CARDINAL TRANSFER PROPERTIES IN EXTENDER MODELS

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Abstract. We prove that if $\mathbf{L}[E]$ is a Jensen extender model, then $\mathbf{L}[E]$ satisfies the Gap-1 morass principle. As a corollary to this and a theorem of Jensen, the model $\mathbf{L}[E]$ satisfies the Gap-2 Cardinal Transfer Property $(\kappa^{++}, \kappa) \to (\lambda^{++}, \lambda)$ for all infinite cardinals κ and λ.

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Jensen isolated $\Diamond$, $\Box$, morass and other combinatorial principles in his work on the fine structure of $\mathbf{L}$ and its applications. One of these applications was to the Gap-1 Transfer Property

$$(\kappa^+, \kappa) \to (\lambda^+, \lambda).$$

The arrow notation means the following. Let $\dot{U}$ be a unary predicate symbol of an arbitrary countable first order language. Suppose that $\mathfrak{A}$ is a structure of this language with $\mathrm{card}(\mathcal{A}) = \kappa^+$ and $\mathrm{card}(\dot{U}^{\mathcal{A}}) = \kappa$. Then there exists $\mathcal{B} \equiv \mathcal{A}$ with $\mathrm{card}(\mathcal{B}) = \lambda^+$ and $\mathrm{card}(\dot{U}^{\mathcal{B}}) = \lambda$.

Vaught proved that $(\kappa^+, \kappa) \to (\omega_1, \omega)$ for all infinite cardinals κ. Chang proved that $(\omega_1, \omega) \to (\lambda^+, \lambda)$ when λ is a regular cardinal. These two are results of $\mathsf{ZFC} + \mathsf{GCH}$ alone. Finally, Jensen added $\Box_\lambda$ to the hypothesis and proved that $(\omega_1, \omega) \to (\lambda^+, \lambda)$ when λ is a singular cardinal. It is interesting to note that Shelah [17] proved later that for singular strong limit λ, there is a principle S_λ very similar to $\Box_\lambda$ such that

- $\Box_\lambda$ implies S_λ,
- S_λ implies Jensen's weak square principle $\Box^*_\lambda$, and
- S_λ is equivalent to $(\omega_1, \omega) \to (\lambda^+, \lambda)$.

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Jensen also proved that the Gap-2 Transfer Property

$$(\kappa^{++},\kappa) \longrightarrow (\lambda^{++},\lambda)$$

holds for all infinite cardinals κ and λ in $\mathbf{L}$. For regular λ, what Jensen really used besides GCH is a λ^+-morass. A combination of $\square_\lambda$ and a λ^+-morass with built-in $\square_{\lambda^+}$ is what he used for singular λ. It is interesting that the "built-in $\square_{\lambda^+}$" is actually obtained purely combinatorially from $\square_\lambda$ and $\square_{\lambda^+}$, again by an argument of Jensen.

This paper is concerned with Jensen's combinatorial principles in models constructed from sequences of extenders. That is, models of the form $\mathbf{L}[E]$. In [15], the authors showed that in $\mathbf{L}[E]$, $\square_\lambda$ holds iff λ is not a subcompact cardinal. Thus, the Gap-1 Transfer Property holds in $\mathbf{L}[E]$. It seems like the methods developed in [15] can be used to generalize Jensen's theorems about the combinatorial structure of $\mathbf{L}$ to $\mathbf{L}[E]$. Here we verify this feeling at least as far as what went into Jensen's work on the Gap-2 Cardinal Transfer Properties is concerned. We will see that in $\mathbf{L}[E]$, there is a λ-morass regardless of whether λ is subcompact or not. By applying results of [15] to Jensen's argument on morasses with built-in square sequences, we also get λ^+-morass with built-in $\square_{\lambda^+}$ when λ is not subcompact. This suffices to conclude that the Gap-2 cardinal transfer property holds in $\mathbf{L}[E]$ for the trivial reason that singular cardinals are not subcompact.

The fundamental reason that the fine structure of $\mathbf{L}[E]$ is trickier than that of $\mathbf{L}$ is the failure of a certain form of condensation. Very roughly, this problem was overcome in [15] by laying out the universe differently, using a sequence of extenders and their fragments, and by showing that condensation holds relative to this replacement sequence. Our replacement method failed exactly when we were working between a subcompact and its successor, which is what led to the characterization of when $\square_\lambda$ holds in $\mathbf{L}[E]$. As the reader will see, we can avoid this sort of failure entirely in the construction of a λ^+-morass.

For the reader who is familiar with a morass construction in $\mathbf{L}$ and with extender models, let us say more to illustrate why an overly naive generalization does not work. Assume $\mathbf{V} = \mathbf{L}[E]$. Let μ be a measurable cardinal and $\alpha = \mu^+$. We discuss a putative construction of an α-morass. Suppose that $\tau < \alpha^+$ is a local successor of α. This means that α is the largest cardinal in J^E_τ. Say $\bar\alpha$ is a local successor of μ

and $\bar{\tau}$ is a local successor of $\bar{\alpha}$. We must decide whether or not $\bar{\tau} \prec \tau$ where $\prec$ is the morass order. Let M be the level of $\mathbf{L}[E]$ over which τ is seen to have cardinality α. By E^M_{top} we mean the top extender of M, which in this case is just $E_{\mathbf{On}\cap M}$. For the sake of this example, we assume that $\mathrm{cr}(E^M_{\mathrm{top}}) = \mu$ and that every element of M is Σ_1-definable in M from parameters $< \alpha$. Let $\sigma : \bar{M} \to M$ be the inverse of the Mostowski collapse of the Σ_1-hull of $\bar{\alpha}$ in M. Assume that $\sigma(\bar{\alpha}) = \alpha$ and $\sigma(\bar{\tau}) = \tau$. Based on the proof in $\mathbf{L}$, we are inclined to set $\bar{\tau} \prec \tau$ and the morass map $\sigma_{\bar{\tau},\tau}$ equal to the restriction $\sigma \restriction \bar{\tau}$. Condensation does not allow us to conclude that $\bar{M}$ is an initial segment of $\mathbf{L}[E]$ but this is not far that from the truth. A more substantial issue arises when we try to prove the morass continuity axioms. Let $\tilde{\tau} = \sup(\sigma''\bar{\tau})$. A morass axiom requires that $\bar{\tau} \prec \tilde{\tau}$ and $\sigma_{\bar{\tau},\tilde{\tau}} = \sigma_{\bar{\tau},\tilde{\tau}}$. Let

$$N = \bigcup \sigma'' M$$

and

$$E^N_{\mathrm{top}} = \bigcup \sigma'' E^N_{\mathrm{top}}.$$

In order to satisfy the continuity axioms, our definitions of both $\{\tau';\, \tau' \prec \tilde{\tau}\}$ and $\{\sigma_{\tau',\tilde{\tau}};\, \tau' \prec \tilde{\tau}\}$ should be based on Σ_1-hulls in N by analogy with how our definitions of $\{\tau';\, \tau' \prec \tau\}$ and $\{\sigma_{\tau',\tau};\, \tau' \prec \tau\}$ were based on M. Let $\tilde{M}$ be the level of $\mathbf{L}[E]$ over which $\tilde{\tau}$ is seen to have cardinality α. The main issue is that $\tilde{M} \neq N$. We can tell this because E^N_{top} is not an extender over N.

In the course of the proof, we will define the terms *protomouse* and *pluripotent mouse.* As a reference for the reader, the structure N above is an example of a protomouse. Premice that look in some ways like M but in other ways like N are called *pluripotent.*

Finally, let us point out that so far our ability to construct $\mathbf{L}[E]$ models from any hypothesis is rather limited; the best result to date, due to Neeman [11], is at the level of a Woodin limit of Woodin cardinals. On the other hand, it has long been known that a λ^+-morass can be added by λ-directed closed forcing. Thus it is relatively consistent that there is a supercompact cardinal λ and a λ^+-morass.

The definition of the morass we are going to construct is in the next section. Modulo this definition, we can state our main result.

Theorem 0.1 (**Main Theorem**)**.** *If* $\mathbf{L}[E]$ *is an extender model, then* $\mathbf{L}[E]$ *satisfies that for all infinite cardinals* κ*, there is a* κ*-morass.*

By the theorem of Jensen mentioned before, we may conclude that $\mathbf{L}[E]$ satisfies $(\kappa^{++}, \kappa) \rightarrow (\lambda^{++}, \lambda)$ for all regular λ. But more is true.

Theorem 0.2. *If* $\mathbf{L}[E]$ *is an extender model, then* $\mathbf{L}[E]$ *satisfies that for all infinite cardinals* κ*, there is a* κ^+*-morass with built-in* $\square_{\kappa^+}$*.*

By Jensen, Theorem 0.2 is a direct consequence of Theorem 0.1 and the following theorem.

Theorem 0.3 ([15])**.** *If* $\mathbf{L}[E]$ *is an extender model, then* $\mathbf{L}[E]$ *satisfies that for all infinite cardinals* κ *that are not subcompact, the principle* $\square_\kappa$ *holds.*

Immediately from the theorems above and and Jensen's argument for obtaining cardinal transfer property we may conclude:

Corollary 0.4. *The Gap-2 Cardinal Transfer Property holds in every extender model.*

The paper is organized as follows. In Section 1, we recall some definitions, useful notions and and properties of extender models. The account here will be, however, very brief and for the technical details we refer the reader to [15], [21] and [22]. [21] contains a detailed description of the fine structure theory and the models we are using in this paper. Moreover, our construction heavily relies on techniques developed in [15] that are not included in [21]. Finally, our notation is identical with that in [15], and thus slightly diverges from that in [21]. The reason for choosing this amended notation is given by the fact that we are dealing with a certain specific situation which allows such amendments and significantly simplifies combinatorial constructions. Finally we note that a construction of Gap-1 Morass in $\mathbf{L}$ based on Jensen's Σ^* fine structure can be found in [20]. This paper may be considered as a warm-up for a reader not familiar with basic aspects of the construction.

1. Preliminaries

1.1. **Morasses.** We begin with the definition of a Gap-1 Morass, which we state for the reader's convenience. Our notion of a Gap-1 Morass is – up to some minor cosmetic amendments that we adopted in order make certain fine structural arguments tidy – identical with the standard definition, see e.g. [4] for a global version or [2] for a local version. Our approach will be closer to that in [4] in that we construct a global version of Gap-1 Morass.

Definition 1.1. *A Gap-1 Morass is a structure characterized by the following data.*

- *A class of ordinals $S(0)$ containing all uncountable cardinals such that $S(0) \cap \kappa$ is a closed unbounded subset of κ whenever κ is an uncountable cardinal;*
- *a sequence $\langle S_\alpha;\, \alpha \in S(0)\rangle$ of closed subsets of (α, α^+) satisfying:*
  - *if α is regular then $S_\alpha = S(0) \cap (\alpha, \alpha^+)$,*
  - *if α is a singular cardinal then S_α is a proper initial segment of $S(0) \cap (\alpha, \alpha^+)$, and*
  - *if α fails to be a cardinal then for any $\beta \in S(0)$, $\beta > \alpha$ we have $\max(S_\alpha) < \beta$;*
- *a tree ordering $\prec$ on the set*

$$S(1) = \bigcup \{S_\alpha;\, \alpha \in S(0)\};$$

- *a system of maps $\langle \sigma_{\bar\tau,\tau};\, \bar\tau, \tau \in S(0) \;\&\; \bar\tau \prec \tau\rangle$.*

The system of maps $\langle \sigma_{\bar\tau,\tau}\rangle$ satisfies the set of axioms stated below. Before stating the axioms, we fix some notation. Given $\alpha \in S(0)$ and $\tau, \tau' \in S(1)$ set:

- *$\alpha(\tau) =$ the unique α such that $\tau \in S_\alpha$;*
- *$\tau <_0 \tau'$ iff $\alpha(\tau) = \alpha(\tau')$ and $\tau < \tau'$;*
- *$\tau(\alpha) = \max(S_\alpha)$;*
- *$\tau_+ =$ the $<_0$-successor of τ.*

Note that the uniqueness of $\alpha(\tau)$ follows from the description of S_α above. The following axioms are called morass axioms.

(M0) *If $\bar\tau \prec \tau$ then $\alpha(\bar\tau) < \alpha(\tau)$ and $\bar\tau$ is $<_0$-minimal, successor, limit iff τ is $<_0$-minimal, successor, limit.*

(M1) *If $\bar\tau \prec \tau$ then $\sigma_{\bar\tau,\tau} : \bar\tau \to \tau$ is an order-preserving map such that*
- *$\sigma_{\bar\tau,\tau} \restriction \alpha(\bar\tau) = \mathrm{id} \restriction \bar\tau$;*
- *$\sigma_{\bar\tau,\tau}(\alpha(\bar\tau)) = \alpha(\tau)$.*

Moreover, these maps commute along the branches in $\prec$, i.e. $\tau^* \prec \bar\tau \prec \tau$ implies that $\sigma_{\bar\tau,\tau} \circ \sigma_{\tau^*,\bar\tau} = \sigma_{\tau^*,\tau}$.

(M2) *Assume $\bar\tau \prec \tau$ and $\bar\tau' < \bar\tau$.*

(a) $\bar{\tau}' \in S_{\alpha(\bar{\tau})} \longleftrightarrow \sigma_{\bar{\tau},\tau}(\bar{\tau}') \in S_{\alpha(\tau)}$.

(b) *If* $\bar{\tau}'_+ <_0 \bar{\tau}$ *then* $\sigma_{\bar{\tau},\tau}(\bar{\tau}'_+) = \sigma_{\bar{\tau},\tau}(\bar{\tau}')_+$.

(c) *If* $\bar{\tau}' <_0 \bar{\tau}$ *and* $\tau' = \sigma_{\bar{\tau},\tau}(\bar{\tau}')$, *then* $\bar{\tau}' \prec \tau'$ *and* $\sigma_{\bar{\tau}',\tau'} = \sigma_{\bar{\tau},\tau} \upharpoonright \bar{\tau}'$.

(M3) *If* $\tau \in S(1)$ *then*

$$C(\tau) = \{\alpha(\bar{\tau});\, \bar{\tau} \prec \tau\}$$

is closed in $\alpha(\tau)$ *and if* τ *is distinct from the largest element of* $S_{\alpha(\tau)}$ *then* $C(\tau)$ *is unbounded in* $\alpha(\tau)$. *If* $C(\tau)$ *is unbounded in* $\alpha(\tau)$ *then*

$$\tau = \bigcup\{\mathrm{rng}(\sigma_{\bar{\tau},\tau});\, \bar{\tau} \prec \tau\}.$$

(M4) *If* $\bar{\tau}$ *is a* $<_0$*-limit and* $\tilde{\tau} = \sup(\mathrm{rng}(\sigma_{\bar{\tau},\tau}))$, *then* $\bar{\tau} \prec \tilde{\tau}$ *and* $\sigma_{\bar{\tau},\tilde{\tau}} = \sigma_{\bar{\tau},\tau}$.

(M5) *Assume* $\bar{\tau}$ *is a limit point of* $S_{\alpha(\bar{\tau})}$, $\bar{\tau} \prec \tau$ *and the map* $\sigma_{\bar{\tau},\tau}$ *is cofinal. Let* $\alpha(\bar{\tau}) < \hat{\alpha} \in S(0)$ *satisfy the following condition:*

(1) *For each* $\bar{\tau}' <_0 \bar{\tau}$ *there is a* $\hat{\tau}' \prec \sigma_{\bar{\tau},\tau}(\bar{\tau}')$ *such that* $\hat{\alpha} = \alpha(\hat{\tau}')$.

Then there is a $\hat{\tau} \prec \tau$ *with* $\alpha(\hat{\tau}) = \hat{\alpha}$.

The above definition describes a *global* version of the Gap-1 Morass. Jensen proved that such a morass exists in $\mathbf{L}$. The restriction of a Gap-1 Morass to $\{\tau < \kappa^+ \,|\, \tau \preccurlyeq \tau' \text{ for some } \tau' \in S_\kappa\}$ is called a *κ-morass.* We can also "localize" the morass by restricting the class of cardinals in $S(0)$. Notice that if we merely require that $\kappa \in S(0)$ we get a κ-morass. Clearly, a κ-morass is interesting primarily when κ is regular. Axioms (M4) and (M5) are called *continuity axioms.* Without them what we would have is a *coarse morass.* Our main result is the following theorem.

Theorem 1.2. *Any extender model* $\mathbf{L}[E]$ *satisfies that there is a Gap-1 Morass.*

This theorem obviously implies Theorem 0.1, and thereby guarantees the Gap-2 cardinal transfer property as explained in the introduction.

1.2. **Extender Models.** We will find a realization of a global Gap-1 Morass in an extender model with Jensen's λ-indexing of extenders; the details regarding this indexing can be found in [21].

We will work in a fixed extender model $\mathbf{L}[E]$. Thus, whenever we say "extender model" or "$\mathbf{L}[E]$", we mean this fixed model. The following lemmata list three basic properties of premice that will be used in the course of our construction. Recall that if E_ν is on the extender sequence of a premouse M with λ-indexing then the initial segment $M||\nu$ is a coherent structure whose top extender is weakly amenable, that is, $M||\nu = \langle J^E_\nu, E_\nu \rangle$ where, letting $\mu = \mathrm{cr}(E_\nu)$ and $\vartheta = \mu^{+M||\nu}$, we have $J^E_\nu = \mathrm{Ult}(J^E_\vartheta, E_\nu)$. We will also need to consider certain coherent structures whose top extender is not weakly amenable; these will be discussed in the next section. The largest cardinal in J^E_ν, i.e. the image of μ under the ultrapower embedding, is denoted by $\lambda(E_\nu)$. Furthermore, we denote the standard parameter of M by p_M and the standard solidity witness for an ordinal α with respect to some parameter $p \in M$ and M is by $W^{\alpha,p}_M$. Recall that if α is such that $\omega\varrho^{n+1}_M \leq \alpha < \omega\varrho^n_M$ then $W^{\alpha,p}_M$ is the transitive collapse of the $\Sigma^{(n)}_1$-hull $\tilde{h}^{n+1}_M(\alpha \cup \{p-(\alpha+1)\})$ where $\tilde{h}^{n+1}_M$ is the $\Sigma^{(n)}_1$-Skolem function for M. We will also consider the so-called *generalized witnesses.* A generalized witness for α with respect to $p \in M$ and M is a pair $\langle Q, r\rangle$ where Q is a J-structure of the same type as M and $r \in Q$ is a finite set of ordinals of the same size as $p-(\alpha+1)$ such that for every $\Sigma^{(n)}_1$ formula $\varphi(v_0, v_1, \ldots, v_\ell)$ and any ordinals $\xi_0, \xi_1, \ldots, \xi_{\ell-1} < \alpha$ we have

$$M \models \varphi(\xi_0, \xi_1, \ldots, \xi_{\ell-1}, p-(\alpha+1)) \quad\Longleftrightarrow\quad Q \models \varphi(\xi_0, \xi_1, \ldots, \xi_{\ell-1}, r).$$

Although the standard solidity witness for α with respect to p and M is unique, there may be many generalized witnesses. If we work with a fixed generalized witness $\langle Q, r\rangle$, we usually denote it by $Q^{\alpha,p}_M$ to indicate that it is a generalized witness for α and it was computed with respect to p and M. Two crucial properties of levels of extender models is the *solidity* (S) and the *universality* (U) of the standard parameter.

(S) Any level of an extender model M is *solid,* i.e. if $W^{\beta,p_M}_M \in M$ whenever $\beta \in p_M$. Equivalently, there is a generalized witness $\langle Q, r\rangle \in M$ for β with respect to p_M and M.

(U) For any level of an extender model M and any $n \in \omega$, the standard parameter p_M is n-universal. That is, if $\bar{M}$ is the transitive collapse of the $\Sigma^{(n)}_1$-hull $\tilde{h}^{n+1}_M(\omega\varrho^{n+1}_M \cup \{p_M\})$ then $\mathcal{P}(\omega\varrho^{n+1}_M) \cap \bar{M} = \mathcal{P}(\omega\varrho^{n+1}_M) \cap M$.

Recall that we have three types of active premice, depending on the distribution of cutpoints of the top extender. Type B premice are those premice whose set of cutpoints is nonempty, but bounded in $\lambda(E^M_{\mathrm{top}})$. The largest element of this set is denoted by λ^*_M and the index of $E^M_{\mathrm{top}}|\lambda^*_M$ is γ_M. In the case of type B premice, we extend the language for premice by a constant denoting γ_M, which has the consequence that all fine structural characteristics are computed relative to γ_M. In the case of type A and C premice, there is no additional parameter used in the computation of these characteristics. To distinguish between the two languages, we will call the language with the additional constant for γ_M the *language for premice* (to have a uniform notation, we can view the additional constant as a singleton predicate in the case of type B premice and the empty predicate for type A and C premice) and the language without the constant the *language for coherent structures.* Obviously, the language for coherent structures is the same as the language for type A and C premice.

For for active premice M we will need to consider the Σ_1-Skolem functions that are computed in the language for coherent structures. To distinguish between the usual Σ_1-Skolem functions (i.e. those computed in the language for premice), we will denote the Σ_1-Skolem function for M computed in the language for coherent structures by h^*_M. In many arguments, it will be crucial to distinguish between h_M and h^*_M. This, of course is non-void only in the case of type B premice.

