CONSISTENCY STRENGTH OF $\neg\Box_{\aleph_\omega}$ AND REFLECTION AT TWO SUCCESSIVE $\aleph_n$

MARTIN ZEMAN

Abstract. We give modest upper bounds for consistency strengths for two well-studied combinatorial principles. These bounds range at the level of subcompact cardinals, which is significantly below a κ^+-supercompact cardinal. All previously known upper bounds on these principles ranged at the level of some degree of supercompactness. We show that using any of the standard modified Prikry forcings it is possible to turn a measurable subcompact cardinal into $\aleph_\omega$ and make the principle $\Box_{\aleph_\omega,<\omega}$ fail in the generic extension. We also show that using Lévy collapse followed by standard iterated club shooting it is possible to turn a subcompact cardinal into $\aleph_2$, and arrange that in the generic extension simultaneous reflection holds at $\aleph_2$ and at the same time every stationary subset of $\aleph_3$ concentrating on points of cofinality ω has a reflection point of cofinality ω_1.

We present two models built using modest large cardinal hypotheses. In the first model the principle $\Box_{\aleph_\omega,<\omega}$ fails. In the second model any family of size ω_1 of stationary subsets of ω_2 concentrating on ordinals of cofinality ω has a common reflection point, and at the same time every stationary subset of ω_3 concentrating on ordinals of cofinality ω reflects at a point of cofinality ω_1. It is of course known that constructing models for these combinatorial situations requires large cardinals, and all models previously known build on the existence of large cardinals of some degree of supercompactness. The natural long-standing open problem is to determine the exact consistency strength. In the past there has been a considerable amount of work done along these lines, and it seems that determining the lower bounds is significantly more demanding than determining the upper bounds. Using relatively simple forcing techniques, we give upper bounds which seem to be not too far from the actual consistency strength, and significantly below any variant of supercompactness.

1. Introduction

Throughout the paper we follow the standard notation from [19]. We will also use the following standard notation. Given $m < n < \omega$ we let

$$S^n_m = \{\xi < \omega_n \mid \mathsf{cf}(\xi) = \omega_m\}.$$

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Also, given regular cardinals $\mu < \lambda < \kappa$, $\mathsf{Refl}(\kappa, \lambda, \mu)$ is the statement

$$\mathsf{Refl}(\kappa, \lambda, \mu) \equiv \text{Every stationary subset of } \kappa \text{ concentrating on points of cofinality } \mu \text{ reflects at a point of cofinality } \lambda$$

The best known lower bound for the failure of $\square_{\aleph_\omega, <\omega}$ is a non-tame mouse [30], and it is believed that using the methods developed in [29] the bound to be strengthened to the level "$\mathsf{AD}_{\mathbb{R}} + \Theta$ regular. This result can be viewed as a culmination of development as recorded in [37, 26, 27, 31, 32, 33, 38]. The best upper bounds are at the level of κ^+-supercompactness [2], and is a part a different line of development culminating in [8, 7]. The best known result for simultaneous reflection at ω_2 accompanied with $\mathsf{Refl}(\omega_3, \omega_1, \omega)$ at is the $\mathsf{AD}_{\mathbb{R}}$-hypothesis, that is, a proper class of strong cardinals and a proper class of Woodin cardinals; this follows from the work in [23]. The best upper bound is a κ^{++}-supercompact cardinals [20].

The gap between Woodin and supercompact cardinals is of course immense, and, as of today, no fine structural model is known that would allow at least some nontrivial instance of supercompactness, say κ^+-supercompactness. Recent work of Woodin [39, 40] shows many of obstacles that need to be addressed when considering such models. Also, in the work cited above, actually much more is obtained than the combinatorial situations whose consistency strengths we are trying to approximate in this paper. Thus, the failure of $\square_{\aleph_\omega}$ in [2] is obtained as a consequence of a stronger property that λ-indecomposable ultrafilters exist in the generic extension at $\aleph_\omega$ for $\lambda < \aleph_\omega$. Another construction [25] which starts form a significantly stronger hypothesis that there is a sequence $\langle \kappa_n \mid n \in \omega \rangle$ of cardinals that are $(\sup_n \kappa_n)^+$-supercompact achieves stationary reflection at $\aleph_\omega$, and thereby $\neg\square_{\aleph_\omega}$. In [20] the so-called full reflection is achieved, that is every stationary subset of S^2_0 reflects at almost all ordinals in S^2_1, and every stationary subset of S^3_0 reflects at almost all points in S^3_1 where of course "almost all" is in the sense of the club filter.

In this paper we we focus on the combinatorial situations formulated above directly, which allows to lower the upper bound for consistency strength down to the level that is compatible with fine structural extender models based on extenders of superstrong type or weaker. Our starting point is the independent observation of Burke [3] and Jensen [22, 35] that a much weaker large cardinal property than κ^+-supercompactness of κ is needed to guarantee the failure of $\square_\kappa$. This large cardinal property is today called subcompactness. An important fact about subcompactness is that it can be witnessed by extenders of type described above. This makes the existence of subcompact cardinals possible in Mitchell-Steel and Jensen fine structural extender models developed in [28, 37] and [21]; see also [41]. These models exist, granting that the iterability conjecture holds, and Jensen proved in [22] that under certain circumstances subcompact cardinals do exist in these models.

The analysis in [34, 35] shows that in any extender model with Jensen's λ-indexing of extenders [21], subcompact cardinals are precisely those cardinals κ for which $\square_\kappa$ fails in the model. The same is true in Mitchell-Steel models, by a similar kind of analysis, or just by quoting [35] combined with the work of Fuchs [14, 15]. This gives rise to a natural conjecture that subcompactness is the right candidate for the consistency strength of the failure of $\square_\lambda$ at a singular cardinal λ. We give an evidence that supports this conjecture by showing that cardinals at the level of subcompactness are sufficient to obtain models with various situations where square fails at small singular cardinals. The situation with the reflection principle at two successive $\aleph_n$ is less clear. Related, but weaker results are obtained in [4] from large cardinal hypotheses that are weaker than a superstrong cardinal, and it is conceivable that simultaneous reflection at ω_2 along with $\mathsf{Refl}(\omega_3, \omega_1, \omega)$ is of large cardinal strength lower than a superstrong.

Given a cardinal κ, let

$$\mathcal{S}_\kappa = \{x \in [\kappa]^{<\kappa} \mid \mathsf{otp}(x) \text{ is a cardinal and } x \cap \kappa \in \kappa\}, \tag{1}$$

and

$$\mathcal{S}^*_\kappa = \{x \in [H_{\kappa^+}]^{<\kappa} \mid x \cap \kappa \in \kappa \;\&\; \langle x, \in\rangle \simeq \langle H_{\mu^+}, \in\rangle \text{ for some } \mu < \kappa\}. \tag{2}$$

Here of course "$\simeq$" means "isomorphic to". We now give the definition of a subcompact cardinal. This definition is formally different from Jensen's definition in [22], but the two definitions are equivalent, and the current definition is more convenient for our purposes here.

Definition 1.1. *A cardinal κ is subcompact if and only if the set $\mathcal{S}^*_\kappa$ is stationary.*

Hence we may without loss of generality assume that $\langle x, \in\rangle \prec \langle H_{\kappa^+}, \in\rangle$ whenever $x \in \mathcal{S}^*_\kappa$. Any such x gives rise to an elementary embedding $\sigma : H_{\mu^+} \to H_{\kappa^+}$ which inverts the Mostowski collapsing isomorphism where $\mu = x \cap \kappa$. Notice that μ is the critical point of the superstrong extender of length κ derived from σ. Thus, if $\mathcal{S}^*_\kappa$ contains some x as above then μ is superstrong, and in fact 1-extendible. It follows by the elementarity of σ that κ is weakly compact, but it is easy to see that κ is not necessarily measurable. If $\mathcal{S}_\kappa$ is stationary then $\Box_{\kappa,<\kappa}$ fails; Lemma 2.6 makes it possible to run Jensen's argument from [22]. Thus, although the statement "$\mathcal{S}_\kappa$ is stationary" is seemingly weaker than κ being subcompact, the analysis in [35] shows that in an extender model the two are equivalent. We do not know if the two statements are equivalent in ZFC, but we believe they are not. However, it sounds plausible that the two statements are equiconsistent modulo ZFC. We are now ready to formulate the results of this paper precisely.

Theorem 1.2. *Assume κ a measurable cardinal such that $\mathcal{S}_\kappa$ is stationary and $2^\kappa = \kappa^+$. Then there is a generic extension of $\mathbf{V}$ in which $\kappa = \aleph_\omega$, $\kappa^{+\mathbf{V}} = \aleph_{\omega+1}$ and $\Box_{\aleph_\omega,<\omega}$ fails.*

Given regular cardinals $\mu < \lambda$ we denote $\lambda \cap \mathsf{cof}(\mu) = \{\xi < \lambda \mid \mathsf{cf}(\xi) = \mu\}$. It follows that $\omega_n \cap \mathsf{cof}(\omega) = S^n_0$ and $\omega_n \cap \mathsf{cof}(<\omega_{n-1}) = S^n_0 \cup \cdots \cup S^n_{n-2}$.

Theorem 1.3. *Assume GCH and κ is subcompact. If $1 < n < \omega$ then there is a forcing extension of $\mathbf{V}$ satisfying the following.*

(a) *Simultaneous reflection at ω_n, that is, if $\langle S_\xi \mid \xi < \omega_{n-1}\rangle$ is a family of stationary subsets of $\omega_n \cap \mathsf{cof}(<\omega_{n-1})$ then there is an ordinal ν of cofinality ω_{n-1} such that all S_ξ reflect at ν.*

(b) *If $S \subseteq \omega_{n+1} \cap \mathsf{cof}(<\omega_{n-1})$ then S has a reflection point of cofinality ω_{n-1}.*

Theorem 1.2 is proved in Section 2, and Theorem 1.3 is proved in Section 3.

2. Failure of square

In this section we give a proof of Theorem 1.2. This theorem is the best possible result if κ^+ is not collapsed in the forcing extension. By the result in [10], necessarily $\Box_{\aleph_\omega,\omega}$ holds in the generic extension, and actually in any extension in which κ is ω-cofinal and κ^+ is preserved. See [9] for a detailed discussion of this phenomenon at higher cofinalities. As $\neg\Box_{\aleph_\omega,<\omega}$ can be obtained under the large cardinal hypothesis in Theorem 1.2 which is of highly local nature (it does not seem to have influence beyond κ^+), this may be considered an indication that $\neg\Box_{\aleph_\omega,\omega}$ has higher consistency strength, associated with a large cardinal axiom whose influence is beyond κ^+.

We now describe the forcing. This is a standard variant of modified Prikry forcing with guiding generic at κ which turns κ into $\aleph_\omega$. We only give the definition of the forcing and list its main properties that we are going to use. For details see [17, 16, 11]; some basic information can also be found in [19]. Assuming κ is measurable, let

- U be a normal measure on κ.
- $j : \mathbf{V} \to \mathbf{M}$ be the ultrapower embedding associated with $\mathsf{Ult}(\mathbf{V}, U)$.
- $F \in \mathbf{V}$ be a filter generic for $\mathsf{Coll}(\kappa^{++}, j(\kappa))^{\mathbf{M}}$ over $\mathbf{M}$.

The filter F is obtained by the standard construction of a descending chain of length κ^+ in $\mathsf{Coll}(\kappa^{++}, j(\kappa))^{\mathbf{M}}$ hitting every dense set in $\mathbf{M}$. Here the closure of $\mathbf{M}$ under κ-sequences in $\mathbf{V}$ guarantees that the poset $\mathsf{Coll}(\kappa^{++}, j(\kappa))^{\mathbf{M}}$ is κ^+-closed in $\mathbf{M}$, and since $\mathcal{P}(\mathsf{Coll}(\kappa^{++}, j(\kappa)))^{\mathbf{M}} \subseteq V^{\mathbf{M}}_{j(\kappa)+1}$, and $\mathsf{card}^{\mathbf{V}}(V^{\mathbf{M}}_{j(\kappa)+1}) = 2^\kappa$, the assumption $2^\kappa = \kappa^+$ guarantees that we only need to hit κ^+ many dense sets.

Let $\vartheta < \kappa$ be an infinite ordinal. Conditions in $\mathbb{P}_\vartheta$ are tuples

$$p = \langle p_{-1}, \delta_0, p_0, \dots, \delta_{n-1}, p_{n-1}, h\rangle$$

satisfying the following.

- $(\delta_0, \dots, \delta_{n-1})$ is an increasing sequence of strongly inaccessible cardinals, and $\vartheta < \delta_0$.
- p_{-1} is a condition in $\mathsf{Coll}(\vartheta^{++}, \delta_0)$.
- p_k is a condition in $\mathsf{Coll}(\delta_k^{++}, \delta_{k+1})$ whenever $0 \le k < n-2$.
- p_{n-1} is a condition in $\mathsf{Coll}(\delta_{n-1}^{++}, \kappa)$.
- h is a function such that:
  - $\mathrm{dom}(h) \in U$ and $\delta_{n-1}, p_{n-1} \in V_{\min(\mathrm{dom}(h))}$.
  - $h(\alpha)$ is a condition in $\mathsf{Coll}(\alpha^{++}, \kappa)$ whenever $\alpha \in \mathrm{dom}(h)$.
  - $j(h)(\kappa) \in F$.

We write s_p for the sequence $(p_{-1}, \delta_0, p_0, \dots, \delta_{n-1}, p_{n-1})$ and call s_p the lower part of p, and h_p for h and call h_p the upper part of p. So $p = \langle s_p, h_p\rangle$. The number n is called the length of p.

