GLOBAL SQUARE SEQUENCES IN EXTENDER MODELS

MARTIN ZEMAN

Abstract. We present a construction of a global square sequence in extender models with λ-indexing.

AMS Subject Classification: 03E05, 03E45, 03E55
Keywords: Global quare sequence, fine structure, extender model.

August 19, 2009

1. Introduction

The purpose of this paper is to present a construction of sequences related to most commonly used global square principles in an extender model with Jensen's λ-indexing introduced in [6]. Straightforward, but technical modifications of these constructions give the same results in extender models with Mitchell-Steel indexing introduced in [10] (see also [15]). The main advantage of using λ-indexing is the relative simplicity and cleanness of the all constructions.

The basic result on global square sequences in extender models with λ-indexing was announced in [13] (Theorem 21). Constructions of global square sequences in lower level extender models were given by Welch [16] in Dodd-Jensen core models, Wylie [17] in extender models for measures of Mitchell order 0 and Zeman [20] in models up to one strong cardinal. Jensen-Zeman [7] gives a construction of a condensation-coherent global square sequence in models for measures of order 0, which actually goes through in any extender model whose extenders are generated by their normal measures[1]; this construction is based on certain ideas from [5]. The construction in this paper builds on a combination of techniques from [20],[14] and [19]. Extender models with λ-indexing were introduced by Jensen in [6], and basic facts about these models can also be found in [18]. For some interesting applications of global square sequences coming from $\mathbf{L}[E]$-models, see [2, 8, 12]. The former two papers focus on the use of fine structural global square sequences to obtain lower bounds for consistency strengths of various stationary reflection principles. In [12] fine structural global square sequences are used to determine a lower bound for the consistency strength of the restricted proper forcing axiom $\mathsf{PFA}(\mathfrak{c}^+ - \mathsf{linked})$ in the following sense: If $\mathsf{PFA}(\mathfrak{c}^+ - \mathsf{linked})$ holds in a generic extension via proper forcing over a fine structural model M then M must contain the so-called Σ^2_1-indescribable cardinal 1-gap, which is stronger than the existence of many subcompact cardinals. This is a remarkable result, as it was proved earlier in [11] show that a Σ^2_1-indescribable 1-gap suffices for obtaining a proper forcing extension of a model satisfying GCH where $\mathsf{PFA}(\mathfrak{c}^+ - \mathsf{linked})$ holds. Paper [21] is

Research partially supported by the NSF grants DMS-0204728 and DMS-0500799.

[1] Equivalently, in any extender model that satisfies the anti-large cardinal requirement that there is no extender with two generators.

a sequel to this paper where the current methods are further extended to give a characterization of stationary reflection at inaccessible cardinals similar to that in $\mathbf{L}$, that is, in terms of coherent club sequences. Paper [9] is a sequel to both the current paper, [21] and [14] where the methods developed in these three papers are further extended to constructions of nonthreadable square sequences at successor cardinals in extender models.

Recall that given a class $\mathcal{S}$ of singular ordinals, a sequence $\langle C_\alpha;\, \alpha \in \mathcal{S}\rangle$ is a global square sequence with domain $\mathcal{S}$, briefly a $\square^{\mathcal{S}}$-sequence, just in case that each C_α is a closed unbounded subset of α of order type strictly smaller than α such that $\lim(C_\alpha) \subseteq \mathcal{S}$ and the sets C_α are coherent in the sense that $C_{\bar\alpha} = C_\alpha \cap \bar\alpha$ whenever $\bar\alpha \in \lim(C_\alpha)$. The principle $\square^{\mathcal{S}}$ postulates the existence of a global square sequence whose domain is $\mathcal{S}$. We write briefly $\square$ if $\mathcal{S}$ is the class of all singular ordinals. Given a class A of singular ordinals, a $\square^{\mathcal{S}}(A)$-sequence is a $\square^{\mathcal{S}}$-sequence which satisfies the additional condition that $\lim(C_\alpha) \cap A = \varnothing$ for all $\alpha \in \mathcal{S}$. By the result of Burke and Jensen (proved independently) [1, 14], the principle $\square_\kappa$ fails whenever κ is subcompact. Since any global square sequence whose domain are all singular ordinals yields a $\square_\kappa$-sequence for each cardinal κ, it is not possible to have such a global square sequence in an extender model in general. However, it turns out that subcompact cardinals constitute the only limitation on domains of global square sequences in extender models. Our main theorem can be formulated as follows.

Theorem 1.1 (Main theorem)**.** *Let* $\mathbf{M}$ *be an extender model with* λ*-indexing and*

$$\mathcal{S}^* = \text{Singular Ordinals of } \mathbf{M} - \bigcup\{(\kappa,\kappa^+);\ \kappa \textit{ is subcompact in } \mathbf{M}\}$$

Then in $\mathbf{M}$*, there is a global square sequence with domain* $\mathcal{S}^*$*. In fact, for any class* $A \subseteq \mathbf{On}$ *there is a class* $A' \subseteq A$ *such that for every regular* κ*,*

$$A \cap \kappa \textit{ is stationary} \implies A' \cap \kappa \textit{ is stationary}$$

and $\square^{\mathcal{S}^*}(A')$ *holds. Consequently, any extender model with* λ*-indexing which contains no subcompact cardinals satisfies the principle* $\square$*.*

Constructions of canonical global square sequences in extender models are all based on Jensen's original construction in $\mathbf{L}$, and are carried out on two disjoint classes: the class of all singular cardinals, and that of all ordinals which fail to be cardinals. Notice that ordinals in the latter class are elements of intervals (κ,κ^+) where κ is a cardinal, so in the case of this class such constructions give rise to $\square_\kappa$-sequences. A typical construction of this kind was presented in [14]. Thus, to obtain a global square sequence, it suffices to focus on the class of all singular cardinals. Our construction will give rise to a version of $\square^{\mathcal{S}}(A')$-sequence where the sets are fully coherent; the precise formulation is stated in the theorem below. Theorem 1.1 is a consequence of the fact that $\square_\kappa(A')$-sequences can be combined with a $\square^{\mathcal{S}}(A')$-sequence, where $\mathcal{S}$ is the class of all singular cardinals, into a single global square sequence $\square(A')$; this is the easier implication in Jensen's result that $(\forall\kappa)\square_\kappa$ together with $\square^{\mathcal{S}}$ is equivalent to $\square$. To obtain $A' \cap (\kappa,\kappa^+)$ from a given class A, we can simply choose $A' \cap (\kappa,\kappa^+)$ to be a stationary subset of $A \cap (\kappa,\kappa^+)$ such that the order types of sets in the $\square_\kappa$-sequence for $\alpha \in A'$ are the same, so our construction will only focus on the less obvious task of obtaining $A' \cap$ Singular Cardinals. We are now ready to state Theorem 1.2. This theorem is the actual result we are going to prove; Theorem 1.1 is a direct consequence of Theorem 1.2 and the above considerations.

Theorem 1.2. *The following is true in any extender model with λ-indexing. Let $\mathcal{S}$ be the class of all singular cardinals and $A \subseteq \mathcal{S}$ be a class. There are a class $A' \subseteq A$ such that for all inaccessible κ,*

$$A \cap \kappa \text{ is stationary} \implies A' \cap \kappa \text{ is stationary}$$

and a sequence $\langle C_\alpha; \alpha \in \mathcal{S}\rangle$ satisfying the following.

(a) *Each C_α is a closed subset of $\alpha \cap \mathcal{S}$; if $\mathrm{cf}(\alpha) > \omega$ then C_α is unbounded in α. (If α is ω-cofinal, it may happen that $C_\alpha = \varnothing$.)*
(b) *$C_{\bar\alpha} = C_\alpha \cap \bar\alpha$ whenever $\bar\alpha \in C_\alpha$. (That is, we have full coherency.)*
(c) $\mathrm{otp}(C_\alpha) < \alpha$*.*
(d) $\lim(C_\alpha) \cap A' = \varnothing$*.*

Section 3 is devoted to the technical tools used in the construction of a sequence as in the previous lemma and Section 4 to the actual construction. The big picture here is similar to that in the construction of a $\square_\kappa$-sequence, but the details differ, in some instances quite significantly. For this reason, several basic notions and technical lemmata have to be amended to the current context. In order to keep the account brief, we state several technical lemmata in Section 3 without proofs whenever these proofs are straightforward amendments of proofs in [14, 18]. However we do present proofs of lemmata where the amendments are not negligible. In order to keep this paper self-contained, we also state those technical lemmata whose formulations alone do not require any substantial amendment; it is however still the case that the proofs of these lemmata may require amendments. The sources [14, 19] and [18] should be considered a prerequisite here. We will be using notions and results from these sources without further notice. Let us stress that throughout the proof, we work in a fixed extender model $\mathbf{L}[E]$ with λ-indexing of extenders.

2. Outline of the construction

In this section we attempt to give a big picture of the construction and explain the important points in possibly general terms. It is mentioned in the introduction that the construction builds on that of $\square_\kappa$-sequence presented in [14], so we begin with refreshing the main points here as well as the main points in the construction of a global square sequence in $\mathbf{L}$. We then isolate the key issues that need to be addressed in order to modify the ideas from [14] to obtain tools for construction of a global square sequence on singular cardinals in an extender model.

The initial idea, due to Jensen, is to construct a square sequence $\langle C_\tau\rangle_\tau$ (no matter whether local or a global) so that the sets C_τ consist of critical points of certain canonical embeddings. One of the key tasks is then the choice of a structure N_τ to each τ in a canonical way that would allow to extract C_τ with all requisite properties as a set of critical points of suitable elementary maps $\sigma_{\bar\tau,\tau} : N_{\bar\tau} \to N_\tau$. In the construction of a $\square_\kappa$-sequence in $\mathbf{L}$, we consider ordinals $\tau \in (\kappa, \kappa^+)$ which look like local successor cardinals, that is, κ is the largest cardinal in J_τ, and the structures N_τ are taken to be collapsing levels for τ in $\mathbf{L}$. Thus τ is regular in N_τ, but is definably (in parameters) singularized, and therefore collapsed, over N_τ. From now on assume for simplicity that there is a singularizing/collapsing partial function $f : \kappa \to \tau$ that is Σ_1-definable over N_τ in parameters. This is equivalent to saying that κ is the first projectum of N_τ and to the fact that the Skolem hull of the first projectum together with the standard parameter $\tilde{h}^1_{N_\tau}(\kappa \cup \{p_{N_\tau}\})$ is the entire N_τ; here p_{N_τ} is the standard parameter. We then define D_τ to be the set of critical point of maps $\sigma_{\bar{\tau},\tau}$ as above with maximal possible degree of elementarity that are the identity on κ and preserve standard parameters. This degree of elementarity is Σ_0, as it follows easily that the only Σ_1-preserving map that is the identity on κ and has p_{N_τ} in its range is the identity map. The sets D_τ are are closed, unbounded whenever τ has uncountable cofinality, and satisfy the strong form of coherency. This can be quite easily verified due to the uniform definition of D_τ. However, these sets may have order type larger than κ. In order to obtain C_τ we "thin out" the sets D_τ by a careful analysis of countable elementary substructures of N_τ. This can be done in a sufficiently uniform way that preserves closeness and coherency, and additionally guarantees that the order type of C_τ does not exceed κ.

We now outline how to extend Jensen's argument to an arbitrary extender model $\mathbf{L}[E]$, as is done in detail in [14]. Analogously as before we consider ordinals τ that are local cardinal successors in $\mathbf{L}[E]$ and attempt to define N_τ to be a singularizing/collapsing level of $\mathbf{L}[E]$ for τ. There is a minor issue concerning preservation of standard parameters under maps $\sigma_{\bar{\tau},\tau}$ which is resolved by additional requirement that $\mathrm{rng}(\sigma_{\bar{\tau},\tau})$ contains solidity witnesses for elements of the standard parameter of N_τ. This introduces some non-uniformity into the construction which causes that the sets D_τ are not fully coherent; here a simple combinatorial manipulation is used to turn the sets D_τ into fully coherent sets B_τ. We then use the same thinning out procedure as before to produce sets C_τ; this time thinning out the sets B_τ instead of D_τ. The just described scenario can really be made to work, up to one important point, namely the proof that the sets B_τ, hence D_τ, are cofinal whenever τ has uncountable cofinality. Here is an explanation why. We actually show that even proving that D_τ is nonempty is highly nontrivial. Assume that N_τ is an active premouse, the critical point of the top extender F of N_τ is strictly smaller than κ and there is a function collapsing τ to κ that is Σ_1-definable over N_τ in parameters. If D_τ were nonempty, there would be a Σ_0-preserving map $\sigma_{\bar{\tau},\tau} : N_{\bar{\tau}} \to N_\tau$ whose critical point is $\bar{\tau}$ and which preserves standard parameters. As we have seen in the previous paragraph, such a map cannot be Σ_1-preserving, and therefore cannot be cofinal. On the other hand $\sigma_{\bar{\tau},\tau}$ will turn out to be sufficiently elementary to guarantee that N_τ is of the form $\langle J^E_{\bar{\nu}}, \bar{F}\rangle$ where $\bar{F}$ is an extender. Since $\sigma_{\bar{\tau},\tau}$ is the identity below κ, critical points of F and $\bar{F}$ agree; denote the common critical point by μ. As $\sigma_{\bar{\tau},\tau}$ is not cofinal, there is some set $a \subseteq \mu$ such that $a \in N_\tau$ but $F(a) \notin \mathrm{rng}(\sigma_{\bar{\tau},\tau})$. Clearly $a \in N_{\bar{\tau}}$, as N_τ and $N_{\bar{\tau}}$ agree up to κ and κ is a cardinal in $\mathbf{L}[E]$. But the value $\bar{F}(a)$ cannot be defined, as the degree of elementarity of $\sigma_{\bar{\tau},\tau}$ would force that $F(a) = \sigma_{\bar{\tau},\tau}(\bar{F}(a))$ and we have seen that $F(a)$ is not in the range of $\sigma_{\bar{\tau},\tau}$. We thus conclude that $\bar{F}$ is an extender on $J^E_{\bar{\nu}}$ that fails to measure all subsets of μ that are elements of $J^E_{\bar{\nu}}$. As a consequence, $N_{\bar{\tau}}$ is not the right choice of the singularizing structure for $\bar{\tau}$, since its top extender measures all subsets of its critical point that are elements of $N_{\bar{\tau}}$. In fact, we have just seen that any structure that would qualify as a candidate for the canonical singularizing structure for $\bar{\tau}$ has top extender that is not total, which means that such structure is not an initial segment of any extender model. Structures of this kind are called protomice. It is now clear that in order to generalize Jensen's construction to extender models, it is necessary to introduce protomice in the construction.

A closer look at protomice reveals that protomice constitute a form of encoding singularizing/collapsing $\mathbf{L}[E]$-levels: Given a protomouse $M = \langle J^E_\nu, F\rangle$, over which τ is Σ_1-definably singularized (in parameters), let M^* be the longest initial segment of M on which F is total. The fine structural ultrapower N of M^* by F is precisely the singularizing level of $\mathbf{L}[E]$ for τ. On the other hand, the ultrapower map π is the inverse of the Mostowski collapsing isomorphism obtained by collapsing the fine structural Skolem hull $\tilde{h}^{n+1}_{N_\tau}(\mu \cup \{r\})$ where $\mu = \mathrm{cr}(F)$, r is the image of the standard parameter of M^* under π and n is largest such that the n-th projectum of N_τ is strictly above κ. In other words, we form the fine structural Skolem hull of $\mu \cup \{r\}$ whose degree of elementarity is $\Sigma^{(n)}_1$. It can be proved that r is a top segment of the standard parameter of N. It is then obvious that the extender derived from π is F, and is fully determined by the pair (μ, r). Thus there is a simple way of obtaining the singularizing $\mathbf{L}[E]$-level from a singularizing protomouse and vice versa. Moreover, there is a nice translation procedure which enables to translate all fine structural information between the two structures. All of this makes it plausible that replacing $\mathbf{L}[E]$ levels with protomice should be a successful approach, but this approach involves solving two crucial issues. First, given τ we have to decide whether the canonical singularizing structure for τ will be an $\mathbf{L}[E]$-level or a protomouse. This decision has to be made before we start the construction, that is, before we see what kind of structures and embeddings we have to consider when defining D_τ. Second, whereas the way from a protomouse to the corresponding $\mathbf{L}[E]$-level is unique, that is, each protomouse uniquely determines the corresponding $\mathbf{L}[E]$-level, the converse is not true in general. It is possible to have an $\mathbf{L}[E]$-level which can be converted into many protomice, as there may be many pairs (μ, r) as above. Such pairs are called divisors of N. To be more precise, in our official definition of a divisor we require the divisors to have the form (μ, q) where q is the bottom part of the standard parameter, that is, $q = p_{N_\tau} - r$ as this definition is more suitable for computations. Of course both (μ, r) and (μ, q) carry the same amount of information, so for the purpose of this informal outline we will consider both to be divisors. To summarize, if we decide to choose a protomouse as the canonical singularizing structure for τ, we face the question which protomouse should be the canonical one. Equivalently, we have to make a choice of a divisor. In order to keep the coherency of the sequence of sets we are constructing, the choice of divisors must be made so that it is preserved under the embeddings $\sigma_{\bar{\tau},\tau}$ which we know have low degree of elementarity.

A careful examination of divisors reveals that for each singularizing $\mathbf{L}[E]$-level N_τ there is a collection of divisors that are easily identified and are preserved under weakly preserving embeddings. We will call these divisors strong. We omit the definition in this outline, as it is not helpful in understanding the big picture. We then show that if N_τ has a strong divisor at all then it has one with largest possible μ, and this divisor is unique for N_τ. We will consider this divisor canonical for N_τ. It will turn out that although canonical divisors are not literally preserved under Σ_0-elementary maps, such maps preserve enough information about them that our construction can be carried out. The construction has the following format. Given τ, we check whether the singularizing $\mathbf{L}[E]$-level N_τ has a strong divisor. If the answer is negative, we let the canonical singularizing structure be N_τ. If the answer is affirmative, we let the canonical singularizing structure M_τ be the protomouse determined by the canonical divisor for N_τ. This splits the set of the ordinals τ

we are considering into two disjoint subsets $\mathcal{S}^0$ and $\mathcal{S}^1$. If our extender model $\mathbf{L}[E]$ contains enough large cardinals, both these sets may be stationary. On each of these sets we then carry out the constructions of D_τ and B_τ. We will be able to show that D_τ, and therefore also B_τ is unbounded in τ whenever τ has uncountable cofinality, closed on a tail-end, and contained in $\mathcal{S}^i$ whenever $\tau \in \mathcal{S}^i$. We then show that the tail-ends can be chosen in a way that they constitute a coherent sequence, which will make it possible to run the thinning out procedure in a manner similar to that in $\mathbf{L}$ to produce the square sequence C_τ.

