DODD PARAMETERS AND λ-INDEXING OF EXTENDERS

MARTIN ZEMAN

Abstract. We study generalizations of Dodd parameters and establish their fine structural properties in Jensen extender models with λ-indexing. These properties are one of the key tools in various combinatorial constructions, such as constructions of square sequences and morasses.

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Typical combinatorial constructions in extender models, such as constructions of square sequences, build on the analysis of Dodd parameters of certain active levels of the models. Dodd parameters come into play whenever we try to construct combinatorial objects that involve some form of "coherency". The reason why they have to be considered is that the standard fine structural characteristics of premice are typically computed relative to a certain additional constant. This impairs the uniformity of the description of the combinatorial objects we are constructing, and thereby complicates the verification of the "coherency" conditions.

In this paper we will deal with Jensen extender models with the so-called λ-indexing introduced in [2]. The relevant background and notation can also be found in [17]. We establish the relationship between the Dodd parameter and the standard parameter for type B levels of extender models and prove the solidity theorem for the Dodd parameters. These results are formulated in Theorems 1.1 and 1.2. The actual formulations of both theorems are adjusted so that they can be immediately applied in [12] and [13], where the facts about the Dodd paremeters are used as a black box. Dodd parameters were originally introduced in [1] in connection with constructions of models for strong cardinals. For extender models of Mitchell-Steel type [7], the Dodd solidity was established by Steel [9, 15], and was used in a substantial way in combinatorial constructions in [9, 10] and in the

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proof of the weak covering lemma for Steel's core model [5, 6]. Dodd parameters were also considered in [4] in constructions of extender models. The results of the current paper differ from those in [9] and [15] in several respects. The most obvious difference is that, here we deal with models with λ-indexing, which means that the relationship between the standard paramter and the Dodd parameter is different from that in models with Mitchell-Steel indexing. In models with λ-indexing, the fact that the Dodd parameter is distinct from the standard parameter is already a large cardinal axiom beyond a *super*strong cardinal. In models with Mitchell-Steel indexing, it is a relatively modest large cardinal axiom well below one strong cardinal. We show that, in the models with λ-indexing, there is a canonical uniform way of conversion between the standard parameter and the Dodd parameter. Moreover, Proposition 1.3 shows that, under a certain relatively modest smallness condition, this conversion is particularly simple (although Proposition 1.3 does not seem to be relevant if we are interested in combinatorial constructions in their full generality). Secondly, in this paper we are considering slightly generalized versions of Dodd parameters; such generalizations are not needed for the basic construction of a square sequence, but they occur in more delicate constructions of square sequences with some amount of condensational coherency, and even in the construction of an ordinary Gap-1 Morass [13].

The only "smallnes condition" used in the fine structure theory of models with λ-indexing is iterability. This means that, modulo iterability, our fine structure theory is developed for models satisfying any large cardinal axiom that is witnessed by extenders in the sense of [17]. (That is, extenders whose sup of generators does not exceed the image of the critical point.) Let us also mention that the strategy of the proof of our main results is completely different from that used in the proof of Dodd solidity in Mitchell-Steel models in [15]. This was caused by our effort to avoid any *direct* reference to iterability in our arguments by substituting comparison arguments by applications of the condensation lemma and the solidity theorem for the standard parameters. However, we did not succeed completely in this effort and the paper does contain a version of a comparison argument after all.

We briefly recall some notation and terminology from [17]. Given two primitive recursively closed ordinals $\kappa < \lambda$, an extender at (κ, λ) is a map $F : V \to \mathcal{P}(\lambda)$

with $V \subset \mathcal{P}(\kappa)$ which preserves primitive recursive definitions; we of course require that V is closed under such definitions. The ordinal κ is called the *critical point* of F and denoted by $\mathrm{cr}(F)$, and λ is called the *length* of F and denoted by $\lambda(F)$. If $\bar\lambda < \lambda$ is primitive recursively closed, we define $F \,|\, \bar\lambda : V \to \mathcal{P}(\bar\lambda)$ to be the map $x \mapsto F(x) \cap \bar\lambda$; this map is obviously an extender. Typically, V is a set of the form $\mathcal{P}(\kappa) \cap M$ where M is some acceptable J-structure. So we can form an ultrapower $\mathrm{Ult}(M, F)$ of M by F. An ordinal $\bar\lambda \in (\kappa, \lambda)$ is a *cutpoint* of F just in case that for every $x \in \mathrm{dom}(F)$ that codes a function from κ into κ (in the obvious way based on Gödel pairing $\prec, \succ$), $F(x) \cap \bar\lambda$ codes a *total* function from $\bar\lambda$ into $\bar\lambda$. More precisely, $\prec\alpha, \zeta\succ \in F(x)$ implies $\zeta < \bar\lambda$ for every $\alpha < \bar\lambda$. In the language of ultrapowers this says that $\pi(f)(\alpha) < \bar\lambda$ for every function $f : \kappa \to \kappa$ from the model F is applied to and every $\alpha < \bar\lambda$; here π is the associated ultrapower map. A J-structure M is *coherent* just in case that M is of the form $\langle J^A_\nu, F\rangle$ and $J^A_\nu = \mathrm{Ult}(J^A_\vartheta, F)$ where $\kappa = \mathrm{cr}(F)$ is the largest cardinal in J^A_ϑ and $\lambda(F)$ is the image of κ under the associated ultrapower map, and thus the largest cardinal in J^A_ν. The extender F measures all subsets of κ that are in J^A_ϑ, but not necessarily all subsets in $\mathcal{P}(\kappa) \cap M$, since the option $\vartheta < \kappa^{+M}$ is allowed. Thus, F need not be weakly amenable with respect to M, but M is always an amenable structure. If M is a coherent structue, we often write $\lambda(M)$ for $\lambda(F)$.

If $M = \langle J^E_\nu, F\rangle$ is a potential premouse in the sense of [17], Chapter 9 (see also [12]), M is of type A if F has no cutpoints, of type B if the set of all cutpoints of F is nonempty and bounded in $\lambda(M) \overset{\text{def}}{=} \lambda(F)$, and of type C if the set of all cutpoints of F is unbounded in $\lambda(M)$. A *premouse* is a potential premouse that satisfies the initial segment condition, which will be discussed in a little while. If M is a type B potential premice, the largest cutpoint of F is denoted by λ^*_M. Letting $\kappa = \mathrm{cr}(F)$ and $\tau = \kappa^{+M}$, the height of $\mathrm{Ult}(J^E_\tau, F \,|\, \lambda^*_M)$ is denoted by γ_M. In the genaral fine structure theory for premice, the *language for coherent structures* contains just predicate symbols denoting E and F, and the *language for premice* contains an additional constant symbol denoting $F \,|\, \lambda^*_M$. In the case of type A or C premice, there is no difference between the two languages. The initial segment condition requires that $F \,|\, \bar\lambda \in M$ whenever $\bar\lambda$ is a cutpoint of F. In the case of type B premice, this is equivalent to the condition $F \,|\, \lambda^*_M \in M$. This is in general weaker than the requirement that $F \,|\, \lambda_M^*$ is on the E-sequence, which considerably simplifies proofs of iterability in the constructions of extender models. However, if M *is* weakly iterable in the sense of [17] (see below), then γ_M is the index of $F \,|\, \lambda_M^*$. For the purpose of doing combinatorics, it is more convenient to use γ_M instead of $F \,|\, \lambda_M^*$ as the additional parameter, so our expanded language of premice will contain a constant symbol $\dot{\gamma}$ denoting γ_M. This makes no difference from the fine structural point of view, since $F \,|\, \lambda_M^*$ and γ_M are lightface $\Sigma_1(M)$-definable from each other in the language for coherent structures. For type B premice M, we shall stick to the following convention. The Σ_1-Skolem function computed in the language for premice will be denoted by h_M. The Σ_1-Skolem function computed in the language for coherent structures, i.e. in the language with *no additional constants,* will be denoted by h_M^*. For type A and C premice, there is obviously no difference between h_M and h_M^*, since the corresponding language for premice does not have any additional constant symbol. (Or, alternatively, if we want to have a uniform definition of the language for premice, we might introduce an additional constant symbol $\dot{\gamma}$ and interpret it always as 0. Obviously, the difference between the two languages will be merely cosmetic.) Any cutpoint of F is a limit cardinal in M; since $J_\theta^E = V_\theta^M$ for any such cardinal θ, the coherency condition combined with the initial segment condition guarantee that that $F \,|\, \lambda$ is a superstrong extender in M whenever λ is a cutpoint of F. Thus, the presence of cutpoints is a very strong large cardinal hypothesis.

Let us now state the Condensation Lemma from [17] we will refer to throughout the paper. This lemma is true for *weakly* iterable premice. Recall that a premouse M is weakly iterable just in case every countable premouse that is elementarily embeddable into M is (ω_1+1, ω_1)-iterable in the sense of [17], Chapter 9. Such premice are automatically solid.

Lemma 0.1. *Let $\bar{M}$ and M be premice of the same type where M is weakly iterable and let $\sigma : \bar{M} \to M$ be an embedding which is both cardinal preserving and $\Sigma_0^{(n)}$-preserving (with respect to the language of premice!), and such that $\sigma \restriction \omega\varrho_{\bar{M}}^{n+1} = \mathrm{id}$.*

Then $\bar{M}$ is weakly iterable and, consequently, solid. Furthermore, if $\sigma \restriction \nu = \mathrm{id}$ and $\bar{M}$ is sound above ν, then one of the following holds:

(a) *$\bar{M}$ is the core of M above ν and σ is the associated core map.*

(b) *$\bar{M}$ is a proper initial segment of M.*

(c) *$\bar{M} = \mathrm{Ult}^*(M\,||\,\zeta, E^M_\alpha)$ for some ζ and α such that $\nu \le \zeta < \mathrm{ht}(M)$, $\alpha \le \omega\zeta$ and $\nu = \kappa^{+M\,||\,\zeta}$ where $\kappa = \mathrm{cr}(E^M_\alpha)$; moreover, ζ is maximal with these properties. Also, E^M_α has a single generator κ.*

(d) *$\bar{M}$ is a proper initial segment of $\mathrm{Ult}(M, E^M_\nu)$.*

If M satisfies the assumptions of the Condensation Lemma and is additionally *sound above ν*, then (a) can be reformulated as follows:

(a′) $\bar{M} = M$ and $\sigma = \mathrm{id}$.

Throughout the paper, we will try to avoid any *direct* reference to iterability whenever possible and substitute it by an application of the Condensation Lemma. This approach is useful in many situations, as unlike the weak iterability, solidity is a *first* order property of premice.

1. Dodd Parameters

We first introduce the parameters d^α_M, which are generalizations of the Dodd parameters. Our context is adjusted to the needs of combinatorial constructions of e.g. [12, 13, 14] and [16]. Thus, the traditional notions of the Dodd projectum and Dodd parameter (for the definitios, see e.g. [12] or [9]) will actually not be considered. Recall that the canonical well-ordering of all finite sets of ordinals is denoted by $<^*$.

Definition. *Let M be an active premouse and $\kappa = \mathrm{cr}(E^M_{\mathrm{top}})$ and let α be an ordinal satisfying*

$$\kappa^{+M} \le \alpha.$$

The parameter d^α_M is the $<^*$-least finite set of ordinals d such that $M = h^*_M(\alpha \cup \{d\})$, if defined. We write d_M for $d^{\omega\varrho^1_M}_M$ whenever $\kappa < \omega\varrho^1_M$.

Obviously, d^α_M need not be defined for each M and α, and its existence guarantees that $\omega\varrho^1_M \le \alpha$. However, d^α_M *is* defined, granting that $\omega\varrho^1_M \le \alpha$ and M is 1-sound above α, which is precisely the situation we are interested in. It is easy to see that if $\bar\alpha \le \alpha$ and $d^{\bar\alpha}_M$ is defined then so is d^α_M and $d^\alpha_M = d^{\bar\alpha}_M - \alpha$. For 1-sound M with $\kappa < \omega\varrho^1_M$, the Dodd parameter in the traditional sense is just d_M. Clearly, $d^\alpha_M = \varnothing$ if M is a type C premouse. If M is a 1-sound type A premouse then easily

$d_M^\alpha = p_M^1 - \alpha$, and the notions of Dodd solidity and solidity coincide. Thus, the notion is non-trivial only for type B premice.

Dodd solidity witnesses are defined in the obvious way. Given an ordinal $\beta \geq \alpha$ and a parameter $p \in [\mathbf{On} \cap M]^{<\omega}$, the standard Dodd solidity witness ${}^*W_M^{\beta,p}$ for β with respect to M and p is the transitive collapse of the hull $h_M^*(\beta \cup \{p - (\beta+1)\})$. Thus, the only difference between the standard solidity witness in the usual sense $W_M^{\beta,p}$ and the standard Dodd solidity witness ${}^*W_M^{\beta,p}$ is that ${}^*W_M^{\beta,p}$ is computed with no reference to γ_M. Similarly, a pair $\langle Q, r\rangle$ is a generalized Dodd solidity witness for β with respect to M and p just in case that Q is transitive, $r \in Q$, and for every Σ_1-formula in the language for coherent structures $\varphi(x, v_1, \dots, v_\ell)$ and every $\xi_1, \dots, \xi_\ell < \beta$ we have

$$M \models \varphi(p - (\beta+1), \xi_1, \dots, \xi_\ell) \;\longrightarrow\; Q \models \varphi(r, \xi_1, \dots, \xi_\ell)$$

The property of being a generalized Dodd solidity witness is Π_1 in the language for coherent structures. We will often make use of the following fact:

(1) The existence of generalized witnesses is equivalent to the existence of standard witnesses.

This is expressed in a somewhat sloppy way, but the meaning of (1) is obvious. It says that if $\beta \in \mathbf{On} \cap M$ and $p \in [\mathbf{On} \cap M]^{<\omega}$ are arbitrary, then $W_M^{\beta,p} \in M$ if and only if M contains some generalized witness for β with respect to M and p, and the corresponding fact is true for Dodd solidity witnesses as well. In [17], this is formulated as Lemma 1.12.3. The lemma assumes that p is a good parameter and $\beta \in p$. However, it is easy to see that this assumption is superfluous: Recall that the heart of the proof of Lemma 1.12.3 is to show the "if" implication. In that proof, the assumptions on β and p are used to show that M has a cardinal larger than β. It is argued that β is the critical point of the witness map and, consequently, the image of β under the witness map is a cardinal in M. But if β and p are arbitrary, then either the witness map has a critical point $\geq \beta$, or it does not have a critical point at all. In the former case, we still get a cardinal in M above β, and exactly as in the proof of Lemma 1.12.3 conclude that $W_M^{\beta,p} \in M$. In the latter case, $W_M^{\beta,p}$ is an initial segment of M. But it must be a *proper* initial segment, as $W_M^{\beta,p}$ is embeddable into some generalized witness that is an element of M.

Definition. *We say that M is* Dodd solid above α *just in case d^α_M is defined and ${}^*W^{\beta,d^\alpha_M}_M \in M$ for every $\beta \in d^\alpha_M$. This is equivalent to the requirement that each $\beta \in d^\alpha_M$ has a generalized Dodd solidity witness with respect to M and d^α_M that is an element of M.*

The reader familiar with the notion of Dodd solidity in the traditional sense will immediately notice that the above definition does not contain any requirement corresponding to the requirement in the traditional definition that the restriction of E^M_{top} to $(\bar\alpha \cup d^\alpha_M) - (\bar\alpha + 1)$ is an element of M. for any $\bar\alpha$ strictly smaller than the Dodd projectum. The reason why we do not need this requirement is that we only need to consider situations where the Dodd projectum agrees with the first projectum, and so the corresponding clause is automatically satisfied, as follows from the elementary properties of projecta.

In all relevant cases, the Dodd parameter d^α_M can be computed from the standard parameter of M and an additional finite set of ordinals, which we denote by e^α_M. This is, however, not immediately clear from the definition of e^α_M.

Definition. *Let M be a type B potential premouse and $\alpha \in$ **On**. The $<^*$-least finite set of ordinals e such that $\gamma_M \in h^*_M(\alpha \cup \{p_M - \alpha, e\})$ is denoted by e^α_M.*

Obviously, e^α_M is always defined and $e^\alpha_M \subset \gamma_M + 1$. The main result of this paper is summarized in the following two theorems.

Theorem 1.1. *Let M be a weakly iterable type B premouse and let $\kappa = \text{cr}(E^M_{\text{top}})$. Assume further that M is sound above α where $\alpha \geq \max\{\kappa^{+M}, \omega\varrho^1_M\}$. Then d^α_M is defined and:*

(a) $d^\alpha_M = (p_M - \alpha) \cup e^\alpha_M$.
(b) *M is Dodd solid above α.*

Theorem 1.2. *Let M be a weakly iterable type B premouse and let $\alpha \geq \kappa^{+M}$ where $\kappa = \text{cr}(E^M_{\text{top}})$. Assume that d is a parameter satisfying:*

- $d \cap \alpha = \varnothing$.
- $M = h^*_M(\alpha \cup \{d\})$.
- *Every $\beta \in d$ has a generalized Dodd solidity witness with respect to M and d that is an element of M.*

Then

(a) *$d = d^\alpha_M$;*

(b) *$d^\alpha_M = (p_M - \alpha) \cup e^\alpha_M$;*

(c) *M is sound above α.*

Under a certain smallness condition, the conversion between d^α_M and $p_M - \alpha$ simplifies as follows.

Proposition 1.3. *Assume there is no inner model with a cardinal which is both subcompact and superstrong. Then (a) in Theorem 1.1 can be reformulated as follows: $d^\alpha_M = p_M - \alpha$ or $d^\alpha_M = (p_M \cup \{\lambda^*_M\}) - \alpha$ or $d^\alpha_M = (p_M \cup \{\gamma_M\}) - \alpha$. Moreover, the second possibility can only occur if $\lambda^*_M = \max(d^\alpha_M)$.*

Subcompactness is a large cardinal property slightly stronger than 1-extendibility. More precisely, if κ is subcompact then there are many 1-extendible cardinals below κ, see [11] or [12] or for a definition. It can be shown [18] that either of the three possibilities named in the above proposition can occur, and actually well below that smallness condition. We don't know whether the smallness condition can be omitted, but we believe it cannot. Let us also note that there are obvious reformulations for type A premice. As has been mentioned above, Theorem 1.1 does not say anyghing new in this case. In Theorem 1.2, only (a) needs an argument; (b) and (c) then follow directly from (a). But (a) is nothing else but Lemma 1.12.5 from [17] in this case.

