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$\diamond$ AT MAHLO CARDINALS

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Abstract. Given a Mahlo cardinal κ and a regular ε such that $\omega_1 < \varepsilon < \kappa$ we show that $\diamond_\kappa(\mathrm{cf} = \varepsilon)$ holds in **V** provided that there are only non-stationarily many $\beta < \kappa$ with $\mathrm{o}(\beta) \geq \varepsilon$ in **K**.

§1. Introduction. The principle $\diamond_\kappa(A)$ postulates the existence of a sequence $\langle S_\alpha;\ \alpha \in A \ \&\ S_\alpha \subset \alpha\rangle$ which tightly approximates every $X \subset \kappa$ in the sense that $X \cap \alpha = S_\alpha$ for stationarily many α. If A is the set of all $\alpha < \kappa$ of a fixed cofinality ε, we write $\diamond_\kappa(\mathrm{cf} = \varepsilon)$ and if $A = \kappa$ we write simply $\diamond_\kappa$. The principle was introduced by Jensen in [Je72]. The behaviour of $\diamond_\kappa$ at successors has been extensively studied and it is known that $\diamond_{\kappa^+}$ holds in **V** for many $\kappa > \omega$.

In this paper we shall focus on $\diamond_\kappa$ at inaccessibles, where the situation is much less clear. Jensen and Kunen have shown in [JK??] that the principle holds at subtle cardinals and in fact that $\diamond_\kappa$(regulars) holds. This suggests that violating the principle at large cardinals will be problematic. First of all, if we want to violate $\diamond_\kappa$(regulars) we have to look for a smaller κ than subtle. It has turned out that this is really possible: Woodin has constructed a model with $\neg\diamond_\kappa$(regulars) for a weakly compact κ and Hauser [Ha92] has obtained a similar result for indescribables. Nevertheless, these constructions do not kill the full $\diamond_\kappa$. Using Radin forcing, Woodin [Cu95] finally succeeded in violating $\diamond_\kappa$ at a Mahlo cardinal κ starting from an assumption that κ is hypermeasurable with order slightly more than κ^{++}. That a strong large cardinal assumption must be used for this task follows from Jensen's result [Je91]: the failure of $\diamond_\kappa(\mathrm{cf} = \omega_1)$ implies the existence of $0^\#$. The main result of this paper is the following strengthening of Jensen's result.

THEOREM 1.1. ($\neg 0^\P$) *Let κ be Mahlo and ε be regular such that $\omega_1 < \varepsilon < \kappa$. The failure of $\diamond_\kappa(\mathrm{cf} = \varepsilon)$ implies that $\mathrm{o}^K(\beta) \geq \varepsilon$ for stationarily many $\beta < \kappa$.*

To prove this theorem, we shall follow the same strategy as in [Je91] but will focus on the points of the construction typical for the core model context. We shall also use the following two theorems of Mitchell.

THEOREM 1.2. ([Mi87]) *Let β be a strong limit singular cardinal of cofinality $\varepsilon > \omega$ which is regular in K. Then $\mathrm{o}^K(\beta) \geq \varepsilon$.*

THEOREM 1.3. ([Mi92]) *Let β be a strong limit singular cardinal of uncountable cofinality. If $\mathrm{o}^K(\beta) < \beta^{++}$, then $2^\beta = \beta^+$.*

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We shall work in Jensen's core model for nonoverlapping extenders, as a reference see [Je?a], [Je97] or [Ze?]. A brief description of our mice and an overview of our fine structural terminology can be also found in [JZ]. Otherwise we use the standard set-theoretic notation, see [JC78].

§2. **The proof.** In this section we describe a construction which, under the assumption that we have c.u.b. many $\beta < \kappa$ with $o^K(\beta) < \varepsilon$, yields a $\diamondsuit'_\kappa(\mathrm{cf} = \varepsilon)$-sequence. The existence of such sequence is, by a theorem of Kunen, equivalent to the existence of a $\diamondsuit_\kappa(\mathrm{cf} = \varepsilon)$-sequence. Recall that $\langle S_\alpha; \alpha \in A\rangle$ is a $\diamondsuit'_\kappa(A)$-sequence if and only if each S_α is a collection of subsets of α, its cardinality is at most that of α and for every $X \subset \kappa$ there are stationarily many $\alpha \in A$ such that $X \cap \alpha \in S_\alpha$.

To commence with, we fix a Mahlo cardinal κ, a set $A \subset \kappa$ such that

$$V_\alpha = J^A_\alpha \text{ whenever } \alpha \leq \kappa \text{ and } \mathrm{card}(V_\alpha) = \alpha$$
$$\mathscr{P}(\alpha) \subset J^A_{\alpha^+} \text{ whenever } \alpha \text{ is as above and } 2^\alpha = \alpha^+$$

and a regular ε such that $\omega_1 < \varepsilon < \kappa$. The set A with the above properties can be obtained by a standard coding argument. Set

$$G := \{\alpha < \kappa;\ \mathrm{card}(V_\alpha) = \alpha \text{ and } \mathrm{cf}(\alpha) = \varepsilon\}$$

and for each $\alpha \in G$ fix a set C_α of order type ε which is cofinal in α. For $\alpha \in G$ we define[1]

$$\eta_\alpha :\simeq \text{ the least } \eta \text{ such that } \alpha \text{ is singular in } \boldsymbol{K}\|(\eta+1)$$

(1)
$$S_\alpha := \begin{cases} \mathscr{P}(\alpha) \cap \bigcup_{y \subset C_\alpha} J^{E^K, A\cap\alpha, y}_{\eta_\alpha+1} & \text{if } \eta_\alpha \text{ is defined} \\ \emptyset & \text{otherwise} \end{cases}$$