Ordering $\le$ in $\mathbb{P}$ is defined as follows. Given two conditions

$$\begin{aligned} p &= \langle p_{-1}, \delta_0, p_0, \dots, \delta_{n-1}, p_{n-1}, h\rangle \\ q &= \langle q_{-1}, \varepsilon_0, q_0, \dots, \varepsilon_{m-1}, q_{m-1}, g\rangle \end{aligned}$$

we let $p \le q$ just in the case where the following are satisfied.

- $m \le n$.
- $\delta_k = \varepsilon_k$ whenever $0 \le k \le m-1$.
- $p_k \supseteq q_k$ whenever $-1 \le k \le m-1$.
- $\delta_k \in \mathrm{dom}(g)$ and $p_k \supseteq g(\alpha)$ whenever $m \le k < n$.
- $\mathrm{dom}(h) \subseteq \mathrm{dom}(g)$ and $h(\alpha) \supseteq g(\alpha)$ whenever $\alpha \in \mathrm{dom}(h)$.

Of course, the relation "$\supseteq$" in the above description of $\le$ should be viewed as the extension in the sense of forcing in the corresponding collapse poset. The direct extension $\le^*$ is defined as follows. We let $p \le^* q$ precisely when

- $m = n$ and $p \le q$.

The following two facts comprise basic properties of the poset $\mathbb{P}_\vartheta$ we will use.

Fact 2.1. *Let $\mathbb{P}_\vartheta$ be the posed defined above.*

(a) *If $\{\langle s, h_\xi\rangle \mid \xi < \lambda\}$ is a family of conditions in $\mathbb{P}_\vartheta$ with common stem s and $\lambda < \kappa$ then there is a common lower bound for all $\langle s, h_\xi\rangle$.*
(b) *$\mathbb{P}_\vartheta$ is κ^+-c.c.*
(c) *The $\leq^*$-ordering in $\mathbb{P}$ is ϑ^{++}-closed.*
(d) *Let $p = \langle p_{-1}, \delta_0, p_0, \dots, \delta_n, p_n, h\rangle$ be a condition in $\mathbb{P}$. Then the poset $\mathbb{P}_\vartheta/p$ is isomorphic to the product*

$$\mathsf{Coll}(\vartheta^{++}, \delta_0) \times \mathsf{Coll}(\delta_0^{++}, \delta_1) \times \cdots \times \mathsf{Coll}(\delta_{n-1}^{++}, \delta_n) \times \mathbb{P}_{\delta_n}/(p_n, h).$$

Here $\mathbb{P}_\vartheta/p$ denotes the poset $\mathbb{P}_\vartheta$ below the condition p.

(e) *Let G be a filter generic for $\mathbb{P}_\vartheta$ over $\mathbf{V}$. Let*

$$\delta_k = \text{the unique } \delta_k^p \text{ such that there is a condition } p \in G \text{ where } p = \langle p_{-1}, \delta_0^p, p_0, \dots, \delta_{n-1}^p, p_{n-1}, h\rangle \text{ and } \delta_k^p = \delta_k,$$

and

$$G_k = \{p_k \mid (\langle p_{-1}, \delta_0^p, p_0, \dots, \delta_{n-1}^p, p_{n-1}, h\rangle \in G \ \&\ k < n)\}$$

for $k \in \omega$. Then

(i) *$\langle \delta_k \mid k \in \omega\rangle$ is a Prikry sequence for the Prikry forcing $\mathbb{P}_U$ associated with U, that is, for every $A \in U$ we have $\delta_k \in A$ for all but finitely many $k \in \omega$.*
(ii) *G_k is generic for $\mathsf{Coll}(\delta_k^{++}, \delta_{k+1})$ over $\mathbf{V}$ for all $k \in \omega$. Here we let $\delta_{-1} = \vartheta$.*

Here (a) follows from the fact that $\mathbb{P}_\vartheta$ is defined using the guiding generic F, precisely, that $j(h_\xi)(\kappa) \in F$ for all $\xi < \lambda$. Clauses (b) – (e) then follow by simple standard considerations. The following is the Prikry property for $\mathbb{P}_\vartheta$. See e.g. [17] for a proof.

Fact 2.2. *Let φ be a formula in the forcing language and $p \in \mathbb{P}_\vartheta$. Then there is $p' \leq^* p$ such that p' decides φ.*

Corollary 2.3. *Let G be a filter generic for $\mathbb{P}_\vartheta$ over $\mathbf{V}$.*

(a) *$\mathbb{P}_\vartheta$ does not add any bounded subset of ϑ^+.*
(b) *Let $\langle \delta_k \mid k \in \omega\rangle$ be the Prikry sequence as in Fact 2.1. Then all cardinals of $\mathbf{V}$ which are collapsed by $\mathbb{P}_\vartheta$ are precisely those in the intervals $(\delta_k^{++}, \delta_{k+1}]$ for $k \in \omega \cup \{-1\}$. Here we let $\delta_{-1} = \vartheta$. Thus, $(\delta_k^{+1+i})^{\mathbf{V}} = (\vartheta^{+2k+3+i})^{\mathbf{V}[G]}$ for all $k \in \omega$ and $i \in \{0,1\}$, and $\kappa = (\vartheta^{+\omega})^{\mathbf{V}[G]}$.*
(c) *If $\vartheta = \omega$ then $\omega_1^{\mathbf{V}} = \omega_1^{\mathbf{V}[G]}$, $\omega_2^{\mathbf{V}} = \omega_2^{\mathbf{V}[G]}$, $(\delta_k^{+1+i})^{\mathbf{V}} = \aleph_{2k+3+i}^{\mathbf{V}[G]}$ for $k \in \omega$ and $i \in \{0,1\}$, $\kappa = \aleph_\omega^{\mathbf{V}[G]}$, and $\kappa^{+\mathbf{V}} = \aleph_{\omega+1}^{\mathbf{V}[G]}$.*

Proof. The conclusions follow in a straightforward way from Fact 2.1(a) – (c) and and the Prikry property, combined with appropriate factoring from Fact 2.1(d) where needed. □

We will make use of the following classical fact on preservation of stationary sets under sufficiently closed forcing.

Fact 2.4. *Let μ be an infinite cardinal.*

(a) *Let $S \subseteq \mu^+ \cap \mathsf{cof}(\omega)$ be a stationary set. Then the stationarity of S is preserved by ω_1-closed forcing.*
(b) *Assume μ is strong limit, $\square^*_\mu$ holds, and $\rho < \mu$ is regular. Let $S \subseteq \mu^+ \cap \mathsf{cof}(< \rho)$ be a stationary set. Then the stationarity of S is preserved under ρ-closed forcing.*

Corollary 2.5. *Assume $\mu < \kappa$ is a cardinal, $S \subseteq \mu^+$ is a stationary set, and one of the following holds.*
(a) $S \subseteq \mu^+ \cap \mathsf{cof}(\omega)$.
(b) $S \subseteq \mu^+ \cap \mathsf{cof}(\omega_1)$, *$\mu$ is strong limit, and $\square^*_\mu$ holds.*

Then S remains stationary in the generic extension via $\mathbb{P}_\omega$.

Proof. Given a generic filter G for $\mathbb{P}_\omega$, let $p = \langle p_{-1}, \delta_0, p_0, \dots, \delta_n, p_n, h\rangle \in G$ be a condition such that $\delta_{n-1} \leq \mu < \delta_n$. It suffices to show that $\mathbb{P}_\omega/p$ preserves the stationarity of S. By Fact 2.1(d), the poset $\mathbb{P}_\omega/p$ is isomorphic to the product

$$\mathsf{Coll}(\omega_2, \delta_0) \times \mathsf{Coll}(\delta_0^{++}, \delta_1) \times \cdots \times \mathsf{Coll}(\delta_{n-1}^{++}, \delta_n) \times \mathbb{P}_{\delta_n}/(p_n, h).$$

By Fact 2.3(a) the poset $\mathbb{P}_{\delta_n}/(p_n, h)$ does not add any bounded subset of δ_n^{++}, so it does not change any of the posets $\mathsf{Coll}(\delta_k^{++}, \delta_{k+1})$ for $k \in \{-1, 0, \dots, n-1\}$, and in particular does not add any subset of μ^+. Hence it suffices to verify that the product $\mathsf{Coll}(\omega_2, \delta_0) \times \mathsf{Coll}(\delta_0^{++}, \delta_1) \times \dots \mathsf{Coll}(\delta_{n-1}^{++}, \delta_n)$, which is the same in $\mathbf{V}$ and in the generic extension via $\mathbb{P}_\omega/(p_n, h)$, preserves the stationarity of S. That $\mathsf{Coll}(\delta_{n-1}^{++}, \delta_n)$ preserves the stationarity of S follows from Fact 2.4. It also follows that the cofinality of μ^+ in the generic extension via $\mathsf{Coll}(\delta_{n-1}^{++}, \delta_n)$ is at least δ_{n-1}^{+} (equal to δ_{n-1}^{+} when $\delta_{n-1} = \mu$ and equal to δ_{n-1}^{++} otherwise), and the product $\mathsf{Coll}(\omega_2, \delta_0) \times \mathsf{Coll}(\delta_0^{++}, \delta_1) \times \cdots \times \mathsf{Coll}(\delta_{n-2}^{++}, \delta_{n-1})$ is the same in $\mathbf{V}$ and the generic extension via $\mathsf{Coll}(\delta_{n-1}^{++}, \delta_n)$. So it suffices to check that this product preserves the stationarity of S, but this follows easily from the fact that the product is δ_{n-1}^{+}-c.c. (in fact of size $< \delta_{n-1}^{+}$). ◻

Lemma 2.6. *Let θ be large regular and let X be an elementary substructure of H_θ such that $x \cap \kappa^+ \in \mathcal{S}_\kappa$. Let $\mu = X \cap \kappa$ and $\tau = \sup(X \cap \kappa^+)$. The following hold.*
(a) *Let $\alpha \in \lim(X) \cap \kappa^+$ and $\alpha' = \min(X - \alpha)$. If $\alpha < \alpha'$ then $\mathsf{cf}(\alpha') = \kappa$ and $\mathsf{cf}(\alpha) = \mathsf{cf}(\mu)$.*
(b) *$V_\mu \subseteq X$, $\mathsf{card}(V_\mu) = \mu$ and $\mathsf{otp}(X \cap \kappa^+) = \mu^+$. Consequently, $\mathsf{cf}(\tau) = \mu^+$.*

Proof. We begin with the proof of (a). Let $\gamma' = \mathsf{cf}(\alpha')$. Assume for a contradiction that $\gamma' < \kappa$. By elementarity and the fact that $\alpha' \in X$ we conclude that $\gamma' \in X$, hence $\gamma' < \mu$. Let $f : \gamma' \to \alpha'$ be a strictly increasing cofinal map. Again using elementarity and the fact that $\gamma', \alpha' \in X$ we may assume that $f \in X$. But then, since $\gamma' \subseteq X$, we conclude that $\mathrm{rng}(f) \subseteq X$. Hence $X \cap \alpha'$ is cofinal in α' which means that $\alpha' = \alpha$. Contradiction.

Since $\mathsf{cf}(\alpha') = \kappa$, pick some cofinal strictly increasing function $f : \kappa \to \alpha'$. As before we may assume that $f \in X$. Then $f[\mu] \subseteq X$, so in fact $f[\mu] \subseteq X \cap \alpha$ as $X \cap [\alpha, \alpha') = \varnothing$ by our assumption $\alpha < \alpha'$. We show that $f[\mu]$ is cofinal in α. So pick some $\zeta < \alpha$; since $\alpha \in \lim(X)$ we may assume $\zeta \in X$. Since f maps κ cofinally into α' we have

$$H_\theta \models (\exists \eta < \kappa)(f(\eta) > \zeta).$$

As $f, \kappa, \zeta \in X$ and X is elementary, there is some $\eta \in X$ such that $f(\eta) > \zeta$. Such η is below μ. We have thus proved that f maps μ cofinally into α, which shows that $\mathsf{cf}(\alpha) = \mathsf{cf}(\mu)$. This completes the proof of (a).

To see (b), notice that for each $\zeta \in X - \kappa$ there is a surjection $g : \kappa \to \zeta$, and by elementarity g may be considered an element of X. Using elementarity again we conclude that g maps μ onto $\zeta \cap X$. Let $\vartheta = \mathsf{otp}(X \cap \kappa^+)$ and $e : \vartheta \to X \cap \kappa^+$ be the unique isomorphism. If $e(\bar\zeta) = \zeta$ then $e^{-1} \circ g$ maps μ onto $\bar\zeta$. It follows that there are no cardinals in the interval (μ, ϑ). Moreover, ϑ is a cardinal since $X \cap \kappa^+ \in \mathcal{S}_\kappa$. It follows that $\vartheta = \mu^+$. That $\mathsf{card}(V_\mu) = \mu$ follows immediately from the assumption that κ is strongly inaccessible and from elementarity – just notice that for each $\alpha < \mu$ the cardinal $\mathsf{card}(V_\alpha)$ is in $X \cap \kappa = \mu$, and some bijection $g : \mathsf{card}(V_\alpha) \to V_\alpha$ is an element of X. This shows $V_\alpha \subseteq X$ for all $\alpha < \mu$, so $V_\mu \subseteq X$. Now if $f : \kappa \to V_\kappa$ is a bijection such that $f \in X$, and such a bijection exists by the strong inaccessibility of κ, then $f \restriction \mu : \mu \to V_\mu$ is a bijection as well. $\Box$

Proof of Theorem 1.2. Assume there is a condition $p = (s_p, h_p) \in \mathbb{P}_\omega$ and a $\mathbb{P}_\omega$-name $\dot C$ such that $p \Vdash_{\mathbb{P}_\omega}$ "$\dot C$ is a $\Box_{\kappa,<\omega}$-sequence". We want $\dot C$ to include enumerations of its "c-sets", so technically we make the requirement that p forces the following statements:

- $\dot C$ is a a partial function from $(\kappa^+ \cap \lim) \times \omega$ such that $\mathrm{dom}(\dot C)(\alpha, -)$ is a nonzero integer.
- $\dot C(\alpha, i)$ is a closed unbounded subset of α whenever $\langle \alpha, i\rangle \in \mathrm{dom} C$.
- For every $\langle \alpha, i\rangle \in \mathrm{dom}(\dot C)$ and every $\bar\alpha \in \lim(\dot C(\alpha, i))$ there is some $j \in \omega$ such that $\dot C(\alpha, i) \cap \bar\alpha = \dot C(\bar\alpha, j)$.
- $\mathsf{otp}(\dot C(\alpha, i) < \kappa$ whenever $\langle \alpha, i\rangle \in \mathrm{dom}(\dot C)$.