We now turn to the proof of the two theorems; as a by-product we will get a proof of Proposition 1.3. Throughout the proof of Theorem 1.1, we will assume that $\alpha < \lambda^*_M$. Roughly speaking, we can do this without loss of generality, since there is less work to be done for larger α. A closer examination of the entire situation reveals that the only more serious issue connected with this assumption occurs when $\gamma_M \in e^\alpha_M$, but we will explain at the beginning of the proof of Lemma 1.6 why the assumption is harmless.

We fix the following notation. E will be the extender sequence of M, F will be the top extender of M, $\kappa = \mathrm{cr}(F)$ and $\tau = \kappa^{+M}$. Furthermore, if π is the ultrapower map associated with $\mathrm{Ult}(J^E_\tau, F)$ and $f : {}^n\kappa \to \kappa$ or $f : {}^n\kappa \to \mathcal{P}(\kappa)$ is in M (here n is a natural number), we will often write $F(f)$ for $\pi(f)$. This is possible, since f is uniformly recursively encodable into a subset of κ in either of these cases.

Both theorems are proved by induction on the height of M. Thus, our argument will make use of the following induction hypothesis:

(2) If M^* is a weakly iterable premouse with $\mathrm{ht}(M^*) < \mathrm{ht}(M)$ then Theorems 1.1 and 1.2 hold of M^*.

Proof of Theorem 1.1. Since M is sound above α, the following holds of $(p_M - \alpha) \cup e^\alpha_M$:

$$h^*_M(\alpha \cup \{p_M - \alpha, e^\alpha_M\}) \supset h^*_M(\alpha \cup \{p_M - \alpha, \gamma_M\}) = h_M(\alpha \cup \{p_M - \alpha\}) = M.$$

This computation does not make use of the minimality of e^α_M. It follows that d^α_M is defined and $d^\alpha_M \leq^* (p_M - \alpha) \cup e^\alpha_M$. To see the equality, it suffices to verify:

$$\text{(3)} \qquad {}^*W_M^{\beta,(p_M-\alpha)\cup e^\alpha_M} \in M \text{ whenever } \beta \in (p_M - \alpha) \cup e^\alpha_M.$$

Indeed, the assumption $d^\alpha_M <^* (p_M - \alpha) \cup e^\alpha_M$ would lead to a contradiction as follows: Fix $\beta \in [(p_M-\alpha)\cup e^\alpha_M]-d^\alpha_M$ such that $[(p_M-\alpha)\cup e^\alpha_M]-(\beta+1) = d^\alpha_M-(\beta+1)$. Let $W = {}^*W_M^{\beta,(p_M-\alpha)\cup e^\alpha_M}$ and $\sigma : W \to M$ be the canonical map; this map is Σ_1-preserving in the language for coherent structures. Obviously $\alpha \cup \{d^\alpha_M\} \subset \mathrm{rng}(\sigma)$, so $\mathrm{rng}(\sigma) = M$. Consequently, $W = M$ and $\sigma = \mathrm{id}$. But (3) guarantees that $W \in M$, a contradiction.

The verification of (3) will take the most part of the paper. As usual, we will start with the largest $\beta \in (p_M - \alpha) \cup e^\alpha_M$ and gradually move downward. The assumptions of Theorem 1.1 require merely that M is sound above α. However, we can without loss of generality assume:

(4) M is *fully* sound.

Although most of the argument will go through without this assumption, the two comparison arguments we describe below do make use of it.

To see (4), notice that if $M^* = \mathrm{core}(M)$, $\sigma_c : M^* \to M$ is the core map and $\alpha^* = (\sigma_c^{-1})''\alpha$, then $\sigma_c(\gamma_{M^*}) = \gamma_M$ and $\sigma(e^{\alpha^*}_{M^*}) = e^\alpha_M$. This follows from the preservation properties of σ_c by a straightforward reflection argument: Obviously $\gamma_M \in h^*_M(\alpha \cup \{p_M, \sigma_c(e^{\alpha^*}_{M^*})\})$, as $\gamma_{M^*} \in h^*_M(\alpha^* \cup \{p_{M^*}, e^{\alpha^*}_{M^*}\})$. So $e^\alpha_M \leq^* \sigma_c(e^{\alpha^*}_{M^*})$. On the other hand, if this inequality were strict, M would satisfy the statement

$$(\exists e <^* \sigma_c(e^{\alpha^*}_{M^*}))(\exists x \subset \sigma(\alpha^*))(x \text{ is finite } \;\&\; \gamma_M = h^*_M(\xi, p_M \cup e)).$$

The preservation properties of σ_c would then yield $\gamma_{M^*} = h^*_{M^*}(x, p_{M^*} \cup e^*)$ for some $e^* <^* e^{\alpha^*}_{M^*}$ and some finite $x \subset \alpha^*$, a contradiction. Once we have proved that ${}^*W^{\beta,(p_{M^*}-\alpha^*)\cup e^{\alpha^*}_{M^*}}_{M^*} \in M^*$ for $\beta \in p_{M^*} \cup e^{\alpha^*}_{M^*}$, the preservation properties of σ_c will imply that $\sigma_c({}^*W^{\beta,(p_{M^*}-\alpha^*)\cup e^{\alpha^*}_{M^*}}_{M^*})$ is a generalized Dodd solidity witness for $\sigma_c(\beta)$ with respect to M and $p_M \cup e^\alpha_M$, and thereby prove Theorem 1.1.

Lemma 1.4. *${}^*W^{\beta,(p_M-\alpha)\cup e^\alpha_M}_M \in M$ whenever $\beta \in (p_M - \alpha) \cup e^\alpha_M$ and $\beta > \gamma_M$.*

Proof. Since $e^\alpha_M \subset \gamma_M + 1$, we have

$$(p_M - \alpha) \cup e^\alpha_M - (\gamma_M + 1) = p_M - (\gamma_M + 1).$$

Then easily ${}^*W^{\beta,(p_M-\alpha)\cup e^\alpha_M}_M = W^{\beta,(p_M-\alpha)\cup e^\alpha_M}_M$, and this object is an element of M by the solidity of M. □(Lemma 1.4)

Notice that the above lemma in fact establishes the agreement between d^α_M and p_M above γ_M. We now focus on the interval $[\lambda^*_M, \gamma_M]$. Our next observation is:

Lemma 1.5. *If $\lambda^*_M \leq \beta < \lambda(M)$ then $\lambda^*_M \in h^*_M(\alpha \cup \{\beta\})$ and $h^*_M(\alpha \cup \{\beta\}) \cap \gamma_M$ is cofinal in γ_M.*

Proof. Let $\bar{M}$ be the transitive collapse of $h^*_M(\alpha \cup \{\beta\})$ and $\sigma : \bar{M} \to M$ be the uncollapsing map. Then $\bar{\beta} \overset{\text{def}}{=} \sigma^{-1}(\beta) < \lambda(\bar{M})$ is an upper bound for $(\sigma^{-1})''\lambda^*_M$, so $\bar{\lambda}^* \overset{\text{def}}{=} (\sigma^{-1})''\lambda^*_M \leq \bar{\beta} < \lambda(\bar{M})$. Now $\bar{\lambda}^*$ is a cutpoint of $\bar{F}$, the top extender of $\bar{M}$. Indeed, if $f : \kappa \to \kappa$ is in $\bar{M}$ and $\delta < \bar{\lambda}^*$ then $\sigma(\delta) < \lambda^*_M$, so $\sigma(\bar{F}(f)(\delta)) = F(f)(\sigma(\delta)) < \lambda^*_M$, and, consequently, $\bar{F}(f)(\delta)$ is an element of $(\sigma^{-1})''\lambda^*_M = \bar{\lambda}^*$. The way we defined $\bar{\lambda}^*$ also guarantees that $\lambda^*_M \leq \sigma(\bar{\lambda}^*) \leq \beta$. The latter inequality here ensures that $\sigma(\bar{\lambda}^*) < \lambda(M)$. So $\sigma(\bar{\lambda}^*)$ is a cutpoint of M, since being a cutpoint is a Π_1-property. But it cannot be strictly larger than λ^*_M, as λ^*_M is the largest cutpoint of M. So necessarily $\sigma(\bar{\lambda}^*) = \lambda^*_M$, which proves the first part of the lemma.

Towards the second part of the lemma, recall that the ultrapower embedding $\pi^* : J^E_\tau \to J^E_{\gamma_M}$ associated with $\mathrm{Ult}(J^E_\tau, F \,|\, \lambda^*_M)$ is cofinal and $\pi^*(x) = F(x) \cap \lambda^*_M$ for every $x \subset \kappa$. Thus, the map

$$x \mapsto \text{the least } \zeta \text{ such that } F(x) \cap \lambda^*_M \in S^E_\zeta$$

maps $\mathcal{P}(\kappa) \cap M$ cofinally into γ_M, and is easily seen to be $\Sigma_1(M)$ in the parameter λ^*_M. The rest follows from the facts that $\alpha \geq \tau$ and $\lambda^*_M \in \mathrm{rng}(\sigma)$. □(Lemma 1.5)

The above lemma easily yields:

$$\text{If } (p_M \cup e^\alpha_M) - (\lambda^*_M + 1) \neq \varnothing \text{ then } e^\alpha_M \cap [\lambda^*_M, \gamma_M) = \varnothing. \tag{5}$$

To see this, assume for a contradiction that e^α_M has an element in the interval $[\lambda^*_M, \gamma_M)$. Let β be the largest one. Then $\beta > \lambda^*_M$. To see this, notice that the hypothesis in (5) implies that, assuming $\beta = \lambda^*_M$, there is some $\beta' \in p_M \cup e^\alpha_M$ that is strictly larger than λ^*_M. By Lemma 1.5, $\beta = \lambda^*_M$ must be then in the hull $h^*_M(\alpha \cup \{\beta'\})$, so $h^*_M(\alpha \cup \{p_M \cup e^\alpha_M\}) = h^*_M(\alpha \cup \{p_M \cup (e^\alpha_M - \{\beta\})\})$, and we have a contradiction with the minimality of e^α_M. Since $\beta > \lambda^*_M$, we can find an ordinal $\zeta \in h^*_M(\alpha \cup \{\lambda^*_M\})$ that is larger than β. Obviously, $h^*_M(\alpha \cup \{\lambda^*_M\})$ contains a surjective map $f : \lambda^*_M \to \zeta$, so we can fix a $\xi < \lambda^*_M$ with $f(\xi) = \beta$. Thus, $\beta \in h^*_M(\alpha \cup \{\lambda^*_M, \xi\})$. Let $e = (e^\alpha_M - \{\beta\}) \cup \{\lambda^*_M, \xi\}$. Clearly, $e <^* e^\alpha_M$ and $h^*_M(\alpha \cup \{p_M - \alpha, e\}) \supset h^*_M(\alpha \cup \{p_M - \alpha, e^\alpha_M\}) \ni \gamma_M$. This again contradicts the minimality of e^α_M.

Notice also that

$$\text{If } (p_M \cup e^\alpha_M) - (\lambda^*_M + 1) = \varnothing \text{ then } \lambda^*_M = \max(p_M \cup e^\alpha_M). \tag{6}$$

Since λ^*_M is a cutpoint of F, the map $k : M \,||\, \gamma_M \to M$, defined by $k(\pi^*(f)(\beta)) = \pi(f)(\beta)$ for $f \in {}^\kappa\kappa \cap M$ and $\beta < \lambda^*_M$ (here π^* is as in the proof of Lemma 1.5), has critical point λ^*_M and $k(\lambda^*_M) = \lambda(M)$. This map is Σ_1-preserving (with respect to the language for coherent structures), so $\gamma_M \notin h^*_M(\alpha \cup \{d\})$ for any finite $d \subset \lambda^*_M$. Thus, $(p_M - \alpha) \cup e^\alpha_M \not\subset \lambda^*_M$, which immediately yields (6).

So far we have characterized $d^\alpha_M - (\gamma_M + 1)$ and established the Dodd solidity for this top part of d^α_M. By the above discussion, the next possible element of d^α_M is γ_M. This happens precisely when $\gamma_M \in e^\alpha_M$. Notice that

$$e^\alpha_M = \{\gamma_M\} \tag{7}$$

in this case, and we have to establish (3) for $\beta = \gamma_M$. As this requires a nontrivial amount of effort, we will formulate it as a lemma.

Lemma 1.6. *Assume that $\gamma_M \in e^\alpha_M$. Then ${}^*W^{\gamma_M, p_M}_M \in M$.*

Obviously, we can write p_M instead of $(p_M - \alpha) \cup e^\alpha_M$ in the above lemma. Before going into the proof of this lemma, let us discuss the consequences. For every

$\beta \in (p_M - \alpha) \cap \gamma_M$ we have ${}^*W_M^{\beta,(p_M-\alpha)\cup\{\gamma_M\}} = W_M^{\beta,p_M} \in M$; the membership to M follows from the solidity of M. Due to (7), we can then conclude:

Lemma 1.7. *If $\gamma_M \in e_M^\alpha$ then $d_M^\alpha = (p_M - \alpha) \cup \{\gamma_M\}$ and M is Dodd solid.*

Proof of Lemma 1.6. The proof is based on a comparison argument. This paper builds on the theory presented in [17], which uses fully Σ^*-iterations in comparison arguments, and in this respect it differs from that in [7]. The use of fully Σ^*-iterations has an advantage that the general theory becomes very "clean". Unfortunately, there is also one drawback of this approach, namely, that the comparison argument breaks down in certain special cases when we apply a superstrong extender at the very first step on the "winning" side of the coiteration. Since the comparison described below *does* apply a superstrong extender at the first step on the "winning" side, it is convenient to treat certain situations independently. We will split the entire argument into two cases. Case 1 will deal with the situations which give rise to pathologies in the comparison argument; it will turn out that no comparison argument is needed in this case. In Case 2, we present the comparison argument itself and no pathologies will occur here. Our strategy is analogous to that in the proof of the solidity theorem in [17], Section 9.3. However, as the current situation is somewhat specific, the argument in Case 1 will be shorter than that in [17].

A simple reflection argument guarantees that we can without loss of generality assume that M is countable and (ω_1+1,ω_1)-iterable. We fix an enumeration $\vec{e} = \langle e_i;\, i \in \omega\rangle$ of M and the $\vec{e}$-minimal (ω_1+1,ω_1)-iteration strategy $\mathbb{S}$ guaranteed by Neeman-Steel [8] (for the version for λ-indexing, see also [17], Lemma 9.2.11).

At this point, we should explain one issue. Recall that just before we went into the proof of Theorem 1.1, we made an assumption $\alpha < \lambda_M^*$. We argued that this assumption is harmless, since there is less work to do for larger α. This is really the case, since the same argument we used to identify $d_M^\alpha - (\gamma_M+1)$ and prove the Dodd solidity will also work for $\alpha > \lambda_M^*$, granting that $\gamma_M \notin d_M^\alpha$. Indeed, the argument did not require any knowledge of $\omega\varrho_M^1$ other than the inequality $\omega\varrho_M^1 \le \alpha$, which is always true if d_M^α is defined. However, if $\gamma_M \in d_M^\alpha$ and $\alpha \ge \lambda_M^*$, that argument is not going to work, and the comparison argument we are going to give will make a substantial use of the fact that $\omega\varrho_M^1 \le \lambda_M^*$. But this is always the case, as $\gamma_M \in d_M^\alpha$

necessarily implies $\alpha \leq \gamma_M$, and there are no M-cardinals in the interval $[\lambda^*_M, \gamma_M]$ other than λ^*_M.

Let us now make some settings and also recall some notation that will be used throughout the rest of the proof. We set

$$W = {}^*W_M^{\gamma_M, p_M}, \tag{8}$$

so

$$\mathrm{cr}(E^W_{\mathrm{top}}) = \mathrm{cr}(E^M_{\mathrm{top}}) = \kappa \text{ and } \kappa^{+W} = \kappa^{+M} = \tau. \tag{9}$$

Let

$$\begin{aligned} m &= \begin{cases} \text{the least } i \in \omega \text{ such that } \omega\varrho^{i+1}_M \leq \tau, & \text{if defined,} \\ \omega & \text{otherwise;} \end{cases} \\ M' &= \mathrm{Ult}^*(M, E_{\gamma_M}) \text{ and } \pi : M \to M' \text{ is the associated ultrapower embedding;} \\ \tilde{M} &= \mathrm{Ult}^m(M, E_{\gamma_M}) \text{ and } \tilde{\pi} : M \to \tilde{M} \text{ is the associated ultrapower embedding.} \end{aligned}$$

For basic information on m-ultrapowers see e.g. [17], Section 3.5 or [12]. Denote the collection of all functions used to build $\mathrm{Ult}^m(M, E_{\gamma_M})$ by $\Gamma^m(\kappa, M)$. Working in the language for premice, π is fully Σ^*-preserving and $\tilde{\pi}$ is $\Sigma^{(m)}_0$-preserving and cofinal at the level m. We also have the Łoś theorem for $\Sigma^{(m)}_0$-formulae for both ultrapowers. Moreover, π is $\Sigma^{(i)}_2$-preserving whenever $\omega\varrho^{i+1}_M > \kappa$.

Case 1: $\mathcal{P}(\gamma_M) \cap \mathrm{Ult}^*(M, E_{\gamma_M}) \neq \mathcal{P}(\gamma_M) \cap \mathrm{Ult}(M, E_{\gamma_M})$.

This happens precisely when $m \in \omega$ and $\omega\varrho^{m+1}_M = \tau$. We only show that these two clauses necessarily follow, as only this half of the equivalence is relevant for the argument below. Notice that if $m = \omega$ (that is, $\omega\varrho^\omega_M > \tau$) or $\omega\varrho^{m+1}_M \leq \kappa$ then $\mathrm{Ult}(H^m_M, E_{\gamma_M}) = H^m_{M'}$ where H^m_M and $H^m_{M'}$ is the domain of the m-th reduct of M and M', respectively. So $\mathrm{Ult}(M, E_{\gamma_M})$ and M' agree up to a common cardinal that is larger than γ_M and, consequently, $\mathrm{Ult}(M, E_{\gamma_M})$ and M' contain the same subsets of γ_M.