The techniques used in combinatorial constructions also require to consider the so-called *Dodd* parameters for premice. The issue arises when dealing with type B premice. Unlike the construction of square sequences in [15], the construction of a Gap-1 Morass requires an elaboration on the notion of a Dodd parameter. This is caused by the fact that in certain situations, we have now less control on projecta than we had in the construction of square sequences. We state the crucial definitions and facts below and refer the reader to [22] for the details. Recall that $<^*$ is the canonical well-ordering of finite sets of ordinals which can be viewed as the lexicographic ordering of finite descending sequences of ordinals.

Definition 1.3. *Let M be an active premouse and $\mu = \mathrm{cr}(E^M_{\mathrm{top}})$. Assume that α is an ordinal satisfying $\max\{\mu^{+M}, \varrho^1_M\} \leq \alpha$. The parameter d^α_M is the $<^*$-least finite set of ordinals d such that $M = h^*_M(\alpha \cup \{d\})$. We write d_M for d^α_M if $\alpha = \omega\varrho^1_M$.*

Obviously, d^α_M need not be defined for each M and α. However, it *is* defined granting that M is sound, which is precisely the situation we are interested in. It is easy to see that if $\bar\alpha \leq \alpha$ and both d^α_M and $d^{\bar\alpha}_M$ are defined then $d^\alpha_M = d^{\bar\alpha}_M - \alpha$. For sound M with $\omega\varrho^1_M \geq \mu^{+M}$, the Dodd parameter in the usual sense is just d_M.

For an ordinal $\beta \geq \alpha$ and a parameter $p \in M$, the standard Dodd solidity witness ${}^*W_M^{\beta,p}$ for α with respect p and M is the transitive collapse of the hull $h_M^*(\beta \cup \{p-(\beta+1)\})$. This is a direct analogue to $W_M^{\beta,p}$ with the only difference that there is no reference to γ_M. Similarly, a pair $\langle Q, r\rangle$ is a generalized Dodd solidity witness for β with respect to p and M just in case that Q is transitive, $r \in Q$ is a finite set of ordinals of the same size as $p-(\beta+1)$ and for every Σ_1-formula in the language for coherent structures $\varphi(v_0, v_1, \ldots, v_\ell)$ and every $\xi_0, \xi_1, \ldots, \xi_{\ell-1} < \beta$ we have

$$M \models \varphi(\xi_0, \ldots, \xi_{\ell-1}, p-(\beta+1)) \longleftrightarrow Q \models \varphi(\xi_0, \ldots, \xi_{\ell-1}, r)$$

M is *Dodd solid above* α just in case that d_M^α is defined and ${}^*W_M^{\beta,d_M^\alpha} \in M$ for every $\beta \in d_M^\alpha$. This is equivalent to demanding that for each $\beta \in d_M^\alpha$ there is a generalized Dodd solidity witness $\langle Q, r\rangle \in M$ with respect to d_M^α and M. Although the standard Dodd solidity witness for β with respect to p and M is unique, there may be many generalized Dodd solidity witnesses. Analogously to the case of solidity witnesses, if we work with a specific generalized Dodd solidity witness $\langle Q, r\rangle$, we usually denote it by ${}^*Q_M^{\beta,p}$ to indicate that it is a generalized witness for β and it is computed with respect to p and M. The following facts summarize the properties of d_M^α.

(DP1) If M is a level of an extender model and α is as in Definition 1.3 then
- (a) $d_M^\alpha = (p_M - \alpha) \cup e_M^\alpha$ where e_M^α is the $<^*$-least finite set of ordinals e such that $\gamma_M \in h_M^*(\alpha \cup \{p_M \cup e\})$.
- (b) M is Dodd solid above α.

(DP2) Let M and α be as in Definition 1.3. Assume that for some n, there is a $\Sigma_0^{(n)}$-preserving (with respect to the language for coherent structures!) embedding σ of M into an $\mathbf{L}[E]$-level such that $\sigma \restriction \omega\varrho_M^{n+1} = \mathrm{id}$. Let d be a parameter satisfying
- $M = h_M^*(\alpha \cup \{d\})$;
- for every $\beta \in d$ there is a generalized Dodd solidity witness $\langle Q, r\rangle \in M$ with respect to d and M.

Then
- (a) $d = d_M^\alpha$;

(b) $d^\alpha_M = (p_M - \alpha) \cup e^\alpha_M$ where e^α_M is the $<^*$-least finite set of ordinals e such that $\gamma_M \in h^*_M(\alpha \cup \{p_M \cup e\})$.

(c) M is sound above α.

Finally we state the Condensation Lemma, that will be used many times throughout the entire construction.

Lemma 1.4 (Condensation Lemma). *Let $\bar{M}$ and M be premice of the same type where M is a level of an extender model and let $\sigma : \bar{M} \to M$ be an embedding which is cardinal preserving, $\Sigma^{(n)}_0$-preserving and such that $\sigma \restriction \omega\varrho^{n+1}_{\bar{M}} = \mathrm{id}$. Then $\bar{M}$ is solid and $p_{\bar{M}}$ is k-universal for every $k \in \omega$. Furthermore, if $\bar{M}$ is sound above $\nu = \mathrm{cr}(\sigma)$ then one of the following holds:*

a) *$\bar{M} = M$ and $\sigma = \mathrm{id}$.*

b) *$\bar{M}$ is a proper initial segment of M.*

c) *$\bar{M} = \mathrm{Ult}^*(M||\eta, E^M_\alpha)$ where $\alpha \leq \omega\eta$ and letting $\kappa = \mathrm{cr}(E^M_\alpha)$, the ordinal η is largest with the property $\nu = \kappa^{+M||\eta}$. Additionally, E^M_α has a single generator κ. (Notice that $\omega\varrho^\omega_{M||\eta} = \kappa$ in this case.)*

d) *$\bar{M}$ is a proper initial segment of $\mathrm{Ult}(M, E^M_\nu)$.*

The construction of a morass is analogous to that of a square sequence in that to each ordinal of interest (this ordinal is usually a local successor cardinal) we assign a unique collapsing structure. In **L**, such a collapsing structure is just the corresponding collapsing level. In an extender model, however, we have to pick the collapsing structure from a broader class of structures, but in a way that makes our choice unique. The new kind of structures we have to consider are the so-called *protomice* [15].

1.3. **Protomice.** We need to adapt the account on protomice from [15] (in that paper, we used the term "fragment" instead of "protomouse") for the present purposes. The amendments we will make are needed because the collapsing structures we consider here are of a slightly different kind than those considered in [15]. We will also discuss properties of protomice and divisors which were irrelevant there, but will be of use now. The notion of a protomouse is defined relative to a fixed extender model $W = \mathbf{L}[E]$.

Recall that $M = \langle J^E_\nu, F\rangle$ is a coherent structure just in case that F is an extender such that $J^E_\nu = \mathrm{Ult}(J^E_{\bar{\nu}}, F)$ where, letting $\mu = \mathrm{cr}(F)$, we have $\mathcal{P}(\mu) \cap J^E_{\bar{\nu}} = \mathrm{dom}(F)$

and $\bar{\nu}$ is the least ordinal with this property. Clearly $\bar{\nu} \leq \mu^{+M}$ and μ is the largest cardinal in $J^E_{\bar{\nu}}$. The ordinal $\bar{\nu}$ is denoted by $\vartheta(M)$.

In this section we shall consider a coherent structure of the form $M = \langle J^E_\nu, F\rangle$ whose top extender F is *not* weakly amenable. So F measures only subsets of $\mu = \mathrm{cr}(F)$ from some initial segment $J^E_{\vartheta(M)}$ where $\vartheta(M) < \mu^{+M}$. We denote the collapsing level for $\vartheta(M)$ in M by $N^*(M)$. We say that M is a *protomouse* just in case that $N(M) = \mathrm{Ult}^*(N^*(M), F)$ is a level of W or $\mathrm{Ult}(W, E_\beta)$ for some β.

We will primarily deal with collapsing structures for local successor cardinals of a certain special kind. Recall that an ordinal α is *cardinal-correct in* W just in case every $\beta < \alpha$ is a cardinal in J^E_α iff β is a cardinal in W.

Definition 1.5. *Let τ be an ordinal. We say that N is* the collapsing structure for *τ just in case the following conditions are satisfied:*

- *α is cardinal-correct in W where α is the cardinal predecessor of τ in N (the option $\tau = \mathbf{On} \cap N$ is allowed);*
- *N is the collapsing level for τ in W or in $\mathrm{Ult}(W, E_\alpha)$.*

For τ as in the above definition, the collapsing structure N is uniquely determined. To see this, notice that if $E_\alpha \neq \varnothing$ then $W||\alpha$ is the collapsing level for α in W, so α cannot be a cardinal in any level of W properly extending $W||\alpha$. Thus, the "or" in the above definition is actually an "exclusive or". One crucial property of the collapsing structure N is that $\omega\varrho^\omega_N \leq \alpha$. In our applications, however, we will often have equality here. We will also consider collapsing protomice. Unlike collapsing structures, these need not be unique; therefore we talk about "a" collapsing protomouse in the next definition.

Definition 1.6. *Let $M = \langle J^E_\nu, F\rangle$ be a protomouse and $\tau = \alpha^{+M}$. We say that the protomouse M is a* collapsing protomouse *for τ iff α is cardinal-correct in W, $\mu < \alpha < \lambda(F)$ and $N(M)$ is the collapsing structure for τ.*

Protomice might emerge in condensation-interpolation arguments, if these arguments involve premice whose top extender overlap α. In many cases, the situation will require to consider fine structure of these premice with respect to the language for coherent structures, even if they are type B premice. We will call such premice *pluripotent.*

Definition 1.7. *Let α be cardinal-correct in W and let N be a level of either W or else $\mathrm{Ult}(W, E_\alpha)$. We say that N is* pluripotent at α *just in case the following three conditions are satisfied:*

(a) $E^N_{\mathrm{top}} \neq \varnothing$;
(b) $\mu < \alpha < \lambda(E^N_{\mathrm{top}})$ *where* $\mu = \mathrm{cr}(E^N_{\mathrm{top}})$;
(c) $\omega\varrho^1_N \leq \alpha$.

We will frequently omit the reference to α, as α will be clear from the context. Thus, we will briefly write "pluripotent" instead of "pluripotent at α". The main objects of our interest will be collapsing levels of extender models, in which case α will be implicitly determined.

Definition 1.8. *Let N be the collapsing structure for τ. N is* pluripotent *just in case that N is pluripotent at α where α is the cardinal predecessor of τ in N.*

In the above definition, τ is a suppressed parameter. In the construction of a morass each suitable ordinal τ uniquely determines its collapsing structure N. The construction then depends on the pluripotency of N at the cardinal predecessor of α in N.

Under the above settings we define the notions of a divisor and strong divisor for N. We adopt the following conventions. Let N be the collapsing structure for τ.

- $\alpha(\tau)$ denotes the cardinal predecessor of τ in N;
- $n(\tau)$ is the unique $n \in \omega$ satisfying $\omega\varrho^{n+1}_N \leq \alpha < \omega\varrho^n_N$.

Definition 1.9. *Let N be the collapsing structure for τ. Assume that $\mu \in \mathbf{On}$ and $q \in [\mathbf{On}]^{<\omega}$. The pair (μ, q) is a* divisor *for N iff there is an ordinal $\lambda = \lambda_N(\mu, q)$ such that, setting $\alpha = \alpha(\tau)$, $n = n(\tau)$ and*

$$r = r(\mu, q) = \begin{cases} p_N - \lambda & \text{if } N \text{ is not pluripotent} \\ d_N - \lambda & \text{if } N \text{ is pluripotent,} \end{cases}$$

the following clauses (F1) – (F4) hold.

(F1) $\mu \leq \omega\varrho^{n+1}_N \leq \alpha < \lambda < \omega\varrho^n_N$ *and* $\mu < \alpha$.
(F2) $q = \begin{cases} p_N \cap \lambda, & \text{if } N \text{ is not pluripotent} \\ d_N \cap \lambda, & \text{if } N \text{ is pluripotent.} \end{cases}$
(F3) $\tilde{h}^{n+1}_N(\mu \cup \{r\}) \cap \omega\varrho^n_N$ *is cofinal in* $\omega\varrho^n_N$ *if N* fails *to be pluripotent and* $h^*_N(\mu \cup \{r\})$ *is cofinal in N if N* is *pluripotent.*

(F4) (a) $\lambda\cap\tilde{h}^{n+1}_N(\mu\cup\{r\})=\mu$ *and* $\lambda\in\tilde{h}^{n+1}_N(\mu\cup\{r\})$ *if* N *is* not *pluripotent;*
(b) $\lambda\cap h^*_N(\mu\cup\{r\})=\mu$ *and* $\lambda\in h^*_N(\mu\cup\{r\})=\mu$ *if* N is *pluripotent.*

What makes the present situation different from that in [15] is that we have to allow the possibility that $\omega\varrho^{n+1}_N=\alpha$ and $\omega\varrho^{\omega}_N<\alpha$.

Let us make some trivial remarks. First, observe that if (μ,q) is a divisor for N, then $\mu\le\omega\varrho^{n+1}_N$. Indeed, (F1) together with (F4) imply that μ must be a limit cardinal in N^*, and therefore in N. Hence μ is a cardinal in W. In our setting, $\omega\varrho^{n+1}_N=\alpha$ whenever $\tau<\max(S_\alpha)$. Finally notice two facts. First, $r\cap\alpha=\varnothing$. Second, no pair (μ,q) can be a divisor for N if N is active and $(\mathrm{cr}(E^N_{\mathrm{top}}))^{+N}=\alpha$.

It is straightforward to verify that the arguments in Subsection 2.1 in [15] (subsection titled "Premice and Fragments") go through when working with the above definitions. We can thus conclude that the transitive collapse N^* of $X=\tilde{h}^{n+1}_N(\mu\cup\{r\})$ (resp. $X=h^*_N(\mu\cup\{r\})$) is a premouse that is sound and solid, and in fact N^* is an initial segment of N. For pluripotent N this requires an extra argument showing that γ_N is in X ([15], Lemma 2.4). If π is the associated uncollapsing map, then by [15], Lemma 2.5, we have:

(2) $\pi(p_{N^*})=r$ whenever N is not pluripotent. For pluripotent N we have $\pi(d_{N^*})=r$ and, moreover, if $\lambda<\lambda(E^N_{\mathrm{top}})$ then $q=p_N\cap\lambda$.

It follows that the description of q on the top line in (F2) also covers that on the bottom line of (F2) unless N is a type B premouse and $\lambda=\lambda(E^N_{\mathrm{top}})$; in this case $q=d_N$ and is in general different from p_N. The map π turns out to be the ultrapower embedding associated with the fine ultrapower $\mathrm{Ult}^*(N^*,F)$ where $F=\pi\restriction(\mathcal{P}(\mu)\cap N^*)$. Letting $\vartheta=\mu^{+N^*}$ and $\nu=\pi(\vartheta)$, we see that $M=\langle J^E_\nu,F\rangle$ is a protomouse, $N(M)=N$ and $\vartheta(M)=\vartheta$.

Notation. *Given* N *together with some divisor* (μ,q) *as in the above definition, we let*

$$N(\mu,q)=\textit{the protomouse determined by }(\mu,q). \tag{3}$$

In general, we will write $N^*(\mu,q)$ *for* N^**,* $\vartheta(\mu,q)$ *for* $\mu^{+N^*}=\vartheta(N(\mu,q))$*,* $\lambda(N(\mu,q))$ *for* $\pi(\mu)$*,* $F((\mu,q))$ *for* $\pi\restriction\mathcal{P}(\mu)$*, i.e. for* E^M_{top}*, etc...*

Our construction of a Gap-1 Morass will also require a generalization of the notion of a strong divisor. The reason for this is that if τ is the largest element of S_α (see Definition 1.1) we do not have any control of the bottom part of the standard parameter $p_N \cap \alpha$ where N is the singularizing structure for τ.

Definition 1.10. *Let N be the collapsing structure for τ and $\alpha = \alpha(\tau)$. A divisor (μ, q) for N is* strong above β *where $\alpha \leq \beta < \lambda(N(\mu, q))$ just in case that for every $\xi < \mu$ and every x of the form $\tilde{h}_N^{n+1}(\xi, p_N - \beta)$ (resp. $h^*(\xi, d_N - \beta)$, if N is pluripotent) we have $x \cap \mu \in N^*(\mu, q)$.*

A divisor (μ, q) for N is strong *iff it is strong above $\omega\varrho_N^{n+1}$.*

Of course, this notion trivializes if $\beta > \max(q)$. By (2), the characterization of strongness using $\tilde{h}_N^{n+1}(\xi, p_N - \beta)$ also works for pluripotent N if $\lambda < \lambda(E^N_{\text{top}})$. (With $n = 0$, of course.) In the following we state some properties of strong divisors. We omit the proofs, as they are easy modifications of those in [15].

The property of being strong above β is equivalent to any of (S1) – (S3) below, unless N is pluripotent and $\lambda = \lambda(E^N_{\text{top}})$:

- (S1) If x is of the form $\tilde{h}_N^{n+1}(\xi, p_N - \beta)$ for some $\xi < \mu$, then $x \cap \mu = y \cap \mu$ for some $y \in H_N^n$ of the form $\tilde{h}_N^{n+1}(\xi, r)$. (Here $r = r(N(\mu, q))$.)
- (S2) $\mathcal{P}(\mu) \cap N^*(\mu, q) = \mathcal{P}(\mu) \cap N'(\mu, \beta)$ where $N'(\mu, \beta)$ is the transitive collapse of $\tilde{h}_N^{n+1}(\mu \cup \{p_N - \beta\})$.
- (S3) $|p_{N^*(\mu,q)}| = |p_{N'(\mu,\beta)}|$.

For pluripotent N with $\lambda < \lambda(E^N_{\text{top}})$ we can equivalently write $h_N^*(\xi, d_N - \beta)$ and $h_N^*(\xi, r)$ in (S1), and $h_N^*(\mu \cup \{d_N - \beta\})$ in (S2). If N is pluripotent and $\lambda = \lambda(E^N_{\text{top}})$ then $q = d_N \cap \lambda$, and $N'(\mu, \beta)$ is defined using $d_N - \beta$ instead of $p_N - \beta$. Again, the obvious reformulations of (S1) and (S2) will hold here. We also have the following characterization of strongness in this case, granting that $d_N - \beta \neq \varnothing$:

- (S4) (μ, d_N) is strong above β iff $E^{N'(\mu,\beta)}_{\text{top}} \mid \mu \notin N'(\mu, \beta)$.

Thus, the strongness of (μ, d_N) above β is equivalent to the failure of $N'(\mu, \beta)$ to satisfy the initial segment condition at μ.

The following is a refinement of two crucial properties of strong divisors ([15], Lemmata 2.24 and 2.26). For fixed q and β we have:

- (S5) The set of all μ where (μ, q) is a divisor strong above β is closed in $\omega\varrho_N^{n+1}$ where $n = n(\tau)$.
- (S6) If (μ, q) is a divisor strong above β, $\bar{\mu} \leq \mu$ and $\bar{q}$ is a proper bottom segment of q such that $\bar{q} \cap \beta = p_N \cap \beta$, then $(\bar{\mu}, \bar{q})$ is not a divisor.

Similarly as in the construction of a $\square_\kappa$-sequence in [15] it will be crucial to identify certain divisors that would serve as a "canonical divisors". The morass embeddings with target τ are determined by Skolem hulls over either the collapsing structure for τ or over a collapsing protomouse for τ. Each τ determines uniquely which of the two kinds of collapsing structures should be chosen; the "canonical divisor" guarantees the unique choice of the protomouse.

In the following we explain how to identify the "canonical divisor". Fix a bottom segment q of p_N (resp. of d_N). If there were cofinally many $\mu < \omega\varrho_N^{n+1}$ such that (μ, q) is a strong divisor for N then (F1) – (F4) would also hold for $\mu = \omega\varrho_N^{n+1}$ with the exception of the requirement $\mu < \alpha$ in (F1). The proof of (S5) would also work for $\mu = \omega\varrho_N^{n+1}$ and we could conclude that the pair $(\omega\varrho_N^{n+1}, q)$ satisfy (S1). This is clearly impossible, as N is sound. We thus conclude that there are only boundedly many $\mu < \omega\varrho_N^{n+1}$ such that (μ, q) is strong. Consequently, there is always a *largest* such $\mu < \omega\varrho_N^{n+1}$, granting that N admits at least one strong divisor of the form (μ, q). This is the key toward determining the "canonical divisor".

In our setting, $\omega\varrho_N^{n+1}$ will turn out to be either α or the cardinal predecessor of α in N. If $(p_N - \omega\varrho_N^{n+1}) \cap \alpha = \varnothing$ then "strong above α" has the same meaning as "strong". It follows immediately from the previous paragraph that there is a divisor (μ, q) for N strong above α with largest possible μ, provided that there is at least one strong divisor. If $(p_N - \omega\varrho_N^{n+1}) \cap \alpha \neq \varnothing$ then $\omega\varrho_N^{n+1}$ is the cardinal predecessor of α in N. So again there is a divisor strong above α with the largest possible value of μ, this time for the trivial reason that $\omega\varrho_N^{n+1} < \alpha$. (Notice that $\omega\varrho_N^{n+1}$ may be the largest possible value of μ in this case.)