Proving the three properties of D_τ mentioned in the previous paragraph is the heart of the argument. The proof actually shows that for sufficiently large $\bar\tau$ such that there is a Σ_0-preserving map $\sigma : \bar M \to M_\tau$ where $\bar M$ is a singularizing protomouse for $\bar\tau$ we necessarily have $\bar M = M_{\bar\tau}$. That is, the only protomouse that can be embedded into the canonical protomouse M_τ are those determined by canonical divisors for $N_{\bar\tau}$. Let us also mention the special case where N_τ is as in the example above, that is N_τ is active, the critical point μ of its top extender is strictly smaller than τ and there is a Σ_1-definable function over N_τ in parameters that singularizes τ. If moreover N_τ does not have any strong divisors, following our choice of singularizing structures, we let N_τ be the canonical singularizing structure for τ. In the example above we saw that for each $\bar\tau \in C_\tau$ the canonical singularizing structure for $\bar\tau$ is the protomouse $M_{\bar\tau}$. Hence the maps $\sigma_{\bar\tau,\tau}$ are maps between protomice and a premouse. Fine structural parameters of protomice $\langle J^E_\tau, F\rangle$ are computed relative to predicates E and F, whereas those of premice $\langle J^E_\tau, F\rangle$ are computed relative to predicates E, F and possibly some additional constants. To ensure coherency, in situations like this we have to treat N_τ as a protomouse; we call it "pluripotent premouse". In particular, such situations require to work with the Dodd parameter of N_τ instead of the standard parameter, as the Dodd parameter is precisely the standard parameter computed relative to E and F. We then we have to prove fine structural lemmata which show that switching to Dodd parameters does not do any harm to the coherency of sets we construct.

We are now ready to discuss the construction of a global square sequence on singular cardinals and give a description of the main technical issues that arise when trying to adapt the ideas described above to this construction. As τ, being now a singular cardinal in $\mathbf{L}[E]$, is a limit cardinal in any initial $\mathbf{L}[E]$-segment J^E_β where $\beta \geq \tau$, it will look like an inaccessible cardinal in the singularizing $\mathbf{L}[E]$-level N_τ for τ. We again split the class of all singular cardinals into two disjoint classes $\mathcal{S}^0$ and $\mathcal{S}^1$ where the former consists of those ordinals for which we let the canonical singularizing structure be N_τ, whereas the latter consists of all those ordinals for which we let the canonical singularizing structure be a canonical protomouse. From the point of view of N_τ, the inaccessible cardinal τ is definably (in parameters) singularized over N_τ. In the construction of a $\square_\kappa$-sequence in [14] κ was the largest cardinal below τ, which was a very handy fact on which we heavily relied, even in the definition of a divisor. Hence the first task in the present construction is to modify the definition of a divisor in a way that would make sense even if no cardinal predecessor of τ exists. Of course, this has the consequence that all of the lemmata concerning divisors and relationship between protomice and the associated $\mathbf{L}[E]$-levels have to be reformulated and the proofs adjusted appropriately. In particular, we have to make sure that we have a meaningful notion of strong divisor in the present context and that the auxiliary lemmata can be appropriately modified as well. The condensation lemma for protomice requires a more careful reformulation and we actually include a proof of the lemma in detail, as it contains nontrivial new technical aspects. There are of course numerous minor points that require some work beyond that included in [14]; those which can be treated in a standard way are left to the reader without further comments. However, our construction involves there three points in that need to be addressed a substantially new way; we would like to say more about these points below.

It was explained above that given a divisor (μ, r) for N_τ we can determine the top extender of the associated protomouse by forming the fine structural $\Sigma_1^{(n)}$-elementary hull $\tilde{h}_{N_\tau}^{n+1}(\mu \cup \{r\})$ and subsequent deriving the extender from the inverse of the collapsing isomorphism π. We also said that there is a natural translation procedure that enables to transfer fine structural information between N_τ and the associated protomouse M. The heart of the issue is that the complexity degree of facts that are possible to translate using this procedure must match the degree of elementarity of the Skolem hull we form, or equivalently to the degree of elementarity of the associated ultrapower map. Thus, in our case the translation procedure translates $\Sigma_1^{(n)}$-facts over N_τ into Σ_1-facts over M and vice versa. If τ were a local successor (as it happens in the construction in [14]) then for n as above we would have a singularizing function for τ that is $\Sigma_1^{(n)}$-definable in parameters over N_τ, so the translation procedure tells us that the same function is Σ_1-definable in parameters over M. In the present case τ is a limit cardinal, so it may happen that even though n is largest such that the n-th projectum of N_τ is strictly above τ, the smallest complexity degree of a singularizing function for τ over N_τ is much larger than n. Thus, in this case the protomouse M would not be a singularizing protomouse for τ, as any function that is Σ_1-definable over M would be $\Sigma_1^{(n)}$-definable over N_τ, so such function would not singularize τ. In order to fix this issue one might attempt to increase the elementarity degree of the Skolem hull of $\mu \cup \{r\}$, but this will clash with the fact that N_τ is obtained as a fine structural ultrapower, and it is a general fact about such ultrapowers that the elementarity degree of the ultrapower map is always the largest n such that the n-th projectum of N_τ is above $\pi(\mu)$. Given these two observations, the only feasible approach to deal with the issue seems to be imposing an additional restriction the class $\mathcal{S}^1$ that every $\tau \in \mathcal{S}^1$ must satisfy the following requirement: If n is largest such that the n-th projectum of N_τ is strictly above τ then there is a partial singularizing function for τ that is $\Sigma_1^{(n)}$-definable over N_τ in parameters. Such $\mathbf{L}[E]$-levels N_τ will be called exact. Notice that in the case where τ was a local successor N_τ was automatically exact, so this restriction is actually quite natural. However, whereas in the construction of $\square_\kappa$ the property of being exact was always automatically granted and we did not even have to consider it explicitly, in our current situation we have to develop effort to keep the property of being exact preserved at all sensitive stages in the construction. Moreover, we also have to verify that there is no loss of generality, more precisely, that despite our restriction of $\mathcal{S}^1$ the sets C_τ are defined for every singular cardinal τ.

The two other major issues that arise in the construction are of more technical nature than that discussed above. Both concern the proof that the sets C_τ are closed. In the construction of a $\square_\kappa$-sequence where the ordinals τ were local cardinal successors we were able to show that B_τ were closed on a tail-end, and a part of the argument was an easy observation that if $\bar{\tau}$ is a limit point of B_τ and we have a diagram of structures N_{τ^*} together with the canonical maps $\sigma_{\tau^*,\tau'}$ between them for $\tau^*,\tau' \in B_\tau \cap \bar\tau$ then the direct limit structure $\bar N$ is a singularizing structure for $\bar\tau$. This is the case since there is a common bound on the domains of the singularizing functions of τ^*, namely κ. In the present situation where all ordinals $\tau, \bar\tau$ and τ^* are singular cardinals no such bound exists, and actually B_τ fails to be closed on a tail-end in general if $\tau \in \mathcal{S}^0$. We however prove that the sets C_τ are closed, by a careful examination of the thinning out procedure that is used to extract C_τ from B_τ. This is done in Lemma 4.11. The main idea of the proof appears already in [20], although here we use a more elaborate version. It should be noted that a variant of the same argument applied to $\mathbf{L}$ is implicit in Jensen's original proof, although it has a different form, as the entire setting of his construction is different from that chosen in this paper. Let us finally note that in the remark following the proof of Lemma 4.11 we also discuss a scenario where the layout of entire construction makes it possible to prove that the sets B_τ can be proved to be closed on a tail-end. However, the author believes the layout chosen in this paper makes the construction somewhat simpler.

The remaining major issue we would like to discuss is even more technical. Already the construction of a global square sequence in $\mathbf{L}$ requires an extra technical tool in order to make the techniques from the construction of a $\square_\kappa$-sequence work. The critical issue is the observation that the maps $\sigma_{\bar\tau,\tau}$ have maximal possible degree of elementarity, but are not cofinal in the relevant projectum (for instance, if τ can be singularized via a function that is Σ_1-definable over N_τ then $\sigma_{\bar\tau,\tau}$ is not cofinal in the 0-th projectum, that is, not cofinal in the usual sense). One way of arranging this is imposing the additional requirement on the maps $\sigma_{\bar\tau,\tau}$ that these maps preserve "semi-cofinalities" of $\bar\tau$. The "semi-cofinality" of τ is denoted by α_τ and is the largest ordinal $\alpha < \tau$ such that $\tau \cap \tilde h^{n+1}_{N_\tau}(\alpha \cup \{p_{N_\tau}\}) = \alpha$. The use of "semi-cofinalities" already appears in Jensen's construction in $\mathbf{L}$. The reason for considering "semi-cofinalities" instead of cofinalities is that the maps $\sigma_{\bar\tau,\tau}$ do not, in general preserve definable cofinalities, but they are sufficiently elementary to preserve "semi-cofinalities". The construction in $\mathbf{L}[E]$ relies on preservation of "semi-cofinalities" more heavily than that in $\mathbf{L}$ in several respects. We will now focus on the most important of them which also goes substantially beyond the construction from [14]. This point is the proof that for $\tau \in \mathcal{S}^1$ the sets B_τ are closed on a tail-end (in contrast to the case $\tau \in \mathcal{S}^0$ discussed in the previous paragraph). Superficially, the reason why B_τ are closed on a tail-end is the existence of a common bound on cofinalities of the kind described in the previous paragraph in connection with the construction in [14], and this bound is μ^+ where we recall that μ is the critical point of the top extender of the protomouse M_τ. The major issue we would like to describe arises in verification that for sufficiently large $\bar\tau$, if there is an embedding $\sigma_{\bar\tau,\tau} : \bar M \to M_\tau$ between two protomice and $\bar M$ is a singularizing protomouse for $\bar\tau$ then necessarily $\bar M = M_{\bar\tau}$. The argument is done by contradiction assuming that we have a strictly increasing sequence $\langle \tau_\xi \mid \xi < \gamma\rangle$ cofinal in τ for which we chose the canonical divisor incorrectly. Letting (μ_ξ, q_ξ) be the the corresponding correct canonical divisors, necessarily $\mu_\xi > \mu$ where (μ, q) is the canonical divisor of N_τ. Without loss of generality we may assume that the ordinals μ_ξ constitute an increasing (not necessarily strictly) sequence converging to some ordinal μ'. If $\mu' < \tau$ we will draw the contradiction by arguing that the choice of (μ, q) was incorrect, as N_τ must have a strong divisor of the form (μ', q')

where $\mu' > \mu$. This is a kind of reflection argument. However, if $\mu' = \tau$ then this argument cannot be carried out. The issue is resolved by setting up the construction so that, roughly speaking, the "semi-cofinalities" of elements of B_τ computed relative to their corresponding canonical protomice are all equal to 0. It is then proved that this setting will rule out the possibility $\mu' = \tau$.

3. Technical tools

We begin with a version of the condensation lemma which is useful for combinatorial constructions; this version is slightly more general than that formulated in [14]. This more general formulation helps to better organize the material in the current section. Moreover we obtain formulations of technical lemmata that can be directly applied to broader area of problems than those discussed in our paper.

Recall that if N is an acceptable structure and ν is an ordinal then $\mathrm{core}_\nu(N)$ is the transitive collapse of the hull $\tilde{h}^{n+1}_N(\nu \cup \{p_N\})$ where p_N is the standard parameter of N, $\tilde{h}^{n+1}_N$ is the $\Sigma^{(n)}_1$-Skolem function for N and n is such that $\omega\varrho^{n+1}_N \leq \nu < \omega\varrho^n_N$. In Jensen's fine structure theory these notions can be defined whenever N is an acceptable structure whose parameters can be lengthened; so N need not be a premouse. The associated core map $\sigma : \mathrm{core}_\nu(N) \to N$ is the inverse to the Mostowski collapsing isomorphism. This map is always Σ^*-preserving. We intend to work with initial segments of the model $\mathbf{L}[E]$, however for development of technical lemmata in this section it will be convenient to work with the broader class of premice weakly embeddable into $\mathbf{L}[E]$-levels which still enjoy many of the properties of $\mathbf{L}[E]$-levels.

Definition 3.1. *We say that a premouse N is* **weakly embeddable** *into a premouse N' just in case that there is a finite sequence $\langle (N_i, \sigma_i, k_i) \mid i < m+1 \rangle$ such that each N_i is a premouse, $N_0 = N$, $N_{m+1} = N'$ and for each $i < m+1$*

- *either N_{i+1} is of same type as N_i and $\sigma_i : N_i \to N_{i+1}$ is a $\Sigma^{(k_i)}_0$-preserving map such that $\sigma \restriction \omega\varrho^{k_i+1}_{N_i} = \mathrm{id}$*
- *or else N_i is an initial segment of N_{i+1}, $\sigma_i = \mathrm{id}$ and $k_i = 0$.*

Obviously, if N is weakly embeddable into N', $\bar{N}$ is a premouse of the same type as N and $\sigma : \bar{N} \to N$ is a $\Sigma^{(k)}_0$-preserving embedding such that $\sigma \restriction \omega\varrho^{k+1}_{\bar{N}} = \mathrm{id}$ then $\bar{N}$ is weakly embeddable into N'. Also any initial segment of a premouse weakly embeddable into N' is itself weakly embeddable into N'. The following condensation lemma holds for premice weakly embeddable into $\mathbf{L}[E]$-levels, see [18] for its proof. The lemma can actually be proved for premice which satisfy sufficient iterability hypothesis, however in this paper we try to avoid any direct reference to iterability.

Lemma 3.2 (Condensation lemma)**.** *Assume N is a premouse weakly embeddable into an $\mathbf{L}[E]$-level. Let $\bar{N}$ be a premouse of the same type as N and $\sigma : \bar{N} \to N$ be a $\Sigma^{(n)}_0$-preserving embedding such that $\sigma \restriction \omega\varrho^{n+1}_{\bar{N}} = \mathrm{id}$. Then $\bar{N}$ is solid and $p_{\bar{N}}$ is k-universal for every $k \in \omega$ (briefly universal). Furthermore, if $\bar{N}$ is sound above $\nu = \mathrm{cr}(\sigma)$ then one of the following holds:*

(a) *$\bar{N} = \mathrm{core}_\nu(N)$ and σ is the associated core map. So if N is sound above ν then $\bar{N} = N$ and $\sigma = \mathrm{id}$.*

(b) *$\bar{N}$ is a proper initial segment of N.*

(c) $\bar{N} = \mathrm{Ult}^*(N\,||\,\eta, E^N_\alpha)$ *where* $\alpha \le \omega\eta$ *and* $\eta < \mathrm{ht}(N)$ *is largest such that* ν *is a cardinal in* $N\,||\,\eta$*; moreover,* $\kappa = \mathrm{cr}(E^N_\alpha)$ *is a cardinal predecessor of* ν *in* $N\,||\,\eta$ *and a single generator of* E^N_α*.*

(d) $\bar{N}$ *is a proper initial segment of* $\mathrm{Ult}(N, E^N_\nu)$*.*

It follows immediately from the above definition that any premouse weakly embeddable into an $\mathbf{L}[E]$-level is solid and its standard parameter is universal.

We now introduce some notation that will help us to economize the text. Given an acceptable J-structure M and an ordinal $\tau \in M$, we let $n^*(\tau, M)$ be the largest $n \in \omega+1$ such that $\omega\varrho^n_M > \tau$. Given a singular ordinal τ, we say that an acceptable structure M is a **singularizing structure** for τ just in case that τ is regular in M and there is some $n \in \omega$ and a good $\mathbf{\Sigma}^{(n)}_1(M)$ map that partially maps some ordinal $\delta < \tau$ cofinally into τ. We denote the least such n by $n(\tau, M)$. Obviously $n^*(\tau, M) \le n(\tau, M)$. We will often say briefly that M singularizes τ. If there is a singularizing partial map for τ that is $\Sigma^{(m)}_1(M)$ in a parameter q then obviously $\tau \cap \tilde{h}^{m+1}_M(\delta \cup \{q\})$ is cofinal in τ for a suitably chosen $\delta < \tau$; here $\tilde{h}^{m+1}_M$ is the canonical $\Sigma^{(m)}_1$-Skolem function for M. If $R^{n^*+1}_M \ne \varnothing$ then of course q can be chosen from $R^{n^*+1}_M$. In particular, if M is sound then we can let $q = p_M$.

It may of course happen that $n^*(\tau, M) < n(\tau, M)$, i.e. that $n(\tau, M)$ is not necessarily the *least* n such that $\omega\varrho^{n+1}_M \le \tau$. This leads to serious difficulties in attempts to translate – in a sufficiently uniform way – fine structural information between premice N and protomice (see below) associated with N, an issue which does not occur in constructions of $\square_\kappa$-sequences in [14] where the focus is on singularizing premice for local successors. Fortunately, premice N with $n^*(\tau, N) < n(\tau, N)$ can be avoided in the analysis of extender fragments that arise in the construction of the canonical global $\square$ sequence.

Definition 3.3. *Let* M *be a singularizing structure for* τ*. We say that* M *is* **exact** *for* τ *just in case that* $n^*(\tau, M) = n(\tau, M)$*.*

Since the definition of exactness is restricted to singularizing structures, whenever we say that M is exact for τ, we implicitly require that M is a singularizing structure for τ. Notice that if $n(\tau, M) = 0$ then M is automatically exact. This simple fact will be crucial in dealing with the issue sketched above. It implies that among all protomice, only those associated with exact levels of the extender model are relevant for the construction of a global $\square$ sequence.

