on κ^+ and provide a close link between the study of such ideals and the PCF-theory.

Rinot recently extended the result of this paper to the situation where weak square is replaced by a variant of approachability property and also showed that, relatively to the existence to a supercompact cardinal, it is consistent that $\square^*_\kappa$ fails but $\diamondsuit_{\kappa^+}(S)$ holds for every stationary $S \subseteq \kappa^+$.

1. The argument

We begin with splitting Shelah's argument into two steps. We first isolate a combinatorial statement that alone implies the existence of a $\diamondsuit_\lambda(S)$-sequence in ZFC; we denote this statement by $\bigcirc_\lambda(S)$. This statement is implicit the arguments in Shelah [8] and Komjáth [6]. It turns out that the implication $\bigcirc_\lambda(S) \implies \diamondsuit_\lambda(S)$ is true no matter whether λ is a successor cardinal or not. The second step is a proof that $\bigcirc_\lambda(S)$ holds, which relies on the localized GCH if $\lambda = \kappa^+$ and S concentrates on points of cofinality distinct from $\mathrm{cf}(\kappa)$ which gives the original Shelah's result, and on the weak square if κ is singular and S concentrates on points of cofinality $\mathrm{cf}(\kappa)$ which gives the result in Theorem 0.3. Our approach owes a lot to Komjáth's exposition in [6].

Definition 1.1. *Let λ be a regular cardinal and $S \subseteq \lambda$. We say that the pair $\langle x_\xi \,|\, \xi < \lambda\rangle$, $\langle A_\delta \,|\, \delta \in S\rangle$ witnesses $\bigcirc_\lambda(S)$ iff the following three conditions are met.*

(a) *$\langle x_\xi \,|\, \xi < \lambda\rangle$ is an enumeration of $[\lambda]^{<\lambda}$.*

(b) *$A_\delta \subseteq \delta$ and $\mathsf{card}(A_\delta) < \mathsf{card}(\delta)$ whenever $\delta \in S$.*

(c) *For every $Z \subseteq \lambda$ there is a stationary $S' \subseteq S$ such that for every $\delta \in S'$ there are unboundedly many $\alpha < \delta$ for which there is $\beta < \delta$ satisfying $\alpha, \beta \in A_\delta$ and $Z \cap \alpha = x_\beta$.*

We say that $\bigcirc_\lambda(S)$ holds iff there are $\langle x_\xi\rangle_\xi$ and $\langle A_\delta\rangle_\delta$ as above.

Notice that $\bigcirc_\lambda(S)$ postulates the existence of an enumeration of $[\lambda]^{<\lambda}$ of length λ, so it imposes some constraints on the behaviour of the exponential function below λ. In particular, if $\lambda = \kappa^+$ then $\bigcirc_\lambda(S)$ implies $2^\kappa = \kappa^+$. Notice also that (b) in the above definition stipulates that the cardinality of A_δ is strictly smaller than that of δ, which together with (c) implies that without loss of generality S can be viewed as a set of singular ordinals. Of course, this has a non-trivial meaning only when λ is inaccessible. Finally observe that if there is a pair $\langle x_\xi \,|\, \xi < \lambda\rangle$, $\langle A_\delta \,|\, \delta \in S\rangle$ witnessing $\bigcirc_\lambda(S)$ then for every enumeration $\langle x'_\xi \,|\, \xi < \lambda\rangle$ there is a sequence $\langle A'_\delta \,|\, \delta \in S\rangle$ such that the pair $\langle x'_\xi\rangle_\xi$, $\langle A'_\delta\rangle_\delta$ witnesses $\bigcirc_\lambda(S)$. To see this, pick any $f : \lambda \to \lambda$ such that $x_\beta = x'_{f(\beta)}$ for all $\beta < \lambda$ and let $A'_\delta = A_\delta \cup f[A_\delta]$ for all $\delta \in S$ satisfying $f[\delta] \subseteq \delta$.

Lemma 1.2. *Let λ be a regular cardinal, $S \subseteq \lambda$ and $\bigcirc_\lambda(S)$ hold. Then there is a pair $\langle x_\xi \,|\, \xi < \lambda\rangle$, $\langle A_\delta \,|\, \delta \in S\rangle$ satisfying the following.*

(a) $\langle x_\xi \,|\, \xi < \lambda \rangle$ *is an enumeration of* $[\lambda \times \lambda]^{<\lambda}$.
(b) $A_\delta \subseteq \delta$ *and* $\mathsf{card}(A_\delta) < \mathsf{card}(\delta)$ *whenever* $\delta \in S$.
(c) *For every* $Z \subseteq \lambda \times \lambda$ *there is a stationary* $S' \subseteq S$ *such that for every* $\delta \in S'$ *there are unboundedly many* $\alpha < \delta$ *for which there is* $\beta < \delta$ *satisfying* $\alpha, \beta \in A_\delta$ *and* $Z \cap (\alpha \times \alpha) = x_\beta \cap (\alpha \times \alpha)$.