The above considerations suggest the following way of identifying the "canonical divisor": If there are μ and q such that (μ, q) is strong above α then there is a largest such μ. The associated q is then uniquely determined as the shortest possible. That is, if (μ, q) and $(\bar{\mu}, \bar{q})$ are divisors for N strong above α and μ is largest possible then q is a bottom segment of $\bar{q}$.

Definition 1.11. $(\mu(N), q(N))$ *is the divisor* (μ, q) *for* N *strong above* α *with largest possible* μ*, granting that some such divisor for* N *exists.*

Being a strong divisor is a $\Pi_2^{(n)}$-property. This fact was irrelevant in the proof of $\square_\kappa$, but will be useful here. Given an ordinal α and a finite set of ordinals s, we let $\max^\alpha(s) = \max(s \cup \{\alpha\})$.

Lemma 1.12. *Let N be the collapsing structure for τ, let $\alpha = \alpha(\tau)$ and let (μ, q) be such that $\mu < \alpha$ and q is a bottom segment of p_N resp. of d_N, if N is pluripotent. Let further $r = p_N - (\max^\alpha(q)+1)$ resp. $r = d_N - (\max^\alpha(q)+1)$ if N is pluripotent.*

(a) *The statement "(μ, q) is a divisor for N" is uniformly $Q^{(n)}$. More precisely:*

(a1) *There is a $Q^{(n)}$-formula $\psi(v_1, v_2, v_3, v_4)$ in the language for premice such that for every $\tau, N, \alpha, (\mu, q)$ and r as above we have:*

- *(μ, q) is a divisor for N iff $N \models \psi(\mu, \max^\alpha(q), r, \alpha)$.*

assuming that either N is not pluripotent or $\lambda(\mu, q) < \lambda(E^N_{\text{top}})$.

(a2) *There is a Π_1-formula $\psi(v)$ in the language for coherent structures such that for all τ, N and μ as above we have:*

- *(μ, d_N) is a divisor for N iff $N \models \psi(\mu)$.*

assuming that N is pluripotent and $\lambda(\mu, q) = \lambda(E^N_{\text{top}})$.

(b) *The statement "(μ, q) is a divisor for N strong above α" is uniformly $\Pi_2^{(n)}$. More precisely:*

(b1) *There is a $\Pi_2^{(n)}$-formula $\psi^*(v_1, v_2, v_3, v_4)$ in the language for premice such that for every $\tau, N, \alpha, (\mu, q)$ and r as above we have:*

- *(μ, q) is a divisor for N strong above α iff $N \models \psi^*(\mu, q-\alpha, r, \alpha)$*

assuming that either N is not pluripotent or $\lambda(\mu, q) < \lambda(E^N_{\text{top}})$.

(b2) *There is a Π_2-formula $\psi^*(v_1, v_2)$ in the language for coherent structures such that for every τ, N, α and μ as we have:*

- *(μ, d_N) is a divisor for N strong above α iff $N \models \psi^*(\mu, d_N - \alpha)$*

assuming that N is pluripotent and $\lambda(\mu, q) = \lambda(E^N_{\text{top}})$.

Proof. For (a1), the formula $\psi(\mu, q-\alpha, r, \alpha)$ is the conjunction

$$(\forall u^n)(\exists v^n \supset u^n)(\exists \xi^n)(v^n = \tilde{h}^{n+1}_N(\xi^n, r)) \;\;\&\;\; (\forall \xi^n)(\forall \zeta^n)\tilde{\psi}(\xi^n, \zeta^n, \mu, q-\alpha, r, \alpha)$$

where $\tilde{\psi}(\xi^n, \zeta^n, \mu, q-\alpha, r, \alpha)$ is the $\Sigma_1^{(n)}$-formula

$$[\xi^n < \mu \;\&\; \zeta^n = \tilde{h}^{n+1}(\xi^n, r) \;\&\; \zeta^n < \max^\alpha(q-\alpha)+1] \;\longrightarrow\; \zeta^n < \mu.$$

The first formula in the above conjunction expresses (F3) and the second formula (F4) in the definition of a divisor. Notice that any reference to α is superfluous unless $q \subseteq \alpha$. The case (a2) is easy, as the formula ψ just says that μ is a cutpoint of E^N_{top}.

Regarding (b), we prove only the former part of the statement and leave the latter part where (μ, d_N) is a divisor to the reader. Notice first that in (S1), we can without loss of generality assume that $x, y \in H^n_M$. The formula $\psi^*(\mu, q, r, \alpha)$ is the conjunction of $\psi(\mu, q, r, \alpha)$ and the formula

$$(\forall x^n)(\forall \xi^n < \mu)[x^n = \tilde{h}^{n+1}_N(\xi^n, ((q-\alpha)\cup r) \longrightarrow \tilde{\psi}^*(x^n, \mu, r)]$$

where $\tilde{\psi}^*(x^n, \mu, r)$ is the $\Sigma^{(n)}_1$-formula

$$(\exists \eta^n < \mu)(\exists y^n)(y^n = \tilde{h}^{n+1}_N(\eta^n, r) \ \&\ y^n \cap \mu = x^n \cap \mu)$$

$\square$(Lemma 1.12)

1.4. **Protomice, Fine Structure and Condensation.** We discuss fine structural lemmata for protomice relevant for the construction of a morass. The focus is on translating fine structural information between N and $N(\mu, q)$ where (μ, q) is a fixed divisor for N, as well as on condensation lemmata for protomice. We need two new condensation lemmata. One is a refinement of a lemma in [15] and will be applied to embeddings with critical points "at α^+". The other deals with embeddings with critical points "at α" and will be applied to morass embeddings.

We will consider a coherent structure $M = \langle J^E_\nu, F\rangle$ together with a premouse N with the following properties:

(C1) J^E_ν is a premouse all of whose initial segments are sound and solid and F is an extender at (μ, λ).

(C2) $\mathrm{dom}(F) = \mathcal{P}(\mu) \cap J^E_\vartheta$, $\vartheta = \vartheta(M) < \mu^{+M}$ and N^* is the collapsing level for ϑ in M.

(C3) $\omega\varrho^{n+1}_{N^*} \leq \mu < \omega\varrho^n_{N^*}$.

(C4) $\pi : N^* \xrightarrow[F]{*} N$.

Clauses (C1) – (C4) cover all cases where M is of the form $N(\mu, q)$ for some divisor (μ, q), and in fact are of higher generality. This higher generality will be needed in our arguments. In some cases we will consider M to be a pluripotent mouse instead of a structure as in (C1) – (C4); in all such cases we will explicitly mention this. Let us recall once more time that we adopt the following convention. Whenever we deal with protomice we automatically use the language for coherent structures. So in (C1) – (C4), the language used for M is the language for coherent structures. On the other hand, for N^* we use the language for premice. If $\mu < \lambda(E^{N^*}_{\mathrm{top}})$ or N^* is of type A or B then N^* and N are premice of the same type, so we use the same kind of language for both N and N^*. However, if $\mu = \lambda(E^{N^*}_{\mathrm{top}})$ and N^* is a type C premouse then the language for premice is the same as the language for coherent structures when dealing with N^*. But the associated premouse N from (C4) may turn out to be a type B premouse, in which case the two languages differ. This kind situation will always be treated separately, if the circumstances require it. The issue here is important as it has a substantial effect on verifying coherency in combinatorial constructions. In general, to avoid any possible misunderstandings, we will explicitly say which language we use in all situations where the confusion is likely.

Given a finite $s \subset \lambda$, the standard witness $W^{\beta,s}_M$ is the transitive collapse of the hull $h_M(\beta \cup \{s\})$. Here h_M is the Σ_1-Skolem function with respect to the *language for coherent structures.* A generalized witness for β with respect to M and s is defined in the obvious way; see the beginning of Subsection 1.2.

For M, N as in (C1) – (C4) we obtain refined versions of results in [15], Section 2 summarized in (FS1) – (FS4). The refinement is needed, as in the construction of a morass we also have to consider the case $\omega\varrho^1_M = \mu < \alpha(\tau)$ where M is a collapsing protomouse for τ. (For comparison with the construction in [15], here α plays the same role as κ in [15].)

Notice first that the map $\pi : N^* \to N$ is $\Sigma^{(n)}_0$-preserving and cofinal at the n-th level, as easily follows from the soundness of N^* and general properties of fine ultrapowers. Also, $\omega\varrho^{n+1}_N \leq \lambda$ since $N = \tilde{h}^{n+1}_N(\lambda \cup \{\pi(p_{N^*})\})$. Let $r = \pi(p_{N^*}) - \lambda$. If N is solid then $r = p_N - \lambda$, as follows by a standard argument. Recall that by the definition of R^{n+1}_N, $a \in R^{n+1}_N$ iff $\tilde{h}^{n+1}_N(\omega\varrho^{n+1}_N \cup \{a\}) = N$ and similarly $a \in R^1_M$ iff $h_M(\omega\varrho^1_M \cup \{a\}) = M$.

(FS1) $\omega\varrho^1_M = \omega\varrho^{n+1}_N$.

(FS2) $R^{n+1}_N \neq \varnothing$ just in case that $R^1_M \neq \varnothing$. In fact, if $q \subseteq \lambda$ is a finite set of ordinals then

$$\begin{aligned} r \cup q \in R^{n+1}_N \quad &\Longrightarrow \quad q \cup \{\vartheta(M)\} \in R^1_M \quad \text{and} \\ q \in R^1_M \quad &\Longrightarrow \quad r \cup q \cup \{\mu\} \in R^{n+1}_N, \end{aligned}$$

granting that $\lambda(E^{N^*}_{\mathrm{top}}) > \mu$. If $\lambda(E^{N^*}_{\mathrm{top}}) = \mu$ then R^{n+1}_N is computed in the language for coherent structures.

(FS3) If $s \subset \lambda$ is finite and $\mu^{+M} \leq \beta$ then

$$\tilde{h}^{n+1}_N(\beta \cup \{s \cup r\}) \cap \lambda = h_M(\beta \cup \{s\}) \cap \lambda,$$

granting that $\lambda(E^{N^*}_{\mathrm{top}}) > \mu$. If $\lambda(E^{N^*}_{\mathrm{top}}) = \mu$ then the equality holds with h^*_N in place of $\tilde{h}^{n+1}_N$; in this case $r = \varnothing$.

(FS4) If $\max\{\mu^{+M}, \omega\varrho^1_M\} \leq \beta < \lambda$, $s \subseteq \lambda$ is finite and $W^{\beta,s}_M = \langle J^{\bar{E}}_{\bar{\nu}}, \bar{F}\rangle$ then

$$W^{\beta, s\cup r}_N = \mathrm{Ult}^*(N^*, \bar{F}) \qquad \text{or} \qquad {}^*W^{\beta,s}_N = \mathrm{Ult}^*(N^*, \bar{F}),$$

where the former holds if $\lambda(E^{N^*}_{\mathrm{top}}) > \mu$ and the latter otherwise.

Recall also useful notation from [15]. If $\beta < \lambda$ then β is of the form $\pi(f)(s)$ where $f : {}^{|q|+1}\mu \to \mu$ is an element of N^* and s is the decreasing sequence enumerating $q \cup \{\xi\}$ for some $\xi < \mu$. We will write "$\pi(f)(q,\xi)$" in place of $\pi(f)(s)$. The map f is an element of J^E_ϑ and is essentially a subset of ${}^{|q|+2}\mu$, so by a slight abuse of notation we can write $F(f)$ instead of $\pi(f)$. Thus, the equality

$$\beta = F(f)(q,\xi) \tag{4}$$

has the obvious meaning. We adopt similar notation when dealing with a subset $x \subseteq \lambda$ in place of an ordinal $\beta < \lambda$. In this case f would be a function with $\mathrm{rng}(f) \subset \mathcal{P}(\mu)$, so f would be essentially a subset of ${}^{|q|+3}\mu$. Again, the equation $x = F(f)(q,\xi)$ has the obvious meaning. The following useful proposition was essentially proved in [15]. It specifies how ordinals definable over N can be expressed in the form described in (4) and vice versa.

Proposition 1.13. *Assume (C1) – (C4). Let $a \in N \cap \mathcal{P}(\lambda)$, $r = \pi(p_{N^*})$ and $x \in [\lambda]^{<\omega}$. Then $a \in \tilde{h}^{n+1}_N(\mu \cup \{r \cup x\})$ just in case that there is a function $f \in J^E_\vartheta$ and $\xi < \mu$ such that $a = F(f)(x,\xi)$.*

Now turning to a different topic and no longer assuming (C1) – (C4), we state the two condensation lemmata. The lemmata refer to a fixed extender model $W = \mathbf{L}[E]$. Recall the notion of collapsing structure from Definition 1.5. Recall also the notation introduced below Definition 1.8, in particular that $\alpha(\tau)$ is the cardinal predecessor of τ in the collapsing structure for τ. The first condensation lemma is a slight refinement of Lemma 2.19 from [15].

Lemma 1.14. *Let $\alpha = \alpha(\tau)$. Let $\bar{M}$ be a coherent structure and let M satisfy one of the following:*

(i) *$M = N(\mu, q)$ where N is the collapsing structure for τ and (μ, q) is a divisor for N;*

(ii) *M is the collapsing structure for τ pluripotent at α.*

Let $\sigma : \bar{M} \to M$ be Σ_0-preserving with critical point $\bar{\tau} = \alpha^{+\bar{M}}$ and $\sigma(\bar{\tau}) = \tau$. Assume there is a finite set of ordinals $\bar{q}$ such that:

(a) *$h_{\bar{M}}(\alpha \cup \{\bar{q}\}) = \bar{M}$.*

(b) *For every $\beta \in \bar{q}$ there is a generalized witness $Q^{\beta,\bar{q}}_{\bar{M}} \in \bar{M}$.*

(c) $\sigma(\bar{q}) = \begin{cases} q - \alpha, & \textit{if (i) above applies} \\ d^{\alpha}_{M}, & \textit{if (ii) above applies;} \end{cases}$

Then $\bar{M} \in M$, $\omega\varrho^1_{\bar{M}} = \alpha$ and $\bar{q} = p^1_{\bar{M}}$. Also $\bar{q} = p_{\bar{N}} \cap \lambda(\bar{M})$ if $\lambda(E^{\bar{M}}_{\vartheta(\bar{M})}) < \mu$ and $\bar{q} = d^{\alpha}_{\bar{N}}$ if $\lambda(E^{\bar{M}}_{\vartheta(\bar{M})}) = \mu$ where $\bar{N}$ is the collapsing structure for $\bar{\tau}$. Finally $(\mu, \bar{q})$ is a divisor for $\bar{N}$ and $\bar{M} = \bar{N}(\mu, \bar{q})$.

We omit the proof of this lemma, as it is – up to some minor modifications – the same as the proof of [15], Lemma 2.19.

In the same sense that our first condensation lemma deals with maps that reflect τ while keeping α fixed, our second condensation lemma deals with maps that reflect α.

Lemma 1.15. *Let $\alpha = \alpha(\tau)$. Let $\bar{M}$ and M be coherent structures satisfying:*

- *$M = N(\mu, q)$ where N is the collapsing structure for τ and (μ, q) is a divisor for N;*
- *$\bar{M}$ is* not *a potential premouse, i.e. its top extender is not weakly amenable.*

Assume $\sigma : \bar{M} \to M$ is a Σ_1-preserving embedding with critical point $\bar{\alpha}$ satisfying $\sigma(\bar{\alpha}) = \alpha$ and $\bar{q} \in \bar{M}$ is a finite set of ordinals such that the following conditions are met:

(a) $h_{\bar{M}}(\bar{\alpha} \cup \{\bar{q}\}) = \bar{M}$.
(b) *For every $\beta \in \bar{q}$ there is a generalized witness $Q^{\beta;\bar{q}}_{\bar{M}} \in \bar{M}$.*
(c) $\sigma(\bar{q}) = q - \alpha$.

Let $\bar{N} = N(\bar{M})$ and $\mu = \mathrm{cr}(E^M_{\mathrm{top}})$. Then $\bar{q} = p^1_{\bar{M}} - \bar{\alpha}$. Also $\bar{q} = (p_{\bar{N}} \cap \lambda(\bar{M})) - \bar{\alpha}$ if $\lambda(E^{\bar{M}}_{\vartheta(\bar{M})}) > \mu$ and $\bar{q} = d^{\bar{\alpha}}_{\bar{N}}$ if $\lambda(E^{\bar{M}}_{\vartheta(\bar{M})}) = \mu$. Furthermore, exactly one of the following two clauses holds:

(A) *$\bar{N}$ is sound. In this case $\bar{N}$ is the collapsing structure for $\bar{\tau} = \bar{\alpha}^{+\bar{M}}$ and there is a divisor $(\mu, \bar{q}^*)$ for $\bar{N}$ such that $\bar{q} = \bar{q}^* - \bar{\alpha}$ and $\bar{M} = \bar{N}(\mu, \bar{q}^*)$.*
(B) *$\bar{N}$ is not sound. Let $\bar{N}^* = N^*(\bar{M})$. In this case α is a successor cardinal in $\bar{N}$, so there is an ordinal κ such that $\bar{\alpha} = \kappa^{+\bar{N}}$. Moreover, $\omega\varrho^{n+1}_{\bar{N}} = \kappa$ where $n = n(\tau)$, so $\omega\varrho^{n+1}_{\bar{N}^*} = \kappa < \omega\varrho^n_{\bar{N}^*}$. If $\lambda(E^{\bar{N}^*}_{\mathrm{top}}) > \mu$ then*

$$\bar{N} = \tilde{h}^{n+1}_{\bar{N}}(\kappa \cup \{\bar{r}, \bar{q}, \kappa\})$$

where $\bar{r} = \bar{\pi}(p_{N^*})$ and $\bar{\pi}$ is the ultrapower map coming from $\mathrm{Ult}^*(\bar{N}^*, E^{\bar{M}}_{\mathrm{top}})$. If $\lambda(E^{\bar{N}^*}_{\mathrm{top}}) = \mu$ then $\bar{N} = h^*_{\bar{N}}(\kappa \cup \{\bar{q}, \kappa\})$.

Proof. We first verify that $\bar{N}$ is as in the statement of the lemma. Let $\bar{F}$ and F be the top extenders of $\bar{M}$ and M, respectively. Obviously, $\mathrm{cr}(\bar{F}) = \mu = \mathrm{cr}(F)$. Because neither of $\bar{F}$ and F is weakly amenable, we have $\vartheta(M) < \alpha$ and $\vartheta(\bar{M}) < \bar{\alpha}$. Letting $E = E^W$, notice that $\mathcal{P}(\mu) \cap \bar{M} \subset J^E_{\bar{\alpha}}$ and $\mathcal{P}(\mu) \cap M \subset J^E_{\alpha}$. The fact that $a \subset \mu$ is measured by $\bar{F}$ can be expressed by the Σ_1-formula $(\exists y)(\langle a, y\rangle \in \bar{F})$ and similarly for F. Now σ is Σ_1-preserving, so for $a \in \mathcal{P}(\mu) \cap \bar{M}$ we have

$$a \in \mathrm{dom}(\bar{F}) \longleftrightarrow a \in \mathrm{dom}(F).$$

This immediately implies that $\vartheta(\bar{M}) = \vartheta(M)$.

By the previous paragraph, $N^*(\bar{M}) = N^*(M)$; we will write briefly N^*. Let n be such that $\omega\varrho^{n+1}_{N^*} \le \mu < \omega\varrho^n_{N^*}$. Let $\bar{N} = N(\bar{M}) = \mathrm{Ult}^*(N^*, \bar{F})$ and $N = N(M) = \mathrm{Ult}^*(N^*, F)$ with the associated ultrapower maps $\bar{\pi}$ and π. We define a map $\tilde{\sigma} : \bar{N} \to N$ by

$$\tilde{\sigma}(\bar{\pi}(f)(\beta)) = \pi(f)(\sigma(\beta))$$

for $f \in \Gamma(\mu, N^*)$ and $\beta < \lambda(\bar{F})$. The map $\tilde{\sigma}$ is easily seen to be Σ_0-preserving, which guarantees that $\bar{N}$ is well-founded, and we will consider it to be transitive. Standard methods [21] then show that $\tilde{\sigma}$ is actually $\Sigma_0^{(n)}$-preserving. The definition of $\tilde{\sigma}$ immediately implies $\pi = \tilde{\sigma} \circ \bar{\pi}$ and $\tilde{\sigma} \restriction \lambda(\bar{F}) = \sigma \restriction \lambda(\bar{F})$. This means that $\bar{\alpha} = \mathrm{cr}(\tilde{\sigma})$.