We show:

LEMMA 2.1. *Suppose there are c.u.b. many $\beta < \kappa$ satisfying $\mathrm{o}^K(\beta) < \varepsilon$. Then $\langle S_\alpha\rangle$ is a $\diamondsuit'_\kappa(G)$-sequence.*

PROOF. Note first that the cardinality of each S_α is at most α since C_α has at most $2^\varepsilon < \mathrm{card}(V_\alpha) = \alpha$ subsets and the size of each set of the union in (1) is also $\leq \alpha$. Hence it suffices to prove that, given any $X \subset \kappa$ and a c.u.b. set $C \subset \kappa$, there is an $\alpha \in C \cap G$ such that $X \cap \alpha \in S_\alpha$. By our assumptions we can without a loss of generality suppose that

$$\mathrm{o}^K(\alpha) < \varepsilon \qquad \text{for every } \alpha \in C.$$

Let $\mathscr{Y} \subset J^A_\kappa$ be the smallest set satisfying

a) $\langle \mathscr{Y}, A \cap \mathscr{Y}, E \cap \mathscr{Y}, X \cap \mathscr{Y}, C \cap \mathscr{Y}\rangle \prec \langle J^A_\kappa, E, X, C\rangle$
b) if $\alpha \in \mathscr{Y}$ and $\mathrm{cf}(\alpha) \leq \varepsilon$ then $\mathscr{Y} \cap \alpha$ is cofinal in α
c) if $\alpha \in \mathscr{Y}$ and $\mathrm{cf}(\alpha) \geq \varepsilon$ then $\mathscr{Y} \cap \alpha$ is ε-cofinal

(here $E = E^K$). $\mathscr{Y}$ can be obtained in a straightforward way by iterating the following pair of operations ε-many times:

– closure under elementary substructure

[1] It was pointed to me by Sy Friedman that if η_α in (1) is defined then $S_\alpha = \mathscr{P}(\alpha) \cap J^{E^K, A\cap\alpha, C_\alpha}_{\eta_\alpha+1}$

– adding a short cofinal sequence to every $(<\varepsilon)$-cofinal α + adding an ordinal between $\sup(\mathscr{Y}\cap\alpha)$ and α for every other α. (If $\operatorname{cf}(\alpha)=\varepsilon$ we have to choose such an ordinal to be large enough so that we ultimately obtain a $\mathscr{Y}$ which is cofinal in α, but this can be easily arranged).

Clearly, $\operatorname{card}(\mathscr{Y})=\varepsilon$.The structure from a) can be transitivised to a $\langle J^{\bar{A}}_{\bar{\kappa}},\bar{E},\bar{X},\bar{C}\rangle$. Let σ be the inverse of the collapsing map. Set

$$\bar{\boldsymbol{K}}:=J^{\bar{E}}_{\bar{\kappa}},\quad \tilde{\boldsymbol{K}}:=J^{E^K}_{\kappa}.$$

By the construction of $\mathscr{Y}$,

a) σ is continuous at every point of cofinality $<\varepsilon$
b) $\operatorname{cf}(\alpha)=\varepsilon$ for every α which is regular in $J^{\bar{A}}_{\bar{\kappa}}$, $\alpha\geq\varepsilon$.

It is a well-known fact from the core model theory that if we have an elementary embedding $\sigma:\bar{\boldsymbol{K}}\to\tilde{\boldsymbol{K}}$ constructed as above then $\bar{\boldsymbol{K}}$ is not moved in the coiteration with $\mathbf{K}$. Furthermore, there is a mouse M_0 and an ordinal $\delta\in M_0$ such that M_0 is sound above δ, coiterates with $\bar{\boldsymbol{K}}$ above δ and $\bar{\boldsymbol{K}}$ is not moved in this coiteration (see [Je?a] or [Ze?] for the details). Let $\langle M_i;\ i\leq\theta\rangle$ be the M_0-side of this coiteration, $\langle\nu_i;\ i<\theta\rangle$ be the iteration indices and $\kappa_i:=\operatorname{cr}(E^{M_i}_{\nu_i})$. Then M_i is sound above κ_i for every $\kappa<\theta$. We shall now consider two cases.

Case 1. The sequence of critical points is bounded in $\bar{\kappa}$.

Let $M':=M_\theta$. Then M' is an end-extension of $\bar{\boldsymbol{K}}$. Let $\kappa^*:=\sup\{\kappa_i;\ i<\theta\}$. By our assumption, $\kappa^*<\bar{\kappa}$. Let η be maximal such that $\bar{\kappa}$ is a cardinal in $M:=\widehat{M'\|\eta}$. Then $\omega\varrho^\omega_M<\bar{\kappa}$. For $\eta=\operatorname{ht}(M')$ this follows from the fact that $\omega\varrho^\omega_{M'}\leq\kappa^*$. Set

$$n:=\text{ the maximal } m \text{ such that } \omega\varrho^m_M\geq\bar{\kappa},\quad \bar{\varrho}:=\max\{\kappa^*,\omega\varrho^{n+1}_M\}$$

and pick a $\bar{\beta}<\bar{\kappa}$ such that

$$\bar{\beta}\text{ is a limitpoint of }\bar{C}\text{, inaccessible in }J^{\bar{A}}_{\bar{\kappa}}\text{ and }\bar{\varrho}<\bar{\beta}$$

This is possible by the Mahloness of κ: there is an inaccessible $\beta'>\sigma(\bar{\varrho})$ which is a limitpoint of C; applying σ backwards then yields $\bar{\beta}$ as required. Then

$$\bar{\varrho}\geq\omega\varrho^{n+1}_M\text{ and }M\text{ is sound above }\bar{\varrho}.$$