Proof. Pick a pair $\langle y_\xi \,|\, \xi < \lambda \rangle$, $\langle B_\delta \,|\, \delta \in S \rangle$ witnessing $\bigcirc_\lambda(S)$. Let $f : \lambda \times \lambda \to \lambda$ be a bijection and $C_f = \{\delta < \lambda \,|\, f[\delta \times \delta] = \delta\}$.

To each $\delta \in S$ pick C_δ to be a subset of $\lim(C_f) \cap \delta$ of size strictly smaller than $\mathsf{card}(\delta)$ that is cofinal in δ if such a set exists; let $C_\delta = \varnothing$ otherwise. Letting $x_\beta = f^{-1}[y_\beta]$ and $A_\delta = B_\delta \cup C_\delta$, we obtain a pair $\langle x_\xi \,|\, \xi < \lambda \rangle$, $\langle A_\delta \,|\, \delta \in S \rangle$ as in the conclusion of the lemma. To see this, it suffices to verify clause (c) in the statement of the lemma.

Given any $Z \subseteq \lambda \times \lambda$, let $S' \subseteq S$ be the stationary set obtained by applying $\bigcirc_\lambda(S)$ to $f[Z]$. Let $\delta \in S' \cap \lim(C_f)$. If $\bar\alpha < \delta$, pick $\alpha \in C_\delta$ such that $\bar\alpha \leq \alpha$. Since S' satisfies (c) in Definition 1.1 with $f[Z], y_\beta$ and B_δ in place of Z, x_β and A_δ, there are $\alpha', \beta \in B_\delta$ such that $\alpha \leq \alpha'$ and $f[Z] \cap \alpha' = y_\beta$. Then $f[Z] \cap \alpha = y_\beta \cap \alpha$ and the conclusion follows immediately from the fact that $\alpha \in C_f$. $\square$

With the statement $\bigcirc_\lambda(S)$ in hand, one can reformulate the first step in Shelah's argument into the following proposition. It reduces the proof of $\diamondsuit_\lambda(S)$ to the proof of $\bigcirc_\lambda(S)$ and works even for cardinals λ that are not successors, which is slightly more than Shelah has originally proved. The second setp in Shelah's argument can be then viewed as a proof of $\bigcirc_{\kappa^+}(S)$ from the localized GCH. We will show how to obtain $\bigcirc_{\kappa^+}(S)$ from the additional assumption that $\square^*_\kappa$ holds in situations where localized GCH does not seem to suffice.

Proposition 1.3. *Let* λ *be regular and* $S \subseteq \lambda$ *be stationary. Then*

$$\bigcirc_\lambda(S) \implies \diamondsuit_\lambda(S)$$

Proof. Let $\langle x_\xi \,|\, \xi < \lambda \rangle$, $\langle A_\delta \,|\, \delta \in S \rangle$ be a pair satisfying the conclusion of Lemma 1.2. For $x \subseteq \lambda \times \lambda$ we write $(x)_\xi$ to denote $\{\zeta < \lambda \,|\, \langle \xi, \zeta \rangle \in x\}$. Consider sequences $\langle X_\xi, C_\xi \,|\, \xi < \theta \rangle$ of length $\theta \leq \lambda$ such that $X_\xi \subseteq \lambda$, C_ξ is closed unbounded in λ and, letting

$$V^\delta_\xi = \{\langle \alpha, \beta \rangle \in A_\delta \times A_\delta \,|\, (\forall \eta < \xi)(X_\eta \cap \alpha = (x_\beta)_\eta \cap \alpha)\},$$

for every $\xi < \theta$ and $\delta \in S \cap C_\xi$ either $\mathrm{dom}(V^\delta_{\xi+1})$ is bounded in δ or else $V^\delta_\xi \supsetneqq V^\delta_{\xi+1}$. Notice that the non-strict inclusion $V^\delta_\xi \supseteq V^\delta_{\xi'}$ holds anyway whenever $\xi \leq \xi'$.