Recall that forming ultrapowers of premice may give rise to coherent structures whose top predicates are extender fragments. This is typical in interpolation arguments if the target premouse is so-called pluripotent. For our present purposes, we will use the following definition of pluripotency.

Definition 3.4. *Let* N *be a premouse and* $\tau \in N$ *be an ordinal. We say that the pair* (N, τ) *is* **pluripotent** *just in case that* N *is active with* $\mathrm{cr}(E^N_{\mathrm{top}}) < \tau$ *and* $n^*(\tau, N) = 0$*. If* τ *is clear from the context, we will briefly say that* N *is pluripotent.*

We now introduce the notion of a divisor; again, we just adapt the notion from [14] to the current context. Recall that if N is active then h^*_N is the Σ_1-Skolem function for N computed in the language for coherent structures. Obviously $h^*_N = \tilde{h}^1_N$ whenever N is a premouse of type A or C, but the two Skolem functions may differ for type B premice. For $\alpha \ge \max\{\omega\varrho^1_N, \mathrm{cr}(E^N_{\mathrm{top}})^{+N}\}$ we let d^α_N be the $<^*$-least finite set of ordinals d with $h^*_N(\alpha \cup \{d\}) = N$ if such a d exists; here $<^*$ is the canonical well-ordering of finite sets of ordinals. If d^α_N exists, we say that N is Dodd sound above α. It is proved in [19] that if d^α_N exists then also d^β_N exists for all $\beta \geq \alpha$ and $d^\beta_N = d^\alpha_N - \beta$. We let $d_N = d^\theta_N$ where $\theta = \max\{\omega\varrho^1_N, \mathrm{cr}(E^N_{\mathrm{top}})^{+N}\}$. This notation slightly diverges from that in [19], but the two notations agree if $\mathrm{cr}(E^N_{\mathrm{top}}) < \omega\varrho^1_N$ which will be the case in our applications. In general, d_N need not exist, but it is proved in [19] that d_N exists whenever N is a sound premouse weakly embeddable into a level of $\mathbf{L}[E]$.

Definition 3.5. *Let N be a sound premouse weakly embeddable into an $\mathbf{L}[E]$-level, $\tau \in N$ be a limit cardinal in N and $n = n^*(\tau, N) \in \omega$. Granting that (N, τ) is not pluripotent, a pair (μ, q) is a* **divisor** *for (N, τ) if and only if there is an ordinal λ such that, setting $r = p_N - q - \tau$ and $Y = \tilde{h}^{n+1}_N(\mu \cup \{r\})$, the following holds:*

(a) $\mu < \tau \leq \lambda < \omega\varrho^n_N$;
(b) $q = (p_N \cap \lambda) - \tau$;
(c) $Y \cap \omega\varrho^n$ *is cofinal in* $\omega\varrho^n_N$;
(d) $\lambda = \min(\mathbf{On} \cap Y - \mu)$.

Granting that (N, τ) is pluripotent and $\mathrm{cr}(E^N_{\mathrm{top}}) < \omega\varrho^1_N$, a pair (μ, q) is a divisor for (N, τ) if and only if there is an ordinal λ such that, setting $r = d_N - q - \tau$ and $Y = h^*_N(\mu \cup \{r\})$, clauses (a) – (d) above hold with the current λ, r, Y and with d_N in place of p_N in (b).

Throughout the paper, if we say that (μ, q) is a divisor for (N, τ) we implicitly assume that N is sound premouse weakly embeddable into a level of $\mathbf{L}[E]$ and τ is a limit cardinal in N. Given a pair (N, τ) and a divisor (μ, q) for (N, τ), the hull Y from the above definition collapses to a premouse $N^* = N^*(\mu, q)$, giving rise to the $\Sigma^{(n)}_1$-preserving map $\pi = \pi_N(\mu, q) : N^* \to N$ and the extender fragment

$$F = F_N(\mu, q) \stackrel{\mathrm{def}}{=} \pi \restriction (\mathcal{P}(\mu) \cap N^*). \tag{1}$$

The notation is analogous to that in [14] in that we expose the pair N and the divisor (μ, q) with any object whose dependency on N and (μ, q) we want to stress. For instance, we write $r_N(\mu, q)$ for r and $\lambda_N(\mu, q)$ for λ. We can drop τ here since once we know that (μ, q) is a divisor for (N, τ) then λ, r, Y, F and all other objects of interest do not depend on τ. Recall also that, by definition, $\nu_N(\mu, q) = \lambda_N(\mu, q)^{+N}$ and $\vartheta_N(\mu, q) = \mu^{+N^*}$.

The proofs of Lemmata 2.1 – 2.3 in [14] go through with the present definition of a divisor, so N^* is a proper initial segment of N (and thus a proper level of $N \,||\, \mu^+$), $N = \mathrm{Ult}^*(N^*, F)$, and π is the associated fine ultrapower map. The structure

$$N(\mu, q) \stackrel{\mathrm{def}}{=} \langle J^{E^N}_\nu, F \rangle$$

is the **protomouse** associated with (N, τ) and the divisor (μ, q).

If (N, τ) is pluripotent, the above definition of a divisor makes use of the Dodd parameter and the Σ_1-Skolem function h^*_N in place of p_N and $\tilde{h}^{n+1}_N$. This may look unnatural at first glance, since the definition using the language of premice makes sense also for pluripotent pairs (N, τ). The reason why we use the language of coherent structures here is that if N is a type B premouse then the definition using the language of premice does not cover all situations we encounter in our main construction. In particular, it does not cover the case where $\lambda_N(\mu, q) = \lambda_N$. Notice also that this case is peculiar since it can happen that N and $N^*(\mu, d_N)$ are premice of different types, an issue that does not occur in the remaining case where

$\lambda_N(\mu, q) < \lambda_N$ or, equivalently, $\mu < \lambda_{N^*}$. If $\mu < \lambda_{N^*}$, it is irrelevant which language we choose for the definition of a divisor, because the proofs of Lemmata 2.4 and 2.5 in [14] go through in the current setting. The fact that the choice of language does not affect the definition of a divisor if $\mu < \lambda_{N^*}$ will be essential in the proof of coherency of the square sequences we construct in the next section.

We now state a fact concerning the relationship between the Dodd parameter and standard parameter that will be useful in our main construction and which gives a rigorous background to the remarks from the previous paragraph. If (N, τ) is pluripotent where N is a sound mouse embeddable into a level of $\mathbf{L}[E]$ with $\mathrm{cr}(E^N_{\mathrm{top}}) < \omega\varrho^1_N$ then $d_N = p^1_N \cup e_N$ where, letting λ^*_N be the largest cutpoint of E^N_{top} and γ_N be the index of $E^N_{\mathrm{top}} \restriction \lambda^*_N$, the parameter e_N is the $<^*$-least finite set of ordinals e such that $\gamma_N \in h^*_N(\omega\varrho^1_N \cup \{p^1_N \cup e\})$, see [19]. From the above remarks we get the following lemma, whose proof can be extracted from the the proofs of Lemmata 2.4 and 2.5 in [14].

Lemma 3.6. *If (N, τ) is pluripotent where N is a sound premouse embeddable into an $\mathbf{L}[E]$-level and (μ, q) is a divisor for (N, τ) with $\lambda_{N^*(\mu,q)} > \mu$ then*

- $e_N \in h^*_N(\mu \cup \{p_N - q - \tau, \gamma_N\})$,
- $\gamma_N \in h^*_N(\mu \cup \{d_N - q - \tau)\})$ *and*
- $(p^1_N \cap \lambda) - \tau = q = (d_N \cap \lambda) - \tau$.

The same is also true of pairs (μ, q) such that $\mu = \tau$ and (a) – (d) in Definition 3.5 are met with $\lambda > \tau$. (Such pairs will not be considered divisors in the rest of the text, but we will need to deal with them occasionally.)

We will consider coherent structures whose initial segments are premice. Given such a coherent structure $M = \langle J^{E^M}_\nu, F\rangle$ we let $\mu_M = \mathrm{cr}(F)$ and $\lambda_M = \lambda(F)$, so F is an extender at (μ_M, λ_M). We also let ϑ_M be the largest ordinal $\vartheta \le \mu^+_M$ such that F measures all sets in $\mathcal{P}(\mu_M) \cap J^{E^M}_\vartheta$ and μ_M is the largest cardinal in $J^{E^M}_\vartheta$. Finally $N^*(M)$ is the collapsing level of J^E_ν for ϑ_M if $\vartheta_M < \mu^{+M}_M$. Granting that $N^*(M)$ is defined, we let $N(M) = \mathrm{Ult}^*(N^*(M), F)$ and π_M be the associated ultrapower embedding. In general, it is not clear that this ultrapower is well-founded, but if it is, we consider $N(M)$ to be transitive. We say that M is a **potential protomouse** if $\vartheta_M < \mu^+_M$ and $N(M)$ is transitive. M is a **protomouse** just in case that M is a potential protomouse and $N(M)$ is a premouse weakly embeddable into an $\mathbf{L}[E]$-level. We do not require that $N(M)$ is sound, but it is straightforward to verify that $N(M)$ is automatically sound above λ_M. If $\mu_M < \tau \le \lambda_M$ then, letting $r_M = \pi_M(p_{N^*(M)})$, $n = n^*(\mu_M, N^*(M))$ and $Y_M = \tilde{h}^{n+1}_{N(M)}(\mu_M \cup \{r_M\})$, clauses (a) – (d) in Definition 3.5 hold with $\lambda_M, \mu_M, r_M, Y_M$ and $q_M = (p_{N(M)} \cap \lambda_M) - \tau = p_{N(M)} - r_M - \tau$ in place of λ, μ, r, Y and q. For sound $N(M)$ this means that $M = N(M)(\mu_M, q)$. The requirement on the soundness of $N(M)$ is in a sense superfluous here since we could easily generalize the definition of a divisor to premice that are not sound. This would however, cause a non-uniformity in the definition of a divisor, which relies on the notion of Dodd parameter, so in order to generalize the definition of a divisor we would first need to generalize the notion of Dodd parameter appropriately and develop some of its properties. Since this will not be needed in our application and would make the text unnecessary complicated we prefer to restrict the definition of a divisor for sound premice. In the construction of the canonical global $\square$ sequence we will often work with singularizing protomice for ordinals τ. Only protomice M with $n(\tau, M) = 0$ will be relevant for our construction; this situation is parallel to that in the construction of $\square_\kappa$ in [14]. By the remarks made above, such protomice are automatically exact.

The method of translation between the definability over protomice and the definability over their associated premice is analogous to that discussed in Section 2.2 of [14]. The following is our basic conversion lemma.

Lemma 3.7. *Let M be a potential protomouse and $N = N(M)$. Let further $n = n^*(\mu_M, N^*(M))$, $p \in R^{n+1}_{N^*(M)}$, $r = \pi_M(p)$ and $q \subset \lambda_M$ be finite.*

(a) *If A is $\Sigma^{(n)}_1(N)$ in $r \cup q$ then there is some A^* that is $\Sigma_1(M)$ in q and ϑ_M satisfying $A \cap \lambda = A^* \cap \lambda$.*

(b) *If A is $\Sigma_1(M)$ in $c \in M$ then there is some ${}^*A$ that is $\Sigma^{(n)}_1(N)$ in c, r and ϑ_M satisfying $A \cap \lambda = {}^*A \cap \lambda$.*

For N we use the language for premice unless $N^*(M)$ is active with $\lambda_{N^*(M)} = \mu$, in which case we use the language for coherent structures. For M we always use the language for coherent structures.

The proof of the above lemma can easily be extracted from the proofs of Lemmata and Corollaries 2.7 – 2.12 of [14]. This lemma, as well as the conversion lemmata stated below, can be formulated in a broader context, not just for potential protomice. The current formulations, however, suffice for our later applications. Parallel to Lemmata 2.13 and 2.14 we also have the following conversion concerning definable singletons.

We first give some details concerning the abuse of notation we will use to simplify the notation. Let M be a coherent structure with top extender F and $\mu = \mathrm{cr}(F)$. We write $F(f)$ for $\pi_M(f)$ where $f : \mu \to \mu$ or $f : \mu \to \mathcal{P}(\mu)$. In either case f can be viewed as a subset of $\mu \times \mu$, so coding by Gödel tuples $\langle \eta, \eta' \rangle \mapsto \prec\eta, \eta'\succ$ makes it possible to replace f by its natural code $a_f \subseteq \mu$. Since obviously $a_{\pi_M(f)} = F(a_f)$, it is harmless to trade the correct notation "$F(a_f)$" for somewhat sloppy, but more intuitive "$F(f)$". Since coding by Gödel tuples allows us to code finite sets of ordinals by ordinals, we usually write $f(x)$, resp. $F(f)(x)$ where f is as above and x is a finite set of ordinals. The correct way of writing this would be $f(\eta_x)$, resp. $F(f)(\eta_x)$ where $\eta_x = \prec\eta_0, \cdots, \eta_{\ell-1}\succ$ and $\langle \eta_0, \cdots, \eta_{\ell-1} \rangle$ is the descending enumeration of x. Finally we write $f(x, \eta)$, resp. $F(f)(x, \eta)$ for $f(\prec\eta_x, \eta\succ)$, resp. $F(f)(\prec\eta_x, \eta\succ)$.

Lemma 3.8. *Let M, N, n and r be as in the previous lemma. Let further $\mu = \mu_M$, $q \subset \lambda_M$ be finite, $\zeta < \lambda_M$ and $y \subseteq \lambda_M$.*

(a) *If ζ (resp. y) is $\Sigma^{(n)}_1(N)$-definable from r and q as a singleton then there is some $f : \mu \to \mu$ (resp. $f : \mu \to \mathcal{P}(\mu)$) in $N^*(M)$ such that $\zeta = F(f)(q)$ (resp. $y = F(f)(q)$).*

(b) *If $\zeta = F(f)(q)$ for some $f : \mu \to \mu$ in $N^*(M)$ (resp. $y = F(f)(q)$) for some $f : \mu \to \mathcal{P}(\mu)$ in $N^*(M)$ then ζ (resp. y) is $\Sigma^{(n)}_1(N)$ definable from r, q and some $\xi < \mu$ as a singleton.*

The languages are chosen the same way as in the previous lemma.

Lemma 3.9. *Let $M = \langle J^E_\nu, F \rangle$ be a coherent structure such that F is a total extender on M. Let $\mu = \mu_M$, $q \subset \lambda_M$ be finite, $\zeta < \lambda_M$ and $y \subseteq \lambda_M$.*

(a) *If ζ (resp. y) is $\Sigma_1(M)$-definable from q as a singleton then there is some $f:\mu\to\mu$ (resp. $f:\mu\to\mathcal{P}(\mu)$) in $J^{E^M}_{\vartheta_M}$ such that $\zeta=F(f)(q,\mu)$ (resp. $y=F(f)(q,\mu)$).*

(b) *If $\zeta=F(f)(q)$ for some $f:\mu\to\mu$ in $J^{E^M}_{\vartheta_M}$ (resp. $y=F(f)(q)$) for some $f:\mu\to\mathcal{P}(\mu)$ in $J^{E^M}_{\vartheta_M}$) then ζ (resp. y) is $\Sigma_1(M)$ definable from q and f as a singleton.*

The previous two lemmata give us a tool for identifying divisors for premice over their associated protomice; this tool corresponds to Lemma 2.15 of [14]. Notice that the characterization of definability in Lemma 3.8 has always the same form over M, no matter whether $\mu_M<\lambda_{N^*(M)}$ or $\mu_M=\lambda_{N^*(M)}$. The same is true of the characterization over $N(\mu,q)$ of divisors for N below. This obviously brings many advantages in applications.

Lemma 3.10. *Assume (μ,q) is a divisor for (N,τ), $n=n^*(\tau,N)\in\omega$, and $F=F_N(\mu,q)$. Let $\mu\le\mu'<\tau$ and q' be a bottom part of q, that is, $q'=q\cap\lambda'$ for some $\lambda'\le\max(q)+1$ where $\max(\varnothing)=\tau$. Finally let $r'=q-q'$. Then (μ',q') is a divisor for (N,τ) if and only if the following condition is satisfied for every $f:\mu\to\mu$ in $N^*(\mu,q)$ and every $\xi<\mu'$:*

$$F(f)(r',\xi)\le\max(q')\quad\Longrightarrow\quad F(f)(r',\xi)<\mu'.$$

Recall that solidity witnesses (no matter whether generalized or standard) are computed in the language for premice in the case of premice and in the language for coherent structures in the case of protomice. In certain arguments we will need to consider structures associated with coherent structures M that are defined analogously as standard witnesses $W^{\alpha,q}_M$ with the only exception that α is smaller than the ultimate projectum of M and q is allowed to contain ordinals smaller than this projectum. Given an acceptable structure M, a number $n\in\omega$, a finite set q of ordinals in M and an ordinal $\alpha\in M$, the structure $W^{n,\alpha,q}_M$ is the transitive collapse of the hull $\tilde{h}^{k+1}_N(\alpha\cup\{q-(\alpha+1)\})$ where k is such that $\omega\varrho^{k+1}_M\le\alpha<\omega\varrho^n_M$ if $\omega\varrho^{n+1}_M\le\alpha$ and $k=n$ if $\alpha<\omega\varrho^{k+1}_M$. We call this structure the **standard n-witness** for α with respect to M and q. If (N,τ) is pluripotent, standard Dodd witnesses are defined in the same manner, but in the language for coherent structures instead of that for premice, and with $n=0$. Standard Dodd witnesses for N are denoted by ${}^*W^{\alpha,q}_N$. If (N,τ) is pluripotent, we get the following consequence of Lemma 3.6.

(2) If N and (μ,q) are as in Lemma 3.6, $\mu\le\alpha<\min(d_N-q)$ and p is a parameter such that $d_N-q\subseteq p$ then ${}^*W^{\alpha,p}_N=W^{0,\alpha,p}_N$.

The next lemma summarizes conversions between fine structural characteristics of N and its associated protomice. These conversions correspond to Lemma 2.16 and Corollary 2.17 in [14]. They easily follow from Lemmata 3.7 – 3.9.