Set

$$\beta:=\sup(\sigma''\bar{\beta}),\quad \varrho:=\sigma(\bar{\varrho}).$$

Clearly β is a limitpoint of C of cofinality ε and $\operatorname{card}(V_\beta)=\beta$. Consequently,

$$\beta\in G\cap C.$$

To complete the proof in this case we have to show that $X\cap\beta\in S_\beta$. The map $\sigma\restriction J^{\bar{E}}_{\bar{\beta}}:J^{\bar{E}}_{\bar{\beta}}\to J^{E}_{\beta}$ is Σ_0-preserving and cofinal. We shall construct its canonical extension $\tilde{\sigma}:\widehat{M}\to\widehat{N}$ using the upwards extension of embeddings techniques. Since $\operatorname{cf}(\beta)>\omega$, such an extension always exists, is $\Sigma^{(n)}_0$-preserving and $\widehat{N}$ is an acceptable structure. By the fact that $\widehat{M}$ is a singularizing structure for $\bar{\beta}$ (i.e., $\bar{\beta}$ is regular in $\widehat{M}$ but is definably singularized over $\widehat{M}$), $\tilde{\sigma}$ is cofinal (= cofinal at the level of n-th reducts). What the structure $\widehat{N}$ looks like depends on whether the following condition holds

$$\operatorname{cr}(E^M_{\text{top}})<\bar{\beta}\ \&\ n=0.$$

If it fails, then $\widehat{N}$ is an expansion of a premouse, so our notation is consistent. Moreover, N is coiterable with **K** hence N is a mouse. If (1) holds, in most cases we obtain a structure $\widehat{N} = \langle J_\alpha^{E'}, F\rangle$ which looks like an expansion of a premouse but whose top extender is not weakly amenable. Such structures are called in literature protomice. To every protomouse we can assign a mouse Q as follows. Let N^* be the maximal initial segment of $\widehat{N}$ such that F measures every subset of $\mu = \mathrm{cr}(F)$ which is a member of N^*. Then $\widehat{Q} := \mathrm{Ult}^*(\widehat{N}^*, F)$. Moreover, for any set of ordinals $Z \subset \widehat{N}$, Z is $\Sigma_1(\widehat{N})$ if and only if Z is $\Sigma_1^{(m)}(\widehat{Q})$ where m is maximal such that $\omega\varrho_{N^*}^m > \mu$. The upwards extension of embeddings techniques can be applied also to protomice. It follows that if we have a situation as above where $\mathrm{cf}(\bar{\beta}) > \omega$, then $\widehat{N}$ is really a protomouse, i.e., its corresponding premouse Q is iterable. Detailed exposition of the upwards extension of embeddings techniques and protomice can be found in [Ze?]. In the sequel we shall focus on the case when $\widehat{N}$ is a mouse. The proof in the other case is similar but contains one more step: all of the information proved about $\widehat{N}$ must be translated to the language of $\widehat{Q}$ because this mouse (and not the protomouse) will be used in the verification that $X \cap \beta \in S_\beta$.

From now on assume that $\widehat{N}$ is a mouse. It follows from the construction of $\tilde{\sigma}$ that every $x \in \widehat{N}$ is $\Sigma_1^{(n-1)}(\widehat{N})$ in parameters from $\mathrm{rng}(\tilde{\sigma}) \cup \beta$. Since M is sound above $\bar{\beta}$, each member of $\mathrm{rng}(\tilde{\sigma})$ is $\Sigma_1^{(n)}(\widehat{N})$ definable from $\tilde{\sigma}(p_M)$ and some $\xi < \beta$. Taken together, the universal good $\Sigma_1^{(n)}(\widehat{N})$ function $\tilde{h}_N^{n+1}$ satisfies

$$\widehat{N} = \tilde{h}_N^{n+1}(\beta \cup \{\tilde{\sigma}(p_M)\}) \tag{2}$$

hence $\omega\varrho_N^{n+1} < \beta$. We next observe that the preservation degree of $\tilde{\sigma}$ is high enough to guarantee that whenever $\nu \in p_M$, $\tilde{\sigma}(W_M^\nu)$ is a generalized witness for ν in $\widehat{N}$ w.r.t. $\tilde{\sigma}(p_M)$ (W_M^ν is the standard witness for ν in $\widehat{M}$ w.r.t. p_M). It is a general fact from the fine structure theory that this together with (2) implies that N is sound above β and $p_N - \beta = \tilde{\sigma}(p_{\bar{M}}) - \beta$. Furthemore, $\widehat{N}$ is a singularizing structure for β. This can be seen as follows. β is regular in $\widehat{N}$ since $\beta = \tilde{\sigma}(\bar{\beta})$ and $\bar{\beta}$ is regular in $\widehat{M}$; β is $\Sigma_1^{(n)}(\widehat{N})$ singularized since the universal function $\tilde{h}_M^{n+1}$ maps $\bar{\varrho} \cup \{p_M\}$ cofinally to $\bar{\beta}$ and $\tilde{\sigma}$ is sufficiently preserving.

We have already mentioned above that N is coiterable with **K**. In fact, the way N has been constructed guarantees that the coiteration is above β. Because of this (and universality of **K**), $\mathscr{P}(\beta) \cap \Sigma_1^{(n)}(\widehat{N}) \subset \boldsymbol{K}$. In particular, β is singular in **K**. Hence η_β is defined. β is Σ^* singularized over $\widehat{\boldsymbol{K}||\eta_\alpha}$ but it can happen that β is singular in the expansion. However, if Q is the maximal initial segment of this expansion in which β is regular, then $\widehat{Q}$ is a singularizing structure for β. Note that N coiterates with Q above β. Since both are singularizing structures for β, they coiterate simply to a common mouse. But then $Q = N$ since both are above β.