The crucial observation is that any sequence $\langle X_\xi, C_\xi \,|\, \xi < \theta \rangle$ as above has length strictly below λ. Assume for a contradiction that this fails, that is, there is such a sequence with $\theta = \lambda$. Let S' come from the application of Lemma 1.2 to the pair $\langle x_\xi \rangle_\xi$, $\langle A_\delta \rangle_\delta$ and to set

$$Z = \{\langle \xi, \zeta \rangle \,|\, \zeta \in X_\xi\},$$

and let $\delta \in S' \cap \triangle\{C_\xi \,|\, \xi < \lambda\}$ be such that $\delta > \kappa$ if $\lambda = \kappa^+$ and δ is a cardinal if λ is inaccessible. We have arbitrarily large $\alpha < \delta$ for which there exists $\beta < \delta$ such that $\alpha, \beta \in A_\delta$ and $Z \cap (\alpha \times \alpha) = x_\beta \cap (\alpha \times \alpha)$, so for each $\xi < \delta$ the set $\mathrm{dom}(V^\delta_\xi)$ is unbounded in δ. Since $\delta \in S \cap C_\xi$ whenever $\xi < \delta$, from the properties of the sequence $\langle X_\xi, C_\xi \,|\, \xi < \theta \rangle$ we obtain $V^\delta_\xi \supsetneqq V^\delta_{\xi'}$ whenever $\xi < \xi' < \delta$. This is a contradiction, as $V^\delta_\xi \subseteq A_\delta \times A_\delta$ and $\mathsf{card}(A_\delta) < \mathsf{card}(\delta)$.

Pick a sequence $\langle X_\xi, C_\xi \mid \xi < \theta\rangle$ as in the previous paragraph which has no proper extension. Then $\theta < \lambda$. Letting

$$D_\delta = \bigcup\{(x_\beta)_\theta \cap \alpha \mid \langle\alpha,\beta\rangle \in V_\theta^\delta\},$$

the sequence $\langle D_\delta \mid \delta \in S\rangle$ is a $\diamondsuit_\lambda(S)$-sequence. To see this, pick an arbitrary $X \subseteq \lambda$ and a closed unbounded $C \subseteq \lambda$. There exists some $\delta \in S \cap C$ such that $\mathrm{dom}(V_\theta^\delta)$ is unbounded in δ and $X \cap \alpha = (x_\beta)_\theta \cap \alpha$ for all $\langle\alpha,\beta\rangle \in V_\theta^\delta$, as otherwise we could extend the sequence $\langle X_\xi, C_\xi \mid \xi < \theta\rangle$ by letting $X_\theta = X$ and $C_\theta = C$, in contradiction with its maximality. But then $X \cap \delta = D_\delta$. □

We now focus on proofs of $\bigcirc_\lambda(S)$. The point of introducing $\bigcirc_\lambda(S)$ is that it is often easier to give a direct proof of $\bigcirc_\lambda(S)$ than a direct proof of $\diamondsuit_\lambda(S)$. This is clear from Shelah's argument in [8] which in our notation is a proof of $\bigcirc_{\kappa^+}(S)$. As Proposition 1.3 also holds for inaccessible λ, our hope was that Shelah's argument may be used for proofs of $\diamondsuit_\lambda(S)$ for inaccessible λ. It seems, however, that for inaccessible λ the proofs of $\diamondsuit_\lambda(S)$ may require more new ideas. For instance, the proofs of $\diamondsuit_\lambda(S^\lambda_\varepsilon)$ for a Mahlo cardinal λ in [5] and [9] can be easily modified to give proofs of $\bigcirc_\lambda(S^\lambda_\varepsilon)$, but introducing a $\bigcirc_\lambda(S^\lambda_\varepsilon)$-sequence into the construction does not seem to enable any strengthening of the results or a simplification of the construction in [9]. For inaccessibles λ that are not Mahlo it is not clear either whether an argument using $\bigcirc_\lambda(S^\lambda_\varepsilon)$ may work. It is certainly clear that constructions of a $\bigcirc_\lambda(S^\lambda_\varepsilon)$-sequence from "below" as in Propositions 1.4 and 1.5 will not work, essentially for the same reason why constructions of $\diamondsuit_\lambda(S^\lambda_\varepsilon)$ from "below" cannot work, as described in [9]. Analogously as in [9], given any fixed $\bigcirc_\lambda(S^\lambda_\varepsilon)$-witness $\langle x_\beta \mid \beta < \lambda\rangle$, $\langle A_\delta \mid \delta < \lambda\rangle$, there is a $< \lambda$-distributive forcing that "kills" such a witness. On the other hand, any construction of a $\bigcirc_\lambda(S^\lambda_\varepsilon)$-witness from "below" would give rise to the same witness in the ground model and in the generic extension.