The requirement that each $\mathrm{dom}(\dot C)(\alpha, -)$ is a nonzero integer in the first clause above of course expresses that the sets on the sequence denoted by $\dot C$ are required to be nonempty. Let θ be regular large, and let $X \prec H_\theta$ be such that $\mathbb{P}_\omega, p, \dot C \in X$ and $X \cap H_{\kappa^+} \in \mathcal{S}_\kappa$. Such an X exists as we assume that $\mathcal{S}_\kappa$ is stationary. Let $\tau = \sup(X \cap \kappa^+)$ and $\mu = X \cap \kappa$. By Lemma 2.6(b) we have $\mathsf{cf}(\tau) = \mu^+$.

Case 1. $\mathsf{cf}(\mu) > \omega$.

Let $d = X \cap \lim(X) \cap \mathsf{cof}(\omega) \cap \tau$. By Lemma 2.6 the set d is closed under ω-limits and $\mathsf{otp}(d) = \mu^+$. (To see the former, if $\alpha \in (\lim(X) \cap \tau) - X$ then $\mathsf{cf}(\alpha) = \mathsf{cf}(\mu) > \omega$ by (a) of the lemma.)

To each $\alpha \in d$ we can pick some condition $\langle s_\alpha, h_\alpha\rangle \le p$, some integer $n_\alpha > 0$, and some finite sequence of ordinals $\langle \gamma_{\alpha,i} \mid i < n_\alpha\rangle$ in ${}^{n_\alpha}\kappa$ such that $\langle s_\alpha, h_\alpha\rangle$ forces

- $\mathrm{dom}(\dot C(\alpha, -)) = n_\alpha$, and
- $\mathsf{otp}(\dot C(\alpha, 0)) = \gamma_{\alpha,0}$ & $\ldots$ & $\mathsf{otp}(\dot C(\alpha, n_\alpha - 1)) = \gamma_{\alpha,n_{\alpha-1}}$.

This is the place in the argument where we it is crucial that $\dot C$ is forced to be a $\Box_{\kappa,<\omega}$-sequence, that is, each $\mathrm{dom}(\dot C)(\alpha, -)$ is forced to be finite. If we tried to run the argument assuming that $\dot C$ is forced to be a $\Box_{\kappa,\omega}$-sequence, the argument would break down here, as we would not be able to pick infinite sequences $\langle \gamma_{\alpha,i} \mid i \in \omega\rangle$ in the ground model; in fact such sequences typically do not have a bound below κ in the ground model. Since $\mathbb{P}_\omega, p, \dot C \in X$, we may find s_α, h_α and $\gamma_{\alpha,i}$ as above in X, and for these objects we then have $s_\alpha \in H_\kappa \cap X$ and $\gamma_{\alpha,i} < \mu$ for all α and i. Since κ is strongly inaccessible there is a bijection $g : \kappa \to H_\kappa$; again by elementarity we may assume that this bijection is in X. Then $g \restriction \mu$ is a bijection between μ and $H_\kappa \cap X$, so there are at most μ many lower parts $s_\alpha \in H_\kappa \cap X$ as above.

Recall that $\mathsf{cf}(\tau) = \mu^+$ by Lemma 2.6. Using the pigeonhole principle we obtain some $s \in H_\mu$, some $n \in \omega$, a sequence $\langle \gamma_i \mid i < n \rangle$ of ordinals smaller than μ, and a stationary $E \subseteq d$ such that $s_\alpha = s$, $n_\alpha = n$, and $\gamma_{\alpha,i} = \gamma_i$ for all $\alpha \in E$ and $i < n$. By Fact 2.1(a) there is some upper part h such that the condition $\langle s, h \rangle$ is a lower bound for all $\langle s, h_\alpha \rangle$ for $\alpha \in E$. So

$$\langle s, h \rangle \Vdash_{\mathbb{P}_\omega} \text{“}\mathrm{dom}(\dot{C}(\alpha, -)) = n \;\&\; \mathsf{otp}(\dot{C}(\alpha, i)) = \gamma_i\text{”} \tag{3}$$

for all $\alpha \in E$ and $i < n$.

Pick a filter G generic for $\mathbb{P}_\omega$ over $\mathbf{V}$ such that $\langle s, h \rangle \in G$. Let $c = \dot{C}^G(\tau, 0)$. By Corollary 2.5(a) the set E remains stationary in $\mathbf{V}[G]$. More precisely, the corollary is applied to the set $g^{-1}[E]$ where $g : \mu^+ \to \tau$ is any normal map in $\mathbf{V}$ mapping μ^+ cofinally into τ. In particular $E \cap \lim(c)$ is stationary. By the coherency of the sequence $\dot{C}^G$, for each $\alpha \in E \cap \lim(c)$ there is some $i_\alpha < n$ such that $c \cap \alpha = \dot{C}^G(\alpha, i_\alpha)$. This splits the stationary set $E \cap \lim(c)$ into n pieces. So we can find some $i < n$ and a stationary $E' \subseteq E \cap \lim(c)$ such that $i_\alpha = i$ for all $\alpha \in E'$. Combining this with (3), for each $\alpha \in E'$ we obtain $\mathsf{otp}(c \cap \alpha) = \mathsf{otp}(\dot{C}^G(\alpha, i)) = \gamma_i$. We thus conclude that arbitrarily large proper initial segments of c are all of the same order type. As c is cofinal in τ, this is impossible.

Case 2. $\mathsf{cf}(\mu) = \omega$.

The proof in this case is the same in Case 1, with the only difference that this time we let $d = X \cap \lim(X) \cap \mathsf{cof}(\omega_1) \cap \tau$ and need a new argument to prove that the set E remains stationary in the generic extension $\mathbf{V}[G]$ via $\mathbb{P}_\omega$. For this, it suffices to show that $g^{-1}[E]$ remains stationary in the generic extension where $g : \mu^+ \to \tau$ is any normal map in $\mathbf{V}$ mapping μ^+ cofinally into τ such that $g(\xi)$ is a successor ordinal whenever ξ is a successor ordinal. For such g we have $g^{-1}[E] \subseteq \mu^+ \cap \mathsf{cof}(\omega_1)$. Our intention is to apply Corollary 2.5(b). That μ is strong limit is given by Lemma 2.6(b), so it suffices to verify $\square^*_\mu$. Now κ is strongly inaccessible, which implies the existence of a $\square^*_\kappa$-sequence $\langle D_\alpha \mid \alpha \in \lim \cap \kappa^+ \rangle$, and by elementarity we may assume that this sequence is an element of X. Let H be the transitive collapse of X and $\langle \bar{D}_\alpha \mid \alpha \in \lim \cap \mu^+ \rangle$ be the image of $\langle D_\alpha \mid \alpha \in \lim \cap \kappa^+ \rangle$ under the Mostowski collapsing isomorphism. Then $H \models$ “$\langle \bar{D}_\alpha \mid \alpha \in \lim \cap \mu^+ \rangle$ is a $\square^*_\mu$-sequence”, but the property of being a $\square^*_\mu$-sequence is sufficiently absolute that $\langle \bar{D}_\alpha \mid \alpha \in \lim \cap \mu^+ \rangle$ is a $\square^*_\mu$-sequence in the sense of $\mathbf{V}$. The rest of the proof goes through exactly as in Case 1. $\square$

Let us make some concluding remarks. First, we could run the proof of Theorem 1.2 under the stronger assumption that κ is subcompact; this argument would be a bit simpler as Case 2 in the proof would become vacuous, and we would not need Lemma 2.6. We feel however that the formulation of Theorem 1.2 is more satisfying using the stationarity of $\mathcal{S}_\kappa$. Given a regular cardinal κ let

$$\mathcal{S}^-_\kappa = \{x \in [\kappa^+]^{<\kappa} \mid \mathsf{otp}(x) \text{ is a cardinal.}\} \tag{4}$$

Obviously $\mathcal{S}_\kappa \subseteq \mathcal{S}^-_\kappa$, so the statement “$\mathcal{S}^-_\kappa$ is stationary” is further weakening of subcompactness. An argument similar to that in [13] which shows that the Chang Conjecture $(\omega_3, \omega_2) \to (\omega_2, \omega_1)$ implies the failure of $\square_{\omega_2}$ can be used to show that the stationarity of $\mathcal{S}^-_\kappa$ implies the failure of $\square_\kappa$, and a variation on the argument with forcing similar to the proof of Theorem 1.2 can be used to produce a model for $\neg\square_{\aleph_\omega}$. Notice the resemblance between the Chang Conjecture and the requirement that $\mathcal{S}^-_\kappa$ is stationary. If we additionally assume that there is a stationary $\mathcal{S} \subseteq \mathcal{S}^-_\kappa$ such that the sets in $\mathcal{S}$ are closed under ω-limits (or some other fixed cofinality $< \kappa$), we obtain $\neg\square_{\kappa,<\omega}$ in $\mathbf{V}$ and $\neg\square_{\aleph_\omega,<\omega}$ in the generic extension. The forcing argument in this case is essentially the same as the proof of Theorem 1.2. We do not know if the stationarity of $\mathcal{S}^-_\kappa$ alone implies the existence of some $\mathcal{S}$ as above, but we believe it does not. Also, we do not know if $\neg\square_{\kappa,2}$ follows from the stationarity of $\mathcal{S}^-_\kappa$ alone, as we do not know if an analogue of Lemma 2.6 can be proved for structures in $\mathcal{S}^-_\kappa$. We could have formulated Theorem 1.2 with the hypothesis "there are stationarily many $x \in \mathcal{S}^-_\kappa$ that are closed under ω-limits" in place of the stationarity of $\mathcal{S}_\kappa$, but this hypothesis is not very appealing and does not seem to yield a significantly stronger theorem. As in the case of $\mathcal{S}_\kappa$, the characterization of $\square_\kappa$ in [35] shows that in an extender model $\mathcal{S}^-_\kappa$ is stationary precisely when κ is subcompact, but in the general ZFC context the stationarity of $\mathcal{S}^-_\kappa$ seems to be weaker than subcompactness.

3. Reflection at two successive $\aleph_n$

In this section we give a proof of Theorem 1.3. The model is constructed following the standard strategy by first using Lévy collapse to turn κ into ω_n, and then performing iterated club shooting as in [18] to make all non-reflecting subsets of $\omega_{n+1} \cap \mathsf{cof}(< \omega_{n-1})$ non-stationary. See also [6] for details concerning iterated club shooting. The proofs of Theorem 1.2 and Theorem 1.3 differ significantly in the way the stationarity of $\mathcal{S}_\kappa$, resp. $\mathcal{S}^*_\kappa$ is used. In the proof of Theorem 1.2, the stationarity of $\mathcal{S}_\kappa$ arranged the failure of $\square_{\kappa,<\omega}$ already in the ground model, and the proof showed that this situation is preserved under forcing with $\mathbb{P}_\omega$. In the case of Theorem 1.3 the stationarity of $\mathcal{S}^*_\kappa$ does not imply reflection at κ^+ in the ground model; in fact the stationarity of $\mathcal{S}^*_\kappa$ is consistent with the existence of densely many nonreflecting stationary subsets of κ^+, as is showed in [5]. The stationarity of S^*_κ is used in the proof of Theorem 1.3 to guarantee that the iterated club shooting is (ω_{n+1}, ∞)-distributive; in [18] the Mahloness of the cardinal which became ω_2 after collapsing was sufficient for this purpose in the situation described there. Here we use terminology and notation consistent with [6], hence a forcing is (ρ, ∞)-distributive just in the case where it does not add any sequences of length $< \rho$. The conclusion (a) in Theorem 1.3 on simultaneous reflection follows immediately from the fact that the club shooting does not add any subsets of ω_n along with the classical result of Baumgartner [1] that Lévy collapsing a weakly compact cardinal arranges simultaneous reflection. Obviously the same argument achieves this situation where ω_n is replaced with a successor of arbitrary regular cardinal; however we phrase the proposition for ω_n, as the case of small regular cardinals is of particular interest.

It should be stressed that the reflection point in (b) in Theorem 1.3 has cofinality ω_{n-1}, that is, the cofinality preceding the maximal possible cofinality. We do not see if the argument we are using can be modified to obtain reflection points of cofinality ω_n. Also, this argument does not seem to give any kind of simultaneous reflection at ω_{n+1}, as is explained in the example at the end of this section, or reflection for sets concentrating on $\omega_{n+1} \cap \mathsf{cof}(\omega_{n-1})$.