Now fix an increasing sequence $\langle\bar{\vartheta}_\xi;\ \xi < \varepsilon\rangle$ cofinal in $\bar{\beta}$. Then $\langle\vartheta_\xi;\ \xi < \varepsilon\rangle$ is cofinal in β, where $\vartheta_\xi := \sigma(\bar{\vartheta}_\xi)$. For each $\xi < \varepsilon$ we thus obtain $\tilde{\sigma}(\bar{X} \cap \bar{\vartheta}_\xi) = X \cap \vartheta_\xi$. Pick sequences ζ_ξ, $\bar{\zeta}_\xi$ such that

$$\begin{aligned} \bar{X} \cap \bar{\vartheta}_\xi &= \text{the } \bar{\zeta}_\xi\text{-th element of } J_{\bar{\beta}}^{\bar{A}} \text{ under the well-ordering } <_{J_{\bar{\beta}}^{\bar{A}}} \\ X \cap \vartheta_\xi &= \text{the } \zeta_\xi\text{-th element of } J_\beta^A \text{ under the well-ordering } <_{J_\beta^A}. \end{aligned}$$

Clearly $\bar{\zeta}_\xi < \bar{\beta}$, $\zeta_\xi < \beta$ and $\tilde{\sigma}(\bar{\zeta}_\xi) = \zeta_\xi$. By (2) we can choose a sequence $\langle \bar{\delta}_\xi;\ \xi < \varepsilon\rangle$ such that

$$\bar{\delta}_\xi < \bar{\varrho} \text{ and } \tilde{h}^{n+1}_M(\bar{\delta}_\xi, p_M) = \bar{\zeta}_\xi \text{ for every } \xi < \varepsilon.$$

Setting $\delta_\xi := \tilde{\sigma}(\bar{\delta}_\xi)$ we obtain

$$\delta_\xi < \varrho \text{ and } \tilde{h}^{n+1}_N(\delta_\xi, \tilde{\sigma}(p_{\bar{M}})) = \zeta_\xi \text{ for } \xi < \varepsilon.$$

Note that if we know ζ_ξ we can reconstruct $X \cap \vartheta_\xi$ inside J^A_β. Hence, in order to reconstruct $X \cap \beta$ we need the sequence $\langle \delta_\xi;\ \xi < \varepsilon\rangle$, the function $h(-, p_N) \restriction \beta$ where h is from (2) and J^A_β. But h is definable over $\widehat{N}$ hence $h \restriction \beta$ is definable over $\boldsymbol{K}||\eta_\beta$. Furthemore, $\langle \delta_\xi;\ \xi < \varepsilon\rangle \in J^A_\beta$. Consequently, all of the objects we need are elements of $J^{E.A\cap\beta}_{\eta_\beta+1}$. The reconstruction uses just very simple operations which can be carried out inside this structure, hence

$$X \cap \beta \in \mathscr{P}(\beta) \cap J^{E.A\cap\beta}_{\eta_\beta+1} \subset S_\beta.$$

Case 2. The sequence of critical points in the coiteration is unbounded in $\bar{\kappa}$.

We shall follow the same strategy as in the Case 1 but we have to overcome several new obstacles. Fix $i^* < \theta$ such that

a) the iteration is simple from i^* on
b) there is a fixed $n \in \omega$ such that $\omega\varrho^{n+1}_{M_i} \leq \kappa_i < \omega\varrho^n_{M_i}$

and a $\bar{\beta} < \bar{\kappa}$ satisfying

c) $\bar{\beta}$ is inaccessible in $J^{\bar{A}}_{\bar{\kappa}}$
d) $\bar{\beta}$ is a limitpoint of $\bar{C}$
e) $\bar{\beta}$ is a limitpoint of $\{\kappa_i;\ i^* \leq i < \theta\}$

This is possible, since we can without a loss of generality assume that the set of ordinals satisfying c) and d) but not e) is bounded in $\bar{\kappa}$. If it were not, we could find arbitrarily large such $\bar{\beta}$ and an $i := i_{\bar{\beta}}$ such that c), d) holds and $\kappa^*_i := \sup\{\kappa_{i'};\ i' < i\} < \bar{\beta} \leq \kappa_i$. Then M_i would be sound above κ^*_i and $\bar{\beta}$ would be regular in $\widehat{M_i}$ since $\mathscr{P}(\bar{\beta}) \cap \widehat{M_i} = \mathscr{P}(\bar{\beta}) \cap \widehat{M_0} \subset \bar{\boldsymbol{K}}$ (the latter follows by acceptability since $\bar{\boldsymbol{K}}$ is a proper segment of $\widehat{M_0}$ and $\bar{\kappa}$ is a cardinal in $\widehat{M_0}$) and $\bar{\beta}$ is inaccessible in $\bar{\boldsymbol{K}}$. Hence $\widehat{M_i}$ would be a singularizing structure for $\bar{\beta}$. Then, setting $M := M_i$ we could proceed as in the Case 1. Set

$$i_{\bar{\beta}} := \text{ the least } i \text{ such that } \kappa_i \geq \bar{\beta}, \qquad M := M_{i_{\bar{\beta}}}.$$

By the above argument, $\bar{\beta}$ is regular in $\widehat{M}$. We shall again construct an extension $\tilde{\sigma}$ of $\sigma \restriction J^{\bar{E}}_{\bar{\beta}}$ to the whole $\widehat{M}$. If $\bar{\beta}$ is $\Sigma^{(n)}_1$ singularized over $\widehat{M}$ we can proceed as in the Case 1 and construct the canonical extension of $\sigma \restriction J^{\bar{E}}_{\bar{\beta}}$ to $\widehat{M}$. In fact, the situation is now simpler, since $\bar{\beta}$ is not overlapped by any extender, i.e., $\mathrm{o}^M(\mu) < \bar{\beta}$ for every $\mu < \bar{\beta}$. This means that the liftup will be a mouse which coiterates with $\mathbf{K}$ above β. Moreover, it is easy to see that $N = \boldsymbol{K}||\eta_\beta$ where $\tilde{\sigma} : \widehat{M} \to \widehat{N}$.