As an immediate consequence we have $\omega\varrho^1_M > \kappa$, so

$$\pi : M \to M' \text{ is } \Sigma_2\text{-preserving with respect to the language of premice.} \tag{10}$$

Let $\tilde{\sigma} : \tilde{M} \to M'$ be the canonical embedding defined by

$$\tilde{\sigma} : \tilde{\pi}(f)(\xi) \mapsto \pi(f)(\xi)$$

for $f \in \Gamma^m(\kappa, M)$ and $\xi < \lambda^*_M$. Recall that f need not be an element of M, but it has a functionally absolute $\Sigma_1^{(m-1)}$-definition in some parameter from M. More precisely, there is a $\Sigma_1^{(m-1)}$-formula $\varphi(v_0, v_1, v_2)$ such that φ defines a partial function by $(a_1, a_2) \mapsto b$ iff $Q \models \varphi(b, a_1, a_2)$ whenever Q is an acceptable J-structure for the language of premice, and $y = f(\xi)$ just in case that $M \models \varphi(y, \xi, p)$ for some parameter $p \in M$. Then $\pi(f)$ is the function defined by φ over M' in the parameter $\pi(p)$. Standard arguments then yield:

- $\tilde{\sigma} : \tilde{M} \to M'$ is $\Sigma_0^{(m)}$-preserving with respect to the language for premice;
- $\mathrm{cr}(\tilde{\sigma}) = \gamma_M^{+\tilde{M}}$;
- $\omega\varrho_{\tilde{M}}^{m+1} \leq \gamma_M$ and $\tilde{M}$ is sound above γ_M.

The first clause follows from the Łoś theorem for $\Sigma_0^{(m)}$-formulae by the standard argument. To see the second clause, notice that $\tilde{\sigma}$ is the identity up to $\gamma_M + 1$ by definition, so $\mathrm{cr}(\tilde{\sigma}) \geq \gamma_M^{+\tilde{M}}$. Our assumption guarantees that $\mathrm{Ult}(M, E_{\gamma_M})$ and M' have different power sets of γ_M. Since $\mathcal{P}(\gamma_M) \cap \tilde{M} = \mathcal{P}(\gamma_M) \cap \mathrm{Ult}(M, E_{\gamma_M})$, we see that M' has more subsets of γ_M than $\tilde{M}$. Consequently, $\gamma_M^{+\tilde{M}} < \gamma_M^{+M'}$. Since $\tilde{\sigma}(\gamma_M^{+\tilde{M}}) = \gamma_M^{+M'}$, the conslusion follows. The third clause is a standard consequence of the soundness of M, the two facts that $\omega\varrho_M^{m+1} = \tau$ and $\tilde{\pi}(\tau) = \gamma_M$, and general properties of m-ultrapowers.

The three clauses established in the previous paragraph enable us to apply the Condensation Lemma to $\tilde{\sigma} : \tilde{M} \to M'$. This is possible, as M', being a Σ^*-ultrapower of M by an internal extender, is itself $(\omega_1 + 1, \omega_1)$-iterable. We want to infer that the conclusion (b) is the case, that is, we want to show:

(11) $\tilde{M}$ is a proper initial segment of M'.

We first focus on ruling out (a). Notice that $\omega\varrho_{M'}^{m+1} \leq \gamma_M$ and M' is sound above γ_M; this follows similarly as in the case of $\tilde{M}$ above. (a$'$) would then imply that $\tilde{M} = M'$, which contradicts the assumption that $\tilde{M}$ and M' have distinct power sets of γ_M. To see that (c) and (d) in the Condensation Lemma fail is easy, as both (c) and (d) would imply that the cardinal predecessor of $\mathrm{cr}(\tilde{\sigma})$ in $\tilde{M}$ is a limit cardinal in $\tilde{M}$. In the present situation, however, this cardinal predecessor is equal to γ_M, which is a successor cardinal in $\tilde{M}$. □(11)

The induction hypothesis (2) guarantees that the following statement is true:

(12) Let R be a type B proper initial segment of M and α_R be an ordinal such that $\alpha_R \geq \max\{(\mathrm{cr}(E^R_{\mathrm{top}})^{+R}, \omega\varrho^1_R\}$. Then

- R is Dodd solid above α_R;
- $d^\alpha_R = (p_R - \alpha_R) \cup e^\alpha_R$.

By the preservation properties of π, this statement is true with M' in place of M. This combined with (11) implies that $\tilde{M}$ is Dodd solid above $\tilde{\alpha} = \tilde{\pi}(\alpha)$. Letting $\alpha' = \pi(\alpha)$, general properties of fine ultrapowers easily imply that both $\tilde{M}$ and M' are type B premice, $\tilde{\pi}(p_M - \alpha) = p_{\tilde{M}} - \tilde{\alpha}$ and $\pi(p_M - \alpha) = p_{M'} - \alpha'$. Furthermore, $\tilde{\pi}(\gamma_M) = \gamma_{\tilde{M}}$ and $\pi(\gamma_M) = \gamma_{M'}$; as a consequence we obtain that $e^{\tilde{\alpha}}_{\tilde{M}} = \{\gamma_{\tilde{M}}\}$ and $e^{\alpha'}_{M'} = \{\gamma_{M'}\}$. Finally (12) implies that $\tilde{W} \overset{\mathrm{def}}{=} {}^*W^{\gamma_{\tilde{M}}, d^{\tilde{\alpha}}_{\tilde{M}}}_{\tilde{M}} \in \tilde{M}$.

Notice also that $\tilde{\sigma}(p_{\tilde{M}} - \tilde{\alpha}) = p_{M'} - \alpha'$ and $\tilde{\sigma}(\gamma_{\tilde{M}}) = \gamma_{M'}$; this is an immediate consequence of the above calculations and the definition of $\tilde{\sigma}$. Our next step is to show that:

(13) In M', there is a generalized Dodd solidity witness for $\gamma_{M'}$ with respect to M' and $p_{M'}$.

For this, we have to refer to the preservation properties of $\tilde{\sigma}$. We show:

$\tilde{\sigma} : \tilde{M} \to M'$ is Σ_1-preserving with respect to the language of premice.

This is obvious if $m > 0$, so it suffices to focus on the case where $m = 0$. In this case, $\tilde{M}$ is a result of a coarse ultrapower, so $\tilde{\pi}$ is Σ_0-preserving and cofinal. M' is *not* a result of a coarse ultrapower, but it is still true that π' is cofinal. To see this, recall that the assignment

$$x \mapsto \text{the least } \zeta \text{ such that } F(x) \in J^E_\zeta$$

is $\Sigma_1(M)$ and maps $\mathcal{P}(\kappa) \cap M$ cofinally into $\mathrm{ht}(M)$, and similarly

$$x \mapsto \text{the least } \zeta \text{ such that } F'(x) \in J^{E'}_\zeta$$

is $\Sigma_1(M')$ and maps $\mathcal{P}(\kappa') \cap M'$ cofinally into $\mathrm{ht}(M')$ where F' is the top extender of M' and $\kappa' = \mathrm{cr}(F')$. Now notice that π' maps τ cofinally into its image γ_M, which easily yields the cofinality of π'. Once we have established the cofinality of both π and π', the cofinality of $\tilde{\sigma}$ follows immediately.

The calculations following (12), together with the preservation properties of $\tilde{\sigma}$ guarantee that $W' = \tilde{\sigma}(\tilde{W})$ is a generalized Dodd solidity witness for $\gamma_{M'}$ with respect to M' and $p_{M'} - \alpha'$, thus proving (13). Now the statement

$$(\exists W')(W' \text{ is a generalized Dodd solidity witness for } \gamma_{M'} \text{ with respect to } M' \text{ and } p_{M'} \text{ and } M')$$

is easily Σ_2, so (10) then immediately yields that M contains some generalized Dodd solidity witness for γ_M with respect to p_M and M. This completes the discussion in Case 1. □(Case 1)

Case 2: $\mathcal{P}(\gamma_M) \cap \mathrm{Ult}^*(M, E_{\gamma_M}) = \mathcal{P}(\gamma_M) \cap \mathrm{Ult}(M, E_{\gamma_M})$.

Let $\sigma : W \to M$ be the canonical witness map, that is, the uncollapsing map associated with $h^*_M(\gamma_M \cup \{p_M - (\gamma_M + 1)\})$. Then $\gamma_M = \mathrm{cr}(\sigma)$, so $\gamma_M = (\lambda^*_M)^{+W}$ and W is a potential premouse which fails to satisfy the initial segment condition at $\lambda^*_M = \lambda^*_W \overset{\mathrm{def}}{=} \lambda^*$. Notice that the coherent structure determined by $E^W_{\mathrm{top}} \,|\, \lambda^*_W$ is just $M \,||\, \gamma_M$; this follows from the fact that $\sigma \restriction \lambda^* = \mathrm{id}$. Fix the following data:

- $\pi_0 : J^E_\tau \to J^E_{\gamma_M}$ is the canonical ultrapower embedding by $E^W_{\mathrm{top}} \,|\, \lambda^*$;
- $\pi_W : J^E_\tau \to J^E_{\nu_W}$ is the canonical ultrapower embedding by E^W_{top};
- $\sigma_W : M \,||\, \gamma_M \to W$ is the canonical map defined by

  $$\sigma_W : \pi_0(f)(\zeta) \mapsto \pi_W(f)(\zeta)$$

  for $f \in {}^\kappa J^E_\tau \cap J^E_\tau$ and $\zeta < \lambda^*$.

Obviously, σ_W maps $J^E_{\gamma_M}$ cofinally into $J^E_{\nu_W}$. As λ^* is a cutpoint of E^W_{top}, we see that $\mathrm{cr}(\sigma_W) = \lambda^*$ and $\sigma_W(\lambda^*) = \lambda(W)$. Notice also that each $\zeta < \nu_W$ is of the form $\sigma_W(f)(\xi)$ for some $f : \lambda^* \to \gamma_M$ that is an element of $J^E_{\gamma_M}$, and some $\xi < \lambda(W)$. (By the cofinality of π_W, choose $\bar{\zeta}$ so that $\sigma_W(\bar{\zeta}) > \zeta$ and some $\bar{f} \in J^E_\tau$ which maps κ surjectively onto $\bar{\zeta}$. Letting $f = \pi_0(\bar{f})$, we see that $\sigma_W(f) = \pi_W(\bar{f})$ is a surjection of $\lambda(W)$ onto $\sigma_W(\bar{\zeta})$, so one can find some ξ as required above.) Letting $\bar{H}$ be the extender at $(\lambda^*, \lambda(W))$ derived from σ_W, the above observations allow us to conclude that

(14) $J^{E^W}_{\nu_W} = \mathrm{Ult}(J^E_{\gamma_M}, \bar{H})$ and σ_W is the associated ultrapower embedding,

so $\bar{Q} = \langle J^E_{\nu_W}, \bar{H}\rangle$ is a coherent structure (see the introduction). Since $\gamma_M = (\lambda^*)^{+W}$, the extender $\bar{H}$ is weakly amenable with respect to W and it is then easy to verify that $\bar{Q}$ is actually a potential premouse. Moreover, $\bar{Q}$ is a type A potential premouse: Indeed, the fact that λ^* is the largest cutpoint of $E^W_{\rm top}$ guarantees that there are cofinally many ordinals below $\lambda(W)$ that are of the form $\pi_W(f)(\lambda^*)$ for some $f \in {}^\kappa\kappa \cap J^E_\tau$. Thus, the ordinals of the form $\sigma_W(g)(\lambda^*)$ where $g \in {}^{\lambda^*}\lambda^* \cap J^E_{\gamma_M}$ also constitute a cofinal subset of $\lambda(W)$, since the functions g include all functions of the form $\pi_0(f)$ where f is as above. This allows us to conclude:

$$\bar{Q} = \langle J^E_{\nu_W}, \bar{H}\rangle \text{ is a type A premouse.}$$

The amenability of both structures $\langle J^E_{\gamma_M}, E^W_{\rm top} \,|\, \lambda^*\rangle$ and W together with (14) also yield that $W = \mathrm{Ult}(\langle J^E_{\gamma_M}, E^W_{\rm top} \,|\, \lambda^*\rangle, \bar{H})$ and σ_W is the associated ultrapower embedding. So W is fully determined by $\langle J^E_{\gamma_M}, E^W_{\rm top} \,|\, \lambda^*\rangle$ and $\bar{H}$. Also, $\langle J^E_{\gamma_M}, E^W_{\rm top} \,|\, \lambda^*\rangle$, being a coherent structure, is fully determined by J^E_τ and $E^W_{\rm top} \,|\, \lambda^* = E^M_{\rm top} \,|\, \lambda^*$. Thus, to prove that $W \in M$, it suffices to verify that both $E^M_{\rm top} \,|\, \lambda^*$ and $\bar{H}$ are elements of M. That $E^M_{\rm top} \,|\, \lambda^* \in M$ is an immediate consequence of the initial segment condition for M. To see that $\bar{H} \in M$, we show

$$\bar{Q} \in M. \tag{15}$$

Let

- $\pi_M : J^E_\tau \to J^E_{\nu_M}$ be the canonical ultrapower embedding by $E^M_{\rm top}$;
- $\sigma_M : M \,||\, \gamma_M \to M$ be the canonical map defined by

  $$\sigma_M : \pi_0(f)(\zeta) \mapsto \pi_M(f)(\zeta)$$

  for $f \in {}^\kappa J^E_\tau \cap J^E_\tau$ and $\zeta < \lambda^*$.
- H be the extender at $(\lambda^*, \lambda(M))$ derived from σ_M.

A discussion similar to that above shows that

$$Q = \langle J^E_{\nu_M}, H\rangle \text{ is a coherent structure} \tag{16}$$

and

$$M = \mathrm{Ult}(M \,||\, \gamma_M, H) \text{ and } \sigma_M \text{ is the associated ultrapower embedding.} \tag{17}$$

Since $\gamma_M < (\lambda^*)^{+M}$, the extender H is not weakly amenable with respect to M. This means that Q is *not* a potential premouse. Instead, Q is a *protomouse* in the sense of [17, 12] or [13]. The extenders $\bar{H}$ and H are extracted from $E^W_{\rm top}$ and $E^M_{\rm top}$ respectively, so the structures $\bar{Q}$ and Q cannot carry more information than W and M. (Actually, Q and M carry the same amount of information.) As a consequence we get the following, with respect to the language for coherent structures:

$$\sigma : \bar{Q} \to Q \text{ is } \Sigma_0\text{-preserving and cofinal.} \tag{18}$$

Indeed, σ can be viewed as a map from $\bar{Q}$ to Q, since $\bar{Q}$ has the same domain as W and Q has the same domain as M. This also guarantees the cofinality of σ. Thus, to see that σ is Σ_0-preserving, it suffices to show that $\sigma(\bar{H} \cap x) = H \cap \sigma(x)$ for all sufficiently large $x \in \bar{Q}$. By the cofinality of π_W, the set x can be chosen of the form $J^{E^W}_\zeta$ where $\zeta = \pi_W(\bar{\zeta})$ for some $\bar{\zeta} < \tau$. In this case $\bar{H} \cap x$ is of the form $\{\langle \sigma_W(g)(\xi) \cap \lambda^*, \sigma_W(g)(\xi)\rangle ; \xi < \lambda^*\}$ for any surjection $g : \lambda^* \to \mathcal{P}(\lambda^*) \cap J^E_{\zeta_0}$ that is an element of $M \,||\, \gamma_M$; here $\zeta_0 = \pi_0(\bar{\zeta})$. In particular, g can be chosen of the form $\pi_0(f)$ where $f : \kappa \to \mathcal{P}(\kappa) \cap J^E_{\bar{\zeta}}$ is any surjection that is an element of J^E_τ. Then $\sigma_W(g) = \pi_W(f)$, and we have

$$\bar{H} \cap x = \{\langle \pi_W(f)(\xi) \cap \lambda^*, \pi_W(f)(\xi)\rangle ;\ \xi < \lambda^*\}. \tag{19}$$

The abuse of notation we introduced immediately before the beginning of the proof of Theorem 1.1 allows us to write $E^W_{\rm top}(f)$ instead of $\pi_W(f)$, so (19) can be rewritten as $\bar{H} \cap x = \{\langle E^W_{\rm top}(f)(\xi) \cap \lambda^*, E^W_{\rm top}(f)(\xi)\rangle ;\ \xi < \lambda^*\}$, and the following calculation can be easily verified:

$$\begin{aligned}
\sigma(\bar{H} \cap x) &= \sigma(\{\langle E^W_{\rm top}(f)(\xi) \cap \lambda^*, E^W_{\rm top}(f)(\xi)\rangle ;\ \xi < \lambda^*\}) \\
&= \{\langle \sigma(E^W_{\rm top}(f))(\xi) \cap \lambda^*, \sigma(E^W_{\rm top}(f))(\xi)\rangle ;\ \xi < \lambda^*\} \\
&= \{\langle E^M_{\rm top}(f)(\xi) \cap \lambda^*, E^M_{\rm top}(f)(\xi)\rangle ;\ \xi < \lambda^*\} \\
&= H \cap \sigma(x).
\end{aligned}$$

The second equality here follows from the preservation properties of σ and the fact that $\mathrm{cr}(\sigma) = \gamma_M$. To see the third equality, notice that $\sigma(E^W_{\rm top}(f)) = E^M_{\rm top}(f)$. The last clause is a consequence of the fact that $E^M_{\rm top}(f) = \pi_M(f)$ is a surjection of $\lambda(M)$ onto $\mathcal{P}(\lambda(M)) \cap J^E_{\sigma(\zeta)} = \mathcal{P}(\lambda(M)) \cap \sigma(x)$, so it enables us to isolate all $x \subset \lambda(M)$ from $\sigma(x)$ which are in the rangle of H in the same way we did it for $\bar{H}$ in (19).