We now prepare some tools for the construction. Let $\theta > \kappa$ be large regular; we keep this θ fixed throughout the argument. One useful feature of sets in S^*_κ is that they allow to construct end-extendings that are elementary substructures of H_θ with high degree of closure. Such end-extendings will be needed in the proof of the distributivity of the iteration.

Lemma 3.1. *Let θ be regular large. Then there is a stationary set $S^*_\kappa(\theta) \subseteq \mathcal{P}_\kappa(H_\theta)$ such that for every $x \in S^*_\kappa(\theta)$, letting $\mu = x \cap \kappa$, the following holds.*

- ${}^{<\mu}x \subseteq x$, $\mathsf{card}(x) = 2^\mu$, *and*
- $H_{\mu^+} \subseteq H_x$ *and* ${}^\mu H_x \subseteq H_x$ *where H_x is the transitive collapse of x.*

Proof. Fix a function $f : {}^{<\omega}H_\theta \to H_\theta$ and a regular θ' much larger than θ. Throughout the argument we assume that $H_{\theta'}$ is equipped with a well-ordering $\lhd$ whose initial segment well-orders H_θ; this well-ordering will be used to compute Skolem functions, and we will often suppress it in our notation. By induction on $\xi < \kappa^+$ define a sequence of partial functions $h_\xi : {}^{<\omega}H_\theta \to H_\theta$. Along with the functions h_ξ we define languages $\mathcal{L}_\xi$ which are obtained by adding function symbols for h_ξ to the language of set theory. Given an enumeration $\langle \varphi_i \mid i < \kappa\rangle$ of $\mathcal{L}_\xi$-formulae, we say that h is the Skolem function for the $\mathcal{L}_\xi$-structure $(H_\theta, \in, \dots)$ with respect to this enumeration and the well-ordering $\lhd$ iff $h : \mu \times {}^{<\omega}H_\theta \to H_\theta$ is a partial function such that if $\varphi_i(\vec{v})$ is an $\mathcal{L}_\xi$-formula with n free variables of the form $(\exists u)\psi(u, \vec{v})$ and $s \in {}^nH_\theta$ then $(H_\theta, \in, \dots) \models \varphi_i(s)$ implies that $h(i, s)$ is defined and is the $\lhd$-least y such that $(H_\theta, \in, \dots) \models \psi(y, s)$.

- $\mathcal{L}_0$ is the language of set theory enriched with a function symbol $\dot{f}$ for f. Fix an enumeration $\langle \varphi_i \mid i < \kappa\rangle$ of $\mathcal{L}_0$-formulae, and let $h_0 : \mu \times {}^{<\omega}H_\theta$ be the Skolem function for $(H_\theta, \in, f)$ relative to the language $\mathcal{L}_0$ computed with respect to the well-ordering $\lhd$.
- Granting that $\mathcal{L}_{\bar\xi}$ and $h_{\bar\xi}$ have been defined for all $\bar\xi < \xi$, pick a function symbol $\dot{h}_{\xi-1}$ for $h_{\xi-1}$ and let $\mathcal{L}_\xi = \mathcal{L}_{\xi-1} \cup \{\dot{h}_{\xi-1}\}$ if ξ is a successor, and let $\mathcal{L}_\xi = \bigcup_{\bar\xi<\xi} \mathcal{L}_{\bar\xi}$ if ξ is a limit. Then pick an enumeration $\langle \varphi_i \mid i < \kappa\rangle$ of all $\mathcal{L}_\xi$-formulae, and let h_ξ be the Skolem function for the $\mathcal{L}_\xi$-structure $(H_\theta, \in, f, \langle h_{\bar\xi} \mid \bar\xi < \xi\rangle)$ computed relative to this enumeration and the well-ordering $\lhd$.

Since S^*_κ is stationary, we can find an elementary substructure $(Y, \in)$ of $(H_{\theta'}, \in)$ such that $H_\theta, f, \langle h_{\bar\xi} \mid \bar\xi < \kappa^+\rangle \in Y$ and $Y \cap H_{\kappa^+} \in S^*_\kappa$. Let $(H', \in, \bar{h})$ be the transitive collapse of $(Y, \in, h)$ and let σ be the inverse of the Mostowski collapsing isomorphism. Also let $\mu = \kappa \cap Y$ and H be the transitive collapse of $Y \cap H_\theta$, that is, $\sigma[H] = Y \cap H_\theta$ and $\sigma(H) = H_\theta$. Obviously the map σ is fully elementary when viewed as a map $\sigma : (H', \in) \to (H_{\theta'}, \in)$. By construction of Y we have $\mu^{+H'} = \mu^{+H} = \mu^+$ and $H^{H'}_{\mu^+} = H^H_{\mu^+} = H_{\mu^+}$. Finally let $\langle \bar{h}_{\bar\xi} \mid \bar\xi < \mu^+\rangle$ be the preimage of $\langle h_{\bar\xi} \mid \bar\xi < \kappa^+\rangle$ under σ, and

$$X = \{\bar{h}_\xi(i, s) \mid \xi < \mu^+ \ \&\ i < \mu \ \&\ s \in {}^{<\omega}H_{\mu^+} \ \&\ \bar{h}_\xi(i, s) \text{ defined}\}.$$

We show that if $\langle x_\eta \mid \eta < \mu\rangle \in \mathbf{V}$ is such that each x_η is an element of X then the sequence $\langle x_\eta \mid \eta < \mu\rangle$ is actually an element of X. Since H_{μ^+} is obviously contained in X, it is then easy to verify that $x = \sigma[X]$ is as required in the statement of Lemma 3.1, so this will complete the proof.

To each $\eta < \mu$ pick $\xi_\eta < \mu^+$, $i_\eta < \mu$ and $s_\eta \in {}^{<\omega}H_\theta$ such that $x_\eta = \bar{h}_{\xi_\eta}(i_\eta, s_\eta)$. Let $\xi < \mu^+$ be larger than all ξ_η where $\eta < \mu$. Since each statement of the form

$w = \dot{h}_{\xi_\eta}(u, v)$ is a formula in $\mathcal{L}_\xi$ and $\bar{h}_\xi$ is a Skolem function for the $\mathcal{L}_\xi$-structure $(H, \in, \dots)$, we can find $j_\eta < \mu$ such that $x_\eta = \bar{h}_\xi(j_\eta, \langle i_\eta\rangle^\frown s_\eta)$. Since the sequences $\langle\langle i_\eta\rangle^\frown s_\eta \mid \eta < \mu\rangle$, $\langle j_\eta \mid \eta < \mu\rangle$ are elements of H_{μ^+}, they are in $\mathrm{dom}(\sigma)$. Let $\langle j'_\eta \mid \eta < \kappa\rangle = \sigma(\langle j_\eta \mid \eta < \mu\rangle)$ and $\langle\langle i'_\eta\rangle^\frown s'_\eta \mid \eta < \kappa\rangle = \sigma(\langle\langle\langle i_\eta\rangle^\frown s_\eta \mid \eta < \mu\rangle)$. Then there is a sequence $\langle x'_\eta \mid \eta < \kappa\rangle$ such that $x'_\eta = h_{\sigma(\xi)}(j'_\eta, \langle i'_\eta\rangle^\frown s'_\eta)$ whenever the right side is defined. Such a sequence is obviously in H_θ, and its first order properties are described in the $\mathcal{L}_{\sigma(\xi)+1}$-structure $(H_\theta, \in, \dots)$ from parameters in Y using the function $h_{\sigma(\xi)}$. Since $h_{\sigma(\xi)+1}$ is a Skolem function for this structure, we can find some $j' < \kappa$ such that

$$\langle h_{\sigma(\xi)}(j'_\eta, \langle i'_\eta\rangle^\frown s'_\eta) \mid \eta < \kappa\rangle = h_{\sigma(\xi)+1}(j', \langle\langle j'_\eta, i'_\eta\rangle^\frown s'_\eta \mid \eta < \kappa\rangle).$$

By elementarity, some such j' is in $\kappa \cap Y \subseteq \mathrm{rng}(\sigma)$, so pick one and denote it by j^*. Let $j = \sigma^{-1}(j^*)$. Using the elementarity of σ it is then easy to see that $\bar{h}_{\xi+1}(j, \langle\langle j_\eta, i_\eta\rangle^\frown s_\eta\rangle \mid \eta < \mu\rangle)$ is a μ-sequence, and for each $\eta < \mu$ its η-th element is $\bar{h}_\xi(j_\eta, \langle i_\eta\rangle^\frown s_\eta) = x_\eta$. Thus, $\langle x_\eta \mid \eta < \mu\rangle \in X$. □

Next we describe a single step in the iteration, that is, adding a club subset of a regular cardinal disjoint from a given non-reflecting stationary set.

Definition 3.2. *Let $\rho < \lambda$ be regular cardinals and let S be a stationary subset of $\lambda \cap \mathsf{cof}(< \rho)$ with no reflection points of cofinality ρ. Let $\mathbb{Q}_S$ be the poset defined as follows.*

- *Conditions are closed bounded subsets of λ disjoint from S.*
- *Ordering is the end-extension.*

We will refer to this poset as the poset for adding closed unbounded subset of λ disjoint form S, or more vaguely the "club shooting" poset.

Under certain circumstances the poset $\mathbb{Q}_S$ from the above definition is known to satisfy certain amount of distributivity. For instance $\mathbb{Q}_S$ is (ρ, ∞)-distributive, that is, $\mathbb{Q}_S$ does not add any sequences of length $< \rho$, if $\mu^{<\rho} < \lambda$ for all $\mu < \lambda$. So cardinals $\leq \rho$ are not collapsed in the generic extension via $\mathbb{Q}_S$. The (ρ, ∞)-distributivity of $\mathbb{Q}_S$ follows from the assumption that $S \subset \lambda \cap \mathsf{cf}(< \rho)$ is stationary with no reflection points of cofinality ρ. The proof of (ρ, ∞)-distributivity is a folklore, and a variant of this proof will appear below when dealing with the successor steps of the proof that the iteration of club shooting is distributive.

The model in Theorem 1.3 will be obtained by iterating posets of the form $\mathbb{Q}_S$. In the proof of the distributivity of the iteration we will make use of the following general facts about forcing.

Fact 3.3. *Let $M \subseteq N$ be transitive models of ZFC. Let $\mathbb{P} \in M$ be a poset, and assume for every $p \in \mathbb{P}$ there is some $G \in \mathbf{V}$ generic for $\mathbb{P}$ over N such that $p \in G$. Assume $\varphi(v)$ is a Σ_0-formula in the language of set theory and $\dot{a} \in M$ is a $\mathbb{P}$-name. Then*

$$p \Vdash^M_{\mathbb{P}} \varphi(\dot{a}) \iff p \Vdash^N_{\mathbb{P}} \varphi(\dot{a}).$$

Given a regular cardinal ρ and an interval of ordinals X, by $\mathsf{Coll}(\rho, X)$ we denote the Lévy collapse with functions of size $< \rho$ which adds a surjection from ρ onto ξ for every $\xi \in X$. We write $\mathsf{Coll}(\rho, < \kappa)$ for $\mathsf{Coll}(\rho, [0, \kappa))$.

Fact 3.4. *Assume $\rho < \kappa$ are regular cardinals and κ is strongly inaccessible. Let $\mathbb{P} \in H_\kappa$ be a ρ-closed poset and $\mu < \kappa$. Then $\mathbb{P} \times \mathsf{Coll}(\rho, [\mu, \kappa))$ is forcing equivalent to $\mathsf{Coll}(\rho, [\mu, \kappa))$.*

Fact 3.5. *Assume $\rho < \mu$ are regular cardinals and $\mu^{<\rho} = \mu$. Let $S \subseteq \mu^+ \cap \mathsf{cof}(< \rho)$ be a stationary set. Then the stationarity of S is preserved by any ρ-closed forcing.*

Note that Fact 3.5 is a somewhat less sophisticated variant of Fact 2.4, and actually a consequence of a slight generalization of Fact 2.4.

We now begin with the construction of the model in Theorem 1.3. Let G be a generic filter for $\mathsf{Coll}(\omega_{n-1}, < \kappa)$ over $\mathbf{V}$. So the conditions are functions of size $< \omega_{n-1}$. The forcing is κ-c.c. and ω_{n-1}-closed. So in $\mathbf{V}[G]$ we have the following situation.

- $\omega_k^{\mathbf{V}[G]} = \omega_k^{\mathbf{V}}$ for all $k < n$.
- $\omega_n^{\mathbf{V}[G]} = \kappa$.
- $\omega_{n+1}^{\mathbf{V}[G]} = \kappa^{+\mathbf{V}}$.

By GCH in $\mathbf{V}$ we have $2^{\kappa^+} = \kappa^{++}$ in $\mathbf{V}[G]$. This means $\mathbf{V}[G] \models \mathsf{card}(H_{\kappa^{++}}) = \kappa^{++}$.

In $\mathbf{V}[G]$ define the club shooting iteration $\langle \mathbb{P}_\alpha \mid \alpha \leq \kappa^{++} \rangle$, $\langle \dot{\mathbb{Q}}_\alpha \mid \alpha < \kappa^{++} \rangle$. Fix an enumeration $\langle x_\alpha \mid \alpha < \kappa^{++} \rangle$ of $H_{\kappa^{++}}$ such that every $x \in H_{\kappa^{++}}$ is repeated κ^{++} times. Let $\mathbb{P}_0 = \{\varnothing\}$, and assuming that $\mathbb{P}_\alpha$ is already constructed, define $\dot{\mathbb{Q}}_\alpha$ to be a $\mathbb{P}_\alpha$-name for a poset such that:

$\Vdash^{\mathbf{V}[G]}_{\mathbb{P}_\alpha}$ $\dot{\mathbb{Q}}_\alpha$ s the poset for adding a club subset of κ^+ disjoint from x_α if x_α is a stationary subset of $\kappa^+ \cap \mathsf{cf}(< \kappa)$ with no reflection point of cofinality ω_{n-1}, and $\dot{\mathbb{Q}}_\alpha$ is the trivial poset otherwise.