Let us turn to the proof of $\bigcirc_{\kappa^+}(S)$. As already mentioned above, the next proposition can be viewed as the first step in Shelah's argument. We include it, as it is a starting point for our variation with weak square.

Proposition 1.4. *Assume $S \subseteq \kappa^+$ is stationary and disjoint from T_κ. Then*

$$2^\kappa = \kappa^+ \implies \bigcirc_{\kappa^+}(S).$$

Proof. Pick an arbitrary enumeration $\langle y_\xi \mid \xi < \kappa^+\rangle$ of $[\kappa^+]^{\le\kappa}$. The existence of such an enumeration is guaranteed by the localized GCH. Let $g : \varepsilon \times \kappa^+ \to \kappa^+$ be a bijection where $\varepsilon = \mathrm{cf}(\kappa)$. For each $\delta \in S$ pick an increasing (with respect to the inclusion) sequence of sets $\langle A_\iota^\delta \mid \iota < \varepsilon\rangle$ such that $|A_\iota^\delta| < \kappa$ for all $\iota < \varepsilon$ and $\bigcup_{\iota<\varepsilon} A_\iota^\delta = \delta$.

We show that there is an $\iota < \varepsilon$ such that for every $Z \subseteq \kappa^+$ there are stationarily many ordinals $\delta \in S$ satisfying:

(1) For unboundedly many $\alpha < \delta$ there are $\beta < \delta$ such that $\alpha, \beta \in A_\iota^\delta$ and $Z \cap \alpha = (g^{-1}[y_\beta])_\iota$.

It follows that letting $A_\delta = A_\iota^\delta$ and $x_\beta = (g^{-1}[y_\beta])_\iota$, the pair $\langle x_\beta \mid \beta < \kappa^+\rangle$, $\langle A_\iota^\delta \mid \delta \in S\rangle$, witness $\bigcirc_{\kappa^+}(S)$. [1]

Assume for a contradiction there is no ι as in the previous paragraph. Then for every $\iota < \varepsilon$ there is a set $Z_\iota \subseteq \kappa^+$ such that (1) holds only on a non-stationary

[1] See proof of Proposition 1.3 for the notation $(u)_\eta$.

subset of S. Let $Z = \{\langle\iota,\xi\rangle \mid \xi \in Z_\iota\}$ and and $Z' = g[Z]$. The set S' consisting of all $\delta \in S$ such that

- $g[\varepsilon \times \alpha] = \alpha$ for cofinally many $\alpha < \delta$ and
- $(\forall\alpha < \delta)(\exists\beta < \delta)(Z' \cap \alpha = y_\beta)$

is stationary in κ^+. To each $\delta \in S'$ pick a cofinal strictly increasing sequence $\langle\alpha^\delta_\eta \mid \eta < \mathrm{cf}(\delta)\rangle$ such that $g[\varepsilon \times \alpha_\eta] = \alpha_\eta$ for each $\eta < \mathrm{cf}(\delta)$, and to each $\eta < \mathrm{cf}(\delta)$ pick $\beta_\eta < \delta$ such that $Z' \cap \alpha_\eta = y_{\beta_\eta}$. This is possible by the above arrangements for elements of S'.

If $\delta \in S'$ then there is an $\iota(\delta) < \varepsilon$ such that $\alpha_\eta, \beta_\eta \in A^\delta_{\iota(\delta)}$ for cofinally many $\eta < \mathrm{cf}(\delta)$. This follows immediately if $\mathrm{cf}(\delta) < \varepsilon$, as the assignment

$$\eta \mapsto \text{the least } \iota \text{ such that } \alpha_\eta, \beta_\eta \in A^\delta_\iota$$

cannot be cofinal in ε, so in fact $\alpha_\eta, \beta_\eta \in A^\delta_{\iota(\delta)}$ for all $\eta < \mathrm{cf}(\delta)$. If $\mathrm{cf}(\delta) > \varepsilon$ this follows by the pigeonhole principle, namely the inverse image of some A^δ_ι under this assignment must have size $\mathrm{cf}(\delta)$. Applying the pigeonhole principle to the assignment $\delta \mapsto \iota(\delta)$, we obtain a stationary $S'' \subseteq S'$ and a $\iota < \varepsilon$ such that $\iota(\delta) = \iota$ for all $\delta \in S''$.