Now suppose that $\bar{\beta}$ is not $\Sigma^{(n)}_1$ singularized over $\widehat{M}$. Here we have to proceed more carefully since we now don't have enough information even to prove that the above canonical extension exists. We can, however, construct a different extension which will suffice for our later applications. Let $\bar{\tau}$ be the successor of $\bar{\beta}$ in $\widehat{M}$. We first coarsely extend $\sigma \restriction J^{\bar{E}}_{\bar{\beta}}$ to a $\sigma^* : J^{\bar{E}}_{\bar{\tau}} \to J^{E}_{\tau}$ and as a second step we extend σ^* to the whole $\widehat{M}$.

We first verify that there is a σ^* with the above properties. Since $\mathrm{cf}(\bar{\beta})$ is uncountable, we can coarsely extend $\sigma \restriction J^{\bar{E}}_{\bar{\beta}}$ to a Σ_0-preserving and cofinal map $\sigma^* : J^{\bar{E}}_{\bar{\tau}} \to J^{E'}_{\tau}$. The target structure of this map is a Σ_0-mouse, i.e., it is iterable in the following sense: the iterations in which we perform just coarse ultrapowers up to the first truncation point exist. Clearly, β is the largest cardinal in $J^{E'}_{\tau}$. Pick $\zeta < \tau$ such that the ultimate projectum of $J^{E'}_{\tau}||\zeta$ drops to β and an $a \in \mathscr{P}(\beta) \cap \Sigma^*(\widehat{J^{E'}_{\tau}||\zeta})$ which is not a member of $J^{E'}_{\tau}||\zeta$. By a coiteration argument similar to that above, $a \in \boldsymbol{K}$ since the coiteration is above β. Hence there is a ζ' such that $a \notin \widehat{\boldsymbol{K}||\zeta'}$ but $a \in \Sigma^*(\widehat{\boldsymbol{K}||\zeta'})$. Hence $J^{E'}_{\tau}||\zeta$ coiterates with $\boldsymbol{K}||\zeta'$ simply to a common mouse above β. Since both structures are sound, they must be equal which means that $J^{E'}_{\tau}||\zeta$ is a segment of $\mathbf{K}$. Since this holds for arbitrarily large $\zeta < \tau$, the whole $J^{E'}_{\tau}$ is a segment of $\mathbf{K}$.

Now we would like to extend σ^* to the whole $\widehat{M}$. We know that M is sound above $\bar{\beta}$ (thus M singularizes $\bar{\tau}$), hence it is sufficient to show that $\mathrm{cf}(\bar{\tau}) > \omega$. Note that $\bar{\boldsymbol{K}}, \widehat{M}$ contain the same subsets of β (the inclusion $\subset$ has been proved above, the one other is trivial). In particular, the successor of $\bar{\beta}$ in $\widehat{M}$ is equal to $\bar{\beta}^{+\bar{\boldsymbol{K}}}$. Thus, $\sigma(\bar{\tau})$ is a successor of β in $\mathbf{K}$ and the weak covering lemma guarantees that its cofinality is at least $\beta > \varepsilon$. It follows then from the construction of σ that $\mathrm{cf}(\bar{\tau}) = \varepsilon$.

Let $\tilde{\sigma} : \widehat{M} \to \widehat{N}$ be the canonical extension of σ^* to $\widehat{M}$. We know that $\widehat{N}$ is a sound mouse whose $(n+1)$-st projectum drops to β and $\tilde{\sigma}(p_M - \bar{\beta}) = p_N$. Similarly as above we show that $\widehat{N}$ is a segment of $\mathbf{K}$. As before, $\beta = \tilde{\sigma}(\bar{\beta})$ is regular in $\widehat{N}$. By Theorem 1.2, β is singular in $\mathbf{K}$ since it is a singular strong limit cardinal of cofinality ε but $\mathrm{o}^{\boldsymbol{K}}(\beta) < \varepsilon$ since $\beta \in C$. Taken together, η_β is defined and N is a proper segment of $\boldsymbol{K}||\eta_\beta$. In the sequel we show that $X \cap \beta$ can be recovered from N, J^A_β and C_β inside $\widehat{\boldsymbol{K}||\eta_\beta}$ which will complete the proof.