Lemma 3.11. *Let M be a potential protomouse and $n=n^*(\mu_M,N^*(M))$. Then*

(a) $\omega\varrho^{n+1}_N=\omega\varrho^1_M$

where $N=N(M)$. Assume τ is an ordinal satisfying $\max\{\mu^{+M}_M,\omega\varrho^1_M\}\le\tau$. Then the following hold.

(b) *N is a singularizing structure for τ with $n(\tau,N)=n$ just in case that M is a singularizing structure for τ with $n(\tau,M)=0$. Consequently, if M and N are singularizing structures for τ then N is exact for τ if and only if M is exact for τ.*

(c) *$p^1_M - \tau = (p^n_N \cap \lambda_M) - \tau$ if $N^*(M)$ is either passive or active such that $\lambda_{N^*(M)} > \mu_M$, and $p^1_M - \tau = d_N - \tau$ if $N^*(M)$ is active with $\lambda_{N^*(M)} = \mu_M$.*

(d) *N is sound above τ if and only if M is sound above τ, granting that $N^*(M)$ is either passive or active with $\lambda_{N^*(M)} > \mu_M$. N is Dodd sound above τ if and only if M is sound above τ, granting that $N^*(M)$ is active with $\lambda_{N^*(M)} = \mu_M$.*

(e) *Assume that $\vartheta_M \le \alpha < \lambda_M$, $p \in R^{n+1}_{N^*(M)}$, $r = \pi_M(p)$ and $q \subset \lambda_M$ is finite.*

If $N^*(M)$ is either passive or active with $\lambda_{N^*(M)} > \mu_M$ then $W^{n,\alpha,r\cup q}_N = \mathrm{Ult}^*(N^*(M), F^{0,\alpha,q}_M)$ where $F^{0,\alpha,q}_M$ is the top extender of $W^{0,\alpha,q}_M$. Letting $W = W^{0,\alpha,q}_N$, the associated ultrapower embedding $\bar\pi : N^*(M) \to W$ is the inverse to the Mostowski collapsing isomorphism that comes from collapsing the hull $\tilde h^{n+1}_W(\mu_M \cup \{\bar\pi(p)\})$ and $F^{0,\alpha,q}_M$ is the extender derived from this embedding. It follows that $W^{n,\alpha,r\cup q}_N \in N$ if and only if $W^{0,\alpha,q}_M \in M$.

If $N^*(M)$ is active with $\lambda_{N^*(M)} = \mu_M$, the same is true with standard Dodd witnesses for N in place of standard 0-witnesses for N.

(f) *N is solid above τ if and only if M is solid above τ, granting that $N^*(M)$ is either passive or active with $\lambda_{N^*(M)} > \mu_M$. N is Dodd solid above τ if and only if M is solid above τ, granting that $N^*(M)$ is active with $\lambda_{N^*(M)} = \mu_M$.*

Recall that the notion of soundness has a definition in Jensen's fine structure theory that does not depend on the notion of solidity and vice versa, so the way clauses (d) and (f) in the above lemma are formulated does make sense. An acceptable structure M is sound above τ just in case that $\tilde h^n_M(\omega\varrho^n_M \cup \{p^n_M - \tau\}) = M$ where n is such that $\omega\varrho^{n+1}_M \le \tau < \omega\varrho^n_M$; M is solid above τ just in case that $W^{\beta,p^n_M}_M \in M$ for each $\beta \in p^n_M - \tau$. If M is solid then p^n_M is a top segment of p_M, as can be easily seen.

The following two lemmata correspond to Lemma 2.18 in [14]. They make use of "generalized" versions of witnesses $W^{n,\alpha,q}_M$ which are defined in similarly as generalized solidity witnesses. If M, α and q are as in the definition of a standard n-witness above, say that a pair $\langle Q, t\rangle$ is a **generalized n-witness** for α with respect to M and q just in case that Q is an acceptable structure, t is a finite set of ordinals in Q with $|t| = |q - (\alpha+1)|$, and for every $\Sigma^{(n)}_1$-formula $\varphi(v_0, v_1, \dots, v_\ell)$ and any $\xi_1, \dots, \xi_\ell < \alpha$ we have

$$M \models \varphi[q - (\alpha+1), \xi_1, \dots, \xi_\ell] \implies Q \models \varphi[t, \xi_1, \dots, \xi_\ell].$$

For pluripotent premice, generalized Dodd witnesses are defined in the same manner, but in the language for coherent structures instead of that for premice, and with $n = 0$.

Given an acceptable structure M, n-witnesses for M (no matter whether standard or generalized) do not provide us with any information about solidity of M below $\omega\varrho^{n+1}_M$. We introduced them for a different purpose — namely, for identifying canonical divisors. The next two lemmata, as well as the notion of an n-witness/generalized n-witness can be formulated in a more general setting, but we restrict ourselves to a formulation that suffices for our applications. There is a slight overlap in the conclusions of the lemmata, which might be seen as a deficiency in the way the paper is organized. The reason we chose these formulations is that applications of the lemmata in the arguments to come (especially in the proof of

Lemma 4.9) become more natural. The proofs of both Lemma 3.12and Lemma 3.13 follow from Lemma 3.11 in a straightforward manner.

Lemma 3.12. *Let M be a potential protomouse, $n = n^*(\tau, N)$ and $N = N(M)$. Let further $p \in R^{n+1}_M$, $r = \pi_M(p)$ and $q \subset \lambda_M$ be finite. Finally let $\vartheta_M \leq \alpha < \lambda_M$.*

Assume $N^*(M)$ is either passive or active with $\lambda_{N^*(M)} > \mu$, and $\langle Q, t\rangle$ is a generalized n-witness for α with respect to N and $r \cup q$. If H is any transitive admissible structure such that $\langle Q, t\rangle \in H$ then $W^{n,\alpha,r\cup q}_N \in H$ and is Σ_0-definable in H from $\langle Q, t\rangle$ and α. It follows that also $W^{0,\alpha,q}_M \in H$ and is Σ_0-definable in H from the parameters $\langle Q, t\rangle, \alpha$ and ϑ_M.

Assume $N^*(M)$ is active with $\lambda_{N^*(M)} = \mu$ and $\langle Q, t\rangle$ is a generalized Dodd witness for α with respect to N and $r \cup q$. If H is a transitive admissible structure such that $\langle Q, t\rangle \in H$ then also ${}^*W^{\alpha,r\cup q}_N$ is an element of H and is Σ_0-definable in H from $\langle Q, t\rangle$ and α. It follows that $W^{0,\alpha,q}_M$ is and element of H and is Σ_0-definable in H from $\langle Q, t\rangle, \alpha$ and ϑ_M.

In either case, the definition of the standard witness is uniform, that is, there is a single formula φ that serves as a definition of the standard witness from the parameters named above. This formula depends neither on these parameters nor on H.

Lemma 3.13. *Let M be a coherent structure, $\vartheta_M \leq \alpha < \lambda_M$ and $q \subset \lambda_M$ be finite.*

Assume M is a potential protomouse and $\langle Q, t\rangle$ is a generalized 0-witness for α with respect to M and q. If H is a transitive admissible structure and $\langle Q, t\rangle \in H$ then also the standard 0-witness $W^{0,\alpha,q}_M$ is an element of H and is Σ_0-definable in H from $\langle Q, t\rangle, \alpha$ and ϑ_M.

Assume is a premouse and $\langle Q, t\rangle$ is a generalized Dodd witness for α with respect to M and q. If H is a transitive admissible structure and $\langle Q, t\rangle \in H$ then also the standard Dodd witness ${}^*W^{\alpha,q}_M$ is an element of H and is Σ_0-definable in H from the parameters $\langle Q, t\rangle, \alpha$ and ϑ_M.

We will make use of the following condensation lemma for protomice, which corresponds to the conjunction of Lemmata 2.19 and 2.20 in [14]. The proof of this lemma follows the same strategy as the proofs of the above named lemmata in [14]. There are however differences in the verification of the initial segment condition for certain extenders which arise in the argument, as well as in the application of the condensation lemma; for this reason we give a sketch of the proof that focuses on these new issues.

Lemma 3.14. *Let M be either a protomouse or an active premouse weakly embeddable into an $\mathbf{L}[E]$-level and let τ be inaccessible in M such that $\mu_M < \tau \leq \lambda_M$. Suppose further that $\bar{M}$ is a coherent structure, $\bar{\tau} \in \bar{M}$ is an ordinal, $\sigma : \bar{M} \to M$ and the following requirements are met:*

(a) *$\bar{M}$ is sound and solid with $\omega\varrho^1_{\bar{M}} \leq \bar{\tau}$.*
(b) *σ is Σ_0-preserving and non-cofinal.*
(c) *$\mathrm{cr}(\sigma) = \bar{\tau}$ and $\sigma(\bar{\tau}) = \tau$.*

Letting $\bar{N} = N(\bar{M})$ and $\mu = \mu_M = \mu_{\bar{M}}$, the structure $\bar{N}$ is a proper level of M, the pair $(\mu, p_{\bar{M}})$ is a divisor for $\bar{N}$ and $\bar{M} = \bar{N}(\mu, p_{\bar{M}})$.

If $\bar{M}$ is a singularizing structure for $\bar{\tau}$ with $n(\bar{\tau}, \bar{M}) = 0$ then $\bar{N}$ is the level of M singularizing $\bar{\tau}$, and is exact for $\bar{\tau}$.

Proof. Obviously $\bar\tau$ is an inaccessible cardinal in the sense of $\bar M$. The non-cofinality of σ yields $\vartheta_{\bar M} < \vartheta_M$. Since $\bar\tau$ is a limit cardinal in $\bar M$, $\vartheta_{\bar M} < \mu^{+\bar M}$, so $N^*(\bar M)$ is defined and, being a proper level of $M \,||\, \vartheta_M$, is in the domain of π_M. Then $N' = \pi_M(N^*(\bar M))$ is a proper level of M; see the discussion of protomice above. Letting $\bar F$ be the top extender of $\bar M$ and $\bar N = N(\bar M)$ (this ultrapower is well-founded by standard considerations), the canonical embedding $\tilde\sigma : \bar N \to N'$ defined by $\pi_{\bar M}(f)(\alpha) \mapsto \pi_M(f)(\sigma(\alpha))$ extends σ and is $\Sigma_0^{(n)}$-preserving where n is such that $\omega\varrho^{n+1}_{N^*(\bar M)} \le \mu < \omega\varrho^{n}_{N^*(\bar M)}$. The structure $\bar N$ is a potential premouse; the weak amenability of its top predicate (granting that $\bar N$ is active) follows from the fact that the ultrapower embedding $\pi_{\bar M} : N^*(\bar M) \to \bar N$ is the identity on the power set of $\mathrm{cr}(E^{N^*(M)}_{\mathrm{top}})$ whenever $\mathrm{cr}(E^{N^*(\bar M)}_{\mathrm{top}}) < \mu$. To see that $\bar N$ is a premouse we need to check the initial segment condition, which can be done by standard considerations unless $\lambda_{N^*(\bar M)} = \mu$. In order to discuss this case, we need some additional information about $\bar N$. Requirement (a) of the current lemma together with Lemma 3.11(a,c,d,f) yield $\omega\varrho^{n+1}_{\bar N} \le \bar\tau < \omega\varrho^{n}_{\bar N}$ and $\bar N$ is sound and solid above $\bar\tau$, unless $\lambda_{N^*(\bar M)} = \mu$ in which case $\bar N$ is Dodd sound and Dodd solid above $\bar\tau$. This is true regardless of the behavior of $E^{N^*(\bar M)}_{\mathrm{top}}$.

We now complete the verification of the initial segment condition for $\bar N$. Recall that we focus on the case where $\lambda_{N^*(\bar M)} = \mu$. Recall also that $p_{\bar M} - \bar\tau = d_{\bar N} - \bar\tau$ in this case, again by Lemma 3.11(c). If $d_{\bar N} - \bar\tau \neq \varnothing$ then the initial segment condition follows from the fact that the standard Dodd solidity witness W for $\max(d_{\bar N})$ is an element of $\bar N$; this is true because either all cutpoints of $E^{\bar N}_{\mathrm{top}}$ are also cutpoints of E^{W}_{top} or else $\lambda(E^{W}_{\mathrm{top}})$ is the largest cutpoint of $E^{\bar N}_{\mathrm{top}}$. If $d_{\bar N} - \bar\tau = \varnothing$, we first observe that all cutpoints of $E^{\bar N}_{\mathrm{top}}$ are strictly smaller than $\bar\tau$. This is trivial if $\lambda(E^{\bar N}_{\mathrm{top}}) = \bar\tau$. If $\lambda(E^{\bar N}_{\mathrm{top}}) > \bar\tau$, this is a consequence of the soundness of $\bar N$ above $\bar\tau$, which guarantees that every ordinal in $\bar N$ is of the form $E^{\bar N}_{\mathrm{top}}(f)(\delta)$ for some $f : \mathrm{cr}(E^{\bar N}_{\mathrm{top}}) \to \mathrm{cr}(E^{\bar N}_{\mathrm{top}})$ and some $\delta < \bar\tau$. In this case the cutpoints of $E^{\bar N}_{\mathrm{top}}$ are actually bounded below $\bar\tau$; otherwise $\bar\tau$ would also be a cutpoint. In either of the above cases, the initial segment condition for $\bar N$ then follows from the initial segment condition for N' and from the fact that $\tilde\sigma \restriction \bar\tau = \mathrm{id} \restriction \bar\tau$.

We would like to apply the condensation lemma to the map $\tilde\sigma : \bar N \to N'$. The discussion in the previous paragraphs guarantees that the assumptions of the condensation lemma are met unless $N^*(\bar M)$ is active with $\lambda_{N^*(\bar M)} = \mu$. This remaining case requires two additional steps. First, it might happen that N' a type C premouse (this is the case if $N^*(\bar M)$ is of type C), whereas we know that $\bar N$ is of type B whenever $\lambda(E^{\bar N}_{\mathrm{top}}) > \bar\tau$. The issue can be resolved the same way as in [14] by replacing N' by $N' \,||\, \nu'$ where, letting $\lambda' = \sup(\tilde\sigma''\lambda(\bar N))$, ν' is the index of $E^{N'}_{\mathrm{top}} \restriction \lambda'$. From now on assume that N' is the result of such a replacement; then $\tilde\sigma$ is a Σ_0-preserving embedding between two type B premice. The second issue to clear up is the verification of the soundness of $\bar N$ above $\bar\tau$. Here we employ Theorem 1.2 from [19]. Although that theorem is formulated for weakly iterable premice, no weak iterability of $\bar N$ is needed for the proof that $\bar N$ is sound; what suffices is the Dodd soundness and Dodd solidity of $\bar N$ above $\bar\tau$ which we verified above.

The application of the condensation lemma now reduces to ruling out the options (c) and (d). But this is immediate, as both would require that $\bar\tau$ is a successor cardinal in $\bar N$. It follows that $\bar N$ is a level of N', and therefore a proper level of M.

In particular, $\bar{N}$ is sound and solid. By construction, $(\mu, p_{\bar{M}} - \bar{\tau})$ is a divisor for $\bar{N}$ and $\bar{M} = \bar{N}(\mu, p_{\bar{M}} - \bar{\tau})$. Regarding the last conclusion of the current lemma, if $\bar{M}$ is a singularizing structure for $\bar{\tau}$, requirement (a) guarantees the existence of some $\delta < \bar{\tau}$ and some partial function mapping δ cofinally into $\bar{\tau}$ that is $\Sigma_1(\bar{M})$-definable in the parameter $p_{\bar{M}}$. Applying Lemma 3.7, this partial function (or at least its part below $\bar{\tau}$) is $\Sigma_1^{(n)}(\bar{N})$-definable in the parameters $p_{\bar{M}}$, $\pi_{\bar{M}}(p_{N^*(\bar{M})})$ and $\vartheta_{\bar{M}}$; here $n = n^*(\mu, N(\bar{M}))$. Since $\bar{\tau}$ is obviously regular in $\bar{N}$, the premouse $\bar{N}$ is a singularizing structure for $\bar{\tau}$ and $n = n(\bar{\tau}, \bar{N})$, which proves that $\bar{N}$ is exact for $\bar{\tau}$. The remaining parts of the lemma are then clear. □(Lemma 3.14)

Finally we summarize the relevant facts about strong divisors reformulated for the present purposes. We will be only considering strong divisors for pairs (N, τ) where N is an initial level of $\mathbf{L}[E]$ and τ is a limit cardinal in $\mathbf{L}[E]$, as this somewhat simplifies the matters and is sufficient for our application. In particular, if (μ, q) is a divisor for such a pair (N, τ) then q is a bottom part of the standard/Dodd parameter for N, as τ is the ultimate projectum of N. Hence subtracting τ from the standard/Dodd parameter becomes superfluous; for instance (b) in Definition 3.5 reads $q = p_N - \lambda$ in the present situation. Recall the notion of a strong divisor first.

Definition 3.15. *Let N be an $\mathbf{L}[E]$-level, $\tau \in N$ be a limit cardinal in $\mathbf{L}[E]$, $n = n^*(\tau, N) \in \omega$ and (μ, q) be a divisor for (N, τ). We say that (μ, q) is* **strong** *just in case that one of the following holds.*

(a) *N is either passive or active with $\lambda_{N^*(\mu,q)} > \mu$, and*
$$\mathcal{P}(\mu) \cap N'(\mu) = \mathcal{P}(\mu) \cap N^*(\mu, q)$$
where $N'(\mu)$ is the transitive collapse of $\tilde{h}_N^{n+1}(\mu \cup \{p_N\})$.

(b) *N is active with $\lambda_{N^*(\mu,q)} = \mu$, and $\mathcal{P}(\mu) \cap N'(\mu) = \mathcal{P}(\mu) \cap N^*(\mu, q)$ where $N'(\mu)$ is the transitive collapse of $h_N^*(\mu \cup \{d_N\})$.*

Of course, in the latter case we have $q = d_N$ and $N^*(\mu, q) = N \,||\, \nu$ where ν is the index of the extender $E_{\text{top}}^N \restriction \mu$. The following characterization, corresponding to [14], Lemma 2.22 and Lemma 2.23 is essential for further analysis of strong divisors, and is also directly used in the main construction.