Pick $\delta \in S''$. By the above arrangements, there are cofinally many $\alpha < \delta$ for which there are $\beta < \delta$ such that $\alpha, \beta \in A^\delta_\iota$ and $Z' \cap \alpha = y_\beta$. Moreover, the ordinals α can be chosen so that $g[\varepsilon \times \alpha] = \alpha$. It follows that

$$Z \cap (\varepsilon \times \alpha) = g^{-1}[Z' \cap \alpha] = g^{-1}[y_\beta],$$

so $Z_\iota \cap \alpha = (g^{-1}[y_\beta])_\iota$ for all α, β as above. Since this is true of any $\delta \in S''$ we obtained a contradiction to the fact that Z_ι is a coutnerexample to (1). $\square$

The following proposition shows how to apply a standard construction that uses $\square^*_\kappa$ to prove $\bigcirc_{\kappa^+}(T)$.

Proposition 1.5. *Assume κ is singular and $T \subseteq T_\kappa$ is a stationary subset of κ^+ with stationarily many reflection points. Then*

$$2^\kappa = \kappa^+ \;+\; \square^*_\kappa \implies \bigcirc_{\kappa^+}(T).$$

Proof. We elaborate on the argument from the proof of Proposition 1.4. Let $\varepsilon = \mathrm{cf}(\kappa)$. Fix the following objects:

- Sequences $\langle y_\xi \mid \xi < \kappa^+\rangle$, $\langle A^\delta_\iota \mid \iota < \varepsilon\rangle$ and a bijection $g : \varepsilon \times \kappa^+ \to \kappa^+$ as in the proof of Proposition 1.4.
- For each ordinal $\delta < \kappa^+$ an injection $h_\delta : \delta \to \kappa$.
- A $\square^*_\kappa$-sequence $\langle \mathcal{C}_\delta \mid \delta \in \lim\cap(\kappa,\kappa^+)\rangle$. For each δ fix an enumeration $\langle c^\delta_\zeta \mid \zeta < \kappa\rangle$ of the set $\mathcal{C}_\delta$.
- An increasing (with respect to the inclusion) sequence of sets $\langle B_\iota \mid \iota < \varepsilon\rangle$ such that $|\,B_\iota\,| < \kappa$ for each ι and $\bigcup_{\iota<\varepsilon} B_\iota = \kappa \times \kappa$.

For each $\delta \in \lim\cap(\kappa,\kappa^+)$ and $\zeta < \kappa$ define a function $f^\delta_\zeta : \delta \to \kappa \times \kappa$ and a sequence of sets $\langle A^\delta_{\zeta,\iota} \mid \iota < \varepsilon\rangle$ as follows.

$$f^\delta_\zeta(\xi) = \langle \eta, h_\gamma(\xi)\rangle$$

where γ is the least element of c^δ_ζ strictly above ξ and $\eta = \mathrm{otp}(c^\delta_\zeta \cap \gamma) - 1$ [2] and

$$A^\delta_{\zeta,\iota} = (f^\delta_\zeta)^{-1}[B_\iota].$$

[2]Notice that $\mathrm{otp}(c^\delta_\zeta \cap \gamma)$ is a successor ordinal if γ is as above.

Notice that each f^δ_ζ is an injection. By our choice of the sets B_ι we then have $|\,A^\delta_{\zeta,\iota}\,| < \kappa$ and $\bigcup_{\iota<\varepsilon} A^\delta_{\zeta,\iota} = \delta$. We also have the following coherency property for the sets $A^\delta_{\zeta,\iota}$: If $\bar\delta$ is a limit point of c^δ_ζ then there is an ordinal $\bar\zeta < \kappa$ such that

$$A^\delta_{\zeta,\iota} \cap \bar\delta = A^{\bar\delta}_{\bar\zeta,\iota}. \tag{2}$$

To see this notice first that if $\bar\delta$ is a limit point of c^δ_ζ then there is $\bar\zeta < \kappa$ such that $c^\delta_\zeta \cap \bar\delta = c^{\bar\delta}_{\bar\zeta}$, and from the definition of f^δ_ζ we immediately conclude that $f^{\bar\delta}_{\bar\zeta} = f^\delta_\zeta \restriction \bar\delta$. The rest follows immediately from the definition of $A^\delta_{\zeta,\iota}$.