For the argument we are going to do below, we will need the following fine structural fact about $\bar{Q}$. Let $\bar{p} = \sigma^{-1}(p_M - (\gamma_M + 1))$. Then

$$h_{\bar{Q}}(\gamma_M \cup \{\bar{p}\}) = \bar{Q}, \text{ so } \omega\varrho^1_{\bar{Q}} \leq \gamma_M. \tag{20}$$

The proof of (20) contains an ingredient from the proof of (30), namely a construction of an auxiliary funcion f. Although that proof comes later in this paper, we still prefer to give the details of the construction of f there rather than here. Recall that $h^*_{\bar{Q}}$ is the same as $h_{\bar{Q}}$ since $\bar{Q}$ is a type A premouse and $W = h^*_W(\gamma_M \cup \{\bar{p}\})$. Fix an ordinal $\zeta < \nu_W$. As σ_W is cofinal, the argument from the proof of (30) yields a function $f : {}^{1+|\bar{p}|}\lambda^* \to \gamma_M$ that is an element of $J^E_{\gamma_M}$ and such that $\zeta = \sigma_W(f)(\xi, \bar{p})$ for some $\xi < \gamma_M$. Now $f(\xi) = \mathrm{otp}(g(\xi))$ for a suitable $g : \lambda^* \to \mathcal{P}(\lambda^*)$ that is an element of $J^E_{\gamma_M}$, so our abuse of notation (see the previous paragraph) enables us to write $\zeta = \mathrm{otp}(\bar{H}(g)(\xi, \bar{p}))$. This tells us that $\nu_W = \mathbf{On} \cap \bar{Q} \subset h_{\bar{Q}}(\gamma_M \cup \{\bar{p}\})$, so $h_{\bar{Q}}(\gamma_M \cup \{\bar{p}\}) = \bar{Q}$. $\square$(20)

The conclusion (18) combined with the fact that Q and M have the same domain imply that $\langle M, \bar{Q}, \gamma_M \rangle$ is a good phalanx in the sense of [17], Section 9.1. Our strategy now is to compare this phalanx with M and use standard comparison techniques to infer that $\bar{Q} \in M$, and thereby complete the proof of Lemma 1.6. This is not possible verbatim, as it is not clear that the phalanx is embeddedable into a sufficiently iterable premouse in the manner described in [17], Section 9.2. For this reason, the entire comparison argument needs certain amendments which are described below. The first step in our argument is to show that the phalanx is coiterable with M. In fact, we show that the phalanx is normally $(\omega_1 + 1)$-iterable. We show this by embedding iterations of $\langle M, \bar{Q}, \gamma_M \rangle$ into slight modifications of iterations of M. In order to describe how these modifications will be formed, we need the following notation.

Given a type B premouse P, we define a protomouse $Q(P)$ in the same manner we defined Q in (16). More precisely, letting $\sigma_P : P \,||\, \gamma_P \to P$ be the canonical map defined in an analogous way as σ_M, the structure $Q(P)$ is of the form $\langle J^{E^P}_{\nu_P}, H_P \rangle$ where $\nu_P = \mathrm{ht}(P)$ and H_P is a $(\lambda^*_P, \lambda(P))$-extender derived from σ_P. The structures P and $Q(P)$ have the same domain.

Lemma 1.8. *Let P be a type B premouse and F_P be a total extender on P such that $P' = \mathrm{Ult}(P, F_P)$ exists (that is, the corresponding ultrapower is well-founded; P' is transitive). Let $\pi : P \to P'$ be the associated ultrapower embedding. Then $Q(P') = \mathrm{Ult}(Q(P), F_P)$ and $\pi : Q(P) \to Q(P')$ is the associated ultrapower embedding.*

Proof. Recall also P and $Q(P)$ have the same domain, and the same applies to P' and $Q(P')$. So π is clearly an ultrapower map when viewed as a map between the domains of $Q(P)$ and $Q(P')$. Thus, it suffices to show that π retains the preservation properties when viewed as a map between $Q(P)$ and $Q(P')$. This amounts to showing that $\pi(H_P \cap x) = H_{P'} \cap \sigma(x)$ for all $x \in P$ and can be proved the same way as (18). □(Lemma 1.8)

Write $(\bar{Q}_{-1}, \bar{Q}_0)$ for $(M, \bar{Q})$, so from now on we can write $\langle \bar{Q}_{-1}, \bar{Q}_0, \gamma_M \rangle$ instead of $\langle M, \bar{Q}, \gamma_M \rangle$. We have seen that $\bar{Q}$ is embedded into $Q(M)$ via σ. Now if $\bar{Q}_i$ is a model of some iteration of $\langle \bar{Q}_{-1}, \bar{Q}_0, \gamma_M \rangle$ that is on the same branch as $\bar{Q}_0 = \bar{Q}$ and there is no truncation point on this branch, the critical point of the iteration map $\bar{\pi}_{0,i} : \bar{Q} \to \bar{Q}_i$ is *strictly above* γ_M, so $\lambda^{*+\bar{Q}_i} = \gamma_M$. If $\tilde{M}_i$ is the corresponding model of the iteration of M that arises in the course of the copying construction then the critical point of the iteration map $\tilde{\pi}_{0,i} : M \to \tilde{M}_i$ is also above γ_M, but γ_M is not a cardinal in M and $\tilde{M}_i$. The relationship between $\bar{Q}_i$ and $\tilde{M}_i$ resembles to that between $\bar{Q}$ and M, and we will embed $\bar{Q}_i$ into $Q(\tilde{M}_i)$. In all other cases we can imitate the usual copying construction; the treatment of anomalies, however requires a bit of extra care, too.

Let us now proceed with the construction. We shall give a recursive definition of a normal iteration strategy $\bar{\mathbb{S}}$ for $\langle \bar{Q}_{-1}, \bar{Q}_0, \gamma_M \rangle$. Following the standard approach, this iteration strategy will be a pullback of $\mathbb{S}$ under σ, but we have to be a bit careful about the copy maps (recall that $\mathbb{S}$ was fixed at the beginning of this section). Let $\bar{\Im}$ be a normal iteration of $\langle \bar{Q}_{-1}, \bar{Q}_0, \gamma_M \rangle$ according to $\bar{\mathbb{S}}$ with iteration indices $\bar{\nu}_i$, critical points $\bar{\kappa}_i$, premice $\bar{Q}_i$, iteration maps $\bar{\pi}_{ij}$ and the associated tree ordering $<_{\bar{\Im}}$. Starting from $\bar{\Im}$ and the embeddings $\mathrm{id} : \bar{Q}_{-1} \to M$ and $\sigma : \bar{Q} \to Q$, we define a normal iteration $\tilde{\Im}$ of M according to $\mathbb{S}$ with iteration indices $\tilde{\nu}_i$, critical points $\tilde{\kappa}_i$, premice $\tilde{M}_i$, iteration maps $\tilde{\pi}_{ij}$ (so $\tilde{M}_0 = M$) and the associated tree ordering $<_{\tilde{\Im}}$ together with "copy" maps σ_i such that:

(a) $\sigma_{-1} = \mathrm{id} : \bar{Q}_{-1} \to \tilde{M}_0$ and $\sigma_0 = \sigma : \bar{Q}_0 \to Q(\tilde{M}_0)$;

(b) $\sigma_i : \bar{Q}_i \to Q(\tilde{M}_i)$ is Σ_0-preserving and cofinal whenever $0 \leq_{\tilde{\Im}} i$ and there is no truncation point on the branch $[0, i]_{\tilde{\Im}}$;

(c) $\sigma_i : \bar{Q}_i \to \tilde{M}_i$ if at least one of the conditions in (b) fails and $\bar{\kappa}_{i-1} \neq \lambda^*$; in this case, σ_i is fully Σ^*-preserving;

(d) $\sigma_i : \bar{Q}_i \to \tilde{M}_i = \mathrm{Ult}(\tilde{M}_0 \,||\, \gamma_{\tilde{M}_0}, E_{\mathrm{top}}^{Q(\tilde{M}_{i-1})}) = \tilde{M}_{i-1}$ if

(∗) $0 \leq_{\tilde{\Im}} i-1$, there is no truncation point on the branch $[0, i-1]_{\tilde{\Im}}$ and $\bar{\nu}_{i-1} = \mathrm{ht}(\bar{Q}_{i-1})$;

in this case $\bar{\kappa}_{i-1} = \lambda^*$ as $\bar{\pi}_{0,i-1} \upharpoonright (\gamma_M + 1) = \mathrm{id}$, and i is a *strong* anomaly in the sense of [17], Section 9.1; it follows that $\{-1, i\}$ is a maximal branch in $\bar{\Im}$. Regarding the equality $\tilde{M}_i = \tilde{M}_{i-1}$, see below.

(e) $\sigma_i : \bar{Q}_i \to \tilde{\pi}_{0,i}(M \,||\, \gamma_M)$ is Σ_0-preserving and cofinal if the following two conditions are satisfied:

- $\bar{\kappa}_{i-1} = \lambda^*$ and
- the condition (∗) from (d) fails;

as in (d), i is a strong anomaly, so $\{-1, i\}$ is a maximal branch in $\bar{\Im}$.

Clauses (a), (c) and (e) can be verified similarly as in the case where we have a standard embedding of a phalanx into a premouse (see [17], Lemma 9.2.9). Clause (b) is a consequence of Lemma 1.8. Let ξ_{i-1} be the immediate $<_{\tilde{\Im}}$-predecessor of i. If $\sigma_{\xi_{i-1}} : \bar{Q}_{\xi_{i-1}} \to Q(\tilde{M}_{\xi_{i-1}})$ and σ_{i-1} are already given, the lemma allows us to define the embedding $\sigma_i : \tilde{Q}_i \to Q(\tilde{M}_i)$ in the standard way. To see (d), notice that (∗) implies that the extender $E_{\tilde{\nu}_{i-1}}^{Q(\tilde{M}_{i-1})} = E_{\mathrm{top}}^{Q(\tilde{M}_{i-1})}$ is not total on M_0, and the result of corresponding ultrapower is just $\tilde{M}_{i-1}$; this follows from the fact that $\tilde{M}_{i-1}$ agrees with $\tilde{M}_0$ up to $\gamma_{\tilde{M}_0} + 1$. Strictly speaking, $\tilde{\Im}$ is not a normal iteration, as it may contain branches of the form $\{0, i\}$ whenever $E_{\tilde{\nu}_{i-1}}^{Q(\tilde{M}_{i-1})}$ is a top extender of $Q(\tilde{M}_{i-1})$. But obviously, by removing such branches from $\tilde{\Im}$, we obtain a normal iteration. All the remaining issues can be handled in the standard way.

The copying construction described above provides us with the obvious definition of $\bar{\mathbb{S}}$, namely $\bar{\mathbb{S}}(\bar{\Im}) = \mathbb{S}(\tilde{\Im})$ whenever $\bar{\Im}$ is an iteration of limit length. This gives the normal iterability of the phalanx $\langle M, \bar{Q}, \gamma_M \rangle$, and thereby its coiterability with M. The coiteration of the phalanx with M terminates in, say, $\theta + 1 < \omega_1$ steps. To keep the notation visually consistent with that in the previous paragraph, let $\bar{\Im}$ be the iteration on the phalanx side, let $\tilde{\Im}$ be its copy as described above and let $\Im$ be the iteration on the M-side. Denote the premice in $\Im$ by M_i and the iteration maps in $\Im$ by π_{ij}. Thus, the last models in $\bar{\Im}$, $\tilde{\Im}$ and $\Im$ are $\bar{Q}_\theta$, $\tilde{Q}_\theta$ and M_θ, respectively. Also, $\bar{\nu}_i$ are the indices of the coiteration.

The application of the Neeman-Steel Lemma as described in [17], Lemma 9.2.12, goes literally through even in the present context; the details are left to the reader. We thus obtain:

- $0 \leq_{\bar{\Im}} \theta$;
- $\bar{Q}_\theta$ is an initial segment of M_θ;
- $[0, \theta]_{\bar{\Im}}$ has no truncation points.

Notice that if $\bar{Q}_\theta = M_\theta$, then $[0, \theta]_\Im$ necessarily contains a truncation point. This follows from the fact that if the iteration maps $\bar{\pi}_{0,\theta} : \bar{Q}_0 \to \bar{Q}_\theta$ and $\pi : M_0 \to M_\theta$ are total, then they preserve the premice type. Now M is a type B premouse, so M_θ would be of type B as well. On the other hand, $\bar{Q}_0 = M$ is a type A premouse, so $\bar{Q}_\theta$ is of type A. So we have:

> Either $\bar{Q}_\theta$ is a *proper* initial segment of M_θ or else
> there is a truncation point on the main branch of $\Im$.

Now we can proceed in the usual way. If $\bar{Q}_\theta$ is a proper initial segment of M_θ, then $\bar{Q}_\theta$ is sound. So the phalanx side of the coiteration is not moved, as otherwise (20) would imply that $\bar{Q}_\theta$ fails to be sound. This means that $\bar{Q}_\theta = \bar{Q}_0 = \bar{Q}$. Notice that $\bar{\nu}_0 = \gamma_M$, as $E^M_{\gamma_M} \neq \varnothing = E^{\bar{Q}}_{\gamma_M}$. Assume there is a disagreement between the extender sequences of M_1 and $\bar{Q}$. Then $\bar{\nu}_1$ is strictly larger than γ_M and is a cardinal in M_θ. Since $\bar{Q}$ projects to γ_M, we see that $\bar{Q}$ is actually a proper inital segment of $M_\theta \,||\, \bar{\nu}_1 = M_1 \,||\, \bar{\nu}_1$, a contradiction. So $\bar{Q}$ must be a proper initial segment of M_1 and $\theta = 1$. By (20), $\bar{Q}$ can be encoded into a subset b of γ_M that is $\mathbf{\Sigma}_1(\bar{Q})$, and therefore into an element of $M_1 = \mathrm{Ult}^*(M, E_{\gamma_M})$. Now recall that we are in Case 2, so $b \in \mathrm{Ult}(M, E_{\gamma_M})$. We want to see that $b \in M$, as this will enable us to decode b inside M and conclude that $\bar{Q} \in M$. The argument is described in [17], proof of Theorem 9.3.1, Case 1. The point is that in M, there are sequences $\langle g_\xi ; \xi < \gamma_M \rangle$ and $\langle \alpha_\xi ; \xi < \gamma_M \rangle$ such that $g_\xi : \kappa \to \tau$, $\alpha_\xi < \lambda^*_M$ and $\xi = \hat{\pi}(g_\xi)(\alpha_\xi)$ where $\hat{\pi}$ is the ultrapower embedding associated with $\mathrm{Ult}(M, E_{\gamma_M})$. Since $E_{\gamma_M} \in M$ and $b = \hat{\pi}(f)(\hat{\alpha})$ for some f and $\hat{\alpha}$ that also are elements of M, the statement "$\xi \in b$"

can be expressed as $\hat{\pi}(g_\xi)(\alpha_\xi) \in \hat{\pi}(f)(\hat{\alpha})$. Using the Łoś Theorem, this can be expressed internally in M.

Now assume that $\bar{Q}_\theta = M_\theta$. Let $i > 0$ be the least such that $E^{M_i}_{\bar{\nu}_i}$ is an extender. Notice that i exists, as there must be a truncation point on the main branch of $\Im$ (so $\theta > 1$). Obviously, $M_i = M_1$ so $\bar{\nu}_i$ indexes an extender on the M_1-sequence (possibly the top one). Also, let $j+1$ be the last truncation point on that branch. The set b encoding $\bar{Q}$ is $\mathbf{\Sigma}_1(\bar{Q})$, so by standard arguments, b is $\mathbf{\Sigma}_1(M^*_j)$ where M^*_j is the result of the last truncation on the main branch in $\Im$. So $b \in M_{\xi_j}$ where ξ_j is the immediate $<_\Im$-predecessor of $j+1$. Now either $\xi_j = 1$ or else $\xi_j > 1$, in which case $\bar{\nu}_i$ is a cardinal in M_{ξ_j}. In either case, b is an element of M_1, due to the agreement between M_1 and M_{ξ_j}. Now apply the assumption of Case 2, and as in the previous case conclude that $b \in M$ and $\bar{Q} \in M$. □(Lemma 1.6)

From now on we can assume

$$e^\alpha_M \neq \{\gamma_M\}. \tag{21}$$

By (5), $e^\alpha_M \subset \lambda^*_M + 1$.

Lemma 1.9. *Assume that $\lambda^*_M \in e^\alpha_M$. Then ${}^*W^{\lambda^*_M,(p_M-\alpha)\cup e^\alpha_M}_M \in M$.*

Proof. By (6), $\lambda^*_M \in e^\alpha_M$ just in case that $\lambda^*_M = \max(p_M \cup e^\alpha_M)$. Letting $W = {}^*W^{\lambda^*_M,(p_M-\alpha)\cup e^\alpha_M}_M$ and $\sigma : W \to M$ be the canonical witness map, we see that $\mathrm{cr}(\sigma) = \lambda^*_M$ and $\sigma(\lambda^*_M) = \lambda(M)$. This follows from the fact that λ^*_M is the largest cutpoint of E^M_{top}. So W is a coherent structure with $\lambda(W) = \lambda^*_M$ and $E^W_{\mathrm{top}} = F \mid \lambda^*_M$. By the initial segment condition, $F \mid \lambda^*_M = E_{\gamma_M}$, so $W = M \,||\, \gamma_M \in M$.

□(Lemma 1.9)

This takes care of the top part of the Dodd parameter above λ^*_M, and we have:

$$d^\alpha_M - \lambda^*_M = (p_M \cup e^\alpha_M) - \lambda^*_M \quad \text{and} \quad {}^*W^{\beta,d^\alpha_M}_M \in M \text{ whenever } \beta \in d^\alpha_M - \lambda^*_M.$$

Recall that we are in the process of verifying (3) and that we have already done the job for $\beta \geq \lambda^*_M$. From now on we will assume:

$$\beta < \lambda^*_M \text{ and } \beta \in (p_M - \alpha) \cup e^\alpha_M \tag{22}$$

$$d^\alpha_M - (\beta+1) = (p_M \cup e^\alpha_M) - (\beta+1) \tag{23}$$

$${}^*W^{\delta,d^\alpha_M}_M \in M \text{ whenever } \delta \in d^\alpha_M - (\beta+1). \tag{24}$$

We will first discuss the case where $\beta \in p_M - \alpha$.

Lemma 1.10. *Assume $\beta \in p_M - \alpha$ and (22) – (24) hold. Then* ${}^*W_M^{\beta, p_M \cup e_M^\alpha} \in M$.