In this notation x_α si treated as a $\mathbb{P}_\alpha$-name; this makes sense even if x_α is not a $\mathbb{P}_\alpha$-name, as we can meaningfully define evaluations of x_α under generic filters simply by replacing x_α with $\{\langle p, z\rangle \in x_\alpha \mid p \in \mathbb{P}_\alpha \;\&\; z \text{ is a } \mathbb{P}_\alpha\text{-name}\}$. This takes care of the successor steps of the iteration. At limit steps we let

- $\mathbb{P}_\alpha$ is the direct limit if $\mathsf{cf}(\alpha) = \kappa^+$.
- $\mathbb{P}_\alpha$ is the inverse limit if $\mathsf{cf}(\alpha) < \kappa^+$.

Finally we let

$$\mathbb{P} = \mathbb{P}_{\kappa^{++}}.$$

We view conditions in the iteration as partial functions with domains contained in κ^{++}, so in our terminology the notions of domain and support agree. It is obvious from the above that the iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \kappa^{++} \rangle$ is precisely a $< \kappa$-supported iteration. The following is our main lemma.

Lemma 3.6. *$\mathbb{P}$ is (κ^+, ∞)-distributive, that is, $\mathbb{P}$ does not add sequences of length at most κ.*

Proof. Let θ be regular large enough such that H_θ has the $\mathsf{Coll}(\omega_{n-1}, < \kappa)$-name for the enumeration $\langle x_\alpha \mid \alpha < \kappa^{++} \rangle$ as element; denote this name by $\dot{\vec{x}}$. By Lemma 3.1 there is an elementary substructure X of H_θ of size κ^+ such that $X \cap H_{\kappa^+} \in S^*_\kappa$ and $\langle x_\alpha \mid \alpha < \kappa^{++} \rangle \in X$ and, letting H be the transitive collapse of X and $\mu = \kappa \cap X$, the structure H contains H_{μ^+} and is closed under μ-sequences in $\mathbf{V}$. Let $\sigma : H \to H_\theta$ be the inverse of the Mostowski collapsing isomorphism, that is, $\mathrm{rng}(\sigma) = X$. Obviously $\mu = \mathsf{cr}(\sigma)$ and $\sigma(\mu) = \kappa$.

Let $\bar{G} = G \cap \mathsf{Coll}(\omega_{n-1}, <\mu)$ and $G' = G \cap \mathsf{Coll}(\omega_{n-1}, [\omega_{n-1}, \kappa))$; so $G \simeq \bar{G} \times G'$. We can now extend the map $\sigma : H \to H_\theta$ to $H[\bar{G}]$; to simplify the notation denote this extension again by σ. Write

$$\tau = \kappa^+, \quad \bar{\tau} = \mu^+, \quad \text{and} \quad \tilde{\tau} = \sup(\sigma[\bar{\tau}]). \tag{5}$$

We thus have an elementary map

$$\sigma : H[\bar{G}] \to H_\theta[G] \tag{6}$$

with the following properties, which are now easy to verify.

(a) $\mathsf{cr}(\sigma) = \mu$ and $\sigma(\mu) = \kappa$. In particular $\sigma(\omega_k) = \omega_k$ whenever $k < n$; for such k the cardinal ω_k is the same in $H[\bar{G}]$, $\mathbf{V}[\bar{G}]$ and $\mathbf{V}[G]$.
(b) In $\mathbf{V}[\bar{G}]$ the structure $H[\bar{G}]$ is closed under sequences of length μ.
(c) In $\mathbf{V}[G]$ the structure $H[\bar{G}]$ is closed under sequences of length $< \omega_{n-1}$.
(d) σ maps $\bar{\tau}$ cofinally into $\tilde{\tau}$ and is continuous at all limit ordinals of $H[\bar{G}]$-cofinality $< \mu$.
(e) In $H[\bar{G}]$ there is a sequence $\langle \bar{x}_\alpha \mid \alpha < \bar{\tau}^{+H} \rangle$ such that

$$\sigma(\langle \bar{x}_\alpha \mid \alpha < \bar{\tau}^{+H} \rangle) = \langle x_\alpha \mid \alpha < \tau^+ \rangle.$$

(f) Letting $\langle \bar{\mathbb{P}}_\alpha \mid \alpha \leq \bar{\tau}^{+H} \rangle, \langle \dot{\bar{\mathbb{Q}}}_\alpha \mid \alpha < \bar{\tau}^{+H} \rangle$ be the iteration defined in $H[\bar{G}]$ from $\langle \bar{x}_\alpha \mid \alpha < \bar{\tau}^{+H} \rangle$ the same way $\langle \mathbb{P}_\alpha \mid \alpha \leq \tau^+ \rangle, \langle \dot{\mathbb{Q}}_\alpha \mid \alpha < \tau^+ \rangle$ is defined from $\langle x_\alpha \mid \alpha < \tau^+ \rangle$ in $\mathbf{V}[G]$, we have

$$\sigma(\langle \bar{\mathbb{P}}_\alpha \mid \alpha \leq \bar{\tau}^{+H} \rangle, \langle \dot{\bar{\mathbb{Q}}}_\alpha \mid \alpha < \bar{\tau}^{+H} \rangle) = (\langle \mathbb{P}_\alpha \mid \alpha \leq \tau^+ \rangle, \langle \dot{\mathbb{Q}}_\alpha \mid \alpha < \tau^+ \rangle).$$

We now recursively construct sequences $\langle \dot{c}_\alpha \mid \alpha < \bar{\tau}^{+H} \rangle$ and $\langle A_\alpha \mid \alpha \leq \bar{\tau}^{+H} \rangle$ with the following properties.

(A) $\langle \dot{c}_\alpha \mid \alpha < \tau^{+H} \rangle$ and $\langle A_\alpha \mid \alpha \leq \tau^{+H} \rangle$ are both elements of $\mathbf{V}[\bar{G}]$.
(B) Let $\dot{\bar{F}}$ be the canonical $\bar{\mathbb{P}}_\alpha$-name for a generic filter on $\bar{\mathbb{P}}_\alpha$. Then $\dot{c}_\alpha$ is a $\bar{\mathbb{P}}_\alpha$-name such that

$$\Vdash^{\mathbf{V}[\bar{G}]}_{\bar{\mathbb{P}}_\alpha} \dot{c}_\alpha \text{ is a closed unbounded subset of } \bar{\tau} \text{ with } \dot{c}_\alpha \cap \bar{x}_\alpha = \varnothing \text{ whenever } H[\bar{G}][\dot{\bar{F}}] \models \text{“}\bar{x}_\alpha \subseteq \bar{\tau} \cap \mathsf{cof}(<\omega_{n-1}) \text{ is stationary with no reflection points of cofinality } \omega_{n-1}\text{”, and } \dot{c}_\alpha = \varnothing \text{ otherwise.} \tag{7}$$

(C) A_α is a $\bar{\tau}$-closed dense subset of $\bar{\mathbb{P}}_\alpha$ in the sense of $\mathbf{V}[\bar{G}]$.

We say that a condition $p \in \bar{\mathbb{P}}_\alpha$ is active at α just in case where p forces the hypothesis in (7), that is, if and only if

$$p \Vdash^{H[\bar{G}]}_{\bar{\mathbb{P}}_\alpha} \bar{x}_\alpha \subseteq \bar{\tau} \cap \mathsf{cof}(<\omega_{n-1}) \text{ is stationary with no reflection points of cofinality } \omega_{n-1}. \tag{8}$$

Assuming we have constructed $\dot{c}_{\bar{\alpha}}$ for $\bar{\alpha} < \alpha$ we define

$$A_\alpha = \{p \in \bar{\mathbb{P}}_\alpha \mid p \restriction \bar{\alpha} \Vdash^{\mathbf{V}[\bar{G}]}_{\bar{\mathbb{P}}_{\bar{\alpha}}} p(\bar{\alpha}) \in \dot{c}_{\bar{\alpha}} \text{ whenever } \bar{\alpha} \in \mathrm{dom}(p)\}. \tag{9}$$

Obviously $A_0 = \{\varnothing\}$, and A_0 has the desired properties. It follows immediately from the definition of A_α that if $\alpha < \beta$ then $A_\alpha \subseteq A_\beta$, and in fact

$$A_\alpha = \{p \restriction \alpha \mid p \in A_\beta\}.$$

Assuming the set A_α is constructed and satisfies (C) above, we will be able to construct the name $\dot{c}_\alpha$.

First of all we verify the induction hypothesis (C) for α. We first show that A_α is $\bar{\tau}$-closed. Let $\langle p_\xi \mid \xi < \vartheta \rangle \in \mathbf{V}[\bar{G}]$ be a descending chain in A_α where $\vartheta \leq \mu$ is a cardinal in $\mathbf{V}[\bar{G}]$. Since $H[\bar{G}]$ is closed under μ-sequences in $\mathbf{V}[\bar{G}]$, we actually have $\langle p_\xi \mid \xi < \vartheta\rangle \in H[\bar{G}]$, and working inside $H[\bar{G}]$ we construct a lower bound $p' \in A_\alpha$ for $\langle p_\xi \mid \xi < \vartheta\rangle$. We let $\mathrm{dom}(p') = \bigcup\{\mathrm{dom}(p_\xi) \mid \xi < \vartheta\}$ and for $\bar{\alpha} \in \mathrm{dom}(p')$ define the values $p'(\bar{\alpha})$ by recursion. Since $H[\bar{G}]$ is closed under μ-sequences in $\mathbf{V}[\bar{G}]$ this union is in $H[\bar{G}]$, and has size at most μ in $H[\bar{G}]$. So it is a support for a condition in $\bar{\mathbb{P}}_\alpha$. Also, the closure properties of $H[\bar{G}]$ guarantee that p' we are inductively constructing is an element of $H[\bar{G}]$. Assuming $p' \restriction \bar{\alpha}$ was already defined and is a condition in $\bar{\mathbb{P}}_{\bar{\alpha}}$ below all $p_\xi \restriction \bar{\alpha}$ we have

$$p' \restriction \bar{\alpha} \Vdash^{H[\bar{G}]}_{\bar{\mathbb{P}}_{\bar{\alpha}}} p_\xi(\bar{\alpha}) \leq p_\eta(\bar{\alpha}) \text{ whenever } \eta < \xi \text{ and } \bar{\alpha} \in \mathrm{dom}(p_\eta). \tag{10}$$

We claim that if $p' \restriction \bar{\alpha}$ is active at $\bar{\alpha}$ (see (8)) in the iteration $\langle \bar{\mathbb{P}}_\xi, \dot{\bar{\mathbb{Q}}}_\xi \mid \xi < \bar{\tau}^{+H}\rangle$ then

$$p' \restriction \bar{\alpha} \Vdash^{H[\bar{G}]}_{\bar{\mathbb{P}}_{\bar{\alpha}}} \sup\left(\bigcup_{\xi<\vartheta} p_\xi(\bar{\alpha})\right) \notin x_{\bar{\alpha}}. \tag{11}$$

To see this, notice that

$$p' \restriction \bar{\alpha} \Vdash^{\mathbf{V}[\bar{G}]}_{\mathbb{P}_{\bar{\alpha}}} \sup(p_\xi(\bar{\alpha})) \in \dot{c}_\alpha, \tag{12}$$

as we are assuming $p_\xi \in A_\alpha$ and $p' \restriction \bar{\alpha}$ forces $\dot{c}_{\bar{\alpha}}$ to be a closed unbounded subset of $\bar{\tau}$. Since the question about membership to a set is a Σ_0-statement, we can use Fact 3.3 to switch between forcing relation over $H[\bar{G}]$ and $\mathbf{V}[\bar{G}]$. By (10) and the fact (7) that $\dot{c}_{\bar{\alpha}}$ is forced to be a closed set disjoint from $x_{\bar{\alpha}}$ we conclude $p' \restriction \bar{\alpha} \Vdash^{\mathbf{V}[\bar{G}]}_{\mathbb{P}_{\bar{\alpha}}} \sup\bigcup_\xi p_\xi(\bar{\alpha}) \notin x_{\bar{\alpha}}$, so the same statement is forced by $p' \restriction \bar{\alpha}$ over $H[\bar{G}]$. This proves (11). Now working in $H[\bar{G}]$ we can define $p'(\bar{\alpha})$ to be a $\bar{\mathbb{P}}_{\bar{\alpha}}$-name for a condition in $\dot{\bar{\mathbb{Q}}}_{\bar{\alpha}}$ such that

$$p' \restriction \bar{\alpha} \Vdash^{H[\bar{G}]}_{\bar{\mathbb{P}}_{\bar{\alpha}}} p'(\bar{\alpha}) = \text{the closure of } \bigcup_{\xi<\vartheta} p_\xi(\bar{\alpha}),$$

and it is clear from the above that $p' \restriction (\bar{\alpha}+1)$ is a condition in $\bar{\mathbb{P}}_{\alpha+1}$. Also, it follows from (12) that $p' \in A_\alpha$. This completes the proof that A_α is $\bar{\tau}$-closed in the sense of $\mathbf{V}[\bar{G}]$.