Lemma 3.16. *Let N be an $\mathbf{L}[E]$-level with $n^*(\tau, N) \in \omega$, $\tau \in N$ be a limit cardinal in $\mathbf{L}[E]$ and (μ, q) be a divisor for (N, τ). If N is either passive or active with $\lambda_{N^*(\mu,q)} > \mu$ then the following are equivalent.*

(a) *(μ, q) is strong.*

(b) *$N^*(\mu, q)$ is the core of $N'(\mu)$.*

(c) *$|\, p_{N^*(\mu,q)} \,| = |\, p_{N'(\mu)} \,| = r_N(\mu, q)$.*

If N is active and $\lambda_{N^*(\mu,q)} = \mu$ then the following are equivalent.

(e) *(μ, q) is strong.*

(f) *$E_{\text{top}}^{N'(\mu)} \notin N'(\mu)$.*

The next general fact about the structure $N'(\mu)$ is simple, but very useful in applications.

Lemma 3.17. *Let N be an $\mathbf{L}[E]$-level with $n^*(\tau, N) \in \omega$, $\tau \in N$ be a limit cardinal in $\mathbf{L}[E]$ and (μ, q) be a divisor for (N, τ). Let $\pi' : N'(\mu) \to N$ be the inverse to the collapsing map. Then ${\pi'}^{-1}(q)$ is a top segment of $p_{N'(\mu)}$.*

The following notion of closeness will be crucial in identifying strong divisors.

Definition 3.18. *Let $M = \langle J^E_\nu, F\rangle$ be a coherent structure. An ordinal $\vartheta \leq \vartheta_M$ is* **closed** *in M relative to $q \in [\lambda_M]^{<\omega}$ just in case that $F(f)(q,\xi) \cap \mu_M \in J^E_\vartheta$ for every $f : \mu_M \to \mathcal{P}(\mu_M)$ in J^E_ϑ and every $\xi < \mu_M$.*

We obtain a characterization of strong divisors that corresponds to Lemma 2.24 in [14]. As in the case of Lemma 3.10, the advantage of this characterization is that it does not depend on the relationship between $\lambda_{N^*}(\mu, q)$ and μ.

Lemma 3.19. *Let N be an $\mathbf{L}[E]$-level, $\tau \in N$ be a limit cardinal in $\mathbf{L}[E]$ and (μ, q) be a divisor for (N, τ). Then (μ, q) is strong just in case that $\vartheta_{N(\mu,q)}$ is closed in $N(\mu, q)$ relative to $q = p_{N(\mu,q)}$.*

The two lemmata that provide us with the key for the choice of a canonical divisor correspond to Lemma 2.25 and 2.26 in [14]. Given an $\mathbf{L}[E]$-level N, a limit cardinal $\tau \in N$ in $\mathbf{L}[E]$ and a bottom part q of the standard/Dodd parameter of N, we let $\mathcal{D}_q(N, \tau)$ be the set of all $\mu < \tau$ such that (μ, q) is a divisor for (N, τ), and $\mathcal{D}^*_q(N, \tau)$ be the set of $\mu < \tau$ such that (μ, q) is a strong divisor for (N, τ).

Lemma 3.20. *Let N be an $\mathbf{L}[E]$-level, $\tau \in N$ be a limit cardinal in $\mathbf{L}[E]$ and q be a bottom part of the standard/Dodd parameter of N. Then the following holds.*

(a) *$\mathcal{D}^*_q(N, \tau)$ is closed in τ.*

(b) *If τ is inaccessible in N and N is exact for τ then $\mathcal{D}^*_q(N, \tau)$ is bounded in τ.*

Proof. (Sketch.) The proof that $\mathcal{D}^*_q(N, \tau)$ is closed in τ is the same as that of Lemma 2.25 in [14]. The heart of the proof is showing that if μ is a limit point of $\mathcal{D}^*_q(N, \tau)$ then $|p_{N'(\mu)}| = |p_{N^*(\mu,q)}|$ whenever $N^*(\mu, q)$ is either passive or active with $\lambda_{N^*(\mu,q)} > \mu$, and $E^N_{\text{top}} \restriction \mu \notin N'(\mu)$ whenever $N^*(\mu, q)$ is active with $\lambda_{N^*(\mu,q)} = \mu$.

The same proof goes through even if we allow $\mu = \tau$, which we excluded in the definition of a divisor. Since obviously $N'(\tau) = N$, τ cannot be a limit point of $\mathcal{D}^*_q(N, \tau)$ if $q \neq \varnothing$, which guarantees that $\mathcal{D}^*_q(N, \tau)$ is bounded in τ in this case.

If $q = \varnothing$ then $\mathcal{D}_q(N, \tau)$ is bounded in τ, as N is a singularizing structure for τ which is exact for τ. It follows that $\mathcal{D}^*_q(N, \tau) \subseteq \mathcal{D}_q(N, \tau)$ is bounded in τ as well. □(Lemma 3.20)

Lemma 3.21. *Let N be the singularizing $\mathbf{L}[E]$-level for τ where τ is inaccessible in N and let (μ, q) be a strong divisor for N. If q' is a proper bottom part of q then there is no $\mu' \leq \mu$ such that (μ', q') is a divisor for N.*

The canonical divisor $(\mu(N, \tau), q(N, \tau))$ is chosen as follows.

$$\begin{aligned} q(N,\tau) &\simeq \text{ the unique } q \text{ such that } \mathcal{D}^*_q(N,\tau) \neq \varnothing \\ \mu(N,\tau) &\simeq \max(\mathcal{D}^*_q(N,\tau)). \end{aligned}$$

The uniqueness of q is a direct consequence of Lemma 3.21.

If N is pluripotent and does not admit any strong divisors, we let

$$\begin{aligned} q(N,\tau) &= d_N \\ \mu(N,\tau) &= \text{cr}(E^N_{\text{top}}). \end{aligned}$$

In this particular case we will consider $(\mu(N, \tau), q(N, \tau))$ a strong divisor for (N, τ) and let $N(\mu(N, \tau), q(N, \tau)) = N$. Even though $(\mu(N, \tau), q(N, \tau))$ is not a divisor in the usual sense, the pair $(\mu(N,\tau), q(N,\tau))$ satisfies the criterion formulated in Lemma 3.19, namely that $\vartheta_N(\mu(N,\tau), q(N,\tau)) = \vartheta_N$, being the cardinal successor of $\mu(N,\tau)$ in N, is closed in N relative to $q(N,\tau)$, and in fact relative to any finite $q \subset \lambda(E^N_{\mathrm{top}})$.

4. Global square sequence

In this section we give a proof of Theorem 1.2. Recall that $\mathcal{S}$ is the class of all singular cardinals of the extender model $\mathbf{L}[E]$. Given a singular cardinal τ, let N_τ denote the level of $\mathbf{L}[E]$ which singularizes τ. That is, N_τ is of the form $\mathbf{L}[E]\,||\,\alpha$ for the unique α such that τ is regular in $\mathbf{L}[E]\,||\,\alpha$ but singular in $\mathbf{L}[E]\,||(\alpha+1)$. We also allow the situation where $\tau \notin N_\tau$, in which case $\tau = \alpha$. Since τ is a cardinal, the ultimate projectum of N_τ is τ. We say that N_τ is pluripotent just in case the pair (τ, N_τ) is pluripotent. We fix the following notation.

- $n_\tau = n(\tau, N_\tau)$ and $n^*_\tau = n^*(\tau, N_\tau)$.
- $p_\tau = p_{N_\tau}$.
- If N_τ is pluripotent then $d_\tau = d_{N_\tau}$ and $e_\tau = e_{N_\tau}$ (see Lemma 3.6).
- $\varrho_\tau = \varrho^{n_\tau}_{N_\tau}$ and $\mathcal{H}_\tau$ is the domain of $J^{E^N_\tau}_{\rho_\tau}$, that is $\mathcal{H}_\tau = S^{E^N_\tau}_{\omega\varrho_\tau}$.
- $\tilde{h}_\tau = \tilde{h}^{n_\tau+1}_{N_\tau}$.
- If N_τ is pluripotent then $h^*_\tau = h^*_{N_\tau}$.

Given two acceptable J-structures $\bar{M}, M$ for the same language, parameters $\bar{p}, p$ in these structures and an ordinal $\bar{\tau} \in \bar{M}$ such that $\bar{p} \cap \bar{\tau} = \varnothing$ and $\tilde{h}^{n+1}_{\bar{M}}(\bar{\tau} \cup \{\bar{p}\}) = \bar{M}$, there is at most one $\Sigma^{(n)}_0$-preserving map $\sigma : \bar{M} \to M$ satisfying $\sigma \restriction \bar{\tau} = \mathrm{id}$ and $\sigma(\bar{p}) = p$. If such a map exists, it is unique and $\sigma : \tilde{h}^{n+1}_{\bar{M}}(i, \langle x, \bar{p}\rangle) \mapsto \tilde{h}^{n+1}_M(i, \langle x, p\rangle)$ for all $i \in \omega$ and $x \in [\bar{\tau}]^{<\omega}$. Obviously, if $\bar{M} = M$ and $\bar{p} = p$ then $\sigma = \mathrm{id}$; it will be notationally convenient to consider also this trivial case.

- $\sigma^{n,\bar{p},p}_{\bar{M},M,\bar{\tau}} : \bar{M} \to M$ is the unique map σ just described above, if exists.
- $\sigma_{\bar{\tau},\tau} = \sigma^{n_{\bar{\tau}},p_{\bar{\tau}},p_\tau}_{N_{\bar{\tau}},N_\tau,\bar{\tau}}$ whenever $\bar{\tau} \le \tau$ are elements of $\mathcal{S}$ and $\sigma^{n_{\bar{\tau}},p_{\bar{\tau}},p_\tau}_{N_{\bar{\tau}},N_\tau,\bar{\tau}}$ exists.

When we write "$\sigma_{\bar{\tau},\tau}$", we implicitly assume that $\bar{\tau} \le \tau$.

Similarly as it is done in the construction of a $\Box_\kappa$-sequence in [14], we split $\mathcal{S}$ into two disjoint classes $\mathcal{S}^0$ and $\mathcal{S}^1$.

$$\begin{aligned} \mathcal{S}^1 &= \{\tau \in \mathcal{S} \mid N_\tau \text{ is exact for } \tau \text{ and } (q(N_\tau), \mu(N_\tau)) \text{ is defined.}\} \\ \mathcal{S}^0 &= \mathcal{S} - \mathcal{S}^1. \end{aligned}$$

We would like to stress at this point that for *each* $\tau \in \mathcal{S}$ there is a uniquely determined singularizing structure which enables us to define a set C_τ that belongs to the canonical global $\Box^{\mathcal{S}}$-sequence. This is one of the main differences between the construction of the canonical $\Box^{\mathcal{S}}$-sequence and that of the canonical $\Box_\kappa$-sequence, where we had to restrict ourselves on a club of ordinals that do not index extenders. Hence $\Box^{\mathcal{S}}$ always holds in extender models, although it should be mentioned that $\Box^{\mathcal{S}}$ might be quite a weak statement, as the class of all singular ordinals of the model might be very "thin" in comparison with all singular cardinals of $\mathbf{V}$.

For $\tau \in \mathcal{S}^1$ we define:

- $q_\tau = q(N_\tau, \tau)$, $m_\tau = |q_\tau|$ and $\mu_\tau = \mu(N_\tau, \tau)$.
- $M_\tau = N_\tau(\mu_\tau, q_\tau)$.
- $N^*_\tau = N^*_\tau(\mu_\tau, q_\tau)$.
- $r_\tau = p_\tau - q_\tau$. If N_τ is pluripotent then $r^*_\tau = d_\tau - q_\tau$.

- $\lambda_\tau = \lambda_{N_\tau}(\mu_\tau, q_\tau)$, $\vartheta_\tau = \vartheta_{N_\tau}(\mu_\tau, q_\tau)$, $\nu_\tau = \nu_{N_\tau}(\mu_\tau, q_\tau)$ and $F_\tau = F_{N_\tau}(\mu_\tau, q_\tau)$.
- If M_τ is a protomouse, $h_\tau = h_{M_\tau}$. If M_τ is a premouse, $h_\tau = h^*_\tau$.
- If (μ, q) is a divisor for N_τ, we let $h^{(\mu,q)}_\tau = h_{N_\tau(\mu,q)}$. If N_τ is pluripotent and $(\mu, q) = (\mathrm{cr}(E^{N_\tau}_{\mathrm{top}}, d_\tau)$, we let $h^{(\mu,q)}_\tau = h^*_{N_\tau} = h_\tau$.

We first construct a global square sequence $\langle C_\tau \tau \in \mathcal{S}\rangle$ and then we produce the set $A' \subseteq A$ as required by Theorem 1.2. Following the strategy in [14], we now define approximations

$$\mathcal{B} = \langle B_\tau \mid \tau \in \mathcal{S}\rangle \qquad \text{and} \qquad \mathcal{B}^* = \langle B^*_\tau \mid \tau \in \mathcal{S}\rangle$$

to our global square sequence. As before, we work on $\mathcal{S}^0$ and $\mathcal{S}^1$ separately. There are, however some differences between our current construction and the construction of a $\square_\kappa$-sequence. In the construction of a $\square_\kappa$-sequence $\langle C_\tau\rangle_\tau$, the direct limit of the collapsing $\mathbf{L}[E]$-levels for ordinals from $C_\tau \cap \bar\tau$ always results in the collapsing $\mathbf{L}[E]$-level for $\bar\tau$. This is a consequence of the fact that τ^* is the cardinal successor of κ in N_{τ^*} for each $\tau^* \in \mathcal{S}^0$, and is crucial in the verification that the set C_τ arising in the construction is closed. In the case of a global square sequence, it is not automatically true that a direct limit of this kind results in a singularizing structure for the respective ordinal, and we need to introduce an extra tool to enforce this. The idea again comes from Jensen's construction in $\mathbf{L}$ in [4]. Even with this extra tool it is not possible to prove that a tail-end of B_τ is closed if $\tau \in \mathcal{S}^0$. One approach here would be to amend the definition of B_τ so that B_τ would be closed, and indeed this can be done, as explained in Remark 4.13. However, we favor the strategy of working with the sets B_τ defined in the standard way, then define B^*_τ the same way as in [14] and finally thin out B^*_τ to obtain some C^*_τ that is closed on a tail-end and has small order-type. We believe this construction gives us more information about the square sequence than that discussed in Remark 4.13. Unlike in the construction of a $\square_\kappa$-sequence, B^*_τ here will not in general be closed on a tail-end. The thinning out procedure is the same as in [14], but we apply it to a slightly different situation that also requires a new element, namely a proof that the result C^*_τ of thinning is closed. We start with the extra tool mentioned above.

Definition 4.1. *Let $\tau \in \mathcal{S}$. We let*

$$\alpha_\tau = \textit{the largest } \alpha < \tau \textit{ satisfying } \tau \cap \tilde h_\tau(\alpha \cup \{p_\tau\}) = \alpha.$$

Notice that α_τ exists for every $\tau \in \mathcal{S}$. This is true, as the set of all ordinals α satisfying $\tau \cap \tilde h_\tau(\alpha \cup \{p_\tau\}) = \alpha$ is closed and bounded in τ. That this set is closed is obvious; its boundedness in τ follows from the fact that N_τ is a singularizing structure for τ, that is, $\tilde h_\tau(\delta \cup \{p_\tau\})$ is cofinal in τ for some $\delta < \tau$. Also, it may happen that $\alpha_\tau = 0$. Looking at the definition of α_τ, one might get an impression that (α_τ, p_τ) is a divisor for N_τ if this level is exact for τ. Although this might be true in some cases, it is false in many typical cases that arise in the construction of the global square sequence for a simple reason that the hull $\tilde h_\tau(\alpha_\tau \cup \{p_\tau\})$ may be bounded in $\omega\varrho_\tau$. Finally note that any α with $\tau \cap \tilde h_\tau(\alpha \cup \{p_\tau\}$ is closed under Gödel pairing, as it is the critical point of the inverse to the Mostowski collapsing isomorphism associated with $\tilde h_\tau(\alpha \cup \{p_\tau\})$. So $\tilde h_\tau(\alpha \cup \{p_\tau\})$ is the set of all values $\tilde h_\tau(i, \langle \xi, p_\tau\rangle)$ for $\xi < \alpha$ and $i < \omega$. In particular, this is true of $\alpha = \alpha_\tau$.

The following lemma establishes basic facts about maps $\sigma_{\bar\tau,\tau}$ that will be relevant in the construction of our global square sequence. This lemma is an analogue to Lemma 3.2 in [14], but the proof needs a new argument, as the proof in [14] relied on the fact that the critical points of $\sigma_{\bar{\tau},\tau}$ were local cardinal successors, which is false in the present case. We formulate the lemma for the broader class of maps of the form $\sigma^n_{(\bar{N},\bar{\tau}),N_\tau}$ defined at the beginning of this section, as this generality will be needed in several applications.

Lemma 4.2. *Let $\alpha\tau^* < \bar{\tau} < \tau$ be ordinals, $M^*, \bar{M}, M$ be acceptable J-structures for the same language, $p^*, \bar{p}, p$ be parameters in these structures and $n \in \omega$. Assume further that the following requirements are met.*

(i) *α is the largest ordinal $\alpha' < \tau$ such that $\tau \cap \tilde{h}^{n+1}_M(\alpha' \cup \{p\}) = \alpha'$.*
(ii) *α is the largest ordinal $\alpha' < \bar{\tau}$ such that $\bar{\tau} \cap \tilde{h}^{n+1}_{\bar{M}}(\alpha' \cup \{\bar{p}\}) = \alpha'$.*
(iii) *$\tilde{h}^{n+1}_{M^*}(\tau^* \cup \{p^*\}) = M^*$ and $\tilde{h}^{n+1}_{\bar{M}}(\bar{\tau} \cup \{\bar{p}\}) = \bar{M}$.*
(iv) *Both maps $\sigma^* = \sigma^{n,p^*,p}_{M^*,M,\tau^*}$ and $\bar{\sigma} = \sigma^{n,\bar{p},p}_{\bar{M},M,\bar{\tau}}$ exist and $\sigma^*(\tau^*) = \tau = \bar{\sigma}(\bar{\tau})$.*

Then the following holds.