Fix an increasing sequence $\langle \kappa_\iota \,|\, \iota < \varepsilon\rangle$ cofinal in κ. For each $\delta \in T$ and $\iota < \varepsilon$ set $A^\delta_\iota = \bigcup_{\zeta<\kappa_\iota} A^\delta_{\zeta,\iota}$. Notice that $|\,A^\delta_\iota\,| < \kappa$, as $|\,A^\delta_{\zeta,\iota}\,| < |\,B_\iota\,|$ for all $\zeta < \kappa_\iota$. Following the ideas from the proof of Proposition 1.4 we prove: There is an $\iota < \varepsilon$ such that for every $Z \subseteq \kappa^+$ there are stationarily many $\delta \in T$ satisfying:

(3) For unboundedly many $\alpha < \delta$ there are $\beta < \delta$ such that $\alpha, \beta \in A^\delta_\iota$ and $Z \cap \alpha = (g^{-1}[y_\beta])_\iota$.

It follows that letting $x_\beta = (g^{-1}[y_\beta])_\iota$ and $A_\delta = A^\delta_\iota$ where ι is as above, the pair $\langle x_\beta \,|\, \beta < \kappa^+\rangle$, $\langle A_\delta \,|\, \delta \in T\rangle$ witnesses $\bigcirc_{\kappa^+}(T)$.

Assume for a contradiction that no ι as above exists. As in the proof of Proposition 1.4 pick a counterexample Z_ι for each $\iota < \varepsilon$, let $Z = \{\langle \iota, \xi\rangle \in \varepsilon \times \kappa^+ \,|\, \xi \in Z_\iota\}$ and $Z' = g[Z]$. Let C be a closed unbounded subset of κ^+. By our assumption on T, there is a reflection point δ' of T such that:

- δ' is a limit point of C.
- $g[\varepsilon \times \alpha] = \alpha$ for cofinally many $\alpha < \delta'$.
- $(\forall \alpha < \delta')(\exists \beta < \delta')(Z' \cap \alpha = y_\beta)$.

As δ' is a reflection point of T, necessarily $\mathrm{cf}(\delta') > \varepsilon$. Pick an increasing sequence $\langle \alpha_\eta \,|\, \eta < \mathrm{cf}(\delta')\rangle$ cofinal in δ' such that $g[\varepsilon \times \alpha_\eta] = \alpha_\eta$ for each $\eta < \mathrm{cf}(\delta')$. To each $\eta < \mathrm{cf}(\delta')$ assign some $\beta_\eta < \delta'$ satisfying $Z' \cap \alpha_\eta = x_{\beta_\eta}$. It is convenient to pick β_η to be least posssible. Since $\mathrm{cf}(\delta') > \varepsilon$, using the pigeonhole principle we conclude that there is some $\iota' < \varepsilon$ such that $\alpha_\eta, \beta_\eta \in A^{\delta'}_{0,\iota'}$ for cofinally many $\eta < \mathrm{cf}(\delta')$. Let $\delta \in T \cap C \cap \lim(c^{\delta'}_0)$ be a limit point of $\{\alpha_\eta \,|\, \alpha_\eta, \beta_\eta \in A^{\delta'}_{0,\iota'}\}$. Such a δ exists by our choice of δ' and ι', and by the fact that $T \cap \delta'$ is stationary in δ'. Let $\xi < \kappa$ be such that $A^\delta_{\xi,\iota'} = A^{\delta'}_{0,\iota'} \cap \delta$ and let $\iota(\delta) > \iota'$ be such that $\kappa_{\iota(\delta)} > \xi$. The existence of such a ξ follows from (2). Then $A^{\delta'}_{0,\iota'} \cap \delta \subseteq A^\delta_{\iota(\delta)}$, as $B_{\iota'} \subseteq B_{\iota(\delta)}$.

The previous paragraph proves that there is a stationary $T' \subseteq T$ such that for every $\delta \in T'$ there is an $\iota(\delta) < \varepsilon$ such that for cofinally many $\alpha < \delta$ there are $\beta < \delta$ such that $\alpha, \beta \in A^\delta_{\iota(\delta)}$, $Z' \cap \alpha = y_\beta$ and $g[\varepsilon \times \alpha] = \alpha$. The rest of the proof literally follows the proof of Proposition 1.4. We first find a stationary $T'' \subseteq T'$ on which $\iota(\delta)$ stabilize; let ι be the stabilized value. Then we unfold Z' and y_β using g and conclude that for α, β as above we have $Z_\iota \cap \alpha = (g^{-1}[y_\beta])_\iota$. This yields a contradiction with the fact that Z_ι is a counterexample to (3). □

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