The proof that A_α is dense proceeds by induction on α, so assume $A_{\bar{\alpha}}$ is a dense subset of $\bar{\mathbb{P}}_{\bar{\alpha}}$ whenever $\bar{\alpha} < \alpha$, and is $\bar{\tau}$-closed whenever $\bar{\alpha} \leq \alpha$. This can be assumed, as we proved the closure of A_α above. Pick $p \in \bar{\mathbb{P}}_\alpha$; we find a $p' \in A_\alpha$ below p. First consider the case where α is a successor ordinal, say $\alpha = \bar{\alpha}+1$. If there is some condition below $p \restriction \bar{\alpha}$ which is active at $\bar{\alpha}$, pick some such $p^* \in \bar{\mathbb{P}}_{\bar{\alpha}}$ and an ordinal γ such that $p^* \leq p \restriction \bar{\alpha}$ and $p^* \Vdash^{\mathbf{V}[\bar{G}]}_{\bar{\mathbb{P}}_{\bar{\alpha}}} \max(p(\bar{\alpha})) < \check{\gamma} \;\&\; \check{\gamma} \in \dot{c}_{\bar{\alpha}}$. This can be done as $\dot{c}_{\bar{\alpha}}$ is forced by p^* over $\mathbf{V}[\bar{G}]$ to be a closed unbounded subset of $\bar{\tau}$. Arguing similarly as in the proof above that A_α is $\bar{\tau}$-closed and relying on Fact 3.3, we can find a $\bar{\mathbb{P}}_{\bar{\alpha}}$-name $q \in H[\bar{G}]$ for a condition in $\bar{\mathbb{Q}}_{\bar{\alpha}}$ such that $p^* \Vdash^{H[\bar{G}]}_{\bar{\mathbb{P}}_{\bar{\alpha}}} q = p(\bar{\alpha}) \cup \{\check{\gamma}\}$. We then have

$$p^* \Vdash^{H[\bar{G}]}_{\bar{\mathbb{P}}_{\bar{\alpha}}} \check{\gamma} = \max(q) \notin x_{\bar{\alpha}}.$$

By the induction hypothesis $A_{\bar{\alpha}}$ is dense in $\bar{\mathbb{P}}_{\bar{\alpha}}$, so we can find some $\bar{p} \in A_{\bar{\alpha}}$ below p^*. It is then easy to verify that letting $p' = \bar{p}^\frown\langle q\rangle$, the function p' is a condition in $\bar{\mathbb{P}}_\alpha$ below p. By construction then, $p' \in A_\alpha$. If no condition in $\bar{\mathbb{P}}_{\bar{\alpha}}$ below $p \restriction \bar{\alpha}$ is active at $\bar\alpha$ then $p \restriction \bar\alpha \Vdash^{H[\bar G]}_{\bar{\mathbb{P}}_{\bar\alpha}} \dot{\bar{\mathbb{Q}}}_{\bar\alpha} = \{\varnothing\}$, so it suffices to pick some $\bar p \in A_{\bar\alpha}$ below $p \restriction \bar\alpha$ and let $p' = \bar p^\frown\langle p(\alpha)\rangle$.

If α is a limit we focus on the case where $\gamma = \mathsf{cf}(\alpha) \le \mu$, as for $\mathsf{cf}(\alpha) = \mu^+$ the conclusion easily follows from the properties of direct limits and the induction hypothesis. Pick a normal sequence $\langle \alpha_\xi \mid \xi < \gamma\rangle$ converging to α. Given a condition $p \in \bar{\mathbb{P}}_\alpha$, construct a descending chain $\langle p_\xi \mid \xi < \gamma\rangle$ such that the following are met

- $p_\xi \in A_{\alpha_\xi}$
- $p_0 \le p \restriction \alpha_0$
- $p_{\xi+1} \le p_\xi \cup p \restriction [\alpha_\xi, \alpha_{\xi+1})$
- $p_\xi \le p_{\bar\xi}$ for all $\bar\xi \le \xi$ for limit ξ.

Passing through the limit steps is guaranteed by the induction hypothesis, as all sets A_{α_ξ} are μ-closed and dense in the respective posets. Since $A_{\alpha_\xi} \subseteq A_\alpha$ and we have already proved that A_α is μ-closed, the sequence $\langle p_\xi \mid \xi < \gamma\rangle$ has a lower bound p' in A_α. Obviously then $p' \le p$. This completes the proof of (C).

Once we verified (C) for α we have the following immediate consequence.

(13) $\bar{\mathbb{P}}_\alpha$ is $(\bar\tau, \infty)$-distributive in $H[\bar G]$ and $\mathbb{P}_{\sigma(\alpha)}$ is (τ, ∞)-distributive in $\mathbf{V}[G]$.

To see (13) it suffices to show that $\bar{\mathbb{P}}_\alpha$ is $(\bar\tau, \infty)$-distributive in $H[\bar G]$ and apply the elementarity of σ. Now if $p \in \bar{\mathbb{P}}_\alpha$ and $\langle D_\xi \mid \xi < \mu\rangle \in H[\bar G]$ is a sequence of open dense subsets of $\bar{\mathbb{P}}_\alpha$ we can construct a descending chain $\langle p_\xi \mid \xi < \mu\rangle$ in $\bar{\mathbb{P}}_\alpha$ such that $p_0 \le p$, $p_{2\cdot\xi} \in A_\alpha$ and $p_{2\cdot\xi+1} \in D_\xi$ for all $\xi < \mu$. Since A_α is τ-closed in $\mathbf{V}[\bar G]$, the construction can be carried out, and at the very end we can pick $p' \in A_\alpha$ below this chain. Then $p' \le p$ and p' is in the intersection of all D_ξ. This proves the $(\bar\tau, \infty)$-distributivity of $\bar{\mathbb{P}}_\alpha$ in $H[\bar G]$.

Now we can construct $\dot c_\alpha$. Given any generic $\bar F$ for $\bar{\mathbb{P}}_\alpha$ over $\mathbf{V}[\bar G]$ we can construct, working inside $\mathbf{V}[\bar G][\bar F]$, a cofinal descending chain $\langle p_\xi \mid \xi < \bar\tau\rangle \in \mathbf{V}[\bar G][\bar F]$ in $\bar F \cap A_\alpha$ in a similar fashion as the chains constructed in the proof of (13). First, the size of $\bar F$ in $\mathbf{V}[\bar G][\bar F]$ is $\bar\tau$, so we have an enumeration $\langle f_\xi \mid \xi < \bar\tau\rangle$ of $\bar F$ all of whose proper initial segments are in $\mathbf{V}[\bar G]$. This last conclusion follows from the fact that $\mathbf{V}[\bar G][\bar F]$ is a generic extension of $\mathbf{V}[\bar G]$ via $\bar{\mathbb{P}}_\alpha$ and $A_\alpha \in \mathbf{V}[\bar G]$ is a $\bar\tau$-closed subset of $\bar{\mathbb{P}}_\alpha$, as is guaranteed by the induction hypothesis (C) above. If $p_{\bar\xi} \in \bar F$ has been constructed for all $\bar\xi < \xi$ in a way that $p_{\bar\xi} \le f_{\bar\xi}$, first find $p'_\xi \in \bar F$ such that $p'_\xi \le p_{\bar\xi}, f_\xi$ for all $\bar\xi < \xi$. If ξ is a successor ordinal this is easy, as it suffices to let p'_ξ be a lower bound for $p_{\xi-1}, f_\xi$ in $\bar F$. If ξ is a limit, then using the genericity of $\bar F$ over $\mathbf{V}[\bar G]$ and the fact that $\langle p_{\bar\xi} \mid \bar\xi < \xi\rangle \in \mathbf{V}[\bar G]$ first pick some lower bound $\tilde p_\xi \in \bar F$ for all $p_{\bar\xi}$ and then a lower bound $p'_\xi \in \bar F$ for $\tilde p_\xi, f_\xi$. Now, using the density of A_α in $\bar{\mathbb{P}}_\alpha$ pick $p_\xi \le p'_\xi$ in $\bar F \cap A_\alpha$.

Let $\tilde G$ be generic for $\mathsf{Coll}(\mu, [\mu, \kappa))$ over $\mathbf{V}[\bar G][\bar F]$. Since forcing with $\bar{\mathbb{P}}_\alpha$ over $\mathbf{V}[\bar G]$ is equivalent to forcing with A_α over $\mathbf{V}[\bar G]$, by (C) and Fact 3.4 we can find a filter G^* generic for $\mathsf{Coll}(\mu, [\mu, \kappa))$ over $\mathbf{V}[\bar G]$ such that $\mathbf{V}[\bar G][\bar F][\tilde G] = \mathbf{V}[\bar G][G^*]$. Let G' be generic for $\mathsf{Coll}(\omega_{n-1}, <\kappa)$ over $\mathbf{V}$ such that $G' \simeq \bar G \times G^*$, and let $\sigma' : H[\bar G] \to H_\theta[G']$ be the natural extension of $\sigma : H \to H_\theta$ to $H[\bar G]$.

Write $\langle x'_\xi \mid \xi < \tau^+\rangle = \sigma'(\langle \bar x_\xi \mid \xi < \bar\tau^{+H}\rangle)$, $\langle \mathbb{P}'_\xi \mid \xi \le \tau^+\rangle = \sigma'(\langle \bar{\mathbb{P}}_\xi \mid \xi \le \bar\tau^{+H}\rangle)$ and $\langle \mathbb{Q}'_\xi \mid \xi < \tau^+\rangle = \sigma'(\langle \bar{\mathbb{Q}}_\xi \mid \xi < \tau^{+H}\rangle)$. Also write $p^*_\xi = \sigma'(p_\xi)$ for $\xi < \bar\tau$ where p_ξ were constructed above. Then $\langle p^*_\xi \mid \xi < \bar\tau\rangle$ is a descending sequence in $\mathbb{P}'_{\sigma(\alpha)}$. We construct a lower bound $p^* \in \mathbb{P}'_{\sigma(\alpha)}$ for this sequence. We let

$$\mathrm{dom}(p^*) = \bigcup_{\xi<\bar\tau} \mathrm{dom}(p^*_\xi)$$

and observe that the size of this set is at most κ, so it is a support for a condition in $\mathbb{P}'_{\sigma(\alpha)}$. We then define $p^*(\bar\alpha)$ by recursion on $\bar\alpha < \sigma(\alpha)$. Assume $p^* \upharpoonright \bar\alpha$ has been defined and is below all $p^*_\xi \upharpoonright \bar\alpha$ where $\xi < \bar\tau$. Let $\dot\delta$ be a name of the ordinal such that

$$p^* \upharpoonright \bar\alpha \Vdash^{\mathbf{V}[G']}_{\mathbb{P}'_{\bar\alpha}} \dot\delta = \sup\{\max(p^*_\xi(\bar\alpha)) \mid \xi < \bar\tau\}$$

where we understand that $\max(\varnothing) = 0$. If $\bar p \le p^* \upharpoonright \bar\alpha$ in $\mathbb{P}'_{\bar\alpha}$ is active at $\bar\alpha$ then, since the conditions p^*_ξ constitute a descending chain in $\mathbb{P}'_{\bar\alpha}$, we obtain

$$p^* \upharpoonright \bar\alpha \Vdash^{\mathbf{V}[G']}_{\mathbb{P}'_{\bar\alpha}} \langle p^*_\xi(\bar\alpha) \mid \xi < \bar\tau\rangle \text{ is a descending chain in } \dot{\mathbb{Q}}'_{\bar\alpha} = \dot{\mathbb{Q}}_{x'_{\bar\alpha}}. \tag{14}$$

Here recall that $\dot{\mathbb{Q}}_{x'_{\bar\alpha}}$ is a name for the poset adding a closed unbounded subset disjoint from the set named by $x'_{\bar\alpha}$. If $\bar p$ forces that the chain $\langle p^*_\xi(\bar\alpha) \mid \xi < \bar\tau\rangle$ is not eventually constant, there is a $\mathbb{P}'_{\bar\alpha}$-name $\dot g \in \mathbf{V}[G']$ for a function such that

$$\bar p \Vdash^{\mathbf{V}[G']}_{\mathbb{P}'_{\bar\alpha}} \mathrm{dom}(\dot g) \text{ is a cofinal subset of } \bar\tau \text{ and } \dot g \text{ is strictly increasing and cofinal in } \dot\delta.$$

By (13) the cofinality of $\bar\tau$ is forced by $\bar p$ to by ω_{n-1}, and by the properties of $\dot g$ the cofinality of $\dot\delta$ is forced by $\bar p$ to be ω_{n-1} as well. As $\bar p \Vdash_{\mathbb{P}'_{\bar\alpha}} x'_{\bar\alpha} \subseteq \tau \cap \mathsf{cof}(<\omega_{n-1})$, we conclude that $\bar p \Vdash_{\mathbb{P}'_{\bar\alpha}} \dot\delta \notin x'_{\bar\alpha}$. Taking this observation into account, we can construct a $\mathbb{P}'_{\bar\alpha}$-name $p^*(\bar\alpha) \in \mathbf{V}[G']$ for a condition in $\dot{\mathbb{Q}}'_{\bar\alpha}$ such that

$$p^* \upharpoonright \bar\alpha \Vdash^{\mathbf{V}[G']}_{\mathbb{P}'_{\bar\alpha}} p^*(\bar\alpha) = \bigcup_{\xi<\bar\tau} p^*_\xi(\bar\alpha) \cup \{\dot\delta\}.$$

Then letting $p^* \upharpoonright (\bar\alpha+1) = (p^* \upharpoonright \bar\alpha)^\frown\langle p^*(\bar\alpha)\rangle$ we have $p^* \upharpoonright (\bar\alpha+1) \in \mathbb{P}'_{\bar\alpha+1}$ and $p^* \upharpoonright (\bar\alpha+1) \le p^*_\xi \upharpoonright (\bar\alpha+1)$ for all $\xi < \bar\tau$.