(a) *$\sigma^*, \bar{\sigma}$ are not $\Sigma^{(n)}_1$-preserving hence not cofinal at the n-th level.*
(b) $\sup(\omega\varrho_\tau \cap \mathrm{rng}(\sigma^*)) < \sup(\omega\varrho_\tau \cap \mathrm{rng}(\bar{\sigma}))$.
(c) $\mathrm{rng}(\sigma^*) \subseteq \mathrm{rng}(\bar{\sigma})$.

Proof. Regarding (a), assume σ^* is $\Sigma^{(n)}_1$-preserving. Then $\tilde{h}^{n+1}_M(\tau^* \cup \{p\}) = \mathrm{rng}(\sigma^*)$, hence $\tau \cap \tilde{h}^{n+1}_M(\tau^* \cup \{p\}) = \tau^*$ which contradicts the maximality of α_τ.

Regarding (b), assume for a contradiction that $\geq$ is the case. Let $x \subseteq \tau^*$ be finite and $i \in \omega$. Assume $\tilde{h}^{n+1}_{\bar{M}}(i, \langle x, \bar{p}\rangle)$ is defined and is an element of the n-th reduct of $\bar{M}$; denote the value by $\bar{y}$. Then $\bar{M} \models \psi[z, \bar{y}, i, \langle x, \bar{p}\rangle]$ where $\psi(v^n, u^0_0, u^0_1, u^0_2)$ is a $\Sigma^{(n)}_0$-formula and $(\exists v^n)\psi$ is a functionally absolute definition of the $\Sigma^{(n)}_1$-Skolem function $\tilde{h}^{n+1}_{\bar{M}}$. It follows that $M \models \psi[\bar{\sigma}(z), y, i, \langle x, p\rangle]$ where $y = \bar{\sigma}(\bar{y})$ and both $\bar{\sigma}(z), y$ are elements of the n-th reduct of M. By our assumption $\geq$ holds in (b) instead of $<$, so there is an ordinal $\zeta < \omega\varrho^n_{M^*}$ such that $\bar{\sigma}(z), y \in \sigma^*(S^E_\zeta)$ where we recall that E is the extender sequence of $\mathbf{L}[E]$. It follows that $M \models (\exists v^n \in \sigma^*(S^E_\zeta))(\exists u^n \in \sigma^*(S^E_\zeta))\psi[v^n, u^n, i, \langle x, p\rangle]$ where $\psi(v^n, u^n, u^0_1, u^0_2)$ is a specialization of $\psi(v^n, u^0_0, u^0_1, u^0_2)$ obtained by substituting u^n for u^0_0; see [18], Section 1.6. Since this statement is $\Sigma^{(n)}_0$, it is preserved under σ^* and we conclude that $y^* = \tilde{h}^{n+1}_{M^*}(i, \langle x, p^*\rangle)$ is defined and $\sigma^*(y^*) = y$. If $\bar{y} \in \bar{\tau}$ then $y \in \tau$ by assumption (iv) above and the Σ_0-elementarity of $\sigma_{\bar{\tau}}$ and similarly we obtain that $y^* \in \tau^*$. This means that $\bar{y} = y = y^* < \tau^*$. It follows that $\bar{\tau} \cap \tilde{h}^{n+1}_{\bar{M}}(\tau^* \cup \{\bar{p}\}) = \tau^*$, which contradicts the maximality of α_τ.

Regarding (c), we run the argument from (b) with the roles of τ^* and $\bar{\tau}$ switched and in higher generality, as now we have to consider values of Skolem functions that are not necessarily elements of the n-th reduct. If $y \in \mathrm{rng}(\sigma^*)$ then $y = \sigma^*(y^*)$ where $y^* = \tilde{h}^{n+1}_{M^*}(i, \langle x, p^*\rangle)$ for some finite $x \subseteq \tau^*$. It follows that $M^* \models (\exists w^0)(w^0 = \tilde{h}^{n+1}_{M^*}(i, \langle x, p^*\rangle)$. This statement is a $\Sigma^{(n)}_1$ statement, as it is obtained by substituting the good $\Sigma^{(n)}_1$-function $\tilde{h}^{n+1}_{M^*}$ into the $\Sigma^{(0)}_1$-formula $(\exists w^0)(w^0 = u^0_0)$; see [18], Section 1.8. Hence the above statement can be expressed in the form $M^* \models (\exists v^n)\psi'[v^n, i, \langle x, p^*\rangle]$ where ψ' is a $\Sigma^{(n)}_0$-formula. Now similarly as in the previous paragraph we show that $\bar{M} \models (\exists v^n)\psi'[v^n, i, \langle x, \bar{p}\rangle]$, this time using the inequality $\leq$ in (b). As the translation between this statement and the statement $\bar{M} \models (\exists w^0)(w^0 = \tilde{h}^{n+1}_{\bar{M}}(i, \langle x, \bar{p}\rangle))$ is uniform, latter statement is true. Hence $\bar{y} =$

$\tilde{h}^{n+1}_{\bar{M}}(i, \langle x, \bar{p}\rangle)$ is defined. It is now obvious that $\bar{\sigma}(\bar{y}) = \tilde{h}^{n+1}_{M}(i, \langle x, p\rangle) = \sigma^*(y^*)$. This proves (c). □(Lemma 4.2)

We are now ready to define B_τ for $\tau \in \mathcal{S}^0$.

Definition 4.3. *Let $\tau \in \mathcal{S}^0$. The set B_τ consists of all $\bar{\tau} \in \tau \cap \mathcal{S}^0$ that meet the following requirements.*

(1) *$N_{\bar{\tau}}$ is a premouse of the same type as N_τ.*
(2) *$n_{\bar{\tau}} = n_\tau$.*
(3) *$\alpha_{\bar{\tau}} = \alpha_\tau$.*
(4) *There is a map $\sigma : N_{\bar{\tau}} \to N_\tau$ that is $\Sigma_0^{(n_\tau)}$-preserving with respect to the language of premice and such that:*
   (a) *$\sigma \restriction \bar{\tau} = \mathrm{id} \restriction \bar{\tau}$ and if $\bar{\tau} \in N_{\bar{\tau}}$ then $\sigma(\bar{\tau}) = \tau$.*
   (b) *$\sigma(p_{\bar{\tau}}) = p_\tau$.*
   (c) *If $\beta \in p_\tau$ then there is a generalized witness $Q^*_\tau(\beta) = \langle Q_\tau(\beta), t_\tau(\beta)\rangle$ for β with respect to N_τ and p_τ satisfying $Q^*_\tau(\beta) \in \mathrm{rng}(\sigma)$.*

Since $n^*_\tau \leq n_\tau$, the map σ from (4) is identical with $\sigma_{\bar{\tau},\tau}$. Three basic facts about maps $\sigma_{\bar{\tau},\tau}$ for $\bar{\tau}, \tau \in B_\tau$ are summarized in the following lemma.

Lemma 4.4. *Let $\tau^* < \bar{\tau}$ be two elements of B_τ and $n = n_\tau$.*

(a) *$\sigma_{\bar{\tau},\tau}$ is not $\Sigma_1^{(n)}$-preserving, and therefore is not cofinal at the n-th level.*
(b) *$\mathrm{rng}(\sigma_{\tau^*,\tau}) \subseteq \mathrm{rng}(\sigma_{\bar{\tau},\tau})$ and $\sup(\omega\varrho_\tau \cap \mathrm{rng}(\sigma_{\tau^*,\tau})) < \sup(\omega\varrho_\tau \cap \mathrm{rng}(\sigma_{\bar{\tau},\tau}))$.*
(c) *$(\sigma_{\bar{\tau},\tau})^{-1} \circ \sigma_{\tau^*,\tau} = \sigma_{\tau^*,\bar{\tau}}$, so $\sigma_{\tau^*,\bar{\tau}}$ exists.*
(d) *$\sigma_{\tau^*,\bar{\tau}} : N_{\tau^*} \to N_{\bar{\tau}}$ is not cofinal at the n-th level.*
(e) *$\sigma_{\tau^*,\bar{\tau}}$ witnesses (4) in Definition 4.3 with $(\tau^*, \bar{\tau})$ in place of $(\bar{\tau}, \tau)$.*
(f) *$B_\tau \cap \bar{\tau} = B_{\bar{\tau}} - \min(B_\tau)$.*

Proof. Clause (a) follows from (a) in Lemma 4.2, the first part of (b) follows from (c) in Lemma 4.2 and the second part of (b) follows from (b) in Lemma 4.2. Clauses (c) and (d) follow directly from (b). Clause (e) follows from (b) and (c) via a straightforward verification; this verification uses the fact that generalized witnesses for $\beta \in p_\tau$ with respect to N_τ and p_τ (and similarly in the case of $p_{\bar{\tau}}$ and $N_{\bar{\tau}}$) are characterized by $\Pi_1^{(n)}$-statements; see also the proof of Lemma 3.3 in [14]. Finally $\subseteq$ in (f) follows directly from (c) and (e); $\supseteq$ in (f) follows by (c) and an argument similar to the proof of (e). □(Lemma 4.4)

Given $\tau \in \mathcal{S}^0$, we define the following objects by recursion.

$$\begin{aligned} \tau(0) &= B_\tau \\ \tau(i+1) &\simeq \min(B_{\tau(i)}) \\ \ell_\tau &= \text{the largest } i \text{ such that } B_{\tau(i)} \text{ is defined} \\ B^*_\tau &= B_{\tau(0)} \cup B_{\tau(1)} \cup \cdots \cup B_{\tau(\ell_\tau)}. \end{aligned}$$

By construction, the set B_τ is a tail-end of B^*_τ. The following lemma summarizes basic properties of sets B^*_τ and $\bar{B}^*_\tau$.

Lemma 4.5. *Let $\bar{\tau} \in B^*_\tau \cup \{\tau\}$, $\tau' < \tau^*$ be elements of $B^*_\tau \cap \bar{\tau}$ and $n = n_\tau$. Then:*

(a) *$B^*_{\bar{\tau}} = B^*_\tau \cap \bar{\tau}$.*
(b) *Clauses (1) – (3) in Definition 4.3 hold.*
(c) *$\sigma_{\tau^*,\bar{\tau}}$ exists, is $\Sigma_0^{(n)}$-preserving and if $\tau^* \in N_{\tau^*}$ then $\sigma_{\tau^*,\bar{\tau}}(\tau^*) = \bar{\tau}$.*
(d) *$\sigma_{\tau^*,\bar{\tau}}$ is not $\Sigma_1^{(n)}$-preserving, and therefore is not cofinal at the n-th level.*

(e) $\operatorname{rng}(\sigma_{\tau',\bar{\tau}}) \subseteq \operatorname{rng}(\sigma_{\tau^*,\bar{\tau}})$ *and* $\sup(\omega\varrho_{\bar{\tau}} \cap \operatorname{rng}(\sigma_{\tau',\bar{\tau}})) < \sup(\omega\varrho_{\bar{\tau}} \cap \operatorname{rng}(\sigma_{\tau^*,\bar{\tau}}))$.
(f) $\sigma_{\tau^*,\bar{\tau}} \circ \sigma_{\tau',\tau^*} = \sigma_{\tau',\bar{\tau}}$.

Proof. Clause (a) follows from the definition of B^*_τ and Lemma 4.4(b) by a straightforward verification. The rest follows immediately from Lemma 4.2, Lemma 4.4, Definition 4.3 and the definition of $\sigma_{\tau^*,\bar{\tau}}$. □(Lemma 4.5)

It is not clear that B^*_τ is closed on a tail-end. It will be convenient to work with a closed set, so we add all missing limit points to B^*_τ. For $\tau \in \mathcal{S}^0$ let

$$\begin{aligned}
\bar{B}^*_\tau &= B^*_\tau \cup \lim(B^*_\tau) \cup \{\tau\}\\
\bar{N}_{\bar{\tau}} &= \begin{cases} N_{\bar{\tau}} & \text{if } \bar{\tau} \in B^*_\tau \\ \lim(\langle N_{\tau',\tau^*}, \sigma_{\tau',\tau^*} \mid \tau', \tau^* \in B^*_\tau \cap \bar{\tau}\rangle) & \text{otherwise} \end{cases}\\
\bar{h}_{\bar{\tau}} &= \begin{cases} \tilde{h}_{\bar{\tau}} & \text{if } \bar{\tau} \in B^*_{\bar{\tau}} \\ \tilde{h}^{n_\tau+1}_{\bar{N}_{\bar{\tau}}} & \text{otherwise.} \end{cases}\\
\bar{\varrho}_{\bar{\tau}} &= \varrho^{n_\tau}_{\bar{N}_{\bar{\tau}}}\\
\bar{p}_{\bar{\tau}} &= p_{\bar{N}_{\bar{\tau}}}.
\end{aligned}$$

The direct limit of the diagram $\langle N_{\tau',\tau^*}, \sigma_{\tau',\tau^*} \mid \tau', \tau^* \in B^*_\tau \cap \bar{\tau}\rangle$ in the above definition is of course considered transitive. This possible due to its well-foundedness, which in turn is a consequence of the fact that the map $\bar{\sigma}_{\bar{\tau},\tau} : \bar{N}_{\bar{\tau}} \to N_\tau$ defined by $\bar{\sigma}_{\bar{\tau},\tau} : \bar{\sigma}_{\tau^*\bar{\tau}}, (x) \mapsto \sigma_{\tau^*,\tau}(x)$ is Σ_0-preserving. Here $\bar{\sigma}_{\tau^*\bar{\tau}} : N_{\tau^*} \to \bar{N}_{\bar{\tau}}$ are the direct limit maps for $\tau^* \in B^*_\tau \cap \bar{\tau}$. It follows that $\bar{N}_{\bar{\tau}}$ is an acceptable J-structure, so it does make sense to talk about fine structure of $\bar{N}_{\bar{\tau}}$. Of course, if $\bar{\tau} \in B^*_\tau$ then $\bar{N}_{\bar{\tau}} = N_{\bar{\tau}}$. In general, it is not clear whether $\bar{N}_{\bar{\tau}}$ is a singularizing structure for $\bar{\tau}$. The following lemma summarizes basic facts about $\bar{N}_{\bar{\tau}}$ for $\bar{\tau} \in \bar{B}^*_\tau$.

Lemma 4.6. *Let $\bar{\tau} \in \lim(B^*_\tau)$, $\tau^* \in B^*_\tau \cap \bar{\tau}$ and $n = n_\tau$. Then the following holds.*
(a) $\omega\varrho^{n+1}_{\bar{N}_{\bar{\tau}}} = \bar{\tau}$.
(b) *The direct limit maps $\bar{\sigma}_{\tau^*,\bar{\tau}}$ as well as the map $\bar{\sigma}_{\bar{\tau},\tau}$ are $\Sigma^{(n)}_0$-preserving.*
(c) $\bar{\sigma}_{\tau^*,\bar{\tau}} \restriction \tau^* = \mathrm{id} \restriction \tau^*$ *and if* $\tau^* \in N_{\tau^*}$ *then* $\bar{\sigma}_{\tau^*,\bar{\tau}}(\tau^*) = \bar{\tau}$. *Similarly,* $\bar{\sigma}_{\bar{\tau},\tau} \restriction \bar{\tau} = \mathrm{id} \restriction \bar{\tau}$ *and if* $\bar{\tau} \in \bar{N}_{\bar{\tau}}$ *then* $\bar{\sigma}_{\bar{\tau},\tau}(\bar{\tau}) = \tau$.
(d) $\bar{\sigma}_{\tau^*,\bar{\tau}}(p_{\tau^*}) = \bar{p}_{\bar{\tau}}$ *and* $\bar{\sigma}_{\bar{\tau},\tau}(\bar{p}_{\bar{\tau}}) = p_\tau$.
(e) $\bar{\sigma}_{\tau^*,\bar{\tau}} = \sigma^{n,p_{\tau^*}}_{(\bar{N}_{\tau^*},\tau^*),\bar{N}_{\bar{\tau}}}$ *and* $\bar{\sigma}_{\bar{\tau},\tau} = \sigma^{n,\bar{p}_{\bar{\tau}}}_{(\bar{N}_{\bar{\tau}},\bar{\tau}),\bar{N}_\tau}$.
(f) *$\bar{N}_{\bar{\tau}}$ is an $\mathbf{L}[E]$-level.*
(g) *α_τ is the largest $\alpha < \bar{\tau}$ such that $\bar{\tau} \cap \bar{h}_{\bar{\tau}}(\alpha \cup \{\bar{p}_{\bar{\tau}}\}) = \alpha$.*