To see that $\bar{N}$ is as in the conclusion of the lemma we apply the Condensation Lemma (Lemma 1.4). In the following we verify that $\bar{N}$ satisfies the assumptions of the lemma. Assume first that $\lambda(E^{N^*}_{\mathrm{top}}) > \mu$. Let $\bar{r} = \bar{\pi}(p_{N^*})$. Clause (a) in the statement of the current lemma together with (FS3) guarantee that

$$\tilde{h}^{n+1}_{\bar{N}}(\bar{\alpha} \cup \{\bar{r}, \bar{q}\}) = \tilde{h}^{n+1}_{\bar{N}}(\lambda(\bar{M}) \cup \{\bar{r}\}) = \bar{N}.$$

This alone implies that $\omega\varrho^{n+1}_{\bar{N}} \leq \bar{\alpha}$, so $\bar{N}$ is solid by the first part of the Condensation Lemma. The solidity of $\bar{N}$, the above fact on the hulls and (FS4) then altogether yield that $\bar{N}$ is sound above $\bar{\alpha}$ ([21], Lemma 1.12.5). This completes the verification in the case where $\lambda(E^{N^*}_{\mathrm{top}}) > \mu$. Now assume $\lambda(E^{N^*}_{\mathrm{top}}) = \mu$. In this case we apply (DP2) additionally to (FS4) to conclude that $\bar{q} = d^{\bar{\alpha}}_{\bar{N}}$ and $\bar{N}$ is sound above $\bar{\alpha}$. However, we need to do some additional work to see that $\bar{N}$ is a solid premouse. Notice that it is not automatically clear that $\bar{N}$ satisfies the initial segment condition; the issue arises since $\bar{N}$ is a type B potential premouse, whereas N^* might have been a type C premouse. Here we use the latter part of (FS4) asserting that ${}^*W^{\beta,\bar{q}}_{\bar{N}} \in \bar{N}$ for $\beta = \max(\bar{q})$. This β is obviously $\geq \lambda^*(\bar{N})$, which is sufficient to guarantee that $E^{\bar{N}}_{\mathrm{top}} \mid \lambda^*(\bar{N}) \in \bar{N}$. Also, $\tilde{\sigma}$ is not necessarily an embedding between two premice, since it only preserves Σ_0 formulae in the language of coherent structures if N^* is of type C. To take care of this, notice that if $\tilde{N} = N||\tilde{\gamma}$ where $\tilde{\gamma}$ is the index of $E^N_{\mathrm{top}} \mid \tilde{\lambda}$ and $\tilde{\lambda} = \sup(\tilde{\sigma}''\lambda(\bar{F}))$ then $\tilde{\sigma} : \bar{N} \to \tilde{N}$ is as needed. This completes the verification in the remaining case where $\lambda(E^{N^*}_{\mathrm{top}}) = \mu$.

We are now ready to apply the Condensation Lemma to $\bar{N}$ and $\tilde{\sigma}$. If follows that one of the clauses (b), (c) or (d) of the Condensation Lemma must apply. To see that (a) cannot be true notice that either $\omega\varrho^{n+1}_{N} = \alpha$ in which case $\bar{N}$ and N have different $(n+1)$-st projecta or else α is a successor cardinal in N (here we use the fact that α is cardinal-correct in the collapsing structure for τ) in which case we have some κ such that $\kappa^{+\bar{N}} = \bar{\alpha} < \alpha = \kappa^{+N}$. In either case $\bar{N} \neq N$.

Let $\bar{q}^* = p_{\bar{N}} \cap \lambda(\bar{F})$ if $\lambda(E^{N^*}_{\text{top}}) > \mu$ and $\bar{q}^* = d^{\bar{\alpha}}_{\bar{N}}$ otherwise. That $\bar{q}$ is as in the statement of the lemma, i.e. $\bar{q} = \bar{q}^* - \bar{\alpha}$, follows from the above discussion and some easy applications of (FS1) – (FS4). If (a) or (d) of the Condensation Lemma applies then $\bar{N}$ is as in (A) in the statement of the current lemma. That $\bar{M} = \bar{N}(\mu, \bar{q}^*)$, follows from the way we defined $\bar{N}$. If (c) of the Condensation Lemma applies then $\bar{N}$ is as in (B) in the statement of the current lemma. To see this notice that $\bar{N} = \text{Ult}^*(Q, E^N_\delta)$ where Q is the collapsing level for $\bar{\alpha}$ in N, $\delta \leq \text{ht}(Q)$ and E^N_δ has a single generator $\kappa = \text{cr}(E^N_\delta)$. This κ is a cardinal predecessor of $\bar{\alpha}$ in Q, and so in $\bar{N}$. By general properties of fine ultrapowers, no projectum of $\bar{N}$ can be in the interval $(\kappa, \pi_Q(\kappa)]$ where π_Q is the associated ultrapower map. We know that $\omega\varrho^{n+1}_{\bar{N}} \leq \bar{\alpha}$, so necessarily $\omega\varrho^{n+1}_{\bar{N}} \leq \kappa$. Now κ is a cardinal in N and $\bar{N} \in N$, so we actually have $\omega\varrho^{n+1}_{\bar{N}} = \kappa$. As E^Q_δ is an extender from the Q-sequence, $\pi_Q(p_{N^*}) = p_{\bar{N}}$. From properties of fine ultrapowers we then see that $\bar{N} = \tilde{h}^{n+1}_{\bar{N}}(\kappa \cup \{p_{\bar{N}}, \kappa\})$, so $\bar{N}$ is sound above $\kappa + 1$. Notice that $\min(p_{\bar{N}}) \geq \pi_Q(\kappa) > \bar{\alpha}$. If $\lambda(E^{N^*}_{\text{top}}) > \mu$ this together with the equality $\bar{q} = (p_{\bar{N}} \cap \lambda(\bar{F}) - \bar{\alpha}$ proved above tells us that $\bar{q} = p_{\bar{N}} \cap \lambda(\bar{F})$, so the soundness of $\bar{N}$ above $\kappa + 1$ gives the first part of (B). If $\lambda(E^{N^*}_{\text{top}}) = \mu$ then $\bar{\kappa} \stackrel{\text{def}}{=} \text{cr}(E^{\bar{N}}_{\text{top}}) = \text{cr}(E^{\bar{N}^*}_{\text{top}}) < \mu$. (Recall $\bar{N}^* = N^*(\bar{M}) = N^*$.) As Q is sound, d_Q exists and Q is Dodd solid by (DP1). Working in the language for coherent structures, general properties of fine ultrapowers yield $\bar{N} = h^*_{\bar{N}}(\kappa \cup \{\pi_Q(d_Q), \kappa\})$. Using (DP2), we infer $\pi_Q(d_Q) = d^{\bar{\alpha}}_{\bar{N}} - \alpha = \bar{q}$ where the latter equality was established above. The conclusion then follows immediately.

$\square$(Lemma 1.15)

2. The Morass Construction

We now give a proof of Theorem 1.2, that is, we describe a construction of the Gap-1 Morass introduced in Definition 1.1. The construction uses techniques that resemble – to some extent – to the construction of a $\square_\kappa$-sequence in [15], in that we split the class of all ordinals of interest into two disjoint subclasses, say $\mathcal{T}^0$ and $\mathcal{T}^1$, and define collapsing structures on each of these classes separately. However, in contrast with that construction, it is *not* possible to define morass embeddings on each of these classes *separately,* as the construction will force us to consider morass embeddings $\sigma_{\bar{\tau},\tau}$ where $\bar{\tau} \in \mathcal{T}^0$ and $\tau \in \mathcal{T}^1$. Apart from this there are other differences. For instance, in order to verify certain morass axioms for $\sigma_{\bar{\tau},\tau}$ where both $\bar{\tau}$ and τ are in $\mathfrak{T}^0$, we need some information on *all* morass embeddings $\sigma_{\bar{\tau}',\tau'}$ where $\tau' <_0 \tau$, and so we also have to consider ordinals $\tau' \in \mathfrak{T}^1$.

The section is divided into two subsections. In the first subsection we define the morass. The morass axioms are verified in the second subsection.

2.1. **Defining the Morass.** We begin with the definition of $S(0)$ and $S(1)$. Recall that we work in a fixed extender model $W = \mathbf{L}[E]$. In order to avoid discussing certain trivial issues, it will be convenient to restrict ourselves to ordinals with sufficient closure properties.

Definition 2.1. *An acceptable structure J^A_α is* good *just in case that J^E_ζ is a ZFC^--model for arbitrarily large $\zeta < \alpha$.*

We then set:

$$S(0) = \text{the class of all cardinal-correct ordinals } \alpha \text{ such that } J^E_\alpha \text{ is good.}$$

To define S_α for $\alpha \in S(0)$, we proceed by cases.

(a) α is regular. Then $S_\alpha = S(0) \cap (\alpha, \alpha^+)$.

(b) α is a singular cardinal. We put τ into S_α just in case that $\tau \in S(0)$, $\tau > \alpha$ and α is regular in J^E_τ.

(c) α is not a cardinal. We distinguish two cases.

  (c1) $E_\alpha = \varnothing$. We let $\tau \in S_\alpha$ just in case that J^E_τ is good, α is the largest cardinal in J^E_τ and is regular in J^E_τ.

  (c2) $E_\alpha \neq \varnothing$. Let $W'_\alpha = \mathrm{Ult}(W, E_\alpha)$ and $E' = E^{W'_\alpha}$. We let $\tau \in S_\alpha$ just in case that $J^{E'}_\tau$ is good and α is the largest cardinal in $J^{E'}_\tau$.

If (c2) in the above definition applies then $\lambda(E_\alpha)$ is a limit cardinal in W, as α is cardinal-correct. Also, α fails to be a cardinal in $J^E_{\alpha+1}$ in this case. Notice that α is a successor cardinal in $J^{E'}_\tau$, so there is no need to demand the regularity of α in $J^{E'}_\tau$ in (c2).

That $S(0)$ and all S_α satisfy all requirements in the definition of a morass stated above the morass axioms is obvious, unless $E_\alpha \neq \varnothing$. In this case, $\max(S_\alpha) = \alpha^{+W'_\alpha} = \pi_\alpha(\nu)$ where $\nu = \mathrm{cr}(E_\alpha)^{++W}$ and π_α is the ultrapower map associated with $\mathrm{Ult}(W, E_\alpha)$. Since λ is a cardinal in W, the ultrapower map associated with $\mathrm{Ult}(J^E_\alpha, E_\alpha)$ agrees with π_α on all elements of J^E_λ, and since λ is a limit cardinal,

$\nu < \lambda$. If $\alpha' \in S(0)$ is larger than α then $\langle J^E_\alpha, E_\alpha\rangle \in J^E_{\alpha'}$. By the goodness of α', the ultrapower $\mathrm{Ult}(J^E_\alpha, E_\alpha)$ together with its associated ultrapower map are in $J^E_{\alpha'}$ and so $\pi_\alpha \restriction \lambda \in J^E_{\alpha'}$. It follows that $\max(S_\alpha) = \pi_\alpha(\nu) \in J^E_{\alpha'}$. Following the definition of a morass, we set

$$S(1) = \bigcup\{S_\alpha;\, \alpha \in S(0)\}.$$

The next task is to define the morass ordering $\prec$ and morass embeddings $\sigma_{\bar\tau,\tau}$. Both are induced by hulls over canonically chosen collapsing structures for τ. We chose such structures for all $\tau \in S(1) - S^+$ where

$$\tau \in S^+ \longleftrightarrow \tau = \max(S_\alpha) \text{ for some } \alpha \in S(0) \text{ with } E_\alpha \neq \varnothing.$$

We are thus ignoring the elements of S^+ in our construction. This might look suspicious at first glance, but notice that the morass axiom (M3) allows the option that each $\tau \in S^+$ is minimal with respect to the morass ordering $\prec$. Furthermore, it will be clear from the construction that situations where $\tau \prec \tau'$ and $\tau \in S^+$ never occur. Before turning to the definition of our collapsing structures we set:

$$\begin{aligned}
N_\tau &= \text{ the collapsing structure for } \tau \text{ (see Definition 1.5)}\\
E^\tau &= E^{N_\tau}\\
n_\tau &= n(\tau) \quad \text{(see remarks below Definition 1.8)}\\
p_\tau &= p_{N_\tau}\\
d^\beta_\tau &= d^\beta_{N_\tau} \quad \text{whenever applicable}\\
\tilde h_\tau &= \tilde h^{n_\tau+1}_{N_\tau}.
\end{aligned}$$

Notice that these definitions do make sense for *any* τ that is a local successor of α in W or in $\mathrm{Ult}(W, E_\alpha)$ (more precisely: for τ such that α is the largest cardinal in J^E_τ or $J^{E'}_\tau$) and we will occasionally use this fact. Notice also that if τ is not the largest element of S_α, then

$$\omega\varrho^{n_\tau+1}_{N_\tau} = \omega\varrho^\omega_{N_\tau} = \alpha(\tau). \tag{5}$$

Indeed, the ultimate projectum of N_τ cannot drop below $\alpha(\tau)$, as otherwise $\alpha(\tau)$ would be collapsed in any $J^E_{\tau'}$ where $\tau < \tau' \in S_{\alpha(\tau)}$. Thus, in this case we have $p_\tau = p_{N_\tau}$. If τ *is* the largest element of S_α, then the only way the ultimate projectum can drop below α is the following:

$$\omega\varrho^\omega_{N_\tau} = \kappa \text{ where } \kappa^{+N_\tau} = \alpha(\tau). \tag{6}$$

This is the case because α is cardinal-correct in W: As κ is a cardinal in W, any bounded subset $x \in W$ of κ must be an element of $J^E_\kappa \subseteq N_\tau$. However, it does *not* follow that $\omega\varrho^{n_\tau+1}_{N_\tau} = \kappa$. It is straightforward to find examples of ordinals τ such that $\mathrm{cf}(\alpha(\tau)) > \omega$ and there are unboundedly many $\bar\alpha < \alpha(\tau)$ that are of the form $\omega\varrho^{n_{\bar\tau}+1}_{N_{\bar\tau}} = \alpha(\bar\tau)$ for some $\bar\tau$ and at the same time $\omega\varrho^{n_{\bar\tau}+2}_{N_{\bar\tau}} < \bar\alpha$ (in fact, such $\bar\tau$ satisfy $\bar\tau \prec \tau$ for $\prec$ introduced below). Finally recall that if $\omega\varrho^\omega_{N_\tau} < \alpha(\tau)$ then $p_\tau \cap \alpha(\tau)$ has at most one element.

We now turn to the collapsing structures. As already mentioned above, we split $S(1)$ into two disjoint classes $\mathfrak{T}^0$ and $\mathfrak{T}^1$, and define the collapsing structures on each of these classes separately.

- $\mathfrak{T}^0 = S(1) - \mathfrak{T}^1$.
- $\mathfrak{T}^1$ is the class of all $\tau \in S(1)$ satisfying:
  - (a) $\alpha(\tau)$ is a successor cardinal in N_τ, say $\alpha(\tau) = \mu^{+N_\tau}$.
  - (b) There is a q such that (μ, q) is a divisor for N_τ strong above $\alpha(\tau)$.

If $\tau \in \mathfrak{T}^1$ and (μ, q) is as in (b) then (μ, q) is a divisor for N_τ strong above $\alpha(\tau)$ with largest possible μ, so $(\mu, q) = (\mu(N_\tau), q(N_\tau))$; see Definition 1.11. We let

$$(\mu_\tau, q_\tau) = (\mu(N_\tau), q(N_\tau))$$

Definition 2.2. *Given a $\tau \in S(1)$, we define the canonical structure M_τ as follows.*

$$M_\tau = \begin{cases} N_\tau & \text{if } \tau \in \mathfrak{T}^0 \\ N_\tau(\mu_\tau, q_\tau) & \text{if } \tau \in \mathfrak{T}^1 \end{cases}$$

Obviously, M_τ is a collapsing structure for τ. For $\tau \in \mathfrak{T}^1$ the structure M_τ agrees with N_τ up to $\lambda(E^{M_\tau}_{\mathrm{top}})^{+N_\tau}$, that is, far beyond $\alpha(\tau)$. For $\tau \in \mathfrak{T}^1$ we further set:

$$\begin{aligned} \lambda_\tau &= \lambda(N_\tau(\mu_\tau, q_\tau)) \\ \vartheta_\tau &= \vartheta(M_\tau) \\ r_\tau &= r(N_\tau) = p_\tau - q_\tau \\ h_\tau &= h_{M_\tau}. \end{aligned}$$

By the above definition, the only divisors we are considering are those with μ_τ being the cardinal predecessor of $\alpha(\tau)$. This makes the choice of the canonical divisor straightforward when comparing the current situation with that in the construction of a square sequence where we have to allow more possibilities for μ_τ.

The discussion above Definition 1.11 yields the characterization of (μ_τ, q_τ) as the divisor (μ, q) for N_τ strong above $\alpha(\tau)$ with largest possible μ and shortest possible q. This characterization be considered an alternative definition of (μ_τ, q_τ), and indeed the entire definition of a Gap-1 Morass can be dubbed with the choice of the canonical divisor based on this characterization. Notice that if we replace the requirement that (μ_τ, q_τ) is strong above α with the requirement that q_τ is shortest possible, the definition still does make sense, but most likely does not define a Gap-1 Morass. We will namely run into difficulties when trying to verify that the new definition really yields a morass. The difficulties occur in the verification of axioms (M4) and (M5) in the crucial Case 2. In those arguments, the only known way to verify that the interpolation argument yields a protomouse together with its canonical divisor is to use the fact that we start from a protomouse associated with a divisor which is strong above the corresponding $\alpha(\tau)$. And we find it plausible that there are levels N_τ that admit divisors of the form (μ, q) where μ is the cardinal predecessor of $\alpha(\tau)$ and q is shortest possible but (μ, q) is not strong above $\alpha(\tau)$. Whether it is possible to eliminate any reference to strong divisors in the construction of a Gap-1 Morass is unclear, though we find it unlikely.

The next task is to define the morass ordering $\prec$ and the morass embeddings $\sigma_{\bar\tau,\tau}$. Given $\tau \in S(1)$, we state a list of conditions that determine the $\prec$-predecessors of τ. We will treat the two cases $\tau \in \mathcal{T}^0$ and $\tau \in \mathcal{T}^1$ separately. We stress again that the definition of $\prec$ allows the possibility that $\tau \in \mathcal{T}^1$ and $\bar\tau \in \mathcal{T}^0$ for $\bar\tau \prec \tau$.

Definition 2.3. *Given $\tau \in \mathcal{T}^0$ and $\bar\tau \in S(1)$, we let $\bar\tau \prec \tau$ just in case the following hold. Let $(\bar\alpha, \alpha) = (\alpha(\bar\tau), \alpha(\tau))$.*

(T0A) $\bar\tau \in \mathcal{T}^0$.

(T0B) $n_{\bar\tau} = n_\tau$.

(T0C) *There is a $\Sigma_1^{(n_\tau)}$-preserving embedding $\sigma : N_{\bar\tau} \to N_\tau$ satisfying:*

(T0C1) $\bar\alpha = \mathrm{cr}(\sigma)$ *and* $\sigma(\bar\alpha) = \alpha$. *If $\tau \in N_\tau$ then $\bar\tau \in N_{\bar\tau}$ and $\sigma(\bar\tau) = \tau$.*

(T0C2) $\sigma(p_{\bar\tau} - \bar\alpha) = p_\tau - \alpha$.

(T0C3) *σ is Q-preserving.*

Recall that a map $\sigma : \bar{N} \to N$ is Q-preserving just in case that σ preserves all Q-formulae, that is, formulae of the form $(\forall u)(\exists v \supseteq u)\psi(v)$ where $\psi(v)$ is a Σ_1-formula that has no free occurence of u. (T0C3) is superfluous except in the case where $n_\tau = 0$ and τ is the height of N_τ, in which case we need (T0C3) to verify morass axiom (M0). This is the only reason for including (T0C3) in Definition 2.3. Actually, a much weaker property than Q-preservation is sufficient for the verification, so we work with stronger (T0C3) only for convenience.

By Definition 2.2 the structures M_τ and N_τ are equal for $\tau \in \mathcal{T}^0$, so we can write M_τ instead of N_τ in Definition 2.3. Given $\bar{\tau}$ and τ as in Definition 2.3, we have $\sigma : \tilde{h}_{\bar{\tau}}(\xi, p_{\bar{\tau}} - \bar{\alpha}) \mapsto \tilde{h}_\tau(\xi, p_\tau - \alpha)$ whenever $\xi < \bar{\alpha}$. It follows that the map σ is uniquely determined and its range is precisely $\tilde{h}_\tau(\bar{\alpha} \cup \{p_\tau - \alpha\})$.

The requirement that $\sigma(\bar{\tau}) = \tau$ in (T0C1) is superfluous and we included it only for convenience. Indeed, if $\tau \in N_\tau$ then $\bar{\tau} \in N_{\bar{\tau}}$ as required in (T0C1) and $\sigma(\bar{\tau})$ is a cardinal in N_τ, as σ is Σ_1-preserving. So actually $\sigma(\bar{\tau}) = \tau$, as α is the largest cardinal in $J^{E^\tau}_{\sigma(\bar{\tau})}$. In most cases the requirement $\tau \in N_\tau \implies \bar{\tau} \in N_{\bar{\tau}}$ is also superfluous, as one can prove that if $\tau \in N_\tau$ then $\tau \in \operatorname{rng}(\sigma)$. However, if $n_\tau = 0$, N_τ is passive and $\operatorname{ht}(N_\tau)$ is a successor ordinal we do not see how to prove that $\tau \in \operatorname{rng}(\sigma)$ in general. Actually, we believe this is false. It is possible that neither of the requirements on $\bar{\tau}$ and τ is necessary, since we always have $\sigma''\bar{\tau} \subseteq \tau$, no matter whether we impose any requirements on $\bar{\tau}$ and τ or not. Even then, the construction becomes much cleaner if we include them.