Let F be a filter generic for $\mathbb{P}'_{\sigma(\alpha)}$ over $\mathbf{V}[G']$ such that $p^* \in F$. Then $\sigma'[\bar F] \subseteq F$, so we can extend σ' to an elementary embedding $\sigma_F : H[\bar G][\bar F] \to H_\theta[G'][F]$ such that $\sigma_F(\bar F) = F$. In particular we have $\sigma_F(\bar x^{\bar F}_\alpha) = (x'_{\sigma(\alpha)})^F$. If in $\mathbf{V}[\bar G][\bar F]$ the set $\bar x^{\bar F}_\alpha$ is a stationary subset of $\bar\tau \cap \mathsf{cof}(<\omega_{n-1})$ with no reflection points of cofinality ω_{n-1} then by the elementarity of σ_F, in $\mathbf{V}[G'][F]$ the set $(x'_{\sigma(\alpha)})^F \cap \zeta$ is non-stationary whenever $\mathsf{cf}(\zeta) = \omega_{n-1}$. We have already established (13), so again by the elementarity of σ_F the poset $\mathbb{P}'_{\sigma(\alpha)}$ is (τ,∞)-distributive in $\mathbf{V}[G']$, that is, the models $\mathbf{V}[G'][F]$ and $\mathbf{V}[G']$ agree on $\le \mu$-sequences. In particular, $\tilde\tau$ is ω_{n-1}-cofinal in $\mathbf{V}[G'][F]$. Hence $(x'_{\sigma(\alpha)})^F \cap \tilde\tau$ is a non-stationary subset of $\tilde\tau$ in $\mathbf{V}[G'][F]$. Appealing again to the (τ,∞)-distributivity of $\mathbb{P}'_{\sigma(\alpha)}$ in $\mathbf{V}[G']$, the models $\mathbf{V}[G'][F]$ and $\mathbf{V}[G']$ agree on subsets of $\tilde\tau$. It follows that $(x'_{\sigma(\alpha)})^F \cap \tilde\tau \in \mathbf{V}[G']$ and is non-stationary in the sense of $\mathbf{V}[G']$. Since σ is continuous at points of $\mathbf{V}[G']$-cofinality $<\omega_{n-1}$ and $\bar x^{\bar F}_\alpha$ concentrates on ordinals of $\mathbf{V}[G']$-cofinality $<\omega_{n-1}$ also $\bar x^{\bar F}_\alpha$ is non-stationary in the sense of $\mathbf{V}[G']$, as $\sigma[\bar x^{\bar F}_\alpha] \subseteq (x'_{\sigma(\alpha)})^F \cap \tilde\tau$.

We show that $\bar x^{\bar F}_\alpha$ is non-stationary in the sense of $\mathbf{V}[\bar G][\bar F]$. To see this we apply Fact 3.5 with $\rho = \omega_{n-1}$. Since μ is strongly inaccessible in $\mathbf{V}$ we have $\mathbf{V} \models \mu^{<\omega_{n-1}} = \mu$. Since $\mathsf{Coll}(\omega_{n-1}, <\mu)$ is ω_{n-1}-closed in $\mathbf{V}$ and $\bar{\mathbb{P}}_\alpha$ has a μ-dense closed subset in $\mathbf{V}[\bar G]$, namely the set A_α, the models $\mathbf{V}$ and $\mathbf{V}[\bar G][\bar F]$ agree on

$< \omega_{n-1}$-sequences, so we still have $\mu^{<\omega_{n-1}} = \mu$ in $\mathbf{V}[\bar{G}][\bar{F}]$. If $\bar{x}_\alpha^{\bar{F}}$ were stationary in $\mathbf{V}[\bar{G}][\bar{F}]$, it would be a stationary subset of $\bar{\tau} \cap \mathsf{cf}(< \omega_{n-1})$ in this model, hence by Fact 3.5 the poset $\mathsf{Coll}(\omega_{n-1}, [\mu, \kappa))$ would preserve its stationarity. Since G' was constructed so that $\mathbf{V}[G'] = \mathbf{V}[\bar{G}][\bar{F}][\tilde{G}]$ where $\tilde{G}$ is generic for $\mathsf{Coll}(\mu, [\mu, \kappa))$ over $\mathbf{V}[\bar{G}][\bar{F}]$, the set $\bar{x}_\alpha^{\bar{F}}$ would be stationary in $\mathbf{V}[G']$, a contradiction. This completes the proof that $\bar{x}_\alpha^{\bar{F}}$ is non-stationary in $\mathbf{V}[\bar{G}][\bar{F}]$.

To summarize, we proved that for every filter $\bar{F}$ generic for $\bar{\mathbb{P}}_\alpha$ over $\mathbf{V}[\bar{G}]$ if

$H[\bar{G}] \models \bar{x}_\alpha^{\bar{F}}$ is a stationary subset of $\bar{\tau} \cap \mathsf{cof}(< \omega_{n-1})$ with no reflection points of cofinality ω_{n-1}

then $\bar{x}_\alpha^{\bar{F}}$ is non-stationary in $\mathbf{V}[\bar{G}][\bar{F}]$. It follows from general properties of forcing that there is a $\bar{\mathbb{P}}_\alpha$-name $\dot{c}_\alpha$ such that (7) holds, which completes the construction of $\dot{c}_\alpha$. This also closes the induction cycle, as at this point we established clause (C) and also constructed $\dot{c}_\alpha$. Thus, we completed the proof of Lemma 3.6. $\Box$

Lemma 3.7. *The following holds in* $\mathbf{V}[G]$. *Given* $\alpha \leq \tau^+$ *let*

$$D_\alpha = \{p \in \mathbb{P}_\alpha \mid p \restriction \alpha \textit{ determines the value } p(\alpha) \textit{ whenever } \alpha \in \mathrm{dom}(p)\}.$$

Then D_α *is a dense subset of* $\mathbb{P}_\alpha$ *which can be identified with a set of* $< \tau$*-sequences in* $H_\tau[G]$. *Thus, for* $\alpha < \tau^+$ *the set* D_α *is of size* τ.

Proof. We follow the setup in the proof of Lemma 3.6, and prove that the sets $\bar{D}_\alpha$ defined in $H[\bar{G}]$ the same way D_α were defined in $\mathbf{V}[G]$ are dense subsets of $\bar{\mathbb{P}}_\alpha$ which can be identified with sets of $< \bar{\tau}$-sequences in $H[\bar{G}]$ of size $\bar{\tau}$ in the sense of $H[\bar{G}]$. Since obviously $\sigma(\bar{D}_\alpha) = D_{\sigma(\alpha)}$ the conclusion in the lemma follows immediately. Notice that $\bar{D}_\alpha = \{p \restriction \alpha \mid p \in \bar{D}_\beta\}$ whenever $\alpha < \beta$.

By induction on $\alpha < \bar{\tau}^{+H}$ we prove that the set $\bar{D}_\alpha$ is a dense subset of $\bar{\mathbb{P}}_\alpha$. Here we use the properties of the sets A_α and names $\dot{c}_\alpha$ established in the proof of Lemma 3.6. We also make use of the fact that the posets $\bar{\mathbb{P}}_\alpha$ are $(\bar{\tau}, \infty)$-distributive in the sense of $H[\bar{G}]$.

Assume first α is a successor, say $\alpha = \bar{\alpha} + 1$. Let $p \in \bar{\mathbb{P}}_\alpha$. Since $\bar{\mathbb{P}}_{\bar{\alpha}}$ is $(\bar{\tau}, \infty)$-distributive in $H[\bar{G}]$, there is an extension $p_1 \leq p \restriction \bar{\alpha}$ and some $d \in H_{\bar{\tau}}[\bar{G}]$ such that $p_1 \Vdash^{H[\bar{G}]}_{\bar{\mathbb{P}}_{\bar{\alpha}}} p(\bar{\alpha}) = \check{d}$. By the induction hypothesis there is some $p_2 \in \bar{D}_{\bar{\alpha}}$ such that $p_2 \leq p_1$. Then $p' = p_2{}^\frown\langle p(\bar{\alpha})\rangle$ is as required, that is, $p' \in \bar{D}_\alpha$ and $p' \leq p$.

Now assume α is a limit. Again, it suffices to focus on α of $H[\bar{G}]$-cofinality $\leq \mu$, as for α of $H[\bar{G}]$-cofinality $> \mu$ the conclusion follows easily from general properties of direct limits. In $H[\bar{G}]$ pick a normal sequence $\langle \alpha_\xi \mid \xi < \gamma\rangle$ where $\gamma = \mathsf{cf}^{H[\bar{G}]}(\alpha)$. Define descending chains $\langle p_\xi \mid \xi < \gamma\rangle$ and $\langle p'_\xi \mid \xi < \gamma\rangle$ so that the following are satisfied.

- $p'_0 \leq p \restriction \alpha_0$.
- $p'_\xi \in A_{\alpha_\xi}$ is such that $p'_\xi \leq p_{\xi-1}{}^\frown(p \restriction [\alpha_{\xi-1}, \alpha_\xi))$ if ξ is a successor, and $p'_\xi \leq p'_{\bar{\xi}}$ for all $\bar{\xi} < \xi$ if ξ is a limit.
- $p_\xi \in \bar{D}_{\alpha_\xi}$ is such that $p_\xi \leq p'_\xi$.

As before, this construction can be carried out as the sets A_{α_ξ} are $\bar{\tau}$-closed. We then define p' similarly as in the proof of Lemma 3.6. We let

$$\mathrm{dom}(p') = \bigcup_{\xi<\gamma} \mathrm{dom}(p_\xi)$$

and observe that this set is a legal support for a condition in $\bar{\mathbb{P}}_\alpha$. We then define the values $p'(\bar\alpha)$ for $\bar\alpha < \alpha$ by recursion on $\bar\alpha$. Assuming $p' \restriction \bar\alpha$ has been already defined, the condition $p_\eta \restriction \bar\alpha$ decides the value $p_\xi(\bar\alpha)$ whenever $\bar\alpha \in \mathrm{dom}(p_\xi)$ and $\eta \geq \xi$, so $p' \restriction \bar\alpha$, being below all $p_\eta \restriction \bar\alpha$, decides $p_\xi(\bar\alpha)$ the same way. Let $d_{\bar\alpha,\xi} \in H[\bar G]$ be this value, $\delta_{\bar\alpha,\xi} = \max(d_{\bar\alpha,\xi})$,

$$\delta_{\bar\alpha} = \sup_{\xi<\gamma} \delta_{\bar\alpha,\xi}, \qquad \text{and} \qquad d_{\bar\alpha} = \bigcup_{\xi<\gamma} d_{\bar\alpha_\xi} \cup \{\delta_{\bar\alpha}\}.$$

Notice that if $p' \restriction \bar\alpha$ is active at $\bar\alpha$ then so are p_ξ for ξ such that $\bar\alpha \in \mathrm{dom}(p_\xi(\bar\alpha)$. This is true because $p' \restriction \bar\alpha$ and $p_\xi \restriction \bar\alpha$ decide $p_\xi(\bar\alpha)$ the same way, the decision that $p_\xi(\bar\alpha)$ is nonempty is made by a condition $q \in \bar{\mathbb{P}}_{\bar\alpha}$ if and only if q is active at $\bar\alpha$, and because $p' \restriction \bar\alpha$ decides $p_\xi(\bar\alpha)$ to be nonempty. We then let $p'(\bar\alpha)$ be a $\bar{\mathbb{P}}_{\bar\alpha}$-name in $H[\bar G]$ for a condition in $\dot{\bar{\mathbb{Q}}}_{\bar\alpha}$ such that

$$p' \restriction \bar\alpha \Vdash^{H[\bar G]}_{\bar{\mathbb{P}}_{\bar\alpha}} p'(\bar\alpha) = \check{d}_{\bar\alpha}$$

if $p' \restriction \bar\alpha$ is active at $\bar\alpha$, and $p'(\bar\alpha)$ to be a name for the empty set otherwise.