Proof. Let $W = W_M^{\beta, p_M}$ and let $\sigma : W \to M$ be the canonical witness map. Let further

$$e_1 = e_M^\alpha - \beta \quad \text{and} \quad e_2 = e_M^\alpha \cap \beta.$$

Then $\sigma(\beta) > \beta$. By the definition of e_M^α, there is some finite $a \subset \alpha$ such that

$$(\exists e)(\exists \bar{\beta})(e \in [\mathbf{On}]^{<\omega} \ \&\ \bar{\beta} < \sigma(\beta) \ \&\ \gamma_M = h_M^*(a, \{p_M - \{\beta\}, \bar{\beta}, e, e_2\}); \tag{25}$$

this is witnessed by e_1 and β. Since σ is Σ_1-preserving with respect to the language for coherent structures (in fact, with respect to the language for premice) and the objects γ_M, $p_M - \{\beta\}$, a and e_2 are in the range of σ, there is an $e \in \text{rng}(\sigma)$ witnessing (25). Assume e is the $<^*$-least such.

Claim. $e = e_1$, so $e_M^\alpha \in \text{rng}(\sigma)$.

Proof. Suppose this is false. There are two possibilities. First consider the possibility $e_1 <^* e$. If this happens then (25) can be rewritten as

$$(\exists \bar{e})(\exists \bar{\beta})(\bar{e} <^* e \ \&\ \bar{\beta} < \sigma(\beta) \ \&\ \gamma_M = h_M^*(a, \{p_M - \{\beta\}, \bar{\beta}, \bar{e}, e_2\}).$$

It would follow that $\text{rng}(\sigma)$ contains some finite set of ordinals $\bar{e} <^* e$ witnessing (25), which contradicts the minimality of e.

Now consider the case where $e <^* e_1$. Fix $\delta \in e_1 - e$ such that $e_1 - (\delta + 1) = e - (\delta + 1)$. Let ${}^*W = {}^*W_M^{\delta, p_M \cup e_M^\alpha}$ and $\sigma^* : {}^*W \to M$ be the canonical witness map. Since $\delta > \beta$, we can apply (23) and (24) and conclude that $W^* \in M$. On the other hand, the definition of δ guarantees that $p_M \cup e \subset \text{rng}(\sigma^*)$. Since σ^* is Σ_1-preserving with respect to the language for coherent structures, we see that both p_M and $\gamma_M = h_M^*(a, \{p_M - \{\beta\}, \bar{\beta}, e, e_2\})$ are in $\text{rng}(\sigma^*)$; here $\bar{\beta} < \beta$ comes again from (25). But this means that $\text{rng}(\sigma^*) \supset h_M^*(\alpha, p_M \cup \{\gamma_M\}) = M$. So ${}^*W = M$, a contradiction again. □(Claim)

Let q be such that $\sigma(q) = (p_M \cup e_M^\alpha) - (\beta + 1)$. The Dodd solidity witness ${}^*W_M^{\beta, p_M \cup e_M^\alpha}$ is the transitive collapse of

$$Y = h_M^*(\beta \cup \{(p_M \cup e_M^\alpha) - (\beta + 1)\}).$$

By the preservation properties of σ, the structures $(Y, \in)$ and $(h^*_W(\beta \cup \{q\}), \in)$ are isomorphic (this notation suppresses the predicates), so ${}^*W_M^{\beta, p_M \cup e_M^\alpha}$ is the transitive collapse of $h^*_W(\beta \cup \{q\})$. Since M is solid, we know that $W \in M$. Now M is active, so its domain is a ZFC^--model. It follows immediately that both $h^*_W(\beta \cup \{q\})$ and ${}^*W_M^{\beta, p_M \cup e_M^\alpha}$ are in M. □(Lemma 1.10)

It remains to discuss the case where $\beta \in e_M^\alpha$. This will require some effort. In a series of lemmata, we first show that the top extender of ${}^*W_M^{\beta, p_M \cup e_M^\alpha}$ can be factored into two extenders; one of them will be an element of M and the other one will satisfy the initial segment condition. We then apply a comparison argument of the same kind we did in Case 2 in the proof of Lemma 1.6 to conclude that this extender is in M. We will also see that such a factoring implies the existence of a cardinal in M that is both subcompact and superstrong. Thus, under a suitable (and relatively modest) anti-large cardinal assumption, we will be able to conclude that $e_M^\alpha \cap \lambda_M^* = \varnothing$, which will prove Proposition 1.3.

In addition to (21) – (24), from now on we assume:

(26) $\quad \beta \in e_M^\alpha$

(27) $\quad W = {}^*W_M^{\beta, p_M \cup e_M^\alpha}$ and $\sigma : W \to M$ is the canonical witness map

(28) $\quad \gamma = (\sigma^{-1})''\gamma_M$

Obviously, W is a type B potential premouse and $\sigma(\lambda_W^*) = \lambda_M^*$.

Lemma 1.11. $\gamma < (\lambda_W^*)^{+W}$.

Proof. Obviously, $\gamma \leq (\lambda_W^*)^{+W}$. Assume for a contradiction that $\gamma = (\lambda_W^*)^{+W}$. Since σ maps γ cofinally into γ_M, we can apply the Interpolation Lemma ([17], Lemma 3.6.10) to the embedding $\sigma : W \to M$ and find a coherent structure $\tilde{M}$, and maps $\tilde{\sigma}$ and σ' such that:

- $\tilde{\sigma} : W \to \tilde{M}$ is Σ_0-preserving with respect to the language for coherent structures and cofinal;
- $\sigma' : \tilde{M} \to M$ is Σ_0-preserving with respect to the language for coherent structures;
- $\tilde{\sigma} \restriction J_\gamma^{E^W} = \sigma \restriction J_\gamma^{E^W}$;
- $\mathrm{cr}(\sigma') = \gamma_M$ and $\sigma'(\gamma_M) = (\lambda_M^*)^{+M}$;

- $\sigma' \circ \tilde{\sigma} = \sigma$.

The definition of σ guarantees that $(p_M \cup e^\alpha_M) - \{\beta\}$ is contained in $\mathrm{rng}(\sigma)$, and therefore also in $\mathrm{rng}(\sigma')$. But since $\mathrm{cr}(\sigma') = \gamma_M > \beta \geq \alpha$, we also have the inclusion $\alpha \cup \{\beta\} \subset \mathrm{rng}(\sigma')$. It follows that $\alpha \cup \{p_M \cup e^\alpha_M\} \subset \mathrm{rng}(\sigma')$. Now the fact that $\mathrm{cr}(\sigma') > \tau$ (recall that $\tau = \kappa^{+M}$ where $\kappa = \mathrm{cr}(F)$) implies that $\sigma' : \tilde{M} \to M$ is cofinal, and therefore Σ_1-preserving with respect to the language for coherent structures. From this we immediately infer that $\mathrm{rng}(\sigma') = h^*_M(\alpha \cup \{p_M \cup e^\alpha_M\}) = M$, and obtain a contradiction with the next-to-last clause above, which tells us that $\mathrm{rng}(\sigma')$ has a critical point. □(Lemma 1.11)

Before proceeding further, we observe:

(29) $$\gamma_M \notin \mathrm{rng}(\sigma).$$

Otherwise $\gamma_M \in h^*_M(\alpha \cup \{p_M \cup (e^\alpha_M - \{\beta\}) \cup e\})$ for some finite $e \subset \beta$. But $(e^\alpha_M - \{\beta\}) \cup e <^* e^\alpha_M$, which contradicts the minimality of e^α_M. □(29)

Let us also point out that (29) implies the failure of the Initial Segment Condition for W. If W satisfied the Initial Segment Condition then $E^W_{\mathrm{top}} \,|\, \lambda^*_W \in W$, and the preservation properties of σ would yield $\sigma(E^W_{\mathrm{top}} \,|\, \lambda^*_W) = F \,|\, \lambda^*_M$. Since J^E_τ is obviously in the range of σ, also $\gamma_M = \mathrm{ht}(\mathrm{Ult}(J^E_\tau, F \,|\, \lambda^*_M))$ would end up in $\mathrm{rng}(\sigma)$, which is impossible.

Recall that λ^*_W is the largest cardinal in $J^{E^W}_\gamma$. Let N be the level of W collapsing γ and $n \in \omega$ be such that $\omega\varrho^{n+1}_N \leq \lambda^*_W < \omega\varrho^n_N$; these objects exist by Lemma 1.11. Our observation (29) guarantees that $\sigma(\gamma) > \gamma_M$. Letting $N' = \sigma(N)$, we now apply the interpolation lemma to the embedding $\sigma \restriction N : N \to N'$. We obtain a transitive structure $\tilde{N}$ together with maps $\tilde{\sigma}$ and σ' satisfying:

- $\tilde{\sigma} : N \to \tilde{N}$ is $\Sigma^{(n)}_0$-preserving with respect to the language for coherent structures and cofinal at the level n;
- $\sigma' : \tilde{N} \to N'$ is $\Sigma^{(n)}_0$-preserving with respect to the language for coherent structures;
- $\sigma' \circ \tilde{\sigma} = \sigma$;
- $\tilde{\sigma} \restriction J^{E^N}_\gamma = \sigma \restriction J^{E^N}_\gamma$;
- $\gamma_M = \tilde{\sigma}(\gamma) = (\lambda^*_M)^{+\tilde{N}}$ (this has a meaning also when $\gamma = \mathrm{ht}(N)$);
- $\sigma' \restriction \gamma_M = \mathrm{id}$;

- $\mathrm{cr}(\sigma') = \gamma_M$ and $\sigma'(\gamma_M) = \sigma(\gamma) \stackrel{\text{def}}{=} \gamma'$ whenever $\gamma < \mathrm{ht}(N)$.

For $n > 0$, the cofinality of $\tilde\sigma$ at the level n follows from the soundness of N (soundness with respect to the language for premice, of course), as it implies that R^n_N, when computed in the language for coherent structures, is nonempty ([17], Lemma 3.6.3 (h)). The second clause is then an immediate consequence of the cofinality of $\tilde\sigma$ at the level n ([17], Lemma 3.6.10). For $n = 0$, the cofinality of $\tilde\sigma$ follows directly from the pseudoultrapower construction.

Fix the following notation. Assuming that N is active, we let

- $\mu = \mathrm{cr}(E^N_{\mathrm{top}})$ and $\tilde\mu = \tilde\sigma(\mu)$;
- $\vartheta = \mu^{+N}$, $\vartheta' = \tilde\sigma(\vartheta)$ and $\tilde\vartheta = \sup(\tilde\sigma''(\vartheta))$;
- $G = E^N_{\mathrm{top}}$, $\tilde G = E^{\tilde N}_{\mathrm{top}}$ and $G' = E^{N'}_{\mathrm{top}}$.

Lemma 1.12. *N is active,* $\mathrm{ht}(N) = \gamma$, $\vartheta = \beta$ *and* $E_\beta \neq \varnothing$.

Proof. First observe that $\beta = \mathrm{cr}(\sigma)$. This is standard: If not, then $\beta \in h^*_M(\beta \cup \{p_M \cup (e^\alpha_M - \{\beta\})\})$, so $h^*_M(\beta \cup \{p_M \cup (e^\alpha_M - \{\beta\})\}) \supset h^*_M(\beta \cup \{p_M \cup e^\alpha_M\}) = M$. We then arrive at a contradiction exactly as in the proof of (29).

We will split the proof into a sequence of claims in which we rule out all possibilities that are incompatible with the statement of the lemma.

Claim 1. N is active, $n = 0$ and $\tilde\sigma''\vartheta$ is bounded in ϑ'.

Proof. Assume for a contradiction that this fails, so either N is passive or $n > 0$ or $\tilde\sigma''\vartheta$ is cofinal in ϑ'. We first show that $\gamma < \mathrm{ht}(N)$. This is clear if N is passive, as $J^{E^N}_\gamma$ is a ZFC^--model. Now suppose that N is active and $\gamma = \mathrm{ht}(N)$, that is, $E^W_\gamma \neq \varnothing$. In this case we have $n = 0$, so our assumption implies that $\sigma''\vartheta = \tilde\sigma''\vartheta$ is cofinal in ϑ'. Recall that $W \models \mathrm{cf}(\gamma) = \mathrm{cf}(\vartheta)$, that is, in W there is a strictly monotonic function $f : \vartheta \to \gamma$ that is cofinal in γ. Then $\sigma(f) : \vartheta' \to \gamma'$ is strictly monotonic and cofinal in γ', so

$$\gamma' = \sup(\sigma(f)''\vartheta') = \sup(\sigma(f)''(\sigma''\vartheta)) = \sup(\sigma''(f''\vartheta)) = \sup(\sigma''\gamma) = \gamma_M.$$

This contradicts the fact that $\gamma' > \gamma_M$. The second equality here follows from the cofinality of $\sigma''\vartheta$ in ϑ'. So indeed $\gamma < \mathrm{ht}(N)$ and, consequently, $\gamma_M = \mathrm{cr}(\sigma')$.

As a next step we show $\tilde N$ is a premouse, and if N is active then $\tilde N$ is of the same type as N and N'. The key point here is to verify that $\tilde G$ is a total extender on $\tilde N$. This is a Π_2-property. If $n > 0$ then $\tilde G$ is total on $\tilde N$, as $\tilde\sigma$, being a fine pseudoultrapower embedding, is Σ_2-preserving ([17], Lemma 3.6.3 (d)). If $\tilde{\sigma}''\vartheta$ is cofinal in ϑ', a standard argument yields

$$\mathrm{dom}(\tilde{G}) = \bigcup\{\tilde{\sigma}(x);\ x \in J_{\vartheta}^{E^N}\ \&\ x \subset \mathcal{P}(\mu)\} = \mathcal{P}(\tilde{\mu}) \cap \tilde{N}.$$

So far we have seen that $\tilde{N}$ is a potential premouse. That $\tilde{N}$ is a premouse of the same type as N and N' (this includes the verification of the initial segment condition) follows from the preservation properties of $\tilde{\sigma}$ with respect to the language for coherent structures established above and from [17], Lemma 9.1.7. It also follows that the preservation properties of maps $\tilde{\sigma}$ and σ' stated above hold with respect to the language of premice.

The soundness of N and our choice of n guarantee that $N = \tilde{h}_N^{n+1}(\lambda_W^* \cup \{p_N\})$. This together with the solidity of N, the fact that $\tilde{N}$ is a result of a fine pseudoultrapower of N by $\sigma \restriction J_\gamma^{E^N}$, and the preservation properties of the maps $\tilde{\sigma}$ and σ' yields:

- $\tilde{N} = \tilde{h}_{\tilde{N}}^{n+1}(\lambda_M^* \cup \{\tilde{p}\})$ where $\tilde{p} = \tilde{\sigma}(p_N)$;
- $\omega\varrho_{\tilde{N}}^{n+1} \leq \lambda_M^*$;
- $\tilde{N}$ is solid;
- each $\delta \in \tilde{p}$ has a generalized witness with respect to $\tilde{p}$ and $\tilde{N}$, that is an element of $\tilde{N}$.
- $\tilde{p} = p_{\tilde{N}} - \lambda_M^*$, so $\tilde{N}$ is sound above λ_M^*.

The first clause here follows from the construction of fine pseudoultrapowers (note that $p_N \cap \lambda_W^* = \varnothing$, as $\omega\varrho_N^\omega = \lambda_W^*$), and immediately yields the second clause. The third clause follows from the second clause and from the first part of the Condensation Lemma applied to the embedding $\sigma' : \tilde{N} \to N$. The fourth clause is a consequence of the preservation properties of $\tilde{\sigma}$ and the solidity of N. Finally, the last clause is a consequence of the previous two ([17], Lemma 1.12.5).

We now apply the second part of the Condensation Lemma to the embedding $\sigma' : \tilde{N} \to N'$. There are four possibilities to be discussed, and we show that each of them yields a contradiction. This will complete the proof of Claim 1. Clause (a$'$) in the Condensation Lemma is clearly false, as $\sigma' \neq \mathrm{id}$. (We are allowed to use (a$'$), as N' is sound.) To see that clause (b) fails, recall that γ_M, being the critical point of σ', is a cardinal in $\tilde{N}$, so $E_{\gamma_M}^{\tilde{N}} = \varnothing$. On the other hand, N' is an initial segment of M, so $E^{N'}_{\gamma_M} = E_{\gamma_M} = F \mid \lambda^*_M$ by the initial segment condition. Clause (c) is false, as $\tilde{N}$ is sound above λ^*_M, but the fine ultrapower from (c) fails to be. Finally, clause (d) fails, as it would imply that $\omega\varrho^\omega_{\tilde{N}} \geq \gamma_M$, and we have seen that $\tilde{N}$ projects to λ^*_M. □(Claim 1)

Claim 2. $\mu < \beta$.

Proof. Notice first that $\mu < \lambda^*_W$. Otherwise $\vartheta > \lambda^*_W$, so $\tilde{\sigma}''\vartheta$ would be cofinal in $\tilde{\sigma}(\vartheta)$, as follows from general properties of pseudoultrapowers (by Claim 1, $\tilde{N}$ is the result of a coarse ultrapower in this case). This would contradict Claim 1. Assume for a contradiction that $\mu \geq \beta$. Then $\beta < \vartheta < \lambda^*_W$. Since λ^*_W is a cardinal in W, so is ϑ, as follows from the acceptability of W and the fact that ϑ is a cardinal in N. The agreement between σ and $\tilde{\sigma}$ below λ^*_W together with Claim 1 guarantee that $\sigma''\vartheta = \tilde{\sigma}''\vartheta$ is bounded in $\vartheta' = \sigma(\vartheta) = \tilde{\sigma}(\vartheta)$.

Now argue as in the proof of Lemma 1.11. We apply the Interpolation Lemma to the embedding $\sigma : W \to M$; this time we form a pseudoultrapower of W by $\sigma \restriction J^{E^W}_\vartheta$. We obtain an acceptable structure $\tilde{W}$ together with maps σ_0 and σ_1 satisfying:

- $\sigma_0 : W \to \tilde{W}$ is Σ_0-preserving with respect to the language for coherent structures and cofinal;
- $\sigma_1 : \tilde{W} \to M$ is Σ_0-preserving with respect to the language for coherent structures;
- $\sigma_1 \circ \sigma_0 = \sigma$;
- $\sigma_0 \restriction J^{E^W}_\vartheta = \sigma \restriction J^{E^W}_\vartheta$;
- $\mathrm{cr}(\sigma_1) = \tilde{\vartheta}$ and $\sigma_1(\tilde{\vartheta}) = \vartheta'$.