Fix a regular uncountable $\varepsilon^* < \varepsilon$. Recall that $i_{\bar{\beta}}$ is the first iteration index i for which $\kappa_i \geq \bar{\beta}$. Clearly, $i_{\bar{\beta}}$ is ε-cofinal. The map $i \mapsto \tilde{\kappa}_i = \sup\{\kappa_{i'};\ i' < i\}$ is continuous, hence the set

$$S^* = \{i < i_{\bar{\beta}};\ \mathrm{cf}(i) = \varepsilon^* \ \&\ \tilde{\kappa}_i \in \bar{C} \ \&\ \mathrm{card}(V_{\tilde{\kappa}_i}) = \tilde{\kappa}_i \text{ in } J^{\bar{A}}_{\bar{\kappa}}\}$$

is closed under limits of ε^*-sequences. The embedding σ is continuous at limitpoints of cofinality less than ε, hence $\kappa'_i := \sigma(\tilde{\kappa}_i)$ is a strong limit cardinal of cofinality ε^* whenever $i \in S^*$. By the weak covering lemma, $(\kappa'_i)^{+\boldsymbol{K}} = (\kappa'_i)^+$ and by Theorem 1.3,

$$2^{\kappa'_i} = (\kappa'_i)^+ \tag{3}$$

since $\mathrm{o}^{\boldsymbol{K}}(\kappa'_i) < \varepsilon$. The set A was constructed so that $\mathscr{P}(\kappa'_i) \subset J^A_{(\kappa'_i)^+}$ hence there is an ordinal $\zeta(i) < (\kappa'_i)^+$ such that $X \cap \kappa'_i$ is the $\zeta(i)$-th element of $J^A_{(\kappa'_i)^+}$ under $<_A$. We can pull this situation back via σ and obtain a $\bar{\zeta}(i) < \tilde{\kappa}_i^{+J^{\bar{A}}_{\bar{\kappa}}} = \tilde{\kappa}_i^{+\bar{\boldsymbol{K}}}$ which encodes $\bar{X} \cap \tilde{\kappa}_i$ with respect to $\bar{A}$ in the same way.

Recall that $\tilde{\kappa}_i^{+\bar{K}} = \tilde{\kappa}_i^{+M} = \tilde{\kappa}_i^{+M_{i+1}}$ and the iteration map $\pi_{i+1,0}$ from M_{i+1} to M_0 is above $\tilde{\kappa}_i^{+M_{i+1}}$. In particular, $\pi_{i+1,0}(\bar{\zeta}(i)) = \bar{\zeta}(i)$. Note also that every ordinal less than $\tilde{\kappa}_i^{+M_{i+1}}$ can be expressed in the form $\pi_{i,i+1}(f)(\tilde{\kappa}_i)$ for a function $f \in \widehat{M_i}$. For every $i \in S^*$ pick such an f_i to $\bar{\zeta}(i)$. Each such f_i has a preimage $\bar{f}_i \in \widehat{M}_{g(i)}$ under $\pi_{g(i),i}$ for some $g(i) < i$. By Fodor's theorem, there is a stationary set $S' \subset S^*$ on which g is bounded. So suppose without a loss of generality that $\bar{f}_i \in \widehat{M_j}$ for every $i \in S'$ where $j < \min(S')$. Since $\widehat{M_j}$ is sound above $\tilde{\kappa}_j$, there is a $\bar{\delta}(i) < \tilde{\kappa}_j < \bar{\beta}$ such that $\bar{f}_i = \tilde{h}^{n+1}_{M_j}(\bar{\delta}(i), p_{M_j})$. Taken together,

$$\bar{\zeta}(i) = \tilde{h}^{n+1}_{M_{i+1}}(\prec \bar{\delta}(i), \tilde{\kappa}_i \succ, p_{M_{i+1}}) = \tilde{h}^{n+1}_{M}(\prec \bar{\delta}(i), \tilde{\kappa}_i \succ, p_M).$$

We can now proceed as in the Case 1. Applying $\tilde{\sigma}$ we obtain

$$\zeta(i) = \tilde{h}^{n+1}_{N}(\prec \delta(i), \kappa_i^* \succ, p_N)$$

where $\delta(i) := \tilde{\sigma}(\bar{\delta}(i))$. The sequence $\langle \delta(i); i \in S' \rangle$ is bounded in β. Hence, in order to reconstruct $X \cap \beta$ we need an unbounded in β subset of $\{\kappa_i^*; i \in S'\}$, $\tilde{h}^{n+1}_N(-, p_N) \restriction \beta$ and J^A_β since the corresponding set of $\delta(i)$'s, being bounded by κ'_j, is an element of J^A_β. It only remains to eliminate the sequence of κ_i^*'s. Here we use the c.u.b. set C_β which we fixed at the beginning of the proof. C_β contains unboundedly many κ_i^*'s since its preimage $(\sigma^{-1})''C_\beta$ is c.u.b. in $\bar{\beta}$ and $\{\tilde{\kappa}_i; i \in S'\}$ is stationary in $\bar{\beta}$. The former follows from the fact that σ is continuous at the points of cofinality less than ε; to see the latter note that the set in question is a continuous image of the stationary set S'. Hence, if we choose an unbounded $y \subset C_\beta \cap \{\kappa_i^*; i \in S'\}$ then $X \cap \beta \in J^{E.A\cap\beta.y}_{\eta_\beta+1}$. This completes the proof of Lemma 2.1. ⊣

§3. Concluding remarks. The above construction of the $\diamond'_\kappa(\mathrm{cf} = \varepsilon)$ sequence goes through for every regular $\varepsilon > \omega_1$. The reason why we restrict ourselves to such an ε is that we used the Singular cardinal hypothesis at a strong limit cardinal of cofinality less than ε. If $\varepsilon = \omega_1$ the situation becomes slightly more complicated, since we have to use the SCH at ω-cofinal strong limit cardinals. Gitik and Mitchell proved in [MG96] that if β is as above and $2^\beta > \beta^+$ then one of the following holds

a) $\mathrm{o}^K(\beta) \geq \beta^{++}$
b) for every $n \in \omega$ there are cofinally many $\alpha < \beta$ with $\mathrm{o}^K(\alpha) \geq \alpha^{+n}$

It is also mentioned in [Gi96] that the clause b) cannot be eliminated. Taking this for granted, if we want to make our construction work for $\varepsilon = \omega_1$, we have to restrict ourselves to a smaller universe. The assumption that there is no inner model for $\mathrm{o}^K(\mu) = \mu^{++}$ clearly suffices. In fact, under this assumption we obtain a slightly stronger result in the sense that we can more precisely determine the ordinals which are of high Mitchell order in $\boldsymbol{K}$. We don't know whether this stronger version holds for $\varepsilon > \omega_1$ under the weaker assumption "$\neg 0^\P$".