If $p' \restriction \bar\alpha$ is active at $\bar\alpha$ in the iteration $\langle \bar{\mathbb{P}}_\xi, \dot{\bar{\mathbb{Q}}}_\xi \mid \xi < \tau^{+H}\rangle$ and $p'_\xi \in A_{\alpha_\xi}$, the ordinal $\delta_{\bar\alpha,\xi}$ is forced into $\dot c_{\bar\alpha}$ by $p'_\eta \restriction \bar\alpha$ over $\mathbf{V}[\bar G]$ whenever $\eta \geq \xi$, hence also by the condition $p' \restriction \bar\alpha$. It follows that $p' \restriction \bar\alpha \Vdash^{\mathbf{V}[\bar G]}_{\bar{\mathbb{P}}_{\bar\alpha}} \check\delta_{\bar\alpha} \in \dot c_{\bar\alpha}$, and similarly as in the proof of Lemma 3.6 we argue that

$$p' \restriction \bar\alpha \Vdash^{H[\bar G]}_{\bar{\mathbb{P}}_{\bar\alpha}} p'(\bar\alpha) = \check d_{\bar\alpha} \cap x_{\bar\alpha} = \varnothing,$$

that is, $p' \restriction (\bar\alpha + 1)$ is a condition in $\bar{\mathbb{P}}_{\bar\alpha+1}$ and in fact $p' \restriction (\bar\alpha+1) \in \bar D_{\bar\alpha+1}$. □

Corollary 3.8. *The poset $\mathbb{P}$ is τ^+-c.c. in $\mathbf{V}[G]$.*

Proof. By Lemma 3.7, all posets $\mathbb{P}_\alpha$ for $\alpha < \tau^+$ are τ^+-c.c., and the iteration involves direct limit at every α of cofinality τ in $\mathbf{V}[G]$, that is, on a stationary set. □

Combining Lemma 3.6 and Corollary 3.8, we conclude that the poset $\mathbb{P}$ does not collapse cardinals, and does not add bounded subsets of τ. It remains to check that in the extension via $\mathbb{P}$ every stationary subset of $\tau \cap \mathsf{cf}(\omega_{n-1})$ has a reflection point of cofinality ω_{n-1}. Assuming this is false, there is a $\mathbb{P}$-name $\dot S$ in $\mathbf{V}[G]$ and a condition $p \in \mathbb{P}$ such that

$$p \Vdash^{\mathbf{V}[G]}_{\mathbb{P}} \begin{array}{l}\dot S \text{ is a stationary subset of } \tau \cap \mathsf{cof}(<\omega_{n-1}) \text{ with no reflection points}\\ \text{of cofinality } \omega_{n-1}.\end{array}$$

By Lemma 3.7 we can take $\dot S$ to be a canonical name for a subset of τ consisting of pairs $\langle q, \check\xi\rangle$ where $q \in D_{\tau^+}$, so by the chain condition of $\mathbb{P}$ this name is actually a $\mathbb{P}_\alpha$-name for some $\alpha < \tau^+$, and is an element of H_{τ^+}. Since each element of H_{τ^+} appears on the enumeration $\langle x_\alpha \mid \alpha < \tau^+\rangle$ cofinally often, we may without loss of generality assume $\dot S = x_\alpha$ for a suitable α, and p is active at α. Now if F is a filter generic for $\mathbb{P}$ over $\mathbf{V}[G]$, $F_{<\alpha}$ is its projection on $\mathbb{P}_\alpha$, and F_α is its projection on $\dot{\mathbb{Q}}_\alpha^{F_{<\alpha}}$ then $\bigcup F_\alpha$ is a closed unbounded subset of τ disjoint from $x_\alpha^{F_\alpha} = \dot S^F$, a contradiction with the assumption that $\dot S^F$ is stationary. This completes the proof of Theorem 1.3. □

In the following we explain why one cannot expect the proof of Theorem 1.3 to yield simultaneous reflection at τ. We begin with some additional facts.

Fact 3.9. *Let $\rho < \lambda$ be regular, and let $\mathbb{N}(\lambda, 2, \rho)$ be the poset for adding a pair of stationary subsets of $\lambda \cap \mathsf{cof}(< \rho)$ each of which reflects at stationarily many $\alpha < \lambda$ of cofinality $\gamma \in [\rho, \lambda)$, but have no common reflection point. The conditions are pairs (p, q) satisfying the following.*

(a) *p, q are functions such that $\mathrm{dom}(p), \mathrm{dom}(q) = \alpha \cap \mathsf{cof}(< \rho)$ for some $\alpha < \lambda$.*
(b) *If $\bar{\alpha} \leq \alpha$ is of cofinality $\geq \rho$ then there is a closed unbounded $c \subseteq \bar{\alpha}$ such that for every $\xi \in c$ we have $p(\xi) = 0$ if $\xi \in \mathrm{dom}(p)$ and $q(\xi) = 0$ if $\xi \in \mathrm{dom}(q)$.*

The following hold.

(i) *The poset $\mathbb{N}(\lambda, 2, \rho)$ is $< \lambda$-strategically closed.*
(ii) *If E is generic for $\mathbb{N}(\lambda, 2, \rho)$ then, letting*

$$S_0 = \{\xi \in \lambda \cap \mathsf{cof}(< \rho) \mid (\exists (p, q) \in E) p(\xi) = 1\}$$
$$S_1 = \{\xi \in \lambda \cap \mathsf{cof}(< \rho) \mid (\exists (p, q) \in E) q(\xi) = 1\},$$

the pair (S_0, S_1) is as described above.
(iii) *If θ is large regular, $\lambda = \kappa^+$ and $X \in S^*_\kappa(\theta)$ write τ for $\sup(X \cap \kappa^+)$ (see Lemma 3.1). Then for every condition $a \in \mathbb{N}(\theta, 2, \rho) \cap X$ and every $i \in \{0, 1\}$ there is $a' \leq a$ in X such that $a' \Vdash$ "$\check{\tau}$ is a reflection point for $\dot{S}_i$".*

The proof of these facts are standard, and resemble to the proof for the poset for adding a single non-reflecting stationary set; see [6]. In the case of (iii) one also uses the fact that X is closed under $< \mu$-sequences where $\mu = X \cap \kappa$. We now give our example.

Lemma 3.10. *Assume* GCH *holds in* $\mathbf{V}$ *and S^*_κ is stationary. Let $\mathbb{P}$ be a κ^+-strategically closed poset. If G is generic for $\mathbb{P}$ over $\mathbf{V}$ then $(S^*_\kappa)^{\mathbf{V}[G]} = S^*_\kappa$, and is stationary in $\mathbf{V}[G]$.*

We recall that a poset is α-strategically closed if and only if in the game where two players a descending sequence the Even player can play so that each run is of length at least α; however it is not required that Even plays the α-th step, that is, a lower bound for all conditions played before step α.

Proof of Lemma 3.10. Obviously, since $\mathsf{card}(H_{\kappa^+}) = \kappa^+$ under GCH, hence H_{κ^+} and $S^*_\kappa \subseteq H_{\kappa^+}$ are not changed under κ^+-strategically closed forcing. So it is sufficient to verify the stationarity of S^*_κ in the generic extension.

Assuming $\dot{g}$ is a name for a function from ${}^{<\omega}H_{\kappa^+} \to H_{\kappa^+}$ and $p \in \mathbb{P}$ is a condition that forces no $x \in S^*_\kappa$ is closed under $\dot{g}$, we show that S^*_κ was not stationary in $\mathbf{V}$. Fix an enumeration $\langle z_\xi \mid \xi < \kappa^+ \rangle$ of ${}^{<\omega}H_{\kappa^+}$. Then play the game where Even chooses p'_ξ according to his winning strategy, that is, at successor steps ξ the condition $p'_{\xi+1}$ extends p_ξ, and at limit steps ξ the condition p'_ξ is a lower bound for all $\pi_{\bar{\xi}}$ where $\bar{\xi} < \xi$. At each step ξ Odd chooses $p_{\xi+1} \leq p'_{\xi+1}$ which decides the value $\dot{g}(z_\xi)$. After κ^+ steps the players constructed a function $g : {}^{<\omega}H_{\kappa^+} \to H_{\kappa^+}$ such that $p_\xi \Vdash \dot{g}_\xi(z_\xi) = g(z_\xi)$. Since $\mathsf{card}(x) < \kappa$ for every $x \in S^*_\kappa$ we can find a $\xi(x) < \kappa^+$ such that $p_{\xi(x)}$ decides all values $\dot{g}(z)$ where $z \in {}^{<\omega}x$, and since p forces that x is not closed under $\dot{g}$, some such value must be outside of x. We thus conclude that no element of S^*_κ is closed under g. Thus, the function g witnesses that S^*_κ is non-stationary. □

Proposition 3.11. *Under the assumptions of Theorem 1.3 let $\mathbb{N} = \mathbb{N}(\kappa^+, 2, \omega_1)$ and K be generic for $\mathbb{N} * \mathsf{Coll}(\omega_{n-1}, < \kappa) * \mathbb{P}$ where $\mathbb{P}$ is the main forcing in the proof of Theorem 1.3. Then the following hold in $\mathbf{V}[K]$.*

(a) $\omega_k^{\mathbf{V}} = \omega_k^{\mathbf{V}[K]}$ *whenever* $k < n$, $\kappa = \omega_n^{\mathbf{V}[K]}$, *and* $\kappa^{+\mathbf{V}} = \omega_{n+1}^{\mathbf{V}[K]}$.

(b) *Every stationary* $S \subseteq \kappa^+ \cap \mathsf{cf}(<\omega_{n-1})$ *has a reflection point of cofinality* ω_{n-1}.

(c) *Letting* S_0, S_1 *be the pair of sets generically added by* $\mathbb{N}$, *both* S_0, S_1 *are stationary subsets of* $\kappa^+ \cap \mathsf{cof}(\omega)$, *each of them has stationarily many reflection points of cofinality* ω_n, *but they do not have a common reflection point.*

Proof. The only thing to be verified is the stationarity of the sets S_0, S_1, as the rest is either easy or follows easily from Theorem 1.3. Since $\mathsf{Coll}(\omega_{n-1}, <\kappa)$ preserves the stationarity of subsets of κ^+, it suffices to show that the stationarity of S_0, S_1 is preserved under $\mathbb{P}$.

By Lemma 3.10, the set S^*_κ remains stationary after forcing with $\mathbb{N}$. Force with $\mathbb{N} * \mathsf{Coll}(\omega_{n-1}, <\kappa) * \mathbb{P}$. Say $K = E * G * F$ where E is generic for $\mathbb{N}$ over $\mathbf{V}$, G is generic for $\mathsf{Coll}(\omega_{n-1}, <\kappa)$ over $\mathbf{V}[G]$, and F is generic for $\mathbb{P}$ over $\mathbf{V}[E][G]$. Let S_0, S_1 be the sets added by E. We show that S_0 remains stationary in $\mathbf{V}[K]$; by the symmetricity of the situation this proves the proposition. Assume for a contradiction that S_0 is non-stationary in $\mathbf{V}[K]$, so in $\mathbf{V}[K]$ there is a closed unbounded $C \subseteq \kappa^+$ disjoint from S_0. Since $\mathsf{Coll}(\omega_{n-1}, <\kappa)$ is of size κ, it preserves the stationarity of S_0. It follows that C is added by $\mathbb{P}$, that is, $C = \dot{C}^F$ where $\dot{C}$ is a $\mathbb{P}$-name and there is a condition $p \in \mathbb{P}$ that forces $\dot{C}$ to be a closed unbounded subset of κ^+ disjoint from S_0. Now we follow the setup in the proof of Theorem 1.3 where we work with $\mathbf{V}[E]$ in place of $\mathbf{V}$. We construct an elementary embedding $\sigma : H[\bar{G}] \to H_\theta^{\mathbf{V}[E]}[G]$ such that $\sigma \restriction H \in \mathbf{V}[E]$ similarly as before, but we construct it in such a way that $\mathbb{N}, S_0, p$ and $\dot{C}$ are in $\mathrm{rng}(\sigma)$. Letting $\bar{\mathbb{N}}, \bar{S}_0, \bar{p}$ and $\dot{\bar{C}}$ be their preimages under σ, pick a filter $\bar{F}$ generic for $\bar{\mathbb{P}}$ over $\mathbf{V}[E][\bar{G}]$ such that $\bar{p} \in \bar{F}$. Letting $\bar{C} = \dot{\bar{C}}^{\bar{F}}$, the set $\bar{C}$ is a closed unbounded subset of μ^+ disjoint from $\bar{S}_0$ in $H[\bar{G}][\bar{F}]$. Since $H[\bar{G}][\bar{F}] \in \mathbf{V}[E][\bar{G}][\bar{F}]$, we conclude that $\bar{S}_0$ is non-stationary in $\mathbf{V}[E][\bar{G}][\bar{F}]$. On the other hand, by (iii) in Fact 3.9 and the fact that σ is continuous at points of cofinality $<\omega_{n-1}$ and $\sigma \restriction H \in \mathbf{V}[E]$, the set $\bar{S}_0$ is stationary in $\mathbf{V}[E]$. Since $\mathsf{Coll}(\omega_{n-1}, <\mu)$ is of size μ, this set remains stationary in $\mathbf{V}[E][\bar{G}]$. In the model $\mathbf{V}[E][\bar{G}]$, the poset $\bar{\mathbb{P}}$ has a dense subset $A_{\bar{\tau}} \in \mathbf{V}[E][\bar{G}]$ that is closed under descending chains of length $\leq \mu$. Additionally, in this model we have $\mu^{<\omega_{n-1}} = \mu$, as μ is inaccessible in $\mathbf{V}[E]$ and $\mathsf{Coll}(\omega_{n-1}, <\mu)$ is ω_{n-1}-closed. By Fact 3.5, such a poset preserves the stationarity of $\bar{S}_0$. This is a contradiction which completes the proof of Proposition 3.11. □

Remark 3.12. There is a counterexample similar to that above which does a little bit more, and gives an indirect argument along these lines. One can show, via an argument similar to that in the proof of Proposition 3.11 that adding a $\square(\kappa^+)$-sequence in place of a pair of reflecting stationary sets without common reflection point achieves a similar effect. More precisely, if one first adds a $\square(\kappa^+)$-sequence using the standard forcing with initial segments then further forcing with $\mathsf{Coll}(\omega_{n-1}, <\kappa) * \mathbb{P}$ where $\mathbb{P}$ is the main forcing in Theorem 1.3, does not add a thread to this $\square(\kappa^+)$-sequence. By an argument pointed out to us by Magidor, the existence of a $\square(\theta)$-sequence implies the existence of a pair of nonreflecting stationary subsets of θ concentrating on a small cofinality.

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Department of Mathematics, University of California at Irvine, Irvine, CA 92697-3875
E-mail address: mzeman@math.uci.edu