The morass map $\sigma_{\bar{\tau},\tau}$ will be just the obvious restriction of σ, i.e.

$$\sigma_{\bar{\tau},\tau} = \sigma \restriction \bar{\tau}.$$

It will be convenient to disregard the distinction between the two maps and write $\sigma_{\bar{\tau},\tau}$ to denote both. Clearly, there is no danger of confusion.

Because $\sigma_{\bar{\tau},\tau}$ is entirely determined by the hull $\tilde{h}_\tau(\bar{\alpha} \cup \{p_\tau - \alpha\})$, we conclude that

$$\prec \restriction (S(1) \times \mathcal{T}^0) = \prec \restriction (\mathcal{T}^0 \times \mathcal{T}^0) \tag{7}$$

is a tree ordering. Indeed, if $\tau^* \prec \tau$ then $\tau \in \mathcal{T}^0$. Letting $\alpha^* = \alpha(\tau^*)$, we have $\operatorname{rng}(\sigma_{\tau^*,\tau}) \subset \operatorname{rng}(\sigma_{\bar{\tau},\tau})$ or vice versa, depending on whether $\alpha^* \leq \bar{\alpha}$ or $\bar{\alpha} \leq \alpha^*$. It is clear that $\sigma_{\bar\tau,\tau}^{-1} \circ \sigma_{\tau^*,\tau}$ witnesses that $\tau^* \prec \bar\tau$ in the former case and $\sigma_{\tau^*,\tau}^{-1} \circ \sigma_{\bar\tau,\tau}$ witnesses that $\bar\tau \prec \tau^*$ in the latter case.

We now complete the definition of the morass ordering. For $\tau \in \mathfrak{T}^1$, we defined $\mu_\tau, q_\tau, \lambda_\tau$ and h_τ above. If $\tau \in \mathfrak{T}^0$ and N_τ is a pluripotent premouse with $\mathrm{cr}(E_{\mathrm{top}}^{N_\tau})$ being the cardinal predecessor of $\alpha(\tau)$ in N_τ, we let

$$\begin{aligned} \mu_\tau &= \mathrm{cr}(E_{\mathrm{top}}^{N_\tau}) \\ \vartheta_\tau &= \vartheta(N_\tau) \qquad (= \mu_\tau^{+N_\tau}) \\ q_\tau &= d_{N_\tau} \\ \lambda_\tau &= \lambda(E_{\mathrm{top}}^{N_\tau}) \\ h_\tau &= h_{N_\tau}^*. \end{aligned}$$

The morass ordering $\prec \restriction S(1) \times \mathfrak{T}^1$ is defined as follows.

Definition 2.4. *Given a $\tau \in \mathfrak{T}^1$ and a $\bar\tau \in S(1)$, we let $\bar\tau \prec \tau$ just in case (T1A), (T1B) and (T1C) below hold with $(\bar\alpha, \alpha) = (\alpha(\bar\tau), \alpha(\tau))$.*

(T1A) *$\bar\tau$ satisfies either (T1A1) or (T1A2) below.*

(T1A1) *$\bar\tau \in \mathfrak{T}^1$;*

(T1A2) *$\bar\tau \in \mathfrak{T}^0$ and $M_{\bar\tau} = N_{\bar\tau}$ is a pluripotent premouse such that $\mathrm{cr}(E_{\mathrm{top}}^{M_\tau})$ is the cardinal predecessor of $\alpha(\tau)$ in M_τ.*

(T1B) *There is an embedding $\sigma : M_{\bar\tau} \to M_\tau$ that is Σ_1-preserving with respect to the language for coherent structures and such that both (T1B1) and (T1B2) below hold.*

(T1B1) *$\bar\alpha = \mathrm{cr}(\sigma)$ and $\sigma(\bar\alpha) = \alpha$.*

(T1B2) *$\sigma(q_{\bar\tau} - \bar\alpha) = q_\tau - \alpha$.*

(T1C) *If $\bar\tau \in \mathfrak{T}^0$ then there is a $\tilde\tau \in \mathfrak{T}^0$ with $\alpha(\tilde\tau) = \vartheta_\tau$ such that (T1A) and (T1B) hold with $(\bar\tau, \bar\alpha)$ replaced by $(\tilde\tau, \alpha(\tilde\tau))$.*

To motivate the above definition we remark that the following situation is possible: $\bar\tau, \tau \in \mathfrak{T}^0$ and $\bar\tau \prec \tau$ but if we let $\tau' = \sup(\sigma''_{\bar\tau,\tau}\bar\tau)$, then $\tau \in \mathfrak{T}^1$. In this sense, the elements of $\mathfrak{T}^1$ can be viewed as the missing limits. When this situation arises, $M_\tau = N_\tau$ is pluripotent and $\alpha(\tau)$ is the cardinal successor of $\mathrm{cr}(E_{\mathrm{top}}^{N_\tau})$ in N_τ. The same is true with τ replaced by $\bar\tau$. On the other hand, $M_{\tau'}$ is a protomouse that Σ_0-embeds into N_τ.

We can now make remarks similar to those for $\tau \in \mathfrak{T}^0$. As before we can see that $\sigma_{\bar\tau,\tau} : h_{\bar\tau}(\xi, q_{\bar\tau}) \mapsto h_\tau(\xi, q_\tau)$, so $\sigma_{\bar\tau,\tau}$ is uniquely determined and that it is the uncollapsing map associated with its range $h_\tau(\bar\alpha \cup \{q_\tau - \alpha\})$. That $\tau \in M_\tau$ is trivial, since $\lambda_\tau > \tau$. From this we immediately have that $\tau \in \mathrm{rng}(\sigma_{\bar\tau,\tau})$. Notice also that $\mu_{\bar\tau} = \mu_\tau$. The reason for including clause (T1C) will be explained below.

The verification of the fact that $\prec \restriction (S(1) \times \mathfrak{T}^1)$ is a tree ordering requires a little argument. First of all, we have to argue that $\prec$ is transitive. More precisely, we want to see that $\tau^* \prec \bar\tau$ and $\bar\tau \prec \tau$ imply $\tau^* \prec \tau$. The point here is to show that $\sigma' = \sigma_{\bar\tau,\tau} \circ \sigma_{\tau^*,\bar\tau}$ satisfies (T1B) with τ^* in place of $\bar\tau$, and thereby witnesses that $\tau^* \prec \tau$. This is obvious, unless $\tau \in \mathfrak{T}^1$ and $\bar\tau \in \mathfrak{T}^0$ with $N_{\bar\tau}$ of type B. In this case the map $\sigma_{\tau^*,\bar\tau}$ is a map between two *premice,* and the conditions (T0A) – (T0C) hold with $(\tau^*, \bar\tau)$ in place of $(\bar\tau, \tau)$. What we need to see is that (T1A) – (T1C) are satisfied with $(\tau^*, \bar\tau)$ in place of $(\bar\tau, \tau)$. The language for premice is richer than that for coherent structures, so $\sigma_{\tau^*,\bar\tau}$ is automatically Σ_1-preserving in this weaker sense. Comparing the definitions of $\prec$ for $\mathfrak{T}^0$ and $\mathfrak{T}^1$, it is clear that only (T1B2) needs a verification. By (DP1), it suffices to show that $e^{\bar\alpha}_{N_{\bar\tau}} \in \mathrm{rng}(\sigma_{\tau^*,\bar\tau})$ where $\bar\alpha = \alpha(\bar\tau)$, as the facts that $\sigma_{\tau^*,\bar\tau}$ is Σ_1-preserving with respect to the language for coherent structures and $\sigma_{\tau^*,\bar\tau}(\gamma_{N_{\tau^*}}) = \gamma_{N_{\bar\tau}}$ guarantee that $\sigma^{-1}_{\tau^*,\bar\tau}(e^{\bar\alpha}_{N_{\bar\tau}})$ is equal to $e^{\alpha^*}_{N_{\tau^*}}$. Here of course, $\alpha^* = \alpha(N_{\tau^*})$. Now $\sigma_{\tau^*,\bar\tau}$ satisfies (T0C) with $(\tau^*, \bar\tau)$ in place of $(\bar\tau, \tau)$, so we conclude that $\gamma_{N_{\bar\tau}} \in h_{\bar\tau}(\bar\alpha \cup \{p_{N_{\bar\tau}} \cup e\})$ for some $e \in \mathrm{rng}(\sigma_{\tau^*,\bar\tau})$. Let $\bar e$ be the $<^*$-least such e; we claim that $\bar e = e^{\bar\alpha}_{N_{\bar\tau}}$. Otherwise $e^{\bar\alpha}_{N_{\bar\tau}} <^* \bar e$, so $\mathrm{rng}(\sigma_{\tau^*,\bar\tau})$ would contain some $e' <^* \bar e$ such that $\gamma_{N_{\bar\tau}} \in h_{\bar\tau}(\bar\alpha \cup \{p_{N_{\bar\tau}} \cup e'\})$, a contradiction with the minimality of $\bar e$.

We also have to check that $\{\bar\tau \in S(1); \bar\tau \prec \tau\}$ is linearly ordered by $\prec$. That is, given $\tau^* \prec \tau$ and $\bar\tau \prec \tau$, we want to see that τ^* and $\bar\tau$ are compatible. Say, $\tau^* < \bar\tau$. If $\bar\tau \in \mathfrak{T}^1$ or $N_{\bar\tau}$ is not of type B, the conclusion follows by the same argument as in the case $\tau \in \mathfrak{T}^0$. Now assume that $\bar\tau \in \mathfrak{T}^0$ and $N_{\bar\tau}$ is of type B. Letting $\sigma^* = \sigma^{-1}_{\bar\tau,\tau} \circ \sigma_{\tau^*,\tau}$, we want to verify that σ^* witnesses $\tau^* \prec \bar\tau$. This time σ^* is Σ_1-preserving with respect to the language for coherent structures. We show that N_{τ^*} is a type B premouse and $\sigma^*(\gamma_{\tau^*}) = \gamma_{\bar\tau}$; from this it automatically follows that σ^* is Σ_1-preserving with respect to the language for premice. This de facto verifies $\tau^* \prec \bar\tau$, as (T0C2) is an immediate consequence of (DP1) and of the Σ_1-elementarity of σ^*.

So let us turn to the verification that N_{τ^*} is a type B premouse and $\sigma^*(\gamma_{\tau^*}) = \gamma_{\bar\tau}$. As $N_{\bar\tau}$ is a type B premouse, $d_{\bar\tau}^{\alpha(\bar\tau)} \neq \varnothing$ and its largest element $\bar\beta$ is at least $\lambda^*_{N_{\bar\tau}}$. [1] It follows that $d_{\tau^*}^{\alpha(\tau^*)} \neq \varnothing$ and $\sigma^*(\beta^*) = \bar\beta \geq \lambda^*_{N_{\bar\tau}}$ where $\beta^* = \max(d_{\tau^*}^{\alpha(\tau^*)})$. Letting $\lambda^* = \{\xi \in \mathbf{On} \,|\, \sigma^*(\xi) < \lambda^*_{N_{\bar\tau}}\}$, obviously $\lambda^* \leq \beta^* < \lambda(E_{\text{top}}^{N_{\tau^*}})$, and it is easy to verify that λ^* is a cutpoint of $E_{\text{top}}^{N_{\tau^*}}$: For $f : \mu_{\tau^*} \to \mu_{\tau^*}$ in N_{τ^*} and every $\alpha < \lambda^*$ we have $\sigma^*(E_{\text{top}}^{N_{\tau^*}}(f)(\alpha)) = E_{\text{top}}^{N_{\bar\tau}}(f)(\sigma^*(\alpha)) < \lambda^*_{N_{\bar\tau}}$, so $E_{\text{top}}^{N_{\tau^*}}(f)(\alpha) < \lambda^*$. It follows that N_{τ^*} is a type B premouse, since it has a cutpoint and it cannot be a type C premouse for the trivial reason that $n_{\tau^*} = 0$. From the definition of λ^* we get $\sigma^*(\lambda^*) \geq \lambda^*_{N_{\bar\tau}}$, so the equality must hold as $\sigma^*(\lambda^*)$ is a cutpoint of $N_{\bar\tau}$. Thus λ^* is the largest cutpoint of N_{τ^*} [2], and the Σ_1-elementarity of σ^* with respect to the language of coherent structures then yields the equality $\sigma^*(\gamma_{\tau^*}) = \gamma_{\bar\tau}$.

The above discussion verifies that $\prec$ is a tree ordering on $S(1)$. We can now focus on the verification of the morass axioms. Before we turn to this, let us make the following remark. By the definition of $\prec$, any $\prec$-predecessor of an element of $\mathfrak{T}^0$ must again be in $\mathfrak{T}^0$. On the other hand, we allow that a $\prec$-predecessor of an element of $\mathfrak{T}^1$ can be in $\mathfrak{T}^0$. If this happens, say $\bar\tau \in \mathfrak{T}^0$ is a $\prec$-predecessor of some $\tau \in \mathfrak{T}^1$, then the set of all $\prec$-predecessors of τ splits into two $\prec$-intervals $[\hat\tau, \tilde\tau]$ and $(\tilde\tau, \tau)$ where $\tilde\tau$ is given by (T1C), the former interval is contained in $\mathfrak{T}^0$ and the latter in $\mathfrak{T}^1$. We included requirement (T1C) in Definition 2.4 because in situations where $\tau \in \mathfrak{T}^1$, $\bar\tau \in \mathfrak{T}^0$ and $\sigma : M_{\bar\tau} \to M_\tau$ is as in (T1B) we do not have any Condensation Lemma that would allow us to conclude that $M_{\bar\tau}$ is a level of $\mathbf{L}[E]$. We will give more details on this issue below the verification of morass axiom (M3). It is not clear to us whether (T1C) can be omitted.

2.2. **Morass Axioms.** In this section we will systematically go over the morass axioms and verify that the definitions from the section 2.1 actually yield a realization of Gap-1 Morass in W.

Axiom (M0). Recall first that it is E_α alone that decides whether the membership to S_α is determined in W or in $\mathrm{Ult}(W, E_\alpha)$. If $E_\alpha = \varnothing$ then for each τ, the membership "$\tau \in S_\alpha$" is determined by examining the structure J_τ^E (and not $J_\tau^{E'}$,

[1]See section 1.2.

[2]The property of being a cutpoint of N is $\Pi_1(N)$, see[21].

see (c2) in the definition of S_α). Similarly, if $E_\alpha \neq \varnothing$ then for each τ, this membership is determined by examining $J_\tau^{E'}$. That $\sigma_{\bar\tau,\tau}$ has sufficient preservation degree to guarantee (M0) follows immediately unless $\tau \in \mathfrak{T}^0$, $\mathrm{ht}(N_\tau) = \tau$ and $n_\tau = 0$. In this case we use (T0C3): Since $\sigma_{\bar\tau,\tau} : N_{\bar\tau} \to N_\tau$ is Q-preserving, $\bar\tau$ is a $<_0$-successor whenever τ is. $\square$(M0)

Axiom (M1). The first part of (M1) is obvious. The second part of (M1), that is, the equality $\sigma_{\tau^*,\tau} = \sigma_{\bar\tau,\tau} \circ \sigma_{\tau^*,\bar\tau}$ was verified in the discussions following Definition 2.3 and Definition 2.4 where we proved that $\prec$ is a tree ordering. $\square$(M1)

Axiom (M2). Clauses (a) and (b) of (M2) are clear. Clause (c) of (M2) requires a little discussion. Since $\bar\tau' < \bar\tau$, the structure $N_{\bar\tau'}$ is an initial segment of $N_{\bar\tau}$. Since $\bar\tau$ is the cardinal successor of $\alpha(\bar\tau) = \alpha(\bar\tau')$ in $N_{\bar\tau}$, the structure $N_{\bar\tau'}$ is a *proper* initial segment of $J_{\bar\tau}^{E^{\bar\tau}}$. It follows that $N_{\tau'} = \sigma_{\bar\tau,\tau}(N_{\bar\tau'})$ is a proper initial segment of $J_\tau^{E^\tau}$. Thus, any first-order statement over $N_{\bar\tau'}$ is Σ_0-expressible over $J_{\bar\tau}^{E^{\bar\tau}}$ using $N_{\bar\tau'}$ as an additional parameter. Such a statement is preserved under $\sigma_{\bar\tau,\tau}$. Consequently, $n_{\bar\tau'} = n_{\tau'}$ and $\sigma_{\bar\tau,\tau}(p_{\bar\tau'}) = p_{\tau'}$. For pluripotent $N_{\bar\tau'}$ we also have $\sigma_{\bar\tau,\tau}(d_{\bar\tau'}) = d_{\tau'}$. If α is a limit cardinal then so is $\bar\alpha$ and both $\bar\tau$ and τ are in $\mathfrak{T}^0$. Otherwise, letting μ be the cardinal predecessor of α, it follows from the above discussion that the following statements are equivalent:

- there is a $\bar q$ such that $(\mu, \bar q)$ is a divisor for $N_{\bar\tau'}$ strong above $\bar\alpha$;
- there is a q such that (μ, q) is a divisor for $N_{\tau'}$ strong above α.

This means that $\bar\tau' \in \mathfrak{T}^1$ just in case that $\tau' \in \mathfrak{T}^1$, and for $\bar\tau' \in \mathfrak{T}^1$ we have $\sigma_{\bar\tau,\tau}(M_{\bar\tau'}) = M_{\tau'}$ and $\sigma_{\bar\tau,\tau}(q_{\bar\tau'}) = q_{\tau'}$. All this together, the uniqueness of the morass maps discussed below Definition 2.3 and Definition 2.4, and the fact that the map $\sigma_{\bar\tau,\tau} \restriction N_{\bar\tau'} : N_{\bar\tau'} \to N_{\bar\tau}$, resp. $\sigma_{\bar\tau,\tau} \restriction M_{\bar\tau'} : M_{\bar\tau'} \to M_{\bar\tau}$ is fully elementary imply that $\sigma_{\bar\tau',\tau'} = \sigma_{\bar\tau,\tau} \restriction N_{\bar\tau'}$, resp. $\sigma_{\bar\tau',\tau'} = \sigma_{\bar\tau,\tau} \restriction M_{\bar\tau'}$. For $\bar\tau' \in \mathfrak{T}^0$, the Q-condition follows from the full elementarity of $\sigma_{\bar\tau',\tau'}$. For $\bar\tau' \in \mathfrak{T}^1$, clause (T1C) is vacuously true. $\square$(M2)

Axiom (M3). We split our discussion into two cases, depending on whether τ is in $\mathfrak{T}^0$ or $\mathfrak{T}^1$.

CASE 1: $\tau \in \mathfrak{T}^0$. Suppose $\bar{\alpha} < \alpha$ is a limit point of $C(\tau)$. Our aim is to show that $\bar{\alpha} \in C(\tau)$. Because $\prec$ is a tree ordering, we have a linear diagram

$$\langle N_{\tau^*}, \sigma_{\tau^*,\tau'};\ \tau^*, \tau' \prec \tau \ \&\ \alpha(\tau^*), \alpha(\tau') \in C(\tau) \cap \bar{\alpha} \ \&\ \tau^* < \tau' \rangle.$$

Consider the direct limit $\langle \bar{N}, \bar{\sigma}_{\tau^*};\ \tau^* \prec \tau \ \&\ \alpha(\tau^*) \in C(\tau) \cap \bar{\alpha} \rangle$ of this diagram. Let $\bar{\sigma} : \bar{N} \to N_\tau$ be the unique embedding satisfying $\bar{\sigma} \circ \bar{\sigma}_{\tau^*} = \sigma_{\tau^*,\tau}$ for all τ^*. A straightforward argument that involves an inductive computation of projecta in fine structural direct limits [3] yields:

- $\bar{N}$ is well-founded, and we will consider it to be transitive;
- all direct limit maps $\bar{\sigma}_{\tau^*} : N_{\tau^*} \to \bar{N}$ are $\Sigma_1^{(n)}$-preserving where $n = n_\tau$;
- $\mathrm{cr}(\bar{\sigma}_{\tau^*}) = \alpha(\tau^*)$ and $\bar{\sigma}_{\tau^*}(\alpha(\tau^*)) = \bar{\alpha}$ whenever $\tau^* \prec \tau$ and $\alpha(\tau^*) \in C(\tau) \cap \bar{\alpha}$.

From these clauses we easily infer

$$\mathrm{cr}(\bar{\sigma}) = \bar{\alpha}, \tag{8}$$

$$\bar{\sigma}(\bar{\alpha}) = \alpha \tag{9}$$

and

$$\bar{\sigma} \text{ is } \Sigma_1^{(n)}\text{-preserving.} \tag{10}$$

We should stress that all computations are done in the *language for premice.* Our aim is to show that $\bar{N}$ is the collapsing structure for $\bar{\tau} = \bar{\alpha}^{+\bar{N}}$ in the sense of Definition 1.5. The main tool here will be the Condensation Lemma (Lemma 1.4). In the following we verify that the triple $\bar{N}, N_\tau$ and σ satisfies the assumptions of the Condensation Lemma.