Proof. We may without loss of generality assume that $\bar{\tau} \notin B^*_\tau$. Clauses (b) and (c) follow from properties of direct limits by a straightforward verification. From (b),(c) and the soundness of each N_{τ^*} we obtain $\bar{h}_{\bar{\tau}}(\bar{\tau} \cup \{\bar{p}\}) = \bar{N}_{\bar{\tau}}$ where $\bar{p}$ is the common value of $\bar{\sigma}_{\tau^*,\bar{\tau}}(p_{\tau^*})$ for $\tau^* \in B^*_\tau \cap \bar{\tau}$, from which we conclude that $\omega\varrho^{n+1}_{\bar{N}_{\bar{\tau}}} \leq \bar{\tau}$. Since $\bar{\tau}$ is a cardinal in $\mathbf{L}[E]$ and $\bar{N}_{\bar{\tau}}$ agrees with $\mathbf{L}[E]$ up to $\bar{\tau}$, we conclude that $\bar{\tau} = \omega\varrho^{n+1}_{\bar{N}_{\bar{\tau}}}$. This gives (a). Let i be such that $\tau(i+1) < \bar{\tau} < \tau(i)$ and $\tau^* \in B^*_\tau \cap \bar{\tau}$ be such that $\tau(i+1) \leq \tau^*$. As each $\bar{\beta} \in \bar{p}$ is of the form $\bar{\sigma}_{\tau^*,\bar{\tau}}(\beta^*)$ for some $\beta^* \in p_{\tau^*}$, the map $\sigma_{\tau^*,\tau(i)}$ has some generalized witness for $\beta(i) = \sigma_{\tau^*,\tau(i)}(\beta^*)$ with respect to $N_{\tau(i)}$ and $p_{\tau(i)}$ in its range. Since $\operatorname{rng}(\sigma_{\tau^*,\tau(i)}) \subseteq \operatorname{rng}((\sigma_{\tau(i),\tau})^{-1} \circ \bar{\sigma}_{\bar{\tau},\tau})$ and $\beta(i) = (\sigma_{\tau(i),\tau})^{-1} \circ \bar{\sigma}_{\bar{\tau},\tau}(\bar{\beta})$, the same applies to the composition $(\sigma_{\tau(i),\tau})^{-1} \circ \bar{\sigma}_{\bar{\tau},\tau}$. As this composition is $\Sigma^{(n)}_0$-preserving and being a generalized witness as above is a $\Pi_1^{(n)}$-property, it is preserved downwards under $\bar\sigma_{\bar\tau,\tau(i)}$. It follows that for each element of $\bar p$ there is a generalized witness with respect to $\bar N_{\bar\tau}$ and $\bar p$ in $\bar N_{\bar\tau}$, so $\bar p = \bar p_{\bar\tau}$ by Corollary 1.12.4 in [18] and $\bar N_{\bar\tau}$ is sound. Now apply the Condensation Lemma to the triple $\bar\sigma_{\bar\tau,\tau} : \bar N_{\bar\tau} \to N_\tau$. We conclude that $\bar N_{\bar\tau}$ is solid and obtain four options. Option (a) in the Condensation Lemma fails, as N_τ and $\bar N_{\bar\tau}$ have different $(n+1)$-st projecta and options (c) and (d) in the Condensation Lemma fail, as $\bar\tau$ is a limit cardinal in $\bar N_{\bar\tau}$. It follows that $\bar N_{\bar\tau}$ is a proper initial segment of N_τ, and hence a level of $\mathbf{L}[E]$, which proves (f) in the current lemma. Clause (e) then follows immediately. Regarding (g), assume $\alpha_\tau < \alpha < \bar\tau$ and pick some $\tau^* \in B^*_\tau \cap \bar\tau$ such that $\alpha < \tau^*$. As $\alpha_{\tau^*} = \alpha_\tau$, we have some finite $x \subseteq \alpha$ and some $i < \omega$ such that $\alpha \le \bar h_{\tau^*}(i, \langle x, p_{\tau^*}\rangle) < \tau^*$. Since $\bar\sigma_{\tau^*,\bar\tau}$ is $\Sigma_0^{(n)}$-preserving, $\bar\sigma_{\tau^*,\bar\tau} \restriction \tau^* = \mathrm{id}$ and $\bar h_{\tau^*}$ has a functionally absolute $\Sigma_1^{(n)}$-definition, $\alpha < \bar h_{\bar\tau}(i, \langle x, \bar p_{\bar\tau}\rangle) < \tau^* < \bar\tau$. This verifies (g). Finally (h) follows by the same argument as Lemma 4.4(a). □(Lemma 4.6)

We next summarize the basic facts about $\bar B^*_\tau$ relevant for our construction. To unify the notation, for $\tau^* \le \bar\tau$ in $\bar B^*_\tau$ we let

$$\bar\sigma_{\tau^*,\bar\tau} = \sigma^{n_\tau,\bar p_{\tau^*}}_{(\bar N_{\tau^*},\tau^*),\bar N_{\bar\tau}}. \tag{3}$$

Of course, if $\tau^*, \bar\tau \in B^*_\tau \cup \{\tau\}$ then $\bar\sigma_{\tau^*,\bar\tau} = \sigma_{\tau^*,\bar\tau}$ and if $\bar\tau \in \bar B^*_\tau - B^*_\tau$ then $\bar\sigma_{\tau^*,\bar\tau}$ is the same as in Lemma 4.6, so this notation is consistent with that used before.

Lemma 4.7. *Let $\tau', \tau^*, \bar\tau \in \bar B^*_\tau$ such that $\tau' < \tau^*$ and let $n = n_\tau$. Then the following holds.*

(a) $\bar B^*_{\bar\tau} = \bar B^*_\tau \cap \bar\tau$ *whenever* $\bar\tau \in B^*_\tau$.
(b) $\bar\sigma_{\tau^*,\bar\tau}$ *exists and* $\bar\sigma_{\tau^*,\bar\tau}(\tau^*) = \bar\tau$ *whenever* $\tau^* \in \bar N_{\tau^*}$.
(c) $\bar\sigma_{\tau',\tau^*} : \bar N_{\tau'} \to \bar N_{\tau^*}$ *is not* $\Sigma_1^{(n)}$*-preserving hence not cofinal at the n-th level.*
(d) $\mathrm{rng}(\bar\sigma_{\tau',\bar\tau}) \subseteq \mathrm{rng}(\bar\sigma_{\tau^*,\bar\tau})$ *and* $\sup(\omega\bar\varrho_{\bar\tau} \cap \mathrm{rng}(\bar\sigma_{\tau',\bar\tau})) < \sup(\omega\bar\varrho_{\bar\tau} \cap \mathrm{rng}(\bar\sigma_{\tau^*,\bar\tau}))$.
(e) $\bar\sigma_{\tau^*,\bar\tau} \circ \bar\sigma_{\tau',\tau^*} = \bar\sigma_{\tau',\bar\tau}$.
(f) α_τ *is the largest* $\alpha < \bar\tau$ *such that* $\bar\tau \cap \bar h_{\bar\tau}(\alpha \cup \{\bar p_{\bar\tau}\}) = \alpha$.

Proof. Clause (a) follows from the coherency of the sets B^*_τ (see Lemma 4.5(a)) and from the construction of $\bar B^*_\tau$. The remaining clauses follow from Lemma 4.5 and Lemma 4.6 in the straightforward way. □(Lemma 4.5)

In the following we prove that B^*_τ is unbounded in τ whenever τ has uncountable cofinality. We split the proof into two lemmata. It will be convenient to introduce the following approximation to B_τ.

$$\hat B_\tau = \{\bar\tau \in \tau \cap \mathcal{S} \mid \bar\tau \text{ meets the requirements (1) – (4) from Definition 4.3}\} \tag{4}$$

Thus, the only difference between B_τ and $\hat B_\tau$ is that $\hat B_\tau$ is allowed to contain elements that are outside $\mathcal{S}^0$. However, we still require the cardinal τ to be an element of $\mathcal{S}^0$.

Lemma 4.8. *$\hat B_\tau$ is unbounded in τ whenever $\tau \in \mathcal{S}^0$ has uncountable cofinality.*

Proof. This is an interpolation argument similar to that in the proof of Lemma 3.7 in [14]. Given an ordinal $\tau' < \tau$, we want to find some $\tilde\tau \in \hat B_\tau$ such that $\tau' \le \tilde\tau < \tau$. We form a fully elementary countable hull $X \prec N_\tau$ such that $\tau, \tau', p_\tau, \alpha_\tau, \in X$, and for each $\beta \in p_\tau$, some generalized witness $Q^*_\tau(\beta)$ for β with respect to N_τ and p_τ is in X as well. (Of course $\tau \in X$ only if $\tau \in N_\tau$.) We then collapse X to some transitive $\bar N$ and obtain a fully elementary embedding $\sigma : \bar N \to N_\tau$ whose range is

X. Obviously, $\bar{N}$ is a sound premouse of the same type as N_τ and σ can be viewed as a fully elementary, and so Σ^*-preserving embedding with respect to the language for premice. Also $n(\bar{\tau},\bar{N}) = n_\tau$; we denote this common value by n. Let $\bar{\tau},\bar{p},\bar{\alpha}$ be the σ-preimages of $\tau, p_\tau, \alpha_\tau$, respectively.

We let $\tilde{\tau} = \sup(\sigma[\bar{\tau}])$. Obviously $\tilde{\tau} > \tau'$. Since $\bar{\tau}$ is countable and τ has uncountable cofinality, we have $\tilde{\tau} < \tau$. Notice also that $\tilde{\tau}$ is a limit cardinal in the sense of $\mathbf{L}[E]$. Using the interpolation lemma (see [14] and [18], Lemma 3.6.10) we obtain an acceptable J-structure $\tilde{N}$ and maps $\tilde{\sigma} : \bar{N} \to \tilde{N}$ and $\sigma' : \tilde{N} \to N_\tau$ such that $\tilde{\sigma}$ is $\Sigma_0^{(n)}$-preserving, σ' is $\Sigma_0^{(n-1)}$-preserving[2], $\sigma' \restriction \tilde{\tau} = \mathrm{id} \restriction \tilde{\tau}$, and if $\tilde{\tau} \in N_{\tilde{\tau}}$ then $\sigma'(\tilde{\tau}) = \tau$. This is true even if $\mathrm{ht}(\bar{N}) = \bar{\tau}$, in which case $\tilde{\sigma}$ is essentially the same object as σ, up to its codomain. Let us stress that $\tilde{N}$ is constructed as the fine pseudoultrapower (in the sense of [18]) of $\bar{N}$ by $\sigma \restriction J^{\bar{E}}_{\bar{\tau}}$ where $\bar{E} = E^{\bar{N}}$ and the functions used to construct this pseudoultrapower are precisely all total good $\mathbf{\Sigma}_1^{(i)}(\bar{N})$-functions with domains in $J^{\bar{E}}_{\bar{\tau}}$ where $i < n$. Our goal is to show that

$$\tilde{\tau} \in \mathcal{S}, \tilde{N} = N_{\tilde{\tau}} \text{ and } n_{\tilde{\tau}} = n_\tau \tag{5}$$

$$\tilde{\sigma} = \sigma_{\tilde{\tau},\tau} \tag{6}$$

$$\alpha_{\tilde{\tau}} = \alpha_\tau \tag{7}$$

We let $\tilde{p} = \tilde{\sigma}(\bar{p})$. Notice that $\tilde{\sigma}(\bar{\alpha}) = \alpha_\tau$. The following sketch summarizes the proof of (5). The details, which are left to the reader, follow from the general fine structure theory described in [18], Chapters 1–3.

Since $\tau \in \mathcal{S}^0$, the pair (N_τ, τ) is not pluripotent, hence $(\bar{N}, \bar{\tau})$ is not pluripotent either. This means that if $\bar{N}$ is active and $\mathrm{cr}(E^{\bar{N}}_{\mathrm{top}}) < \bar{\tau}$ then $n = n(\bar{\tau},\bar{N}) > 0$, so the map $\tilde{\sigma}$, being a pseudoultrapower map in the sense of [18], is Σ_2-preserving. It follows that $\tilde{N}$ is a potential premouse in the sense of [18], that is, the top extender of $\tilde{N}$ is total on $\tilde{N}$. To see that $\tilde{N}$ is a premouse one has to verify the initial segment condition, and this is immediate if $\bar{N}$ is a type A or type B premouse. If $\bar{N}$ is a type C premouse, the initial segment condition follows from the fact that $\tilde{\sigma}$ maps $\lambda(\bar{N})$ cofinally into $\lambda(\tilde{N})$ if $n = 1$ and that $\tilde{\sigma}$ is $\Sigma_2^{(1)}$-preserving if $n > 1$.

By the elementarity of σ we have $\bar{p} \in R^{n+1}_{\bar{N}}$, which has two immediate consequences. First, $\tilde{\sigma}$ is cofinal at the n-th level hence $\Sigma_1^{(n)}$-preserving and σ' is $\Sigma_0^{(n)}$-preserving. Second, $\tilde{h}^{n+1}_{\tilde{N}}(\tilde{\tau} \cup \{\tilde{p}\}) = \tilde{N}$, so $\omega\varrho^{n+1}_{\tilde{N}} \leq \tilde{\tau}$. Thus $\tilde{N}$ is solid, as follows from the first part of the Condensation Lemma. Since $\tilde{\tau}$ is a cardinal and $\tilde{N}$ agrees with $\mathbf{L}[E]$ up to $\tilde{\tau}$, we conclude that $\omega\varrho^{n+1}_{\tilde{N}} = \tilde{\tau}$ and $\tilde{p} \in R^{n+1}_{\tilde{N}}$. Recall that the property of being a generalized witness for β is a $\Pi_1^{(n)}$-property whenever β is larger than or equal to the $(n+1)$-st projectum of the structure in question. Since the generalized solidity witness $Q^*_\tau(\beta)$ is an element of $\mathrm{rng}(\sigma)$ for each $\beta \in p_\tau$ and $\mathrm{rng}(\sigma') \supseteq \mathrm{rng}(\sigma)$, for each $\tilde{\beta} \in \tilde{p}$ there is a generalized solidity witness with respect to $\tilde{N}$ and $\tilde{\beta}$ in $\tilde{N}$, namely $(\sigma')^{-1}(Q^*_\tau(\sigma'(\tilde{\beta})))$. Then $\tilde{p} = p_{\tilde{N}}$ by Corollary 1.12.4 in [18] and $\tilde{N}$ is sound. We now show that $\tilde{N}$ is an initial segment of N_τ, and therefore an initial segment of $\mathbf{L}[E]$. This is clear if $\tilde{\tau} = \mathrm{ht}(\tilde{N})$, so assume that $\tilde{\tau} \in \tilde{N}$. We have $\tilde{\tau} = \mathrm{cr}(\sigma')$ and the condensation lemma applied to σ' gives us four possibilities. Options (c) and (d) can easily be ruled out since either of them implies that

---

[2] Here we let $n - 1 = 0$ if $n = 0$.

$\tilde\tau$ must be a successor cardinal in $\tilde N$. Hence (a) or (b) is the case, and so $\tilde N$ is an initial segment of N_τ. Notice that in fact (b) holds, as $\omega\varrho^{n+1}_{\tilde N} = \tilde\tau \neq \tau = \omega\varrho^{n+1}_{N_\tau}$.

We now complete the proof of (5). Since $\bar N$ is a singularizing structure for $\bar\tau$ with $n(\bar\tau, \bar N) = n$ by the elementarity of σ, we have some ordinal $\bar\gamma < \bar\tau$ such that $\tilde h^{n+1}_{\bar N}(\bar\gamma \cup \{\bar p\})$ is cofinal in $\bar\tau$. Let $\gamma = \tilde\sigma(\bar\gamma) < \tilde\tau$. As $\tilde\sigma$ is $\Sigma^{(n)}_1$-preserving, $\tilde h^{n+1}_{\tilde N}(\gamma \cup \{p_{\tilde N}\}) \supseteq \tilde\sigma[\tilde h^{n+1}_{\bar N}(\bar\gamma \cup \{\bar p\})]$ and the latter is cofinal in $\tilde\tau$ since $\tilde\sigma$ maps $\bar\tau$ cofinally into $\tilde\tau$. It follows that $\tilde N$ is the singularizing level of $\mathbf{L}[E]$ for $\tilde\tau$ and $n_{\tilde\tau} = n(\tilde\tau, \tilde N) \le n$. To see that $n_{\tilde\tau} = n$, we show that $\tilde\tau \cap \tilde h^{k+1}_{\tilde N}(\delta \cup \{\tilde p\})$ is bounded in $\tilde\tau$ for every $\delta < \tilde\tau$ and $k < n$. Given such a δ, let $\bar\delta = \tilde\sigma^{-1}[\delta]$. Since $\tilde\sigma$ maps $\bar\tau$ cofinally into $\tilde\tau$, the ordinal $\bar\delta$ is strictly smaller than $\bar\tau$. Since $n(\bar\tau, \bar N) = n$, the hull $\tilde h^{k+1}_{\bar N}(\bar\delta \cup \{\bar p\})$ is bounded below $\bar\tau$, say by some $\bar\gamma$. This fact can be expressed in a $\Pi^{(k)}_1$ manner over $\bar N$ as follows.

$$(\forall x^k \in [\bar\delta]^{<\omega})(\forall \zeta^k < \bar\tau)(\forall i^k \in \omega)\big(\zeta^k = \tilde h^{k+1}_{\bar N}(i^k, \langle x^k, \bar p\rangle) \quad\longrightarrow\quad \zeta^k < \bar\gamma\big).$$

The $\Sigma^{(k)}_1$-elementarity of $\tilde\sigma$ then guarantees that the above statement is transferred by $\tilde\sigma$ to $\tilde N$, so $\tilde\tau \cap \tilde h^{k+1}_{\tilde N}(\tilde\sigma(\bar\delta) \cup \{p_{\tilde N}\})$ is bounded by $\tilde\sigma(\bar\gamma) < \tilde\tau$. As $\tilde\sigma(\bar\delta) \ge \delta$, we obtain the desired conclusion, and thereby the equality $n_{\tilde\tau} = n$. This completes the proof of (5). Clause (6) then follows immediately.