PROPOSITION 3.1. *Suppose there is no inner model for* $\mathrm{o}(\mu) = \mu^{++}$. *Let* ε *be an uncountable regular cardinal and* κ *be Mahlo. If* $\diamond_\kappa(\mathrm{cf} = \varepsilon)$ *fails then there are stationarily many* $\beta < \kappa$ *of cofinality* ε *which become regular in* **K**.

To see that this proposition holds it suffices to analyze the proof from the previous section. The point here is that the present assumption automatically guarantees that (3) holds (in the context of the previous section we needed the fact that $o^K(\kappa_i') < \varepsilon$ in order to be able to apply SCH; to guarantee this we arranged the situation so that $\kappa_i' \in C$). Thus, if we define S_α as before, we can repeat the above proof to infer that $\langle S_\alpha \rangle$ is a $\diamondsuit_\kappa'(G)$-sequence provided that there are only nonstationarily many ordinals of cofinality ε which become regular in **K**. The only amendment we need to make in the initial settings is the following: this time we require that $o^K(\alpha) < \varepsilon$ for every $\alpha \in C$ of cofinality ε (thus, not necessarily for every α). The rest of the proof goes through as before.

The above proposition also yields a lower bound for the consistency strength of the failure of $\diamondsuit_\kappa(\mathrm{cf} = \varepsilon)$ for uncountable ε and thus of the failure of $\diamondsuit_\kappa(\text{singulars})$, as follows from normality of the $\diamondsuit$-ideal (the function $\beta \mapsto \mathrm{cf}(\beta)$ is regressive at singulars). Note also that our lower bound is considerably weaker than Woodin's assumption under which he killed $\diamondsuit_\kappa$. This is caused by the fact that we need β to remain singular in **K** in order to keep the size of S_β low. A natural question thus arises:

(4) Can the above lower bound be improved?

Woodin's model also drastically violates the GCH below κ. On the other hand, our method does not give us a better lower bound if we make an additional assumption that the GCH holds below κ – it only enables us to avoid the use of the SCH, so we obtain a result similar to Proposition 3.1.

(5) ($\neg 0^\P$ + GCH below κ) If κ is Mahlo and ε is regular and greater than ω_1 then the statement of Proposition 3.1 holds.

It is, however, not clear whether $\diamondsuit_\kappa$ already follows from the GCH below κ. Hence, we can ask:

(6) Is ($\neg\diamondsuit_\kappa$ + GCH below κ) consistent?

Finally, we are not able to say anything about $\diamondsuit_\kappa(\mathrm{cf} = \omega)$ since our methods break down at several places – even the lifting arguments do not go through in this case. Hence, it would be also of interest whether $\neg\diamondsuit_\kappa(\mathrm{cf} = \omega)$ can be forced without use of a large cardinal assumption.

We close this section with an observation (due to Jensen) that the construction from Section 2 need not necessarily yield a $\diamondsuit_\kappa^*$-sequence if κ is not Mahlo. More precisely, the construction will not yield such a sequence if κ is the first inaccessible or, more generally, if κ has only those large cardinal properties which can be fully characterized over V_κ in a first order fashion. This follows from the fact that every single $\diamondsuit_\kappa$-sequence (and, consequently, $\diamondsuit_\kappa^*$-sequence) can be killed by a forcing which preserves such large cardinal properties. (For technical reasons we formulate the following proposition and its proof in terms of $\diamondsuit_\kappa$-sequences; the proof of a corresponding proposition for $\diamondsuit_\kappa^*$-sequences is similar, but more technical.)

PROPOSITION 3.2. *Let κ be inaccessible and $\mathscr{B} = \langle B_\alpha;\ \alpha < \kappa \rangle$ be a $\diamondsuit_\kappa$-sequence. There is a $(< \kappa)$-distributive, κ^+-c.c. forcing notion $\mathbb{P} = \mathbb{P}(\mathscr{B})$ satisfying the following statement: For any $\mathbb{P}$-generic filter H there is a set $X \subset \kappa$ and a c.u.b. set $C \subset \kappa$ in $V[H]$ such that the pair $\langle X, C \rangle$ witnesses that $\langle B_\alpha \rangle_\alpha$ fails to be a $\diamondsuit_\kappa$-sequence in $V[H]$, i.e., $X \cap \alpha \neq B_\alpha$ for every $\alpha \in C$.*

By the construction from Section 2, the sequence $\mathcal{S} = \langle S_\alpha \rangle_\alpha$ is definable over the structure $\langle V_\kappa, A, E^K \upharpoonright \kappa, \mathscr{C} \rangle$ where $\mathscr{C} = \langle C_\alpha \rangle_\alpha$ is the sequence of c.u.b. sets we fixed at the beginning of the construction. It is also clear that the set A is definable over V_κ. Finally, by the argument of Kunen, starting from $\mathcal{S}$ we can define (over V_κ) a $\diamondsuit_\kappa$-sequence $\mathscr{B}$ using only parameters from V_κ.