Exactly as in the proof of Lemma 1.11 we observe that σ_1 is Σ_1-preserving with respect to the language for coherent structures and $(p_M \cup e^\alpha_M) - \{\beta\} \in \mathrm{rng}(\sigma_1)$. Since $\vartheta > \beta$, we have $\mathrm{cr}(\sigma_1) = \tilde{\vartheta} > \beta$. It follows that $\alpha \cup \{\beta\} \subset \mathrm{rng}(\sigma_1)$, so actually $\alpha \cup \{p_M \cup e^\alpha_M\} \subset \mathrm{rng}(\sigma_1)$. Thus, $\mathrm{rng}(\sigma_1) = M$, which contradicts the fact that σ_1 has a critical point. □(Claim 2)

Claim 3. $\vartheta = \beta$.

Proof. By Claim 2, $\mu < \beta$, so $\vartheta \leq \beta$ (recall that β, being a critical point of σ, is a cardinal in W). If $\vartheta < \beta$, we would have $\vartheta' = \sigma(\vartheta) = \vartheta$ and $\tilde{\sigma}''\vartheta = \sigma''\vartheta = \vartheta$, which would yield an immediate contradiction with Claim 1. □(Claim 3)

Claim 4. $E_\beta \neq \varnothing$.

Proof. Building on the previous claims, we conclude:

- $\tilde{\sigma} : N \to \tilde{N}$ and $\sigma' : \tilde{N} \to N$ are both Σ_0-preserving with respect to the language for coherent structures, and $\tilde{\sigma}$ is cofinal;
- $\mathrm{cr}(\tilde{G}) = \mu$;
- $\mathrm{dom}(\tilde{G}) = \mathcal{P}(\mu) \cap J^E_\beta \subsetneqq \mathcal{P}(\mu) \cap \tilde{N}$;
- $\tilde{N}$ is a coherent structure that is *not* a potential premouse.

Let N^* be the level of $\tilde{N}$ collapsing β. Obviously, $\beta = \mu^{+N^*}$ and $\mathrm{dom}(\tilde{G}) = \mathcal{P}(\mu) \cap N^*$. By the agreement between $\tilde{N}$ and M, the premouse N^* is the level of M collapsing β, so N^* is a proper initial segment of $J^E_{\beta'}$ where $\beta' = \sigma(\beta)$. Let π' be the ultrapower embedding associated with $\mathrm{Ult}(J^E_{\beta'}, G')$ and let $R' = \pi(N^*)$. Let further $\tilde{R} = \mathrm{Ult}^*(N^*, \tilde{G})$ with the associated map $\tilde{\pi}$. Then R' is a level of N', and thus a level of M. To see that $\tilde{R}$ exists (that is, the corresponding ultrapower is well-founded), notice that the assignment $[\delta, f] \mapsto \pi'(f)(\sigma'(\delta))$ is Σ_0-preserving; here $\langle \delta, f \rangle$ appears in the construction of the fine ultrapower, so $\delta < \lambda(\tilde{N})$ and $f \in \Gamma(\mu, N^*)$. This assignment gives rise to the embedding $k : \tilde{R} \to R'$.

The map $\pi' \restriction N^* : N^* \to R'$ is fully elementary in the language for premice, and N^* and R' are of the same type. We would like to obtain an analogous conclusion with $\tilde{R}$ and $\tilde{\pi}$ in place of R' and $\pi' \restriction N^*$. This is, however, not possible in general. Let m be such that $\omega\varrho^{m+1}_{N^*} \leq \mu < \omega\varrho^m_{N^*}$. Notice that, in fact, $\omega\varrho^{m+1}_{N^*} = \mu = \omega\varrho^\omega_{N^*}$.

Now we apply our induction hypothesis (2) to the premouse N', which is a *proper* initial segment of M. We obtain that $d_{N'} = d^{\lambda^*_M}_{N'}$ is defined and N' is Dodd solid. Since fine structural properties of N' are first-order expressible over N', they are preserved under σ. Thus, $d_N = d^{\lambda^*_W}_N$ is defined and N is Dodd solid. The general properties of fine pseudoultrapowers yield:

- $h_{\tilde{N}}(\lambda^*_M \cup \{\tilde{d}\}) = \tilde{N}$ where $\tilde{d} = \tilde{\sigma}(d_N)$;
- $W^{\delta,\tilde{d}}_{\tilde{N}} \in \tilde{N}$ whenever $\beta \in \tilde{d}$.

In the latter case, of course, we first conclude that each $\delta \in \tilde{d}$ has a generalized Dodd solidity witness with respect to $\tilde{d}$ and $\tilde{N}$ that is in $\tilde{N}$, and then use the generalized witness to obtain the standard Dodd solidity witness by applying (1). (Recall: now we work in the language for coherent structures.)

Assume for a contradiction that $E_\beta = \varnothing$. Then obviously $\beta < \mathrm{ht}(N^*)$, and, moreover,

$$\lambda(N^*) > \mu \text{ whenever } N^* \text{ is active.}$$

It follows that $\tilde{R}$ is either passive or is a premouse of the same type as N^*, and $\tilde{\pi}$ is $\Sigma_0^{(m)}$-preserving with respect to the language for premice and cofinal at the level m. The verification of this, as well as that of the following clauses, is standard:

- $k(\tilde{\pi}(f)(\delta)) = \pi'(f)(\sigma'(\delta))$ for $f \in \Gamma(\mu, N^*)$ and $\delta < \lambda(\tilde{N})$;
- $k \restriction J_{\tilde{\nu}}^{E^{\tilde{N}}} = \sigma'$ where $\tilde{\nu} = \mathrm{ht}(\tilde{N})$;
- $k \restriction \gamma_M = \mathrm{id}$;
- $k : \tilde{R} \to R'$ is $\Sigma_0^{(m)}$-preserving with respect to the language for premice;
- $\tilde{h}_{\tilde{R}}^{m+1}(\lambda(\tilde{N}) \cup \{\tilde{\pi}(p_{N^*})\}) = \tilde{R}$;
- every $\delta \in \tilde{\pi}(p_{N^*})$ has a generalized witness with respect to $\tilde{R}$ and $\tilde{\pi}(p_{N^*})$, that is an element of $\tilde{R}$.

Our next task is to show that $\omega\varrho_{\tilde{R}}^{m+1} \le \lambda_M^*$ and $\tilde{R}$ is sound above λ_M^*. This together with the middle two clauses on the above list will enable us to apply the Condensation Lemma and derive a contradiction the same way we did in the proof of Claim 1. Toward the soundness of $\tilde{R}$ above λ_M^*, we first show:

(30) $$\lambda(\tilde{G}) \subset \tilde{h}_{\tilde{R}}^{m+1}(\lambda_M^* \cup \{\tilde{d} \cup \tilde{\pi}(p_{N^*})\})$$

(31) $$W_{\tilde{R}}^{\delta, \tilde{d} \cup \tilde{\pi}(p_{N^*})} \in \tilde{R} \text{ whenever } \delta \in \tilde{d}$$

The former together with the next-to-last clause on the list in the previous paragraph will imply

(32) $$\tilde{h}_{\tilde{R}}^{m+1}(\lambda_M^* \cup \{\tilde{d} \cup \tilde{\pi}(p_{N^*})\}) = \tilde{R},$$

and, consequently, that $\omega\varrho_{\tilde{R}}^{m+1} \le \lambda_M^*$. This inequality together with the middle two clauses from that list guarantee the solidity of $\tilde{R}$. Finally (32) combined with (31), the last clause on the above list, and the solidity of $\tilde{R}$ yield $\tilde{d} \cup \tilde{\pi}(p_{N^*}) = p_{\tilde{R}} - \lambda_M^*$, and thereby the soundness of $\tilde{N}$ above λ_M^*. (That $\tilde{d} \cap \lambda_M^* = \varnothing$, follows from the fact that $d_N \cap \lambda_W^* = \varnothing$.)

Clauses (30) and (31) follow from lemmata established in [12], Section 2, where a more general theory is developed. The proof of (31) makes use of the induction hypothesis (2). In order to show how (2) is used, and also to make this paper self-contained, we give the proof of both.

First focus on (30). Given a $\zeta < \lambda(\tilde{N})$, we have seen above (proof of the current claim, the fourth paragraph) that $\zeta = h_{\tilde{N}}(\xi, \tilde{d})$ for some $\xi < \lambda^*_M$. Let $H(z,x,y,w)$ be a $\Sigma_0(\tilde{N})$-relation that determines the Skolem function $h_{\tilde{N}}$, i.e. such that for every $x, y, w \in J^{E^{\tilde{N}}}_{\tilde{\nu}}$ we have

$$y = h_{\tilde{N}}(x,w) \;\longleftrightarrow\; (\exists z)H(z,y,x,w).$$

We stress that now we work in the language for coherent structures, as $\tilde{N}$ is a coherent structure that is not a potential premouse. Fix $\bar{\beta} < \beta$ such that, letting $\bar{\nu} = \tilde{\pi}(\bar{\beta})$, there is a $z \in J^{E^{\tilde{R}}}_{\bar{\nu}} = J^{E^{\tilde{N}}}_{\bar{\nu}}$ satisfying $H(z, \zeta, \xi, \tilde{d})$. This is possible, since $\tilde{\pi}$ maps β cofinally into $\tilde{\pi}(\beta) = \tilde{\nu}$. Define a function $\tilde{f} : {}^{1+|\tilde{d}|}\lambda(\tilde{N}) \to \lambda(\tilde{N})$ by

$$\tilde{f}(\xi', w) = \begin{cases} \text{the unique } \zeta' < \lambda(\tilde{N}) \text{ with } (\exists z \in J^{E^{\tilde{R}}}_{\bar{\nu}})H(z, \zeta', \xi', w), \text{ if defined;} \\ 0, \text{ otherwise.} \end{cases}$$

Obviously, $\zeta = \tilde{f}(\xi, \tilde{d})$. The function $\tilde{f}$ is Σ_0-definable over $\langle J^{E^{\tilde{R}}}_{\bar{\nu}}, \tilde{G} \cap J^{E^{\tilde{R}}}_{\bar{\nu}} \rangle$. Now $J^{E^{\tilde{R}}}_{\bar{\nu}}$ is in the range of $\tilde{\pi}$, and therefore is an element of $\tilde{h}^{m+1}_{\tilde{R}}(\tilde{\mu} \cup \{\tilde{\pi}(p_{N^*})\})$. The predicate $\tilde{G} \cap J^{E^{\tilde{R}}}_{\bar{\nu}}$ can be expressed as $\{\langle \tilde{\pi}(g)(\eta) \cap \mu, \tilde{\pi}(g)(\eta) \rangle;\, \eta < \mu\}$ where $g \in N^*$ is a surjection of μ onto $\mathcal{P}(\mu) \cap J^{E}_{\bar{\beta}}$, so $\tilde{G} \cap J^{E^{\tilde{R}}}_{\bar{\nu}} \in \tilde{h}^{m+1}_{\tilde{R}}(\mu \cup \{\tilde{\pi}(p_{N^*}), \mu\})$. Taken together, $\tilde{f} \in \tilde{h}^{m+1}_{\tilde{R}}(\mu \cup \{\tilde{\pi}(p_{N^*}), \mu\})$, so $\zeta \in \tilde{h}^{m+1}_{\tilde{R}}(\mu \cup \{\mu, \xi, \tilde{d} \cup \tilde{\pi}(p_{N^*})\})$. This completes the proof of (30) $\square$(30)

Now turn to the proof of (31). Let $\delta \in \tilde{d}$, let $\tilde{W}_\delta = W^{\delta, \tilde{q}}_{\tilde{N}}$ and let $\sigma_\delta : \tilde{W}_\delta \to \tilde{N}$ be the canonical witness map. $\tilde{W}_\delta$ is computed in the language for coherent structures. (Obviously, $\tilde{W}_\delta$ fails to be a potential premouse for the same reason $\tilde{N}$ fails to be.) This map is Σ_1-preserving with respect to the language for coherent structures and its range is precisely $h_{\tilde{N}}(\delta \cup \{\tilde{d} - (\delta+1)\})$. Thus, $\tilde{W}_\delta = h_{\tilde{W}_\delta}(\delta \cup \{\tilde{d}_\delta\})$ where $\sigma_\delta(\tilde{d}_\delta) = \tilde{d} - (\delta+1)$. Denote the top extender of $\tilde{W}_\delta$ by $\tilde{G}_\delta$. Since $\sigma_\delta \restriction \delta = \mathrm{id}$, the domain of $\tilde{G}_\delta$ is exactly $\mathcal{P}(\mu) \cap N^* = \mathrm{dom}(\tilde{G})$. Let $W_\delta = \mathrm{Ult}^*(N^*, \tilde{G}_\delta)$ and let $\tilde{\pi}_\delta : N^* \to W_\delta$ be the associated ultrapower map. The existence of W_δ follows from the preservation properties of the assignment $[\xi, f] \mapsto \tilde{\pi}(f)(\sigma_\delta(\xi))$ where $\xi < \lambda(\tilde{G}_\delta)$

and $f \in \Gamma(\mu, N^*)$. This assignment gives rise to a map $\tilde\sigma_\delta : W_\delta \to \tilde R$. We show:

$$W_\delta = W^{\delta, \tilde d \cup \tilde\pi(p_{N^*})}_{\tilde R}. \tag{33}$$

We have seen above that $\tilde R$ is a premouse of the same type as N^* and $\tilde\pi : N^* \to \tilde R$ is $\Sigma_0^{(m)}$-preserving with respect to the language for premice and cofinal at the level m. The same argument yields the same conclusion for W_δ and $\tilde\pi_\delta : N^* \to W_\delta$. The map $\sigma_\delta : W_\delta \to \tilde R$ is $\Sigma_0^{(m)}$-preserving with respect to the language for premice, as follows by the standard argument based on the Łoś Theorem. We also obtain the cofinality of $\tilde\sigma_\delta$ at the level m; this is a consequence of the cofinality of both $\tilde\pi$ and $\tilde\pi_\delta$ the level m. So $\tilde\sigma_\delta$ is $\Sigma_1^{(m)}$-preserving with respect to the language for premice. The definition of $\tilde\sigma_\delta$ guarantees that $\tilde\sigma_\delta(\tilde\pi_\delta(p_{N^*})) = \tilde\pi(p_{N^*})$ and $\tilde\sigma_\delta \restriction \lambda(\tilde G) = \sigma_\delta \restriction \lambda(\tilde G)$; the latter implies that $\tilde\sigma_\delta(\tilde d_\delta) = \tilde d - (\delta + 1)$. Now since $\tilde W_\delta = h_{\tilde W_\delta}(\delta \cup \{\tilde d_\delta\})$, we can imitate the proof of (32) and infer $\tilde h^{m+1}_{W_\delta}(\delta \cup \{\tilde\pi_\delta(p_{N^*}), \tilde d_\delta\}) = W_\delta$. This together with the preservation properties of $\tilde\sigma_\delta$ just established guarantees that $\mathrm{rng}(\tilde\sigma_\delta) = \tilde h^{m+1}_{\tilde R}(\delta \cup \{\tilde\pi(p_{N^*}), \tilde d - (\delta + 1)\})$, which completes the proof of (33).

Recall that we proved that $\tilde W_\delta \in \tilde N$. We also know that $\tilde W_\delta$ can be encoded into a $\mathbf{\Sigma}_1(\tilde N)$ set $b \subset \delta$. Since $\delta < \tilde\lambda \overset{\text{def}}{=} \lambda(\tilde G)$, we see that $b \in J^{E^{\tilde N}}_{\tilde\lambda} = J^{E^{\tilde R}}_{\tilde\lambda}$. Now $J^{E^{\tilde R}}_{\tilde\lambda}$ is a ZFC^- model, so inside this structure we can both recover $\tilde W_\delta$ and form $\mathrm{Ult}(N^*, \tilde G_\delta)$. It follows that $W^{\delta, \tilde d \cup \tilde\pi(p_{N^*})}_{\tilde R} \in \tilde R$. □(31)

Finally we obtained the soundness of $\tilde R$ above λ^*_M, which allows us to turn to the application of the Condensation Lemma. The embedding in question is $k : \tilde R \to R'$ and, as we have already indicated above, we can then proceed exactly as in the proof of Claim 1, with k in place of σ'. □(Claim 4)

Claim 5. $\mathrm{ht}(N) = \gamma$.

Proof. Assume for a contradiction that the claim is false, that is, $\mathrm{ht}(N) > \gamma$. Define R', π', $\tilde R$, $\tilde\pi$ and k as in the proof of Claim 4. The structure $\tilde R$ is an active potential premouse. The map $k : \tilde R \to R'$ is Σ_0-preserving with respect to the language for coherent structures and cofinal. The cofinality of k follows from the fact that $E^{R'}_{\mathrm{top}}$ and $E^{\tilde R}_{\mathrm{top}}$ have the same critical point that is *below* μ, μ is a cardinal in both R' and $\tilde R$ and $k \restriction \mu = \mathrm{id}$. Claim 4 implies that $N^* = \tilde N \,||\, \beta = M \,||\, \beta$, so $\lambda(\tilde R) = \tilde\pi(\mu) = \lambda(\tilde G)$. Since γ is the cardinal successor of λ^*_W in the sense of N, our assumption $\mathrm{ht}(N) > \gamma$ implies that $\lambda(N) > \lambda^*_W$, and, consequently, $\lambda(\tilde{R}) = \lambda(\tilde{N}) > \lambda^*_M$.

Working in the language for coherent structures, we argue similarly as in the proof of Claim 4 to get the following analogues to (30) and (31).

$$\lambda(\tilde{G}) \subset h^*_{\tilde{R}}(\lambda^*_M \cup \{\tilde{d}\}) \tag{34}$$

$$^*W^{\delta,\tilde{d}}_{\tilde{R}} \in \tilde{R} \text{ whenever } \delta \in \tilde{d}. \tag{35}$$

As before, the proof of the latter uses the Dodd solidity of N, and thus makes use of the induction hypothesis (2). The inclusion (34) immediately implies

$$h^*_{\tilde{R}}(\lambda^*_M \cup \{\tilde{d}\}) = \tilde{R}, \tag{36}$$

which tells us that $d^{\lambda^*_M}_{\tilde{R}}$ is defined and $\omega\varrho^1_{\tilde{R}} \leq \lambda^*_M < \lambda(\tilde{R})$. It follows that $\tilde{R}$ cannot be a type C potential premouse (see the introduction). In fact, $\tilde{R}$ must be a type B premouse, as μ is easily seen to be a cutpoint of $E^{\tilde{R}}_{\mathrm{top}}$.