We begin by noticing:

$$\bar{N} \text{ is a premouse of the same type as } N_\tau. \tag{11}$$

This follows easily if $n_\tau > 0$; see [21], Chapter 9. If $n_\tau = 0$ then N_τ must be a premouse of type A or B and the same applies to all premice N_{τ^*}; namely, they all must be of the same type as N_τ. By construction, $\bar{N}$ must be a type A or B potential premouse. Since premousehood in both these cases is a Π_2-property in the language for premice [21], it is downwards preserved under $\bar{\sigma}$.

[3] This argument goes through abstractly for any acceptable structures, see e.g. preliminaries in [15] or Section 5 in [9])

Let $\bar{p} = \bar{\sigma}_{\tau^*}(p_{\tau^*} - \alpha(\tau^*))$; this value obviously does not depend on τ^*. Also, $\bar{\sigma}(\bar{p}) = p_\tau - \alpha$. This, the soundness of N_τ and (10) together yield

$$\tilde{h}^{n+1}_{\bar{N}}(\bar{\alpha} \cup \{\bar{p}\}) = \bar{N} \tag{12}$$

which in turn gives

$$\omega\varrho^{n+1}_{\bar{N}} \leq \bar{\alpha} < \omega\varrho^{n}_{\bar{N}}.[4] \tag{13}$$

Clauses (8), (10) and (13) enable us to apply the first part of the Condensation Lemma to the embedding $\bar{\sigma} : \bar{N} \to N_\tau$ and conclude:

$$\bar{N} \text{ is solid.} \tag{14}$$

Let $\bar{\beta} \in \bar{p}$ and $\tau^* \prec \tau$ and $\beta^* \in p_{\tau^*}$ be such that $\bar{\sigma}_{\tau^*}(\beta^*) = \bar{\beta}$. Since the map $\bar{\sigma}_{\tau^*}$ is $\Sigma^{(n)}_1$-preserving and N_{τ^*} is solid, $\bar{\sigma}_{\tau^*}(W^{\beta^*,p_{\tau^*}}_{N_{\tau^*}}) \in \bar{N}$ is a generalized witness for $\bar{\beta}$ with respect to $\bar{p}$ and $\bar{N}$. This together with the solidity of $\bar{N}$ and (12) then guarantees ([21], Lemma 1.12.5):

$$\bar{p} = p_{\bar{N}} - \bar{\alpha} \tag{15}$$

It follows that $\bar{N}$ is sound above $\bar{\alpha}$. As an immediate consequence we see that $\bar{N}$ is a collapsing structure for $\bar{\tau} = \bar{\alpha}^{+\bar{N}}$.

The soundness of $\bar{N}$ above $\bar{\alpha}$ allows us to apply the second part of the Condensation Lemma. Our immediate goal is proving that $\bar{N} = N_{\bar{\tau}}$. Notice that it suffices to rule out possibility (c) in the Condensation Lemma, as this will tell us that exactly one of the following holds:

- $E^\tau_{\bar{\alpha}}$ is *not* an extender and $\bar{N}$ is the initial level of N_τ collapsing $\bar{\tau}$.
- $E^\tau_{\bar{\alpha}}$ *is* an extender and $\bar{N}$ is the initial level of $\mathrm{Ult}(N_\tau, E_{\bar{\alpha}})$ collapsing $\bar{\tau}$.

Since N_τ agrees with W below α (so, in particular, $E^\tau_{\bar{\alpha}} = E_{\bar{\alpha}}$), we conclude that $\bar{N}$ is the initial segment of W or $\mathrm{Ult}(W, E_{\bar{\alpha}})$ collapsing $\bar{\tau}$, which means $\bar{N} = N_{\bar{\tau}}$.

Assume for a contradiction that clause (c) in the Condensation Lemma holds. Then $\bar{N}$ is of the form $\mathrm{Ult}^*(Q, E^{N_\tau}_\delta)$ where Q is sound and $\mu \overset{\text{def}}{=} \mathrm{cr}(E^{N_\tau}_\delta)$ is the cardinal predecessor of $\bar{\alpha}$ in both Q and $\bar{N}$. Moreover, $E^{N_\tau}_\delta$ has a single generator, namely μ. It follows that $\bar{N}$ is sound above $\mu + 1$, and no projectum of $\bar{N}$ can lie in the interval $(\mu, \pi(\mu)]$ where π is the associated ultrapower map. Since $\bar{\alpha} < \pi(\mu)$ and

[4]The sharp inequality on the right follows from the fact that $\alpha(\tau^*) < \omega\varrho^n_{N^*_\tau}$ for all τ^*.

(13) holds, we must have $\omega\varrho_{\bar N}^{n+1} \leq \mu$. So $\tilde h_{\bar N}^{n+1}(\mu \cup \{p_{\bar N}, \mu\}) = \bar N$. Since $\bar N$ is the fine ultrapower of Q by an internal measure, $p_{\bar N} = \pi(p_Q)$. So $\min(p_{\bar N} - \mu) \geq \pi(\mu)$. [5] Applying (15), we obtain $\bar N = \tilde h_{\bar N}^{n+1}(\mu \cup \{\bar p, \mu\})$. But $\mu \cup \{\bar p, \mu\} \subseteq \mathrm{rng}(\bar\sigma_{\tau^*})$ for each τ^*, and the fact that $\bar\sigma_{\tau^*}$ is $\Sigma_1^{(n)}$-preserving yields $\mathrm{rng}(\bar\sigma_{\tau^*}) = \bar N$, so $\bar\sigma_{\tau^*} = \mathrm{id}$. This contradicts e.g. (8), and thereby rules out clause (c) in the Condensation Lemma.

Summarizing the above discussion, we have $\bar N = N_{\bar\tau}$ and, by (15), also $\bar p = p_{\bar\tau} - \bar\alpha$. It is also clear that if $\tau \in N_\tau$ then $\tau^* \in N_{\tau^*}$ for all τ^* and $\bar\tau = \sigma_{\tau^*}(\tau^*) \in N_{\bar\tau}$. This, the fact that $\bar\sigma(\bar p) = p_\tau - \alpha$, (8) – (10) and (13) verify (T0B) and (T0C) except the Q-condition, which we verify now. Fix a $\Sigma_1^{(n)}$-formula $\varphi(u, v)$ and some $\bar q \in N_{\bar\tau}$. Pick some τ^* such that $\alpha(\tau^*) \in C(\tau) \cap \bar\alpha$ and $\bar q \in \mathrm{rng}(\bar\sigma_{\tau^*})$. Let $\bar\sigma_{\tau^*}(q^*) = \bar q$ and $\bar\sigma(\bar q) = q$. If $N_{\bar\tau} \models \varphi(\xi, \bar q)$ for cofinally many $\xi < \bar\tau$ then $N_{\tau^*} \models \varphi(\xi, q^*)$ for cofinally many $\xi < \tau^*$ since $\bar\sigma_{\tau^*}$ is $\Sigma_1^{(n)}$-preserving. But then the Q-condition for $\sigma_{\tau^*,\tau}$ guarantees that $N_\tau \models \varphi(\xi, q)$ for cofinally many $\xi < \tau$.

It remains to verify that $\bar\tau \in \mathfrak{T}^0$. Clearly, $\bar\alpha \in S(0)$. Furthermore, the structure $J_{\bar\tau}^{E^{\bar\tau}}$ is good, as follows from the preservation properties of $\bar\sigma$. The ordinal $\bar\alpha$ is the largest cardinal in $J_{\bar\tau}^{E^{\bar\tau}}$ and is regular there. So we conclude that $\bar\tau \in S(1)$ and $\bar\alpha = \alpha(\bar\tau)$. In the following we rule out the possibilitiy $\bar\alpha \in \mathfrak{T}^1$. If $\bar\tau$ were in $\mathfrak{T}^1$ then $\bar\alpha$ would be a successor cardinal in $N_{\bar\tau}$, say $\bar\alpha = \mu^{+N_{\bar\tau}}$ and $(\mu, \bar q)$ would be a divisor for $N_{\bar\tau}$ strong above $\bar\alpha$ where $\bar q$ is some bottom part of $p_{\bar\tau}$ if $N_{\bar\tau}$ fails to be pluripotent and $\bar q = d_{\bar\tau}$ if $N_{\bar\tau}$ is pluripotent. [6] Let $\tau^* \in C(\tau) \cap \bar\alpha$. The pair $(\mu, \bar q - \bar\alpha)$ would be in the range of $\bar\sigma_{\tau^*}$, say $(\mu, \bar q) = \bar\sigma_{\tau^*}(\mu, q^*)$ where q^* is a bottom part of $p_{\tau^*} - \alpha(\tau^*)$ or $q^* = d_\tau - \alpha(\tau^*)$. [7] By (b1) or (b2) of Lemma 1.12 and the fact that $\bar\sigma_{\tau^*}$ is $\Sigma_1^{(n)}$-preserving, the pair (μ, q^*) would be a divisor for N_{τ^*} strong above $\alpha(\tau^*)$. Contradiction, as $\tau^* \in \mathfrak{T}^0$. This completes the verification that $C(\tau)$ is closed.

In the following we show that $C(\tau)$ is unbounded in α, granting that τ is not the largest element of S_α. Let $\tau' \in S_\alpha$ be larger than τ. Then α is regular in

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[5] In fact, $\min(p_{\bar N}) \geq \pi(\mu)$. This follows from the fact that μ is a cardinal in W, as $\bar\alpha$ is cardinal-correct in W. So actually $\omega\varrho_Q^{n+1} = \omega_Q^\omega = \mu = \omega\varrho_{\bar N}^{n+1} = \omega\varrho_{\bar N}^\omega$ and $\bar p = p_{\bar N} = \pi(p_Q)$. However, these facts have no relevance in showing that (c) in the Condensation Lemma does not occur.

[6] The reader may consult (2) in section 1.3.

[7] Recall that $d_N^\beta = d_N - \beta$; see the discussion below Definition 1.3.

$J^{E^{\tau'}}_{\tau'}$. [8] Choose an ordinal $\alpha^* < \alpha$. In $J^{E^{\tau'}}_{\tau'}$, we can carry out the standard elementary chain construction to get a fully elementary substructure $X \prec N_\tau$ such that $(\alpha^*+1) \cup \{\alpha, p_{N_\tau}\} \subset X$ and $\bar\alpha = X \cap \alpha \in \alpha$. Letting $\bar N$ be the transitive collapse of X, the inverse to the associated collapsing map $\bar\sigma : \bar N \to N_\tau$ is fully elementary. This is more than sufficient to transfer all information about the fine structure of N_τ to $\bar N$, as this information is first order expressible over N_τ in the parameters α and p_{N_τ}. Letting $\bar p \in \bar N$ be the $\bar\sigma$-preimage of p_τ, we conclude that $\omega\varrho^{n+1}_{\bar N} = \bar\alpha < \omega\varrho^n_{\bar N}$, all clauses (8) – (12) are satisfied with current $\bar\alpha, \bar\sigma, \bar N$ and $\bar p$, $\bar p = p_{\bar N}$, and $\bar N$ is solid and sound. [9] Here (8) and (9) follow for current $\bar\alpha$ and $\bar\sigma$ from the fact that $X \cap \alpha = \bar\alpha$ and the rest is a direct consequence of the full elementarity of $\bar\sigma$. We apply the Condensation Lemma to $\bar\sigma : \bar N \to N_\tau$ and conclude that $\bar N = N_{\bar\tau}$ where $\bar\tau = \bar\sigma^{-1}(\tau)$ if $\tau \in \bar N$ [10] and $\bar\tau = \mathrm{ht}(\bar N)$ otherwise. This time, the failure of (c) in the Condensation lemma fails follows immediately from the fact that $\bar N$ is sound.

It remains to verify that $\bar\tau \in \mathcal{T}^0$, since the rest goes through as before. But this is an immediate consequence of the full elementarity of $\bar\sigma$: If $\bar\tau$ were in $\mathcal{T}^1$ then $(\mu, \bar q)$ would be a strong divisor for $N_{\bar\tau}$ where μ is a cardinal predecessor of $\bar\alpha$ in $N_{\bar\tau}$. Using the full elementarity of $\bar\sigma$ and (b1) or (b2) of Lemma 1.12 we conclude that $(\mu, \bar\sigma(\bar q))$ is a strong divisor for N_τ, a contradiction with the fact that $\tau \in \mathcal{T}^0$. This shows that $\bar\alpha = \alpha(\bar\tau) \in C(\tau)$ and $\bar\alpha > \alpha^*$, and thereby completes the proof of the unboundedness of $C(\tau)$ in $\alpha(\tau)$.

Once we know that $C(\tau)$ is unbounded in α, it follows that every $\zeta < \tau$ is in the range of some $\sigma_{\tau^*,\tau}$; it suffices to fix some $\xi < \alpha$ such that $\zeta = \tilde h^{n+1}_\tau(\xi, p_\tau)$ and choose τ^* so that $\alpha(\tau^*) > \xi$. As a direct consequence we obtain

$$\tau = \bigcup\{\sigma''_{\tau^*,\tau}\tau^*;\, \tau^* \prec \tau\},$$

which verifies the last clause of (M3) and thus completes Case 1 in the verification of (M3). $\square$((M3), Case 1)

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[8] Recall that $E^\tau = E^{\tau'} \restriction \tau$, see (c2) in the definition of S_α.

[9] The present notation is consistent with that used above.

[10] Since $\bar\sigma$ is fully elementary, $\tau \in \mathrm{rng}(\bar\sigma)$ in this case.

Case 2: $\tau \in \mathfrak{T}^1$. Towards the closedness of $C(\tau)$, we again consider a limit point $\bar{\alpha}$ of $C(\tau)$ and show that it is in $C(\tau)$. This time M_τ is a protomouse, μ_τ is the cardinal predecessor of $\alpha = \alpha(\tau)$ and $\vartheta_\tau < \alpha$. If there is some $\tau^* \prec \tau$ with $\tau^* \in \mathfrak{T}^0$ then, by (T1C), we also have a $\tilde{\tau} \prec \tau$ with $\tilde{\tau} \in \mathfrak{T}^0$ and $\tilde{\alpha} = \alpha(\tilde{\tau}) = \vartheta_\tau$. By the discussion at the end of the previous subsection, the set $\{\tau^*;\, \tau^* \prec \tau\}$ splits into two intervals, namely $[\hat{\tau}, \tilde{\tau}]$ and $(\tilde{\tau}, \tau)$. Accordingly, $C(\tau)$ splits into two intervals, namely $C(\tilde{\tau})$ and $C(\tau) \cap (\tilde{\alpha}, \alpha)$. But we already know that $C(\tilde{\tau})$ is closed, as this falls under Case 1. Thus, no matter whether there are $\tau^* \prec \tau$ that are in $\mathfrak{T}^0$ or not, it suffices to consider the case where $\bar{\alpha} > \vartheta_\tau$, i.e. where $\bar{\alpha} = \alpha(\bar{\tau})$ for some $\bar{\tau} \prec \tau$ such that $\bar{\tau} \in \mathfrak{T}^1$.

We again have a linear diagram [11]

$$\langle M_{\tau^*}, \sigma_{\tau^*,\tau'};\, \tau^*, \tau' \prec \tau \;\&\; \alpha(\tau^*), \alpha(\tau') \in C(\tau) \cap (\tilde{\alpha}, \bar{\alpha}) \;\&\; \tau^* < \tau' \rangle.$$

We form the direct limit of the above diagram with the direct limit structure $\bar{M}$ and the direct limit maps $\bar{\sigma}_{\tau^*} : M_{\tau^*} \to \bar{M}$. We also define the map $\bar{\sigma} : \bar{M} \to M_\tau$ in the canonical way. The following list summarizes relevant information about these objects. Most of the clauses below are obtained in the same manner as in Case 1, but some require an additional explanation. In the following we assume that τ^* is as above, i.e. $\tau^* \prec \tau$ and $\alpha(\tau^*) \in C(\tau) \cap \bar{\alpha}$.

(i) $\bar{M}$ is well-founded and we will consider it to be transitive.

(ii) All maps $\bar{\sigma}_{\tau^*} : M_{\tau^*} \to \bar{M}$ as well as $\bar{\sigma} : \bar{M} \to M_\tau$ are Σ_1-preserving and cofinal, and $\sigma_{\tau^*,\tau} = \bar{\sigma} \circ \bar{\sigma}_{\tau^*}$.

(iii) $\bar{\sigma}_{\tau^*}(\alpha(\tau^*)) = \bar{\alpha}$.

(iv) $\mathrm{cr}(\bar{\sigma}) = \bar{\alpha}$ and $\bar{\sigma}(\bar{\alpha}) = \alpha$.

(v) $\bar{M}$ is a coherent structure and $\vartheta(\bar{M}) = \vartheta_\tau = \vartheta_{\tau^*}$.

(vi) $h_{\bar{\tau}}(\alpha \cup \{\bar{q}\}) = \bar{M}$ for $\bar{q} = \bar{\sigma}_{\tau^*}(q_{\tau^*} - \alpha(\tau^*))$. [12]

(vii) $\bar{\sigma}(\bar{q}) = q_\tau - \alpha_\tau$.

(viii) For every $\beta \in \bar{q}$ there is a generalized witness $Q^{\beta,\bar{q}}_{\bar{M}} \in \bar{M}$ for β with respect to $\bar{q}$ and $\bar{M}$.

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[11] We consider $\tilde{\alpha} = 0$ if $\{\tau^*; \tau^* \prec \tau\} \cap \mathfrak{T}^0 = \varnothing$.

[12] Of course $\bar{\sigma}_{\tau^*}(q_{\tau^*} - \alpha(\tau^*))$ is independent of τ^*.

The maps $\bar{\sigma}_{\tau^*}$ and $\bar{\sigma}$ in (ii) are Σ_1-preserving, as all maps $\sigma_{\tau^*,\bar{\tau}}$ are Σ_1-preserving; the cofinality of $\bar{\sigma}$ is a consequence of our choice of $\bar{\alpha}$ to be larger than ϑ_τ. This choice of $\bar{\alpha}$ also guarantees (v). The preservation properties of the maps $\bar{\sigma}_{\tau^*}$ give the rest. Regarding (vi), we additionally apply the fact that $h_{\tau^*}(\alpha(\tau^*) \cup \{q_{\tau^*}\}) = M_{\tau^*}$ for all τ^* as above. To see the last clause, we apply the fact that for each $\beta^* \in q_{\tau^*}$ there is a generalized witness $Q^{\beta,q_{\tau^*}}_{M_{\tau^*}} \in M_{\tau^*}$ with respect to q_{τ^*} and M_{τ^*}. The rest is standard.

Our aim is to prove that $\bar{M} = M_{\bar{\tau}}$ where $\bar{\tau} = \bar{\alpha}^{+\bar{M}} = \bar{\sigma}_{\tau^*}(\tau^*)$. Here we apply our substitute for the Condensation Lemma, namely Lemma 1.15, to the embedding $\bar{\sigma} : \bar{M} \to M_\tau$. Assumptions (a) – (c) of the lemma were verified above. We will argue that conclusion (A) of the lemma must apply, and thereby obtain that $\bar{M}$ is of the form $\bar{N}(\mu, \bar{q}^*)$ where $\bar{N}$ is the collapsing structure for $\bar{\tau}$ and $\bar{q}^*$ is such that $\bar{q} = \bar{q}^* - \bar{\alpha}$. Thus, $\bar{N} = N_{\bar{\tau}}$ and $\bar{M} = N_{\bar{\tau}}(\mu, \bar{q}^*)$; the lemma also tells us that $\bar{\alpha} = \alpha(\bar{\tau})$ and $\bar{q}^*$ is a bottom part of $p_{\bar{\tau}} - \bar{\alpha}$ or $\bar{q} = d^{\bar{\alpha}}_{\bar{\tau}}$.

To see that conclusion (B) in Lemma 1.15 fails, argue by contradiction. Assuming that (B) holds, we have $\bar{\alpha} \in \tilde{h}^{n+1}_{\bar{N}}(\mu \cup \{\bar{r}, \bar{q}, \mu\})$ where $\bar{N} = N(\bar{M})$, $n = n_\tau$ and $\bar{r} = p_{\bar{N}} - \lambda(E^{\bar{M}}_{\mathrm{top}})$. By Proposition 1.13, we can find some function $f \in J^E_\vartheta$ and $\xi < \mu$ such that $\bar{\alpha} = E^{\bar{M}}_{\mathrm{top}}(f)(\bar{q}, \xi)$ where $E = E^{\bar{M}}$ and $\vartheta = \vartheta(\bar{M}) = \vartheta_\tau$. Let $\tau^* \prec \tau$, $\tau^* < \bar{\tau}$ be large enough such that $\tau^* \in \mathfrak{T}^1$. Then $\vartheta_{\tau^*} = \vartheta$. Since $\bar{\sigma}_{\tau^*}$ is Σ_1-preserving and $f, \xi, \bar{q} \in \mathrm{rng}(\bar{\sigma}_{\tau^*})$, we conclude that $\bar{\alpha} \in \mathrm{rng}(\bar{\sigma}_{\tau^*})$. This is a contradiction, as $\bar{\alpha} = \mathrm{cr}(\bar{\sigma}_{\tau^*})$.