Now force with $\mathbb{P} = \mathbb{P}(\mathscr{B})$. Since $\mathbb{P}$ is $(< \kappa)$-distributive, $V_\kappa = (V_\kappa)^{V[H]}$. Then $A = A^{V[H]}$ and $\boldsymbol{K} = \boldsymbol{K}^{V[H]}$, so by the construction of Section 2, $\mathcal{S} = \mathcal{S}^{V[H]}$. This implies $\mathscr{B} = \mathscr{B}^{V[H]}$. Since κ is inaccessible, the large cardinal properties of κ mentioned in the paragraph preceding Proposition 3.2 remain preserved in $V[H]$, but $\mathscr{B}^{V[H]}$ fails to be a $\diamondsuit_\kappa$-sequence. Hence, the construction of Section 2 does not yield a $\diamondsuit^*_\kappa$-sequence in $V[H]$. It also follows that it is very unlikely that there is a constructive proof of $\diamondsuit_\kappa$ at κ as above.

Proof of Proposition 3.2. Let $\mathbb{P} = \mathbb{P}(\mathscr{B})$ be the set of all pairs $\langle x, c \rangle$ such that

- c is a closed bounded subset of κ and $x \subset \max(c)$,
- $x \cap \alpha \neq B_\alpha$ for every $\alpha \in c$,

ordered as follows

$$\langle x', c' \rangle \leq \langle x, c \rangle \quad \text{if and only if} \quad \begin{array}{l} x', \text{ respecively } c' \text{ is an end-} \\ \text{extension of } x, \text{ respectively } c. \end{array}$$

Clearly, $\mathbb{P}$ is the collection of all approximations of a pair $\langle X, C \rangle$ we are looking for. Given a $\mathbb{P}$-generic filter H, set

$$X = \bigcup \{x;\ \langle x, c \rangle \in H \text{ for some } c\}$$
$$C = \bigcup \{c;\ \langle x, c \rangle \in H \text{ for some } x\};$$

it follows easily that $\langle X, C \rangle$ is as desired. (Note that X, C are unbounded in κ since each $\{\langle x, c \rangle \in \mathbb{P};\ \max(c) \geq \beta\}$ is trivially dense in $\mathbb{P}$.)

We shall verify the $(< \kappa)$-distributivity of $\mathbb{P}$; the rest follows by standard arguments. Let $\lambda < \kappa$ and $\langle D_\xi;\ \xi < \lambda \rangle$ be a sequence of open dense subsets of $\mathbb{P}$. We have to show that their intersection is nonempty. Let $\langle x^\gamma_{-1}, c^\gamma_{-1} \rangle$ be mutually incompatible conditions from $\mathbb{P}$ for $\gamma < \lambda^+$. We can, for instance, take $c^\gamma_{-1} = \{\lambda\}$ and pick x^γ_{-1} to be mutually distinct subsets of λ such that $x^\gamma_{-1} \neq B_\lambda$. For $\xi < \lambda$, define $\langle x^\gamma_\xi, c^\gamma_\xi \rangle$, R_ξ, R^*_ξ and $\langle \bar{x}^\gamma_\xi, \bar{c}^\gamma_\xi \rangle$ for $\gamma \in R_\xi$ inductively as follows.

- $R^*_\xi = \{\gamma < \lambda^+;\ \langle x^\gamma_\xi, c^\gamma_\xi \rangle \text{ is defined }\}$
- $R_\xi = \bigcap_{\zeta < \xi} R^*_\zeta$
- $\langle x^\gamma_{\xi+1}, c^\gamma_{\xi+1} \rangle \in D_{\xi+1}$ is chosen such that $\langle x^\gamma_{\xi+1}, c^\gamma_{\xi+1} \rangle \leq \langle x^\gamma_\xi, c^\gamma_\xi \rangle$ for $\gamma \in R^*_\xi$ and $\max(c^\gamma_{\xi+1}) > \max(c^{\gamma'}_\xi)$ for all $\gamma' \in R^*_\xi$
- $\bar{x}^\gamma_\xi = \bigcup_{\zeta < \xi} x^\gamma_\zeta$, $\bar{c}^\gamma_\xi = \operatorname{cl}\left(\bigcup_{\zeta < \gamma} c^\gamma_\zeta \right)$ for limit ξ and $\gamma \in R_\xi$
- $\langle x^\gamma_\xi, c^\gamma_\xi \rangle \in D_\xi$ is chosen such that $\langle x^\gamma_\xi, c^\gamma_\xi \rangle \leq \langle \bar{x}^\gamma_\xi, \bar{c}^\gamma_\xi \rangle$ for limit ξ and $\gamma \in R^*_\xi$

Then $R^*_{\xi+1} = R^*_\xi$ and $R^*_\xi - R_\xi$ is a singleton or empty for every ξ. To see the latter assume without a loss of generality that ξ is limit. For $\gamma \in R_\xi$, $\langle \bar{x}^\gamma_\xi, \bar{c}^\gamma_\xi \rangle \in \mathbb{P}$ if and only if $\bar{x}^\gamma_\xi \neq B_\alpha$ where $\alpha := \sup(c^\gamma_\xi)$ since all the remaining requirements for membership in $\mathbb{P}$ are satisfied automatically. By construction, the sets $\bar{x}^\gamma_\xi$ are mutually distinct, so there can be at most one $\gamma := \gamma_\xi$ such that $\bar{x}^\gamma_\xi$ is equal to B_α. Thus, either $R^*_\xi = R_\xi$ or else $R^*_\xi = R_\xi - \{\gamma_\xi\}$. It follows easily that $R^*_\xi = \lambda^+ - \{\gamma_\zeta;\ \zeta \leq \xi\}$, so its cardinality is at least λ^+. In particular, there are λ^+-many conditions $\langle x^\gamma_\lambda, c^\gamma_\lambda\rangle$ where $\gamma \in R^*_\lambda$; each such condition lies, by construction, in the intersection of all D_ξ. This completes the proof. $\dashv$(Proposition 3.2)

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