We claim that $\tilde{R}$ is a *premouse,* that is, it satisfies the initial segment condition. Consider two cases. If $\tilde{d} \neq \varnothing$, let $\tilde{\delta} = \max(\tilde{d})$. Then $\tilde{\delta} \geq \lambda^*_{\tilde{R}}$ (to see this, recall that an argument similar to the proof of (30) shows that $\lambda \notin h^*_{\tilde{R}}(\lambda)$ for any cutpoint λ of $E^{\tilde{R}}_{\mathrm{top}}$). So $\sigma_{\tilde{\delta}} \restriction \lambda^*_{\tilde{R}} = \mathrm{id}$ where $\sigma_{\tilde{\delta}} : {}^*W^{\tilde{\delta},\tilde{d}}_{\tilde{R}} \to \tilde{R}$ is the canonical witness map. From this and (35) we immediately get

$$E^{\tilde{R}}_{\mathrm{top}} \restriction \lambda^*_{\tilde{R}} = E^{^*W^{\tilde{\delta},\tilde{d}}_{\tilde{R}}}_{\mathrm{top}} \restriction \lambda^*_{\tilde{R}} \in \tilde{R},$$

which tells us that $\tilde{R}$ satisfies the initial segment condition. Now assume $\tilde{d} = \varnothing$. This means that no $\lambda \geq \lambda^*_M$ can be a cutpoint of $E^{\tilde{R}}_{\mathrm{top}}$. Consequently, $\lambda^*_{\tilde{R}} < \lambda^*_M$. But we know that $k \restriction \gamma_M = \mathrm{id}$, and using this it is easy to see that $E^{\tilde{R}}_{\mathrm{top}} \,|\, \lambda^*_{\tilde{R}} = E^{R'}_{\mathrm{top}} \,|\, \lambda^*_{\tilde{R}}$. By the initial segment condition for R', the extender $E^{R'}_{\mathrm{top}} \,|\, \lambda^*_{\tilde{R}}$ is an element of R'. But R' agrees with $\tilde{R}$ up to λ^*_M and λ^*_M is a cardinal in both structures, so $E^{\tilde{R}}_{\mathrm{top}} \,|\, \lambda^*_{\tilde{R}} \in \tilde{R}$. This proves that $\tilde{R}$ is a premouse.

Our aim is to apply the Condensation Lemma and obtain a contradiction in the same way as in the proof of Claim 4. Since the Condensation Lemma is formulated in the language for premice, we have to replace k by a suitable Σ_1-preserving map (with respect to the language for coherent structures) $\tilde{k} : \tilde{R} \to \hat{R}$ where where $\hat{R}$ is a type B premouse and $\tilde{k}(\gamma_{\tilde{R}}) = \gamma_{\hat{R}}$. The Condensation Lemma cannot be applied to k directly, as R' is a type C premouse. This follows from the elementarity of

π' and the fact that $N^* = M\,||\,\beta$ is a type C premouse; the latter, in turn, is a consequence of the fact that $\lambda(E_\beta) = \mu$ is inaccessible in M. Let $\lambda^* = k(\lambda^*_{\tilde{R}})$ and $\hat{\lambda} = \sup(k''\lambda(\tilde{R}))$. We show:

(37) $\hat{\lambda}$ is the least cutpoint of $E^{R'}_{\rm top}$ larger than λ^*.

To see that $\hat{\lambda}$ is a cutpoint of $E^{R'}_{\rm top}$, it suffices to show that $E^{R'}_{\rm top}(g)(\delta) < \hat{\lambda}$ whenever $\delta < \hat{\lambda}$ and $g : \bar{\mu} \to \bar{\mu}$ is a monotonic function that is an element of N^*; here $\bar{\mu} = \mathrm{cr}(E^{\tilde{R}}_{\rm top}) = \mathrm{cr}(E^{R'}_{\rm top})$. (It is enough to consider just monotonic functions, as every $g' : \bar{\mu} \to \bar{\mu}$ is majorized by such a function.) Now if $\tilde{\delta} < \lambda(\tilde{R})$ is such that $\delta \le k(\tilde{\delta})$, we have:

$$E^{R'}_{\rm top}(g)(\delta) \le E^{R'}_{\rm top}(g)(k(\tilde{\delta})) = k(E^{\tilde{R}}_{\rm top}(g)(\tilde{\delta})) < \hat{\lambda}.$$

Now focus on the proof that no cutpoint of $E^{R'}_{\rm top}$ lies in the interval $(\lambda^*, \hat{\lambda})$. Since $\lambda^*_{\tilde{R}}$ is the largest cutpoint of $E^{\tilde{R}}_{\rm top}$, there are cofinally many ordinals in $\lambda(\tilde{R})$ that are of the form $E^{\tilde{R}}_{\rm top}(g)(\xi)$ for some g as above and some $\xi = \prec\xi_1, \dots, \xi_\ell\succ$ where $\xi_1, \dots, \xi_\ell \le \lambda^*_{\tilde{R}}$ (here $\prec\xi_1, \dots, \xi_\ell\succ$ is the Gödel ℓ-tuple). By the preservation properties of k, there are cofinally many ordinals in $\hat{\lambda}$ that are of the form $E^{R'}_{\rm top}(g)(\xi')$ where g is as above and $\xi' = \prec\xi'_1, \dots, \xi'_\ell\succ$ where $\xi'_1, \dots, \xi'_\ell \le \lambda^*$. □(37)

Let $\hat{R}$ be the initial segment of R' whose top extender is $E^{R'}_{\rm top}\,|\,\hat{\lambda}$. By the above discussion, $\hat{R}$ is a type B premouse and $\lambda^*_{\hat{R}} = \lambda^*$. Let $\hat{k} : \hat{R} \to R'$ be the canonical factor map, so $\hat{k} : E^{\hat{R}}_{\rm top}(g)(\delta') \mapsto E^{R'}_{\rm top}(g)(\delta')$ whenever $g : \bar{\mu} \to \mathcal{P}(\bar{\mu})$ is an element of J^E_τ and $\delta' < \hat{\lambda}$. (This again involves our abuse of notation.) We define the map $\tilde{k} : \tilde{R} \to \hat{R}$ by $\tilde{k} = \hat{k}^{-1} \circ k$. It is easy to check that

$$\tilde{k}(E^{\tilde{R}}_{\rm top}(g)(\delta)) = E^{\hat{R}}_{\rm top}(g)(k(\delta))$$

for g as above and $\delta < \lambda(\tilde{R}) = \lambda(\tilde{G})$. That $\tilde{k}$ is Σ_0-preserving with respect to the language for coherent structures and cofinal follows from the fact that both $\hat{k}$ and k have these properties. As an immediate consequence of the definition of $\tilde{k}$ we have

(38) $$\tilde{k} \restriction \lambda(\tilde{R}) = k \restriction \lambda(\tilde{R}) = \sigma' \restriction \lambda(\tilde{R}),$$

so $\tilde{k}(\lambda^*_{\tilde{R}}) = \lambda^*_{\hat{R}}$. Since $\tilde{R}$ satisfies the initial segment condition, $E^{\tilde{R}}_{\gamma_{\tilde{R}}} = E^{\tilde{R}}_{\rm top}\,|\,\lambda^*_{\tilde{R}}$. Using the preservation properties of $\tilde{k}$ we obtain $E^{\hat{R}}_{\tilde{k}(\gamma_{\tilde{R}})} = E^{\hat{R}}_{\rm top}\,|\,\lambda^*_{\hat{R}}$, which proves that $\tilde{k}(\gamma_{\tilde{R}}) = \gamma_{\hat{R}}$. This means that $\tilde{k} : \tilde{R} \to \hat{R}$ is Σ_1-preserving with respect to the language for premice.

We know that $\omega\varrho^1_{\tilde{R}} \leq \lambda^*_M$, as this follws from (36). We also have $\tilde{k} \restriction \gamma_M = \mathrm{id}$, as this follows from (38). These two facts combined with the preservation properties of $\tilde{k}$ established in the previous paragraph and with the first part of the Condensation Lemma yield the weak iterability of $\tilde{R}$ (in fact they yield (ω_1, ω_1+1)-iterability of $\tilde{R}$, as we are assuming that M is countable). In order to be able to apply the second part of the Condensation Lemma and obtain the desired contradiction, we need to show that $\tilde{R}$ is sound above λ^*_M. Now $\hat{R}$, being a proper initial segment of R', is a proper initial segment of M. Since $\tilde{R}$ is embeddable into $\hat{R}$, we have $\mathrm{ht}(\tilde{R}) \leq \mathrm{ht}(\hat{R}) < \mathrm{ht}(M)$. By the induction hypothesis (2), Theorem 1.2 holds of $\tilde{R}$. We have just shown that $\tilde{R}$ is weakly iterable; that the rest of the assumptions of Theorem 1.2 is satisfied follows from (35) and (36). This guarantees the soundness of $\tilde{R}$ above $\lambda^*_{\tilde{R}}$. Now we can proceed exactly as in the proof of Claim 4 and get a contradiction. $\square$(Claim 5)

It is now obvious that the proof of Lemma 1.12 is merely a direct combination of the five claims we have just established. $\square$(Lemma 1.12)

In view of Lemma 1.12, we can summarize the current state of affairs as follows.

- $F = E^M_{\mathrm{top}}$ and $\bar{F} \stackrel{\mathrm{def}}{=} E^W_{\mathrm{top}}$;
- $\gamma' = \sigma(\gamma)$ and $N' = \sigma(N)$;
- $G = E^N_{\mathrm{top}} = E^W_\gamma$, $G' = E^{N'}_{\mathrm{top}} = E_{\gamma'}$ and $\tilde{G} = G' \cap J^E_{\gamma_M}$;
- $\mu = \mathrm{cr}(G) = \mathrm{cr}(G')$, as $\sigma(\mu) = \mu$;
- $\beta = \mu^{+N} = \mu^{+W}$ is the critical point of σ and $\beta' \stackrel{\mathrm{def}}{=} \sigma(\beta)$.

Since E_β is a total extender in M and $\beta = \mathrm{cr}(\sigma)$, an easy argument shows that, in the sense of M, there are stationarily many ordinals $\bar{\beta} \subset \beta'$ with $E_{\bar{\beta}} \neq \varnothing$. This means that μ is subcompact in M (see [12]). Also, E_{γ_M} witnesses that μ is superstrong in M (see the introduction). These observations give a proof of Proposition 1.3: Under the assumption that no cardinal is both subcompact and superstrong in an inner model, we immediately obtain that β as above cannot exist, so $e^\alpha_M \cap \lambda^*_M = \varnothing$.

Now focus on the general case. The structure $\langle J_\gamma^{E^W}, \bar{F}\,|\,\lambda_W^*\rangle$ is a potential premouse. If $\bar{F}\,|\,\lambda_W^*$ were an element of W, it would be equal to E_γ^W. Consequently, $\sigma(\bar{F}\,|\,\lambda_W^*)$ would be equal to $F\,|\,\lambda_W^* = E_{\gamma_M}$. So $\gamma_M \in \mathrm{rng}(\sigma)$, contradicting (29). It follows that $\bar{F}\,|\,\lambda_W^* \notin W$, so W fails to satisfy the initial segment condition. The structure $\langle J_\gamma^{E^W}, E_\gamma^W, \bar{F}\,|\,\lambda_W^*\rangle$ looks like a bicephalus, and it is not difficult to prove that it is iterable. Because of this, one might attempt to apply the bicephalus lemma to conclude that $\bar{F}\,|\,\lambda_W^* = E_\gamma^W$, and thereby obtain a contradiction. This contradiction would show that β as above does not exist, that is, it would enable us to remove the smallness condition from the assumptions of Proposition 1.3. However, the bicephalus argument is not going to work here, as we will see in a little while that $\langle J_\gamma^{E^W}, \bar{F}\,|\,\lambda_W^*\rangle$ fails to satisfy the initial segment condition.

Let $\bar{\nu} = \mathrm{ht}(W)$ and

- $\bar{\pi} : J_\tau^E \to J_{\bar{\nu}}^{E^W}$ be the ultrapower embedding associated with $\mathrm{Ult}(J_\tau^E, \bar{F})$;
- $\bar{\pi}^* : J_\tau^E \to J_\gamma^{E^W}$ be the ultrapower embedding coming from $\mathrm{Ult}(J_\tau^E, \bar{F}\,|\,\lambda_W^*)$;
- $\bar{k} : J_\gamma^{E^W} \to J_{\bar{\nu}}^{E^W}$ be the canonical map defined by $\bar{k} : \bar{\pi}^*(f)(\xi) \mapsto \bar{\pi}(f)(\xi)$ for $f \in {}^\kappa\kappa \cap J_\tau^E$ and $\xi < \lambda_W^*$; so $\bar{k} \restriction \lambda_W^* = \mathrm{id}$ and $\bar{\pi} = \bar{k} \circ \bar{\pi}^*$;
- $\bar{K}$ be the extender derived from $\bar{k}$.

Using standard arguments we infer that $\mathrm{cr}(\bar{k}) = \lambda_W^*$, $\bar{k}(\lambda_W^*) = \lambda(W)$ and $\bar{K}$ has no cutpoints (see the proof of Lemma 1.6, the discussion at the beginning of Case 2). Since $\bar{k}$ is a cofinal map, there is a unique predicate $\bar{H}$ on $J_{\bar{\nu}}^{E^W}$ such that

$$\bar{k} : \langle J_\gamma^{E^W}, G\rangle \to \langle J_{\bar{\nu}}^{E^W}, \bar{H}\rangle \text{ is } \Sigma_0\text{-preserving.}$$

The cofinality of $\bar{k}$ guarantees that

$$\bar{Q} = \langle J_{\bar{\nu}}^{E^W}, \bar{H}\rangle$$

is a coherent structure, and since $\mathrm{cr}(\bar{k}) = \lambda_W^*$, $\bar{H}$ is an extender with critical point μ, λ_W^* is a cutpoint of $\bar{H}$ and $\lambda(\bar{H}) = \lambda(W)$. It also follows that $\bar{H}$ is a total extender on $J_{\bar{\nu}}^{E^W}$, so $\bar{Q}$ is a potential premouse. And since $\bar{K}$ has no cutpoints (any cutpoint of $\bar{K}$ would be a cutpoint of $\bar{F}$ larger than λ_W^*), there are no cutpoints of $\bar{H}$ above λ_W^*, so λ_W^* is the largest cutpoint of $\bar{H}$. The definition of $\bar{H}$ guarantees that $\bar{H}\,|\,\lambda_W^* = G = E_\gamma^W$, so $\bar{Q}$ satisfies the initial segment condition. It follows that $\bar{Q}$ is a type B premouse and $\lambda_{\bar{Q}}^* = \lambda_W^*$.

Lemma 1.13. $\bar{H} \circ E_\beta = \bar{F}$.

Proof. We first show that $\bar{F} \,|\, \lambda^*_W$ factors in a similar way and then use this information to get the desired conslusion about $\bar{F}$.

CLAIM 1. $\tilde{G} \circ E_\beta = E_{\gamma_M}$. Hence $\mathrm{cr}(E_\beta) = \mathrm{cr}(F) = \kappa < \mu$. Moreover, μ is a cutpoint of both F and $\bar{F}$.

Proof. We use the notation from the proof of Claim 4 in Lemma 1.12. Notice that $N^* = M \,||\, \beta$ in our case. We have the maps $\tilde{\pi} : N^* \to \tilde{R}$, $\pi' \restriction N^* : N^* \to R'$ and $k : \tilde{R} \to R'$. Both $\tilde{R}$ and R' are coherent structures whose top extenders have critical points equal to $\mathrm{cr}(E_\beta) < \mu$. Furthermore, $\lambda(\tilde{R}) = \lambda(\tilde{G}) = \lambda(G') = \lambda(R')$ and this value is λ^*_M. It is now easy to see that $k \restriction (\lambda^*_M + 1) = \mathrm{id}$, so in fact $k = \mathrm{id}$ and $R' = \tilde{R}$. But $\mathrm{ht}(\tilde{R}) = \mathrm{ht}(\tilde{N}) = \gamma_M$ and R' is an initial segment of M, so $E^{\tilde{R}}_{\mathrm{top}} = E_{\gamma_M}$. It follows that $\mathrm{cr}(E_\beta) = \mathrm{cr}(E^{\tilde{R}}_{\mathrm{top}}) = \mathrm{cr}(E_{\gamma_M}) = \mathrm{cr}(F) = \kappa$, which proves the second part of the lemma. It also follows that μ is a cutpoint of $E^{\tilde{R}}_{\mathrm{top}}$, which yields the last sentence in the lemma. Regarding the first part of the lemma, for $x \in \mathcal{P}(\mathrm{cr}(E_\beta)) \cap N^*$ we have:

$$(\tilde{G} \circ E_\beta)(x) = \tilde{G}(E_\beta(x)) = \tilde{\pi}(E_\beta(x)) = E^{\tilde{R}}_{\gamma_M}(\tilde{\pi}(x)) = E_{\gamma_M}(x),$$

which completes the proof. □(Claim 1)

CLAIM 2. $G \circ E_\beta = \bar{F} \,|\, \lambda^*_W$.

Proof. Given any $x \in \mathcal{P}(\kappa) \cap W$, we have

$$\begin{aligned} \sigma((G \circ E_\beta)(x)) &= \sigma(G(E_\beta(x))) = G'(E_\beta(x)) = \tilde{G}(E_\beta(x)) \\ &= (\tilde{G} \circ E_\beta)(x) = E_{\gamma_M}(x) = F(x) \cap \lambda^*_M = \sigma(\bar{F}(x) \cap \lambda^*_W). \end{aligned}$$

The third equality here follows from the fact that $\tilde{G}$ and G' agree on $\mathcal{P}(\mu) \cap J^E_\beta$ and the fifth one follows from Claim 1. □(Claim 2)

The following computation then completes the proof of the lemma:

$$\bar{F} = \bar{K} \circ (\bar{F} \,|\, \lambda^*_W) = \bar{K} \circ G \circ E_\beta = \bar{H} \circ E_\beta.$$

The equality on the left follows from the fact that $\bar{\pi} = \bar{k} \circ \bar{\pi}^*$; this fact also implies $\bar{H}(x) = \bar{k}(G(x)) = \bar{K}(G(x))$ for all $x \in \mathcal{P}(\kappa) \cap W$, which yields the equality on the right. The middle equality comes from Claim 2. □(Lemma 1.13)

Our aim is to prove that $W \in M$, and since W is a coherent structure, this amounts to showing that $\bar{F} \in M$. By Lemma 1.13, it suffices to establish the following.