Recall that $\bar{q}^* = p_{\bar{\tau}} \cap \max(\bar{q} + 1)$ if N^* is passive or active with $\lambda(E^{N^*}_{\mathrm{top}}) > \mu$ and $\bar{q}^* = d_{\bar{N}}$ otherwise. To conclude that $\bar{M} = M_{\bar{\tau}}$, we have to verify that $(\mu, \bar{q}^*)$ is a divisor for $N_{\bar{\tau}}$ strong above $\bar{\alpha}$. Here we apply Proposition 1.13; see also the discussion concerning (4). Let $x \subseteq \lambda(E^{\bar{M}}_{\mathrm{top}})$ be in the hull $\tilde{h}_{\bar{\tau}}(\mu \cup \{p_{\bar{\tau}} - \bar{\alpha}\})$. By Proposition 1.13 we can find a function $f \in J^{E^{\bar{\tau}}}_\vartheta$ and an ordinal $\eta < \mu$ such that $x = E^{\bar{M}}_{\mathrm{top}}(f)(\bar{q}, \eta)$ where $\vartheta = \vartheta(\bar{M}) = \vartheta_\tau$. Since $\bar{\sigma}$ is Σ_1-preserving and the identity on $\bar{\alpha}$, $\bar{\sigma}(x) = E^{M_\tau}_{\mathrm{top}}(f)(q_\tau - \alpha(\tau), \eta)$. Again by Proposition 1.13, now used in the reversed direction, we conclude that $\bar{\sigma}(x) \in \tilde{h}_\tau(\mu \cup \{p_\tau - \alpha(\tau)\})$. As (μ, q_τ) is a divisor for N_τ strong above $\alpha(\tau)$ and $\bar{\sigma}$ is the identity on μ, we apply (S2) and conclude $x \cap \mu = \sigma(x) \cap \mu \in J^{E^\tau}_\vartheta = J^{E^{\bar{\tau}}}_\vartheta$. If N^* is active and $E^{N^*}_{\mathrm{top}} = \mu$ we proceed similarly but work with $d_{\bar{\tau}}$ and d_τ instead of $p_{\bar{\tau}}$ and p_τ. This proves that the divisor

$(\mu, \bar{q}^*)$ for $N_{\bar{\tau}}$ is strong above $\bar{\alpha}$, and thereby completes the proof that $\bar{M} = M_{\bar{\tau}}$. It follows that $\bar{\tau} \in \mathfrak{T}^1$; that $\bar{\tau} \prec \tau$ is witnessed by the map $\bar{\sigma}$. So $\bar{\alpha} = \alpha(\bar{\tau}) \in C(\tau)$, which verifies the closedness of $C(\tau)$.

Finally we prove that $C(\tau)$ is unbounded in α. The structure of the argument is the same as in Case 1. Given $\alpha^* < \alpha$, we pick some $\tau' > \tau$ in S_α and use the elementary chain construction inside $J^{E^{\tau'}}_{\tau'}$ for some to construct a coherent structure $\bar{M}$, and ordinal $\bar{\alpha} < \alpha$, a finite set of ordinals $\bar{q}$ and a fully elementary embedding $\bar{\sigma} : \bar{M} \to M_\tau$ such that:

(i) $\bar{\alpha} = \mathrm{cr}(\bar{\sigma}) > \alpha^*$ and $\bar{\sigma}(\bar{\alpha}) = \alpha$.
(ii) $\vartheta_\tau = \vartheta(\bar{M}) < \bar{\alpha}$.
(iii) $\omega\varrho^1_{\bar{M}} = \bar{\alpha}$.
(iv) $\bar{\sigma}(\bar{q}) = q_\tau$.
(v) $\bar{M} = h_{\bar{M}}(\bar{\alpha} \cup \{\bar{q}\})$.
(vi) For every $\beta \in \bar{q}$ there is a generalized witness $Q^{\beta,\bar{q}}_{\bar{M}} \in \bar{M}$

The first clause here is again guaranteed by construction, as $\alpha \in \mathrm{rng}(\bar{\sigma})$ and is regular in $J^{E^{\tau'}}_{\tau'}$; the rest follows from the full elementarity of $\bar{\sigma}$ and the fact that we put α and q_τ into the range of $\bar{\sigma}$. Exactly as in the above proof of the closedness, using Lemma 1.15 and Lemma 1.12 we then show that $\bar{M} = M_{\bar{\tau}}$ where $\bar{\tau} = \bar{\alpha}^{+\bar{M}}$. To see that conclusion (B) in Lemma 1.15 does not apply in this case is easier than in the proof of closedness, as it is an immediate consequence of (iii) above. It follows that $\tau \in \mathfrak{T}^1$, $\alpha^* < \alpha(\bar{\tau})$ and $\bar{\tau} \prec \tau$. This completes the proof that $C(\tau)$ is unbounded in α in Case 2.

The verification of the rest of (M3) is easy and practically the same as in Case 1.

$\square$(M3)

Remark. Case 2 in the verification of morass axiom (M3) is the only place in the entire construction where we make use of requirement (T1C) from Definition 2.4, and the situation arising in that argument is the very reason why we introduce (T1C) in the definition of morass ordering and morass embeddings. As already indicated at the end of the last section, the issue here is that we do not see how to verify that $\bar{M}$ is a level of M_τ in the situation where $\tau \in \mathfrak{T}^1$, $n_\tau = 0$, $\bar{M}$ is a pluripotent premouse, M_τ is a protomouse such that $\mathrm{cr}(E^{\bar{M}}_{\mathrm{top}}) = \mathrm{cr}(E^{M_\tau}_{\mathrm{top}})$ is the cardinal predecessor of both $\bar{\alpha}$ and $\alpha(\tau)$, and there is a Σ_1-preserving map

$\bar{\sigma} : \bar{M} \to M_\tau$ that also preserves all relevant parameters. This happens because of the lack of any kind of condensation lemma that could be applied in such a situation. The way we approach the issue is demanding the existence of the largest $\tilde{\tau} \in \mathcal{T}^0$ that is a $\prec$-predecessor of τ, as this allows us to apply the standard condensation lemma, namely Lemma 1.4. The largest $\tilde{\tau}$ that is a $\prec$-predecessor of τ is precisely the ordinal $\tilde{\tau}$ satisfying $\vartheta_{\tilde{\tau}} = \vartheta_\tau$ as demanded in (T1C). Of course, the question arises whether (T1C) is not too strong, that is, whether $\tilde{\tau}$ as above exists. We will see in verifications of morass axioms (M4) and (M5) that such $\tilde{\tau}$ exists in all relevant situations.

Axiom (M4). Assume $\bar{\tau} \prec \tau$ and let $\tau' < \tau$ be the supremum of $\sigma''_{\bar{\tau},\tau}\bar{\tau}$. Our aim is to show that $\bar{\tau} \prec \tau'$. [13] We will again consider two cases.

Case 1: $\tau \in \mathcal{T}^0$ and it is *not* the case that N_τ is both pluripotent and $\mathrm{cr}(E^M_{\mathrm{top}})$ is the cardinal predecessor of α in N_τ (equivalently, in W).

Obviously, the same applies to $N_{\bar{\tau}}$. Let $n = n_\tau$. The Interpolation Lemma ([21], Section 3.6) applied to $\sigma_{\bar{\tau},\tau} : N_{\bar{\tau}} \to N_\tau$ yields a transitive structure N' together with maps $\bar{\sigma}$ and σ' satisfying:

- $\bar{\sigma} : N_{\bar{\tau}} \to N'$ is $\Sigma_0^{(n)}$-preserving and cofinal;
- $\sigma' : N' \to N_\tau$ is $\Sigma_0^{(n)}$-preserving;
- $\mathrm{cr}(\bar{\sigma}) = \bar{\alpha}$ and $\bar{\sigma}(\bar{\alpha}) = \alpha$;
- $\mathrm{cr}(\sigma') = \tau'$ and $\sigma'(\tau') = \tau$.

We would like to use the Condensation Lemma to argue that $N' = N_{\tau'}$. For this, we need to verify first that N' is a premouse of the same type as N_τ and $N_{\bar{\tau}}$ (note that both these structures are premice of the same type). This follows from preservation properties of the maps $\bar{\sigma}$ and σ'. Here we verify merely that N' is a *potential premouse,* which reduces to proving that $E^{N'}_{\mathrm{top}}$ is a total extender on N' whenever N' is active. If $n > 0$ this follows from the fact that $\bar{\sigma}$ is Σ_2-preserving. If $n = 0$ and $\mathrm{cr}(E^{N_\tau}_{\mathrm{top}}) \geq \alpha$ we have $\mathrm{cr}(E^{N_{\bar{\tau}}}_{\mathrm{top}}) \geq \bar{\alpha}$, and the conclusion follows from the fact that $\bar{\sigma}$, being a coarse (pseudo)ultrapower ([21], Section 3.6) embedding, maps $\bar{\tau}$ cofinally into τ'. This observation is sort of trivial, but we decided to include it, as this is the place which makes Case 1 different from Case 2, and the issue

[13] We are, of course, omitting the trivial case $\tau' = \tau$.

that arises in Case 2 at this place is the reason for introducing protomice in the entire construction. In the remaining case we have $\mu \stackrel{\text{def}}{=} \text{cr}(E_{\text{top}}^{N_{\bar\tau}}) < \kappa$ where κ is the cardinal predecessor of $\bar\alpha = \text{cr}(\bar\sigma)$ in $N_{\bar\tau}$. It follows that both structures $N_{\bar\tau}$ and N' contain the same subsets of μ and $\bar\sigma$ is the identity on this power set. [14] This verifies that N' is a potential premouse. That N' is of the same type as the two other premice and that it satisfies the initial segment conditions, follows from the lemmata in [21], Section 9.1.

Now, using the soundness and solidity of $N_{\bar\tau}$ and the preservation properties of the embedding $\bar\sigma$, we conclude:

(i) $N' = \tilde{h}_{N'}^{n+1}(\alpha \cup \{\bar\sigma(p_{\bar\tau})\})$;
(ii) $\omega\varrho_{N'}^{n+1} \le \alpha < \omega\varrho_{N'}^{n}$;
(iii) N' is solid;
(iv) $\bar\sigma(p_{\bar\tau}) - \alpha = p_{N'} - \alpha$;
(v) N' is sound above α.

The verification of these facts is standard. Clause (i) follows from the soundness of $N_{\bar\tau}$ and the preservation degree of $\bar\sigma$. Clause (ii) follows from (i), clause (iii) follows from (ii) combined with the first part of the Condensation Lemma [15] , (iv) follows from (i), the solidity of $N_{\bar\tau}$ and the preservation degree of $\bar\sigma$ and (v) follows from (i) and (iv). The arguments are the same as in Case 1 in the verification of (M3).

Now we can apply the second part of the Condensation Lemma to conclude that $N' = N_{\tau'}$. This amounts ruling out clauses (c) and (d) in the Condensation Lemma. To see that (c) fails, notice that (c) would imply that N' is not sound above α which contradicts (v) above. To see that (d) fails, notice that (d) would imply that N' is a proper initial segment of $\text{Ult}(N_\tau, E_{\tau'}^{\tau})$, so its ultimate projectum must be at least τ'. This contradicts (ii) above. It follows that N' is the initial level of N_τ collapsing τ', so $N' = N_{\tau'}$. Obviously, $N_{\tau'}$ is a proper initial segment of N_τ, as we are assuming that $\tau' < \tau$.

[14]Actually, all of this is vacuous: The argument at the beginning of Case 2 shows that $\mu < \bar\alpha$ is never the case if $\tau' < \tau$, but this information does not simplify the proof in the present case.

[15]Recall that this is Lemma 1.4

Summarizing the results, we have just verified (T0B) and (T0C).[16] To see that $\tau' \prec \tau$, it suffices to prove that $\tau' \in \mathfrak{T}^0$. But this follows by exactly the same argument as in the verification of (M3), Case 1.

The verification that $\tau' \prec \tau$ is now complete. The equality $\sigma_{\bar{\tau},\tau'} \restriction \bar{\tau} = \sigma_{\bar{\tau},\tau} \restriction \bar{\tau}$ is an immediate consequence of the fact that $\sigma' \restriction \tau' = \mathrm{id}$. $\Box$((M4), Case 1)

Case 2: Either $\tau \in \mathfrak{T}^0$ and N_τ is pluripotent and $\mathrm{cr}(E^{N_\tau}_{\mathrm{top}})$ is the cardinal predecessor of α in N_τ or else $\tau \in \mathfrak{T}^1$.

Either of these two options yields:

(16) $\bar{\tau} \in \mathfrak{T}^0$, i.e. $N_{\bar{\tau}}$ is a pluripotent premouse.

(17) $\mu_{\bar{\tau}} = \mu_\tau \overset{\mathrm{def}}{=} \mu$ is the cardinal predecessor of $\bar{\alpha}$ in $N_{\bar{\tau}}$ (and of α in N_τ).

In the former case where $\tau \in \mathfrak{T}^0$ this is just obvious. In the latter case where $\tau \in \mathfrak{T}^1$, it is clear that μ is the cardinal predecessor of $\bar{\alpha}$ in $M_{\bar{\tau}}$. What we have to verify is that $M_{\bar{\tau}} = N_{\bar{\tau}}$ is a pluripotent premouse. That is, we have to show that $\bar{\tau} \in \mathfrak{T}^0$. Assume for a contradiction that $\bar{\tau} \in \mathfrak{T}^1$. Then $\vartheta(M_{\bar{\tau}}) = \vartheta(M_\tau)$, since $\vartheta(M_{\bar{\tau}}) < \bar{\alpha}$. It follows that the domains of $E^{M_{\bar{\tau}}}_{\mathrm{top}}$ and $E^{M_\tau}_{\mathrm{top}}$ agree, so $\sigma_{\bar{\tau},\tau}$, being the identity on these domains, maps $M_{\bar{\tau}}$ cofinally into M_τ. Let $H_\tau(x,y,w,z)$ be a $\Sigma_0(M_\tau)$-relation such that for every parameter $p \in M_\tau$ we have

$$y = h_\tau(x,p) \longleftrightarrow (\exists z) H_\tau(x,y,p,z).$$

For $\zeta < \mathrm{ht}(M_\tau)$ define a partial function $g^\zeta_\tau : J^{E^\tau}_\zeta \to J^{E^\tau}_\zeta$ and a sequence of ordinals $\langle \tau(\zeta);\ \zeta < \mathrm{ht}(M_\tau)\rangle$ by

$$y = g^\zeta_\tau(x) \longleftrightarrow (\exists z \in J^{E^\tau}_\zeta) H_\tau(x,y,q_\tau,z)$$

and

$$\tau(\zeta) = \sup(\tau \cap (g^\zeta_\tau)''\alpha).$$

Obviously, each g^ζ_τ is an element of M_τ, so $\tau(\zeta) < \tau$, as $\tau = \alpha^{+M_\tau}$ is a regular cardinal in M_τ. Furthermore, the sequence $\langle \tau(\zeta)\rangle_\zeta$ is monotonic and converges to τ; the latter follows from the fact that $h_\tau(\alpha \cup \{q_\tau\}) = M_\tau$. Let $g^\zeta_{\bar{\tau}}$ and $\bar{\tau}(\zeta)$ be defined in the same way over $M_{\bar{\tau}}$. Then $\sigma_{\bar{\tau},\tau}(g^\zeta_{\bar{\tau}}, \bar{\tau}(\zeta)) = (g^{\sigma_{\bar{\tau},\tau}(\zeta)}_\tau, \tau(\sigma_{\bar{\tau},\tau}(\zeta)))$. Thus, if $\tau^* < \tau$, we can find some ζ such that $\tau(\zeta) \geq \tau^*$ and, by the cofinality of $\sigma_{\bar{\tau},\tau}$, some

[16] We omitted the trivial verification of the Q-condition.

$\bar{\zeta}$ with $\sigma_{\bar{\tau},\tau}(\bar{\zeta}) \geq \zeta$. It follows that $\tau > \sigma_{\bar{\tau},\tau}(\bar{\tau}(\bar{\zeta})) = \tau(\sigma_{\bar{\tau},\tau}(\bar{\zeta})) \geq \tau(\zeta) \geq \tau^*$, which tells us that $\sigma_{\bar{\tau},\tau}$ maps $\bar{\tau}$ cofinally into τ. We have thus reached contradiction with our assumption that $\tau' < \tau$. $\Box$(16),(17)

Following the same strategy as in Case 1, we apply the Interpolation lemma to the map $\sigma_{\bar{\tau},\tau} : M_{\bar{\tau}} \to M_\tau$. We obtain a transitive structure M' and embeddings $\bar{\sigma}$ and σ' satisfying

- $\bar{\sigma} : M_{\bar{\tau}} \to M'$ is Σ_0-preserving and cofinal;
- $\sigma' : M' \to M_\tau$ is Σ_0-preserving;
- $\mathrm{cr}(\bar{\sigma}) = \bar{\alpha}$ and $\bar{\sigma}(\bar{\alpha}) = \alpha$;
- $\mathrm{cr}(\sigma') = \tau'$ and $\sigma'(\tau') = \tau$.

Since $M_{\bar{\tau}}$ is a premouse, the first two clauses immediately imply that M' is a coherent structure and

$$\vartheta(M') = \bar{\alpha} < \alpha = \mu^{+M'}. \tag{18}$$

To see the equality on the left, recall that both $M_{\bar{\tau}}$ and M' are amenable structures, so by the cofinality of $\bar{\sigma}$ we have

$$E^{M'}_{\mathrm{top}} = \bigcup\{\bar{\sigma}(E^{M_{\bar{\tau}}}_{\mathrm{top}} \cap J^{E^{\bar{\tau}}}_{\bar{\pi}(\xi)});\, \xi < \bar{\alpha})\}$$

where $\bar{\pi}$ is the ultrapower embedding associated with $\mathrm{Ult}(J^{E^{\bar{\tau}}}_{\bar{\tau}}, E^{M_{\bar{\tau}}}_{\mathrm{top}})$. The maps $\bar{\sigma}(E^{M_{\bar{\tau}}}_{\mathrm{top}} \cap J^{E^{\bar{\tau}}}_{\bar{\pi}(\xi)})$ and $E^{M_{\bar{\tau}}}_{\mathrm{top}} \cap J^{E^{\bar{\tau}}}_{\bar{\pi}(\xi)}$ have the same domains for each $\xi < \bar{\alpha}$, which implies that also $E^{M_{\bar{\tau}}}_{\mathrm{top}}$ and $E^{M'}_{\mathrm{top}}$ have the same domain, i.e. $\vartheta(M') = \bar{\alpha}$. $\Box$(18)

From the soundness and solidity of $M_{\bar{\tau}}$ above $\bar{\alpha}$ and (DP1) we obtain that $h_{\bar{\tau}}(\bar{\alpha} \cup \{q_{\bar{\tau}}\}) = M_{\bar{\tau}}$ and for each $\beta \in q_{\bar{\tau}} - \bar{\alpha}$ there is a generalized witness $Q^{\beta,q_{\bar{\tau}}}_{M_{\bar{\tau}}} \in M_{\bar{\tau}}$ with respect to $q_{\bar{\tau}}$ and β. Standard methods then yield:

- $M' = h_{M'}(\alpha \cup \{q'\})$ where $q' = \bar{\sigma}(q_{\bar{\tau}} - \alpha)$.
- For each $\beta \in q'$ there is a generalized witness $Q^{\beta,q'}_{M'} \in M'$ with respect to q' and M'.

This verifies the assumptions of Lemma 1.14. It follows that $N' = N(M')$ is the collapsing level for τ' in N and (μ, q') is a divisor for N'. The former easily implies that $N' = N_{\tau'}$. The latter follows from the fact that α is actually the ultimate projectum of N', so q' is a bottom part of $p_{\tau'}$ or $q' = d_{\tau'}$. To see that

$M' = M_{\tau'}$ we verify that (μ, q') is strong for $N_{\tau'}$. [17] First consider the case where $\lambda(E_{\text{top}}^{N^*}) > \mu$ where $N^* = (N_{\tau'})^*(\mu, q')$, i.e. N^* is the collapsing level for $\bar{\alpha}$ in $N_{\tau'}$. This is equivalent to saying that $E_{\bar{\alpha}}^{N_{\tau'}} = \varnothing$. Now use criterion (S2) from the discussion of strong divisors in section 1.3. If $a \subset \mu$ is an element of the transitive collapse of $\tilde{h}_{\tau'}(\mu \cup \{p_{\tau'}\})$ then, using the notation described in (4), a is of the form $E_{\text{top}}^{M'}(f)(q', \xi) \cap \mu$ for some $\xi < \mu$ and a suitable f. Since ξ, f and a are essentially subsets of μ, they are not moved by $\bar{\sigma}$, so

$$\bar{\sigma}(a) = a = E_{\text{top}}^{M'}(f)(q', \xi) \cap \mu = \bar{\sigma}\left(E_{\text{top}}^{M_{\bar{\tau}}}(f)(\bar{q} - \bar{\alpha}, \xi) \cap \mu\right).$$

This tells us that $a = E_{\text{top}}^{M_{\bar{\tau}}}(f)(\bar{q} - \bar{\alpha}, \xi) \cap \mu$, so $a \in M_{\bar{\tau}}$. But $\bar{\alpha}$ is a cardinal in $M_{\bar{\tau}}$, so in fact $a \in J_{\bar{\alpha}}^{E^{\bar{\tau}}} = J_{\bar{\alpha}}^{E^{\tau'}} \subset N^*$. By (S2), (μ, q') is strong for N'. If $E_{\bar{\alpha}}^{N_{\tau'}} \neq \varnothing$, the argument is the same, but uses the version of (S2) for $d_{N_{\tau'}}$.