Lemma 1.14. $\bar{H} \in M$.

Proof. We generalize the argument from the proof of Lemma 1.6. As before, the argument splits into two cases. In the present situation, β plays the role γ_M played in that proof. The case where $\mathcal{P}(\beta) \cap \mathrm{Ult}^*(M, E_\beta) \neq \mathcal{P}(\beta) \cap \mathrm{Ult}(M, E_\beta)$ can be handled in the same way as before, and we will leave the details to the reader. Thus, for the rest of the argument we will assume

$$\mathcal{P}(\beta) \cap \mathrm{Ult}^*(M, E_\beta) = \mathcal{P}(\beta) \cap \mathrm{Ult}(M, E_\beta). \tag{39}$$

We have already constructed $\bar{Q}$; this structure which will play an analogous role as $\bar{Q}$ constructed in the proof of Lemma 1.6. One major difference between the two situations is that in the present case, $\bar{Q}$ is a type B premouse. The next step is to define the structure Q. We have seen in Case 1 in the proof of Lemma 1.13 that μ is a cutpoint of F. This enables us to repeat the construction described in (16), this time with μ in place of λ^*_M and set

$$Q = Q(M, \mu)$$

where, given an active premouse P and a cutpoint υ of E^P_{top},

$$Q(P, \upsilon) = \langle J^{E^P}_{\nu_P}, H_{P,\upsilon} \rangle.$$

Here $\nu_P = \mathrm{ht}(P)$ and, letting ζ_υ be the index of $E^P_{\mathrm{top}} \,|\, \upsilon$ and $\upsilon_{P,\upsilon} : P \,||\, \zeta_\upsilon \to P$ be the canonical factoring map, $H_{P,\upsilon}$ is the extender derived from $\upsilon_{P,\upsilon}$. (Precisely: $\upsilon_{P,\upsilon} : \pi_{P,\upsilon}(f)(\xi) \mapsto \pi_P(f)(\xi)$ for $f \in P$ with $\mathrm{dom}(f) = \mathrm{cr}(E^P_{\mathrm{top}})$ and $\xi < \upsilon$; here $\pi_{P,\upsilon}$ and π_P are the ultrapower maps associated with $\mathrm{Ult}(J^{E^P}_{\tau_P}, E^P_{\mathrm{top}} \,|\, \upsilon)$ and $\mathrm{Ult}(J^{E^P}_{\tau_P}, E^P_{\mathrm{top}})$, respectively, and $\tau_P = (\mathrm{cr}(E^P_{\mathrm{top}})^{+P})$.) The proof of (18) can be literally repeated with present structures $\bar{Q}$ and Q, and the ordinal μ in place of λ^*_M, so we have

$$\sigma : \bar{Q} \to Q \text{ is } \Sigma_0\text{-preserving and cofinal.}$$

Again, here we work in the language for coherent structures. As before, we see that $\langle M, \bar{Q}, \beta \rangle$ is a good phalanx and that the pair of maps $\langle \mathrm{id}, \sigma \rangle$ is an embedding of this phalanx into the pair $\langle M, Q \rangle$ in the sense described in the proof of Lemma 1.6.

We again write $\bar{Q}_0$ for $\bar{Q}$ and $\bar{Q}_{-1}$ for M and compare $\langle M, \bar{Q}, \beta\rangle$ with M. This gives rise to iterations $\bar{\Im}$, $\tilde{\Im}$ and $\Im$ where $\bar{\Im}$ and $\Im$ constitute the coiteration in question and $\tilde{\Im}$ is the "copy" of $\bar{\Im}$ in the sense described in the proof of Lemma 1.6. This works in the present situation, since the obvious generalization of Lemma 1.8 goes through — this time for extenders with critical points above μ (actually, this is true for arbitrary extenders that are weakly amenable with respect to Q). More precisely, we have:

> If F_P is a total extender on P with $\mathrm{cr}(F_P) > \mu$ and $\pi : P \to P'$ is the ultrapower map associated with $\mathrm{Ult}(P, F_P)$, then $Q(P', \mu) = \mathrm{Ult}(Q(P), F_P)$ and $\pi : Q(P, \mu) \to Q(P', \mu)$ is the associated ultrapower map.

Clauses (a) – (e) concerning the "copying" construction from the proof of Lemma 1.6 hold with:

- the current versions of $\bar{Q}_0$ and $\bar{Q}_{-1}$;
- $Q(\tilde{M}_i, \mu)$ in place of $Q(\tilde{M}_i)$;
- β in place of $\gamma_{\tilde{M}_0}$;
- μ in place of λ^*.

Our notation is consistent with that in the proof of Lemma 1.6, so for instance the models of $\bar{\Im}$, $\tilde{\Im}$ and $\Im$ are denoted by $\bar{Q}_i$, $\tilde{M}_i$ and M_i, respectively. The application of the Neeman-Steel Lemma can be carried out exactly as before, so we conclude:

- The coiteration of $\langle M, \bar{Q}, \beta\rangle$ with M terminates at some $\theta + 1 < \omega_1$;
- $0 \leq_{\bar{\Im}} \theta$;
- There is no truncation point on the branch $[0, \theta]_{\bar{\Im}}$;
- $\bar{Q}_\theta$ is an initial segment of M_θ.

The next step in the argument is showing that M wins the coiteration, which is formulated rigorously as the next claim. This was easy to see in the situation from the proof of Lemma 1.6, but requires an argument in the present case.

Claim. Either there is a truncation on the main branch of $\Im$ or else $\bar{Q}_\theta$ is a proper initial segment of M_θ.

Proof. Assume this fails, so $\bar{Q}_\theta = M_\theta$ and both iteration maps $\bar{\pi} : \bar{Q} \to M_\theta$ and $\pi : M \to M_\theta$ along the main branches in $\bar{\Im}$ and $\Im$ are total. The existence of $\bar{\pi}$ guarantees that $\bar{Q}$ is $(\omega_1 + 1, \omega_1)$-iterable. Notice that the coiteration begins with no action on the phalanx side and with the ultrapower by E_β on the M-side.

SUBCLAIM 1. $h^*_{\bar{Q}}(\beta \cup \{\bar{d}\}) = \bar{Q}$ where $\sigma(\bar{d}) = (p_M \cup e^\alpha_M) - (\beta+1)$. Consequently, $\omega\varrho^1_{\bar{Q}} \leq \beta$.

Proof. Choose any ordinal $\zeta < \nu$; we will show that $\zeta \in h^*_{\bar{Q}}(\beta \cup \{\bar{d}\})$. Since $W = h^*_W(\beta \cup \{\bar{d}\})$, the proof of (30) gives us a function $f : {}^{1+|\bar{d}|}\kappa \to \mathcal{P}(\kappa)$, $f \in J^E_\tau$ such that $\zeta = \mathrm{otp}(\bar{F}(f)(\xi, \bar{d}))$ for some $\xi < \beta$ (again, with a slight abuse of notation). By Lemma 1.13, the right side here is equal to $\mathrm{otp}(\bar{H}(f')(\xi, \bar{d}))$ where $f' = E_\beta(f) \in J^E_\beta$. So ζ is $\Sigma_1(M)$-definable from $\bar{d}$ and elements of J^E_β. □(Subclaim 1)

SUBCLAIM 2. $\omega\varrho^1_M \leq \tau$.

Proof. Let $i+1$ be the immediate successor of 0 on the main branch of $\Im$. Recall the ν_i are the indices in the coiteration and κ_i are the critical points in $\Im$. As we have mentioned above, $\nu_0 = \beta$, that is, the first extender applied on the M-side of the coiteration is E_β. It follows that $\nu_i \geq \beta$ and $\kappa_i < \mu = \lambda(E_\beta)$.

Assume that $\omega\varrho^1_M > \tau$. This means that $\omega\varrho^1_{M_{i+1}} > \pi_{0,i+1}(\tau) = \beta$. Since the iteration map $\pi_{i+1,\theta} : M_{i+1} \to M_\theta$ is Σ^*-preserving, $\omega\varrho^1_{M_\theta} \geq \omega\varrho^1_{M_{i+1}} > \beta$. On the other hand, all extenders applied on the main branch of the phalanx side of the coiteration have critical points at least β, so $\omega\varrho^1_{M_\theta} = \omega\varrho^1_{\bar{Q}} \leq \beta$; the inequality on the right here comes from Subclaim 1. Contradiction. □(Subclaim 2)

SUBCLAIM 3. The option $\omega\varrho^1_M \leq \kappa$ is impossible.

Proof. Assume $\omega\varrho^1_M \leq \kappa$. Then both sides of the coiteration are above the first projectum on their main branches. This is clear in the case of $\bar{\Im}$, as we have seen in the proof of Subclaim 2. Let i be as in that claim. If the main branch of $\Im$ failed to be above the first projectum, then necessarily $\kappa_i < \omega\varrho^1_M$, as the critical points ascend along the branches, and the projecta, that are below the critical points, are preserved. Consequently, $\kappa_i < \kappa$, so $i > 0$. But then $\omega\varrho^1_{M_{i+1}} > \lambda(E^{M_i}_{\nu_i}) > \beta$, so $\omega\varrho^1_{M_\theta} > \beta$. As in Subclaim 2, this yields a contradiction.

Since we are assuming that M is sound (see (4)), the conclusion fom the previous paragraph implies that $M = \mathrm{core}(\bar{Q})$. Let $\bar{\sigma} : M \to \bar{Q}$ be the associated core map, $d = (p_M \cup e^\alpha_M) - (\beta+1)$ and $d' = \bar{\sigma}(d)$. Recall that $\sigma(\bar{d}) = d$. We have already proved that ${}^*W^{\delta,d}_M \in M$ for all $\delta \in d$. It follows that for every $\delta' \in d'$, there is a generalized Dodd solidity witness with respect to $\bar{Q}$ and d', that is an element of $\bar{Q}$, namely $\bar{\sigma}({}^*W^{\delta,d}_M)$ where $\delta' = \bar{\sigma}(\delta)$. This together with Subclaim 1 yields the middle inequality in the following formula; the other two inequalities follow from the monotonicity of both $\bar\sigma$ and σ:

$$d \leq^* d' \leq^* \bar d \leq^* d.$$

It follows that $d' = \bar d$, so $h^*_{\bar Q}(\beta \cup \{d'\}) = \bar Q$. In particular, $\bar\sigma(\gamma_M) \in h^*_{\bar Q}(\beta \cup \{d'\})$. Consequently, $\gamma_M \in h^*_M((\bar\sigma^{-1})''\beta \cup \{d\}) \subset h^*_M(\beta \cup \{d\})$. Thus, there is a finite $e \subset \beta$ such that $\gamma_M \in h^*_M(\alpha \cup \{p_M \cup (e^\alpha_M - (\beta+1)) \cup e\})$. This contradicts the minimality of e^α_M, as $(e^\alpha_M - (\beta+1)) \cup e <^* e^\alpha_M$. □(Subclaim 3)

Subclaim 4. The option $\omega\varrho^1_M = \tau$ is impossible.

Proof. The proof is an elaboration on that of Subclaim 3. Assume that $\omega\varrho^1_M = \tau$ and seek for a contradiction. Let i be as in Subclaim 2; we first observe that $i = 0$. Indeed, if $i > 0$, then

$$\omega\varrho^1_{M_\theta} \geq \omega\varrho^1_{M_{i+1}} = \pi_{0,i+1}(\tau) > \pi_{0,i+1}(\kappa_i) = \lambda(E^{M_i}_{\nu_i}) > \beta,$$

which yields a contradiction as in the proof of Subclaim 3. So M_1 is on the main branch of $\Im$ and $\omega\varrho^1_{M_1} = \beta$. Also, M_1 is sound above β, as follows from the soundness of M. Since both branches $[1,\theta]_\Im$ and $[0,\theta]_{\bar\Im}$ are above β, we conclude that $M_1 = \mathrm{core}_\beta(\bar Q)$. Let $\bar\sigma : M_1 \to \bar Q$ be the associated core map.

This time set $d' = \bar\sigma \circ \pi_{0,1}(d)$ (d was defined in the proof of Subclaim 3). We now see that every $\delta' \in d'$ has a generalized Dodd solidity witness with respect to $\bar Q$ and d', that is an element of $\bar Q$. Using Subclaim 1, we argue as in the proof of Subclaim 3 and conclude that $d' = \bar d$. So $h^*_{\bar Q}(\beta \cup \{d'\}) = \bar Q$. In particular, $\bar\sigma \circ \pi_{0,1}(\gamma_M) \in h^*_{\bar Q}(\beta \cup \{d'\})$. It follows that

$$\gamma_M \in h^*_M(((\bar\sigma \circ \pi_{0,1})^{-1})''\beta \cup \{d\}) \subset h^*_M(\beta \cup \{d\}),$$

which yields a contradiction with the minimality of e^α_M as before. □(Subclaim 4)

By Subclaim 2, either $\omega\varrho^1_M = \tau$ or $\omega\varrho^1_M \leq \kappa$. Subclaims 3 and 4 guarentee that neither of these options can occur, which yields a contradiction and thereby completes the proof of the Claim. □(Claim)

The rest of the proof of Lemma 1.14 is standard. By Subclaim 1 above, $\bar Q$ can be encoded into some $\mathbf{\Sigma}_1(\bar Q)$ subset b of β. As in the proof of Lemma 1.6 we show that $b \in M_1$, so $b \in \mathrm{Ult}(M, E_\beta)$ by (39). Then we apply the argument from the end of the proof of Lemma 1.6 to conclude that $b \in M$. Using the fact that J^E_ν is a ZFC^--model, we decode b inside M, so $\bar{Q} \in M$. □(Lemma 1.14)

This also completes the proof of Theorem 1.1. □(Theorem 1.1)

Proof of Theorem 1.2. We verify (a) first. The assumptions of the theorem imply that d^α_M is defined and $d^\alpha_M \leq^* d$. Assuming that $d^\alpha_M <^* d$, we have an ordinal $\beta \in d - d^\alpha_M$ such that $d - (\beta+1) = d^\alpha_M - (\beta+1)$. Then $d^\alpha_M \in \mathrm{rng}(\sigma)$ where $\sigma : {}^*W^{\beta,d}_M \to M$ is the witness map. This implies that $h^*_M(\alpha \cup \{d^\alpha_M\}) \subset \mathrm{rng}(\sigma)$, so $\mathrm{rng}(\sigma) = M$. On the other hand, we are assuming that M contains a generalized Dodd solidity witness for β with respect to M and d. Remark (1) guarantees that ${}^*W^{\beta,d}_M \in M$, so $\mathrm{rng}(\sigma) \neq M$, a contradiction. It follows that $d = d^\alpha_M$. □(a)

We next prove that M is sound above α, that is, $h_M(\alpha \cup \{p_M\}) = M$. Since $h^*_M(\alpha \cup \{d\}) = M$, it suffices to show that $d \in h^*_M(\alpha \cup \{\gamma_M, p_M\})$. Assume this is false and denote the largest element of d that is not in $h^*_M(\alpha \cup \{\gamma_M, p_M\})$ by β. Let β' be the least ordinal that is in $h^*_M(\alpha \cup \{\gamma_M, p_M\}) - \beta$. Then $\beta' > \beta$ and $d \cap (\beta+1)$ witnesses the following statement in M:

$$(\exists x)\big(x \in [\beta']^{<\omega} \ \&\ p_M \cup \{\gamma_M\} = h^*_M(y, (d - (\beta+1)) \cup x)\big)$$

where $y \in [\alpha]^{<\omega}$ is such that $p_M \cup \{\gamma_M\} = h^*_M(y, d)$. It follows that this statement is witnessed by some $d^* \in h^*_M(\alpha \cup \{\gamma_M, p_M\})$; here d^* is a finite subset of β. Thus, we have $p_M \cup \{\gamma_M\} = h^*_M(y, (d - (\beta+1)) \cup d^*)$. Let $\sigma : {}^*W^{\beta,d}_M \to M$ be the canonical witness map. Then $\{y, d'\} \subset \mathrm{rng}(\sigma)$ where $d' = (d - (\beta+1)) \cup d^*$, and therefore $p_M \cup \{\gamma_M\} \in \mathrm{rng}(\sigma)$ as well. Let $\sigma(\bar{p}) = p_M \cup \{\gamma_M\}$. Since p_M is a good parameter for M with respect to the language for premice, we have a set A that is $\Sigma_1(M)$ in $p_M \cup \{\gamma_M\}$ such that $A \cap \omega\varrho^1_M \notin M$; here $\Sigma_1(M)$ is meant with respect to the language for coherent strutures. The map σ is Σ_1-preserving with respect to the language for coherent structures and $\sigma \restriction \omega\varrho^1_M = \mathrm{id}$, so letting $\bar{A}$ be the $\Sigma_1({}^*W^{\beta,d}_M)$-set by the same definition as A in the parameter $\bar{p}$, we have $\bar{A} \cap \omega\varrho^1_M = A \cap \omega\varrho^1_M \notin M$. Now by the assumptions of the theorem, ${}^*W^{\beta,d}_M \in M$, so $\bar{A} \cap \omega\varrho^1_M \in M$ after all. This contradiction shows that $d \in h^*_M(\alpha \cup \{\gamma_M, p_M\})$ and thereby completes the proof. □(c)

Clauses (a) and (c) together with our assumption that M is weakly iterable now allow to apply Theorem 1.1 (actually, just (b) of Theorem 1.1) and obtain (b). $\Box$(Theorem 1.2)

Let us finally remark that although we did apply Theorem 1.1 to M at the end of the above proof, such an application does *not* make the proof of the two main theorems as a whole circular. This is the case, because the proof of Theorem 1.1 does not make any use of an application of Theorem 1.2 to M — the only application of Theorem 1.2 in that proof occurs at the end of the proof of Claim 5 in the proof of Lemma 1.12, where Theorem 1.2 is applied to a premouse of height *strictly* smaller than $\mathrm{ht}(M)$.

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Department of Mathematics, University of California at Irvine, Irvine, CA 92697

E-mail address: `mzeman@math.uci.edu`