We have thus proved that $\bar{\tau} \prec \tau'$. The above discussion verifies (T1A) and (T1B); (T1C) is trivially satisfied, as we can let $\tilde{\tau} = \bar{\tau}$. So $\bar{\sigma} = \sigma_{\bar{\tau}, \tau'}$. $\square$(M4)

Remark. It is this very last part of Case 2 in the verification of morass axiom (M4) where we make use of the fact that the divisors used in our construction are strong. There are only two places in the entire construction of Gap-1 Morass where the requirement on strongness is used in a substantial way; this is one of them. The other one is in the verification of morass axiom (M5) where an interpolation is used. In the case of (M0) – (M3) it is possible to avoid the use of strong divisors and use divisors of the form (μ, q) with shortest possible q instead. However, we do not see how to show that the divisor (μ, q') for N' that arises in the interpolation argument has shortest possible q' among all divisors of the form (μ, q) for N' unless we work with strong divisors. This is caused by the fact that the two interpolation maps fail to have sufficient preservation degree that would allow us to use Lemma 1.12 to transfer the information on strongness of the respective divisors from M_τ or $M_{\bar{\tau}}$ to $M_{\tau'}$.

Notice also that the situation in Case 2 above is a reason for introducing protomice into our construction. If both M_τ and $M_{\bar{\tau}}$ are pluripotent premice such that $\text{cr}(E_{\text{top}}^{M_\tau}) = \text{cr}(E_{\text{top}}^{M_{\bar{\tau}}})$ is the cardinal predecessor of $\alpha(\tau)$ (and so the cardinal predecessor of $\alpha(\bar{\tau})$) then the structure $M_{\tau'}$ that arises in the interpolation argument

[17] "Strong" is in this case equivalent to "strong above α".

is necessarily a protomouse even though the inputs in the argument are premice. We thus arrive at the situation where where $\bar\tau \in \mathfrak{T}^0$ and $\tau' \in \mathfrak{T}^1$. It should be noted that the situation in Case 2 above and the analogous situation in Case 2 in the verification of (M5) constitute the only reason for introducing protomice in our construction. That is, in all other arguments throughout the construction protomice never arise in situations where inputs are premice.

Finally note that the interpolation argument gives rise to situation that is required in (T1C), that is, we have $\bar\tau \prec \tau'$ where $\tau' \in \mathfrak{T}^1$, $\bar\tau \in \mathfrak{T}^0$ and $\vartheta_{\bar\tau} = \vartheta_{\tau'}$. Recall that we included (T1C) in the definition of the morass ordering and morass embeddings since without (T1C) we don't see how to verify axiom (M3) in the case where $\tau \in \mathfrak{T}^1$ and $\bar\tau \in \mathfrak{T}^0$ is such that $\alpha(\bar\tau)$ is a limit point of $C(\tau)$. Recall also that the issue here is that we do not have any condensation lemma allowing us to conclude that if $\bar\sigma : \bar M \to M_\tau$ is a sufficiently preserving map and $\bar M$ is a premouse then $\bar M$ is a level of M_τ. Ideally, we wish we could arrange that all $\prec$-predecessors of elements of $\mathfrak{T}^1$ are again elements of $\mathfrak{T}^1$, but we have seen above this is not possible. So we have to deal with two issues simultaneously: We have to allow $\prec$-predecessors of $\tau' \in \mathfrak{T}^1$ that are in $\mathfrak{T}^0$, and at that same time we must make sure that if some $\tau^* \in \mathfrak{T}^0$ is a $\prec$-predecessor of a $\tau' \in \mathfrak{T}^1$ then we also have a $\prec$-predecessor $\bar\tau$ of τ' that is in $\mathfrak{T}^0$ and satisfies the requirement $\vartheta_{\bar\tau} = \vartheta_{\tau'}$. The argument in Case 2 above as well as that in Case 2 in the verification of (M5) guarantee this.

Axiom (M5). Assume that $\bar\tau, \tau$ and $\hat\alpha$ satisfy the hypothesis of (M5). We will find $\hat\tau$ as in the conclusion of (M5). At some point, our proof will break up into two cases: $\tau \in \mathfrak{T}^0$ and $\tau \in \mathfrak{T}^1$. By our earlier calculations, in the first case, we will have that $\bar\tau, \hat\tau \in \mathfrak{T}^0$. As we pointed out earlier, in the second case both $\bar\tau \in \mathfrak{T}^0$ and $\bar\tau \in \mathfrak{T}^1$ are possible. In both of these subcases, it turns out that $\hat\tau \in \mathfrak{T}^1$. The verification uses the hypothesis of (M5) that the range of $\sigma_{\bar\tau,\tau}$ is unbounded in τ.

To simplify the notation, we will write σ for $\sigma_{\bar\tau,\tau}$ and τ' for $\sigma(\bar\tau')$ whenever $\bar\tau' <_0 \bar\tau$. We will use an interpolation argument to construct $N_{\hat\tau}$ and the map $\sigma_{\hat\tau,\tau}$. In order to satisfy the assumptions of the interpolation lemma, we need to find $\hat\tau$, a structure $J^{\hat E}_{\hat\tau}$ and Σ_0-preserving maps

$$\sigma^0_0 : J^{E^{\bar\tau}}_{\bar\tau} \to J^{\hat E}_{\hat\tau} \quad \text{and} \quad \sigma^0_1 : J^{\hat E}_{\hat\tau} \to J^{E^\tau}_\tau$$

such that σ_0^0 is cofinal and $\sigma_1^0 \circ \sigma_0^0 = \sigma$. Notice that these requirements together with the cofinality of σ imply that σ_1^0 is cofinal as well.

Notice that for each $\bar\tau' <_0 \bar\tau$ the ordinal $\hat\tau'$ as in (M5) is unique. From the argument in the verification of morass axiom (M2) we obtain that if $\bar\tau', \bar\tau'' <_0 \bar\tau$ with, say $\bar\tau' < \bar\tau''$ then $E^{\hat\tau'} = E^{\hat\tau''} \restriction \hat\tau'$ and

$$\sigma_{\bar\tau',\hat\tau'} \restriction J^{E^{\bar\tau}}_{\bar\tau'} \quad = \quad \sigma_{\bar\tau'',\hat\tau''} \restriction J^{E^{\bar\tau}}_{\bar\tau'} \tag{19}$$

$$\sigma_{\hat\tau',\tau'} \restriction J^{E^{\hat\tau'}}_{\hat\tau'} \quad = \quad \sigma_{\hat\tau'',\tau''} \restriction J^{E^{\hat\tau'}}_{\hat\tau'} \tag{20}$$

where $\hat\tau' \prec \tau'$, $\hat\tau'' \prec \tau''$ and $\alpha(\hat\tau') = \hat\alpha = \alpha(\hat\tau'')$, as guaranteed by the hypothesis of (M5). This allows us to set

$$\begin{aligned}
J^{\hat E}_{\hat\tau} &= \bigcup\{J^{E^{\hat\tau'}}_{\hat\tau'} \,;\, \hat\tau' \prec \tau' \text{ for some } \bar\tau' <_0 \bar\tau\} \\
\sigma_0^0 &= \bigcup\{\sigma_{\bar\tau',\hat\tau'} \restriction J^{E^{\bar\tau'}}_{\bar\tau'} \,;\, \bar\tau' \prec \tau' \text{ for some } \bar\tau' <_0 \bar\tau\} \\
\sigma_1^0 &= \bigcup\{\sigma_{\hat\tau',\tau'} \restriction J^{E^{\hat\tau'}}_{\hat\tau'} \,;\, \hat\tau' \prec \tau' \text{ for some } \bar\tau' <_0 \bar\tau\}
\end{aligned}$$

For $\bar\tau' <_0 \bar\tau$ we have $\bar\tau' \prec \hat\tau' \prec \tau'$, and since the maps along the branches of the morass ordering $\prec$ commute,

$$\sigma \restriction \bar\tau' = \sigma_{\bar\tau',\tau'} \restriction \bar\tau' = (\sigma_{\hat\tau',\tau'} \restriction \hat\tau') \circ (\sigma_{\bar\tau',\hat\tau'} \restriction \bar\tau') = (\sigma_0^0 \restriction \bar\tau') \circ (\sigma_1^0 \restriction \hat\tau').$$

As an immediate consequence we obtain

$$\sigma_1^0 \circ \sigma_1^0 = \sigma \restriction \bar\tau \tag{21}$$

and all three maps are cofinal.

We are now ready to apply the Interpolation Lemma. We again consider two separate cases.

Case 1: $\tau \in \mathfrak{T}^0$.

In this case $\bar\tau \in \mathfrak{T}^0$ by (T0A). From the Interpolation Lemma we obtain a coherent structure $\hat N$, and maps σ_0^1 and σ_1^1 where each σ_i^1 is an extension of σ_i^0, and:

- $\sigma_0^1 : N_{\bar\tau} \to \hat N$ is $\Sigma_0^{(n)}$-preserving and cofinal at the level n;
- $\sigma_1^1 : \hat N \to N_\tau$ is $\Sigma_0^{(n)}$-preserving and cofinal at the level n;
- $\sigma_1^1 \circ \sigma_0^1 = \sigma$

where $n = n_\tau = n_{\bar\tau}$. It follows that both maps σ_0^1 and σ_1^1 are $\Sigma_1^{(n)}$-preserving. The cofinality of σ_0^1 is a consequence of the fact that the maps σ and σ_0^1 are cofinal and $\sigma = \sigma_1^1 \circ \sigma_0^1$. The cofinality of both σ and σ_1^1 follows by a standard argument from the fact that these maps are $\Sigma_0^{(n)}$-preserving, map $\bar\tau$ cofinally into τ, resp. $\hat\tau$ and $\bar\tau \subseteq \tilde h_{\bar\tau}(\alpha(\bar\tau) \cup (p_{\bar\tau} - \alpha(\bar\tau)))$. [For σ, the argument runs as follows: Since $\tilde h_{\bar\tau}$ is a good $\Sigma_0^{(n)}(N_{\bar\tau})$ function, it has a functionally absolute $\Sigma_1^{(n)}$ definition, say $y = \tilde h_{\bar\tau}(x, p_{\bar\tau} - \alpha(\bar\tau))$ iff $(\exists z^n)\bar H(z^n, x, y, p_{\bar\tau} - \alpha(\bar\tau))$ where $\bar H$ is $\Sigma_0^{(n)}(N_{\bar\tau})$. If $\omega\varrho = \sup(\sigma''\omega\varrho^n_{N_{\bar\tau}}) < \omega\varrho^n_{N_\tau}$ and H is $\Sigma_0^{(n)}(N_\tau)$ by the same definition as $\bar H$, we could define a function h mapping $\alpha(\tau)$ cofinally into τ by letting $y = h(x)$ iff $(\exists z^n \in J_\varrho^{E^\tau})H(z^n, x, y, p_\tau - \alpha(\tau))$. This function would be an element of N_τ, which is impossible, as it would collapse τ inside N_τ.]

N_τ is a premouse and σ_1^1 is Σ_1-preserving if N_τ is of type A or B and $\Sigma_1^{(1)}$-preserving if N_τ is of type C. Since premousehood is a Π_2-property in the former case and Π_1^1-property in the latter case, $\hat N$ must be a premouse of the same type as N_τ. It may be instructive to see directly that a situation where protomice result from an interpolation argument analogous to that in the verification of (M4) does not arise here. It cannot happen that N_τ and $N_{\bar\tau}$ are pluripotent and their top extenders have a common critical point μ that is the common cardinal predecessor of $\alpha(\tau)$ and $\alpha(\bar\tau)$, as the fact that σ maps $\bar\tau$ cofinally into τ would imply that τ is collapsed inside $\langle J_\zeta^{E^\tau}, E^{N_\tau}_{\mathrm{top}} \cap J_\zeta^{E^\tau}\rangle \in N_\tau$ where ζ is the image of $\vartheta_{\bar\tau}$ under the ultrapower map by $E^{N_\tau}_{\mathrm{top}}$. [18] Using the soundness and solidity of $N_{\bar\tau}$, standard arguments yield:

(i) $\hat N = \tilde h^{n+1}_{\hat N}(\hat\alpha \cup \{\hat p\})$ where $\hat p = \sigma_0^1(p_{\bar\tau} - \bar\alpha)$.

(ii) For every $\beta \in \hat p$ there is a generalized witness $Q^{\beta,\hat p}_{\hat N}$ with respect to $\hat p$ and $\hat N$.

(iii) $\omega\varrho^{n+1}_{\hat N} \le \hat\alpha$, $\hat p = p_{\hat N} - \alpha$ and $\hat N$ is sound above $\hat\alpha$ and solid.

Regarding (iii), that $\omega\varrho^{n+1}_{\hat N} \le \hat\alpha$ is immediate by (i). From this combined with the first part of the Condensation Lemma we obtain the solidity of $\hat N$. Once we know that $\hat N$ is solid, (i) and (ii) yield $\hat p = p_{\hat N} - \alpha$, and thereby also the soundness of $\hat N$ above $\hat\alpha$ ([21], Lemma 1.12.5).

Now we apply the Condensation Lemma to the triple $\sigma_1^1 : \hat N \to N_\tau$ and conclude that $\hat N = N_{\hat\tau}$. Here we proceed exactly as in the discussion of the closedness of

[18] This can be seen by an argument of the same kind as in the previous paragraph.

$C(\tau)$ in (M3), Case 1. It suffices to rule out possibility (c) in the Condensation Lemma. As in that argument we show that (c) would imply $\tilde{h}^{n+1}_{\hat{N}}(\mu \cup \{\hat{p}, \mu\}) = \hat{N}$ where μ is the cardinal predecessor of $\hat{\alpha}$ in $\hat{N}$. Then μ would be the cardinal predecessor of $\alpha(\bar{\tau})$ in $N_{\bar{\tau}}$, so $\mu < \alpha(\bar{\tau}) = \mathrm{cr}(\sigma^1_0)$. Since σ^1_0 is $\Sigma^{(n)}_1$-preserving and $\mu \cup \{\hat{p}, \mu\} \subset \mathrm{rng}(\sigma^1_0)$, we would obtain $\mathrm{rng}(\sigma^1_0) = \hat{N}$. But σ^1_0, being an extension of σ^0_0, has critical point $\bar{\alpha}$, a contradiction.

Once we know that $\hat{N} = N_{\hat{\tau}}$, the verification of the fact that $\hat{\tau} \in \mathcal{T}^0$, as well as that of the Q-condition, is literally the same as in (M3), Case 1. The key point here is that $\sigma^1_1(\mu, \hat{\alpha}, p_{\hat{\tau}} - \hat{\alpha}) = (\mu, \alpha, p_\tau - \alpha)$ where $\mu < \hat{\alpha}$ and that $\sigma^1_1(d^{\hat{\alpha}}_{\hat{\tau}}) = d^{\alpha}_{\tau}$ whenever applicable. $\square$((M5), Case 1)

Remark. We would like to point out that in the above argument, we are unable to prove that $\omega\varrho^{\omega}_{N_{\hat{\tau}}} = \alpha(\hat{\tau})$. Thus, it may happen that this projectum is the cardinal predecessor of $\alpha(\hat{\tau})$, even in the case where $\omega\varrho^{\omega}_{N_{\bar{\tau}}} = \alpha(\bar{\tau})$. It may namely happen that the single ordinal in $p_{N_{\hat{\tau}}} \cap \alpha(\hat{\tau})$ is not in the range of σ^1_0. The situation here can equivalently be described as $\bar{\tau} \prec \hat{\tau}$ where $\hat{\tau}$ is the largest element in $S_{\alpha(\hat{\tau})}$. Verification of (M5) is the only place in the entire construction of a Gap-1 Morass where it may happen that we introduce ordinals that are of the form $\max(S_\beta)$ for some β, even though the inputs are not of this form. Verifications of all other axioms can be arranged so that starting from $\beta < \max(S_\beta)$ we produce an ordinal β' that is smaller than $\max(S_{\beta'})$.

Case 2: $\tau \in \mathcal{T}^1$.

In this case $M_{\bar{\tau}}$ is either a protomouse or else a pluripotent premouse such that $\mathrm{cr}(E^{M_{\bar{\tau}}}_{\mathrm{top}}) = \mu = \mathrm{cr}(E^{M_\tau}_{\mathrm{top}})$ is the common cardinal predecessor of $\alpha(\tau)$ and $\alpha(\bar{\tau})$ and $\vartheta_\tau = \vartheta_{\bar{\tau}} = \alpha(\bar{\tau})$. [19] From the interpolation argument we obtain a coherent structure $\hat{M}$, and maps σ^1_0 and σ^1_1 where σ^1_i is an extension of σ^0_i for $i = 0, 1$, and:

- $\sigma^1_0 : M_{\bar{\tau}} \to \hat{M}$ is Σ_0-preserving and cofinal;
- $\sigma^1_1 : \hat{M} \to M_\tau$ is Σ_0-preserving and cofinal;
- $\sigma^1_1 \circ \sigma^1_0 = \sigma$.

[19] That $\vartheta_\tau = \vartheta_{\bar{\tau}}$ follows by an argument similar to that at the end of the paragraph immediately above (i) in Case 1.

It follows that both maps σ_0^1 and σ_1^1 are Σ_1-preserving. The cofinality of these maps can be verified as in Case 1. [In fact, there is another way of seeing this: Just recall that $\vartheta_{\bar{\tau}} = \vartheta_\tau$ and, by construction, this value is equal to $\vartheta(\hat{M})$. The rest follows from the fact that we deal with coherent structures and both maps σ_0^1 and σ_1^1 agree with the identity map up to ϑ.]

Our aim is to prove that $\hat{M} = M_{\hat{\tau}}$. The main tool we are going to use for this purpose is Lemma 1.15. In the following we verify the assumptions of the lemma. First notice the important fact that $\hat{M}$ is a coherent structure whose top extender is not weakly amenable since $\vartheta(\hat{M}) = \vartheta_{\bar{\tau}} \leq \alpha(\bar{\tau}) < \hat{\alpha} = \mu^{+\hat{M}}$. This can be seen similarly as in the verification of Case 2 in (M4) or directly using the Σ_1-elementarity of σ_1^1. Using an argument analogous to that in the verification of (M4), Case 2 we obtain:

(i) $\hat{M} = h_{\hat{N}}(\hat{\alpha} \cup \{\hat{q}\})$ where $\hat{q} = \sigma_0^1(q_{\bar{\tau}} - \bar{\alpha})$.

(ii) For every $\beta \in \hat{q}$ there is a generalized witness $Q_{\hat{M}}^{\beta,\hat{q}}$ with respect to $\hat{q}$ and $\hat{M}$.

Both clauses follow from the corresponding properties of $M_{\bar{\tau}}$. This verifies assumptions (a) and (b) in Lemma 1.15; assumption (c) is obviously satisfied as $\sigma_1^1 \circ \sigma_0^1 = \sigma$. We can thus apply Lemma 1.15 to the triple $\sigma_1^1 : \hat{M} \to M_\tau$. To get the desired conclusion, we verify that conclusion (B) in the lemma does not apply. Here we proceed as in Case 2 in the verification of (M3), applying Proposition 1.13: Letting $\vartheta = \vartheta(\hat{M}) = \vartheta_{\bar{\tau}} = \vartheta_\tau$ and $E = E^{\hat{M}}$, we find a function $f \in J_\vartheta^E$ and an ordinal $\xi < \mu$ such that $\hat{\alpha} = E_{\text{top}}^{\hat{M}}(f)(\hat{q}, \xi)$. We then get a contradiction by observing that $\hat{\alpha} \in \text{rng}(\sigma_0^1)$, whereas $\hat{\alpha} = \text{cr}(\sigma_0^1)$. Notice that unlike in the case of (M3), here we also have to consider the situation wehre $\bar{\tau} \in \mathfrak{T}^0$. Due to our notation, both cases $\bar{\tau} \in \mathfrak{T}^0$ and $\bar{\tau} \in \mathfrak{T}^1$ are treated the same way.

Finally we have to verify that the divisor $(\mu, p_{\hat{M}} \cap \lambda(E_{\text{top}}^{\hat{M}}))$ for $N_{\hat{\tau}}$ is strong above $\hat{\alpha} = \alpha(\hat{\tau})$, but this can be done exactly as in the verification of (M4), Case 2. $\square$(M5)

This completes the construction of a Gap-1 Morass in $\mathbf{L}[E]$.

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