It remains to prove (7). To see that $\alpha_{\tilde\tau} \le \alpha_\tau$, pick any α' such that $\alpha_\tau < \alpha' < \tilde\tau$ and let $\bar\alpha' = \tilde\sigma^{-1}[\alpha']$. Then $\bar\alpha < \bar\alpha' < \bar\tau$ where the second inequality follows again from the fact that $\tilde\sigma$ maps $\bar\tau$ cofinally into $\tilde\tau$; recall also that $\tilde\sigma(\bar\alpha) = \alpha$. By the full elementarity of σ, there is some finite $x \subseteq \bar\alpha'$ and some $i \in \omega$ such that $\bar\alpha' \le \tilde h^{n+1}_{\bar N}(i, \langle x, p_{\bar N}\rangle) < \bar\tau$. If we apply $\tilde\sigma$ to these inequalities, we obtain $\tilde\sigma(\bar\alpha') \le \tilde h_{\tilde N}(i, \langle \tilde\sigma(x), p_{\tilde\tau}\rangle) < \tilde\tau$. By the definition of $\bar\alpha'$ each element of x is mapped into α', so $\tilde\sigma(x)$ is a finite subset of α'. Since $\tilde\sigma(\bar\alpha') \ge \alpha'$, we see that $\tilde\tau \cap \tilde h_{\tilde N}(\alpha' \cup \{p_{\tilde\tau}\}) \neq \alpha'$. As this is true of any α' satisfying $\alpha_\tau < \alpha' < \tilde\tau$, we conclude that $\alpha_{\tilde\tau} \le \alpha_\tau$. To see the converse, we verify the inclusion $\tilde\tau \cap \tilde h_{\tilde N}(\alpha_\tau \cup \{p_{\tilde\tau}\}) \subseteq \alpha_\tau$. Let $x \subseteq \alpha_\tau$ be finite, $i \in \omega$ and $\zeta = \tilde h_{\tilde N}(i, \langle x, p_{\tilde\tau}\rangle) < \tilde\tau$. Since σ' is $\Sigma^{(n)}_0$-preserving, $\sigma'(\zeta) = \tilde h_\tau(i, \langle x, p_\tau\rangle) < \tau$. It follows that $\zeta < \alpha_\tau$, which completes the proof of (7) and thereby the proof of the entire lemma. □(Lemma 4.8)

Lemma 4.9. *Assume $\tau \in \mathcal{S}^0$ and $\hat B_\tau$ is unbounded in τ. Then $\hat B_\tau$ is almost contained in $\mathcal{S}^0$. It follows that there is some $\hat\tau < \tau$ such that $\hat B_\tau - \hat\tau = B_\tau - \hat\tau$.*

Proof. Let $n = n_\tau$. The premouse $N_{\bar\tau}$ is Σ_0-embeddable into N_τ whenever $\bar\tau \in \hat B_\tau$, so $N_{\bar\tau}$ is active if and only if N_τ is active and $\mathrm{cr}(E^{N_{\bar\tau}}_{\mathrm{top}}) < \bar\tau$ if and only if $\mathrm{cr}(E^{N_\tau}_{\mathrm{top}}) < \tau$. It follows that none of $N_{\bar\tau}$ is pluripotent, as N_τ, being an element of $\mathcal{S}^0$, is not pluripotent. Assume for a contradiction that unboundedly many $\tau \in \hat B_\tau$ fail to be in $\mathcal{S}^0$. For each such $\bar\tau$ the structure $N_{\bar\tau}$ is exact for $\bar\tau$, which means that $\omega\varrho_{\bar\tau} > \bar\tau$. As $\sigma_{\bar\tau,\tau}$ is $\Sigma^{(n)}_0$-preserving, $\omega\varrho_\tau > \tau$ hence N_τ is exact for τ. Find a strictly increasing sequence $\langle \tau_\iota \mid \iota < \gamma\rangle$ cofinal in τ such that each N_{τ_ι} is exact for τ_ι, the sequence $\langle \mu_{\tau_\iota}\rangle_\iota$ is monotonic, which includes the possibility that it is constant, and the parameters q_{τ_ι} have the same number of elements for all $\iota < \gamma$; see the beginning of this section for the definition of μ_{τ_ι} and q_{τ_ι}. The monotonicity of μ_{τ_ι} is obtained by choosing $\tau_{\iota+1}$ with minimal possible value of $\mu_{\tau_{\iota+1}}$ at each step; more details can be found in [14], proof of Lemma 3.9. The number of elements in

q_{τ_ι} can be fixed by the pigeonhole argument. Let μ be the supremum of all μ_{τ_ι} and $q = \sigma_{\tau_\iota,\tau}(q_{\tau_\iota})$; this value obviously does not depend on ι.

For $\mu < \tau$ the proof is similar to the proof of Lemma 3.9 in [14]. We first verify that (μ, q) is a divisor for N_τ. Given $\eta < \mu$ and $i \in \omega$ such that $\zeta = \tilde{h}_\tau(i, \langle \eta, p_\tau \rangle) \leq \max(q_\tau)$, let $\iota < \gamma$ be such that $\eta < \mu_{\tau_\iota}$ and $\zeta_\iota = \tilde{h}_{\tau_\iota}(i, \langle \eta, p_{\tau_\iota} \rangle)$ is defined. This is possible, as the ordinals τ_ι are unbounded in τ. As $(\mu_{\tau_\iota}, q_{\tau_\iota})$ is a divisor for N_{τ_ι}, $\zeta_\iota < \mu_\iota$ and so $\zeta = \sigma_{\tau_\iota,\tau}(\zeta_\iota) < \mu_{\tau_\iota} \leq \mu$. This verifies (d) in Definition 3.5 of a divisor. Since $\tilde{h}_{\tau_\iota}(\mu_{\tau_\iota} \cup \{q_{\tau_\iota}\})$ is unbounded in $\omega\varrho_{\tau_\iota}$ for all $\iota < \gamma$ and $\omega\varrho_\tau$ is the union of all $\sigma_{\tau_\iota,\tau}[\omega\varrho_{\tau_\iota}]$ for $\iota < \gamma$ (this is again a consequence of the fact that τ_ι's are unbounded in τ), the hull $\tilde{h}_\tau(\mu \cup \{q\})$ is unbounded in $\omega\varrho_\tau$, which verifies (c) in the definition of a divisor, and thereby proves that (μ, q) is a divisor for N_τ. Because this divisor cannot be strong, Lemmata 3.16 and 3.17 give us an ordinal $\beta' = \max(p_{N'_\tau(\mu)}) - {\pi'_\tau}^{-1}(q)$; here we follow the notation from those lemmata, so $N'_\tau(\mu)$ is the transitive collapse of $\tilde{h}_\tau(\mu \cup \{p_\tau\})$ and π'_τ is the inverse to the collapsing map. Then $\beta = \pi'_\tau(\beta') \geq \mu$ and, letting $Q^* = \pi'_\tau(\langle W^{\beta',p'}_{N'_\tau(\mu)}, t \rangle)$ where $p' = {\pi'_\tau}^{-1}(q)$ and t is the inverse image of p' under the associated witness map, the pair Q^* is a generalized n-witness for β with respect to N_τ and q; see the text immediately preceding Lemma 3.11. Let $\eta, \xi < \mu$ and $i, j \in \omega$ be such that $Q^* = \tilde{h}_\tau(i, \langle \eta, p_\tau \rangle)$ and $\beta = \tilde{h}_\tau(j, \langle \xi, p_\tau \rangle)$. Pick $\iota < \gamma$ large enough that $\eta, \xi < \mu_{\tau_\iota}$ and both $Q^*_{\tau_\iota} = \tilde{h}_{\tau_\iota}(i, \langle \eta, p_{\tau_\iota} \rangle)$ and $\beta_{\tau_\iota} = \tilde{h}_{\tau_\iota}(j, \langle \xi, p_{\tau_\iota} \rangle$ are defined; the existence of such an ι is again guaranteed by the fact that τ_ι's are unbounded in τ. As $\sigma_{\tau_\iota,\tau}$ is $\Sigma_0^{(n)}$-preserving and the property of being a generalized n-witness is a $\Pi_1^{(n)}$-property, $Q^*_{\tau_\iota}$ is a generalized n-witness for $\beta_{\tau_\iota} = \sigma^{-1}_{\tau_\iota,\tau}(\beta)$ with respect to N_{τ_ι} and q_{τ_ι}. Moreover, $Q^*_{\tau_\iota}, \beta_{\tau_\iota} \in \tilde{h}_{\tau_\iota}(\mu_{\tau_\iota} \cup \{p_{\tau_\iota}\}) = \mathrm{rng}(\pi'_{\tau_\iota})$ by our choice of ι and $\beta_{\tau_\iota} \geq \mu \geq \mu_{\tau_\iota}$. It follows that ${\pi'_{\tau_\iota}}^{-1}(Q^*_{\tau_\iota})$ is a generalized witness for $\beta'_{\tau_\iota} = {\pi'_{\tau_\iota}}^{-1}(\beta_{\tau_\iota})$ with respect to $N'_{\tau_\iota}(\mu_{\tau_\iota})$ and ${\pi'_{\tau_\iota}}^{-1}(q_{\tau_\iota})$. Moreover $\beta'_{\tau_\iota} \geq \mu_{\tau_\iota}$, as $\beta_{\tau_\iota} \geq \mu_{\tau_\iota}$, hence ${\pi'}^{-1}_{\tau_\iota}(q_{\tau_\iota})$ is not the standard parameter of N'_{τ_ι}, but merely a proper top segment of it. By Lemma 3.16, the divisor $(\mu_{\tau_\iota}, q_{\tau_\iota})$ is not strong, which contradicts our assumption.

Now consider the situation where $\mu = \tau$. We claim that $q \neq \varnothing$. Notice that if $q = \varnothing$ then $q_{\tau_\iota} = \varnothing$ for all $\iota < \gamma$. It follows directly from definition of α_{τ_ι} that no pair (μ', p_{τ_ι}) can be a divisor for N_{τ_ι} if $\mu' > \alpha_{\tau_\iota}$, so necessarily $\mu_{\tau_\iota} \leq \alpha_{\tau_\iota} = \alpha_\tau$ for all $\iota < \gamma$. Then $\mu \leq \alpha_\tau$, a contradiction. Once we know that $q \neq \varnothing$, we can proceed similarly as in the previous paragraph. Letting $\beta = \max(q)$, the standard solidity witness $\langle W^{\beta,p_\tau}_{N_\tau}, t \rangle$ is of the form $\tilde{h}_\tau(\eta, p_\tau)$ for some $\eta < \tau$, this time by the soundness of N_τ. Choose ι large enough such that $\eta < \mu_\iota$ and $\tilde{h}_{\tau_\iota}(\eta, p_\tau)$ is defined. Similarly as above we then show that $(\mu_{\tau_\iota}, q_{\tau_\iota})$ fails to be strong. Contradiction. □(Lemma 4.9)

Corollary 4.10. *Assume $\tau \in \mathcal{S}^0$ has uncountable cofinality. Then B_τ is unbounded in τ. Consequently, B^*_τ is unbounded in τ.*

We are now ready to define the sets C^*_τ which constitute a very close approximation to our global square sequence. Each C^*_τ will be a closed subset of $\bar{B}^*_\tau$ cofinal in τ for uncountably cofinal τ, the successor points of C^*_τ will be in B^*_τ and C^*_τ will have small order type. It will soon turn out that C^*_τ is almost contained in B^*_τ whenever $\tau \in \mathcal{S}^0$ has uncountable cofinality. The definition of C^*_τ proceeds by recursion; along with C^*_τ we also define ordinals τ_ι and ξ^τ_ι by recursion on ι. In what follows we will be making use of the properties of $\bar{B}^*_\tau$ summarized in Lemma 4.7. Recall also the definitions of parameters $\bar{N}_{\bar\tau}, \bar{h}_{\bar\tau}, \bar{\varrho}_{\bar\tau}$ and $\bar{p}_{\bar\tau}$ that are stated immediately above Lemma 4.6. Recall also that if $\bar\tau \in B^*_\tau$ then $\bar{h}_{\bar\tau} = \tilde{h}_{\bar\tau}$ and the versions of the remaining parameters with "bars" agree with those without "bars", so for instance $\bar{N}_{\bar\tau} = N_{\bar\tau}$ and $\bar\varrho_{\bar\tau} = \varrho_{\bar\tau}$.

$$\begin{aligned}
\tau_0 &= \min(B^*_\tau \cup \{\tau\})\\
\xi^\tau_\iota &= \text{the least } \xi < \tau \text{ such that } h_\tau(\{\xi\} \cup \{p_\tau\}) \not\subseteq \mathrm{rng}(\bar\sigma_{\tau_\iota,\tau})\\
\tau_{\iota+1} &= \text{the least } \bar\tau \in B^*_\tau \cup \{\tau\} \text{ such that } h_\tau(\{\xi^\tau_\iota\} \cup \{p_\tau\}) \subseteq \mathrm{rng}(\sigma_{\bar\tau,\tau})\\
\tau_\iota &= \sup(\{\tau_{\bar\iota} \mid \bar\iota < \iota\}) \text{ for limit } \iota\\
\iota_\tau &= \text{the least } \iota \text{ such that } \tau_\iota = \tau
\end{aligned}$$

We let

$$C^*_\tau = \{\tau_\iota \mid \iota < \iota_\tau\}.$$

C^*_τ is obviously a closed subset of $\bar{B}^*_\tau$. An easy induction on ι using Lemma 4.7(d) yields that both $\langle \tau_\iota \mid \iota < \iota_\tau\rangle$ and $\langle \xi^\tau_\iota \mid \iota < \iota_\tau\rangle$ are strictly increasing. By Corollary 4.10 and the fact that each $h_\tau(\{\xi\} \cup \{p_\tau\})$ is countable, C^*_τ is unbounded in τ whenever τ has uncountable cofinality. We stress that successor points in C^*_τ are elements of B^*_τ, whereas the membership of limit points of C^*_τ to B^*_τ is not clear at this point. The following lemma summarizes some less obvious facts about C^*_τ.

Lemma 4.11. *The following is true of the sets C^*_τ.*

(a) *If $\bar\tau > \alpha_\tau$ is a limit point of C^*_τ then $\bar\tau \in \mathcal{S}$, $\bar{N}_{\bar\tau} = N_{\bar\tau}$, $n_{\bar\tau} = n_\tau$ and $\alpha_{\bar\tau} = \alpha_\tau$.*

(b) *If $\bar\tau \in C^*_\tau \cap B^*_\tau$ then $C^*_{\bar\tau} = C^*_\tau \cap \bar\tau$.*

(c) *$\mathrm{otp}(C^*_\tau) < \tau$ for all $\tau \in \mathcal{S}^0$.*

Proof. Regarding (a), from Lemma 4.6 (f) and (c) we know that $\bar{N}_{\bar\tau}$ is an $\mathbf{L}[E]$-level in which $\bar\tau$ is inaccessible where we allow the option $\bar\tau = \mathrm{ht}(\bar{N}_{\bar\tau})$. Once we have proved that $n_{\bar\tau} = n_\tau$, it follows from Lemma 4.6(g) that $\alpha_{\bar\tau} = \alpha_\tau$. Let $n = n_\tau$. The following simple argument shows that $\bar\tau \cap \tilde{h}^{k+1}_{\bar{N}_{\bar\tau}}(\delta \cup \{\bar{p}_{\bar\tau}\})$ is bounded in $\bar\tau$ whenever $\delta < \bar\tau$ and $k < n$. If $\tau^* \in B^*_\tau \cap \bar\tau$ is larger than δ then $\tau^* \cap \tilde{h}^{k+1}_{N_{\tau^*}}(\delta \cup \{p_{\tau^*}\})$ is bounded in τ^* by some β, so for each $i \in \omega$ the $\Pi^{(k)}_1$-statement

$$(\forall\zeta^k)(\forall x^k)[(\zeta^k = \tilde{h}^{k+1}_{N_{\tau^*}}(i, \langle x^k, p_{\tau^*}\rangle) \ \&\ x^k \in [\delta]^{<\omega} \ \&\ \zeta^k < \tau^*) \longrightarrow \zeta^k < \beta]$$

holds in N_{τ^*}. As $\bar\sigma_{\tau^*,\bar\tau}$ is $\Sigma^{(k)}_1$-preserving, it transfers this statement to $\bar{N}_{\bar\tau}$, hence $\bar\tau \cap \tilde{h}^{k+1}_{\bar{N}_{\bar\tau}}(\delta \cup \{\bar{p}_{\bar\tau}\})$ is bounded by β. In order to complete the proof of (a) it is therefore sufficient to show that $\bar\tau \cap \bar{h}_{\bar\tau}(\delta \cup \{\bar{p}_{\bar\tau}\})$ is cofinal in $\bar\tau$ for some $\delta < \bar\tau$.

We first observe that $\xi^\tau_\iota < \tau_\iota$ for every $\iota < \iota_\tau$. Otherwise we would have $\tilde{h}_\tau(i, \langle\xi, p_\tau\rangle) \in \mathrm{rng}(\bar\sigma_{\tau_\iota,\tau})$ for all $\xi < \tau_\iota$ and $i \in \omega$ hence $\tilde{h}_\tau(\tau_\iota \cup \{p_\tau\}) \subseteq \mathrm{rng}(\bar\sigma_{\tau_\iota,\tau})$ as every finite $x \subseteq \tau_\iota$ can be coded as a single ordinal $\xi < \tau_\iota$ using the Gödel pairing function due to the fact that τ_ι is primitive recursively closed. This would mean that $\tau \cap \tilde{h}_\tau(\tau_\iota \cup \{p_\tau\}) = \tau_\iota$ which contradicts the maximality of α_τ.

We next show that $\bar{h}_{\bar\tau}(\xi^\tau_{\bar\iota} \cup \{\alpha_\tau, \bar\tau\} \cup \{\bar{p}_{\bar\tau}\})$ is cofinal in $\bar\tau$ where $\bar\iota$ is such that $\bar\tau = \tau_{\bar\iota}$. An important technical tool here is the observation that if $\bar\tau \in \bar{B}^*_\tau$, $X \subseteq \bar{N}_{\bar\tau}$ and $\tilde{h}_\tau(\bar\sigma_{\bar\tau,\tau}[X] \cup \{p_\tau\}) \subseteq \mathrm{rng}(\bar\sigma_{\bar\tau,\tau})$ then

$$\tilde{h}_\tau(\bar\sigma_{\bar\tau,\tau}[X] \cup \{p_\tau\}) = \bar\sigma_{\bar\tau,\tau}[\bar{h}_{\bar\tau}(X \cup \{\bar{p}_{\bar\tau}\})]. \tag{8}$$

This holds because $\tilde{h}_\tau(\bar{\sigma}_{\bar{\tau},\tau}[X] \cup \{p_\tau\})$ is a $\Sigma_1^{(n)}$-elementary substructure of N_τ, so there are witnesses to existential quantifiers of the form $(\exists v^n)$ inside the substructure. More precisely, if $y \in \tilde{h}_\tau(\bar{\sigma}_{\bar{\tau},\tau}[X] \cup \{p_\tau\})$ then $y = \tilde{h}_\tau(i, \langle x, p_\tau \rangle)$ for some finite $x \subseteq \bar{\sigma}_{\bar{\tau},\tau}[X]$ and $i \in \omega$ hence $N_\tau \models (\exists v^n)\psi[v^n, y, i, \langle x, p_\tau \rangle]$ where the formula $(\exists v^n)\psi(v^n, u_0^0, u_1^0, u_2^0)$ constitutes a functionally absolute definition of $\tilde{h}_\tau$ and ψ is a $\Sigma_0^{(n)}$-formula. By the $\Sigma_1^{(n)}$-uniformization, see [18], Lemma 3.1.4, some witness z to the existential quantifier $(\exists v^n)$ has the form $\tilde{h}_\tau(j, \langle x, p_\tau \rangle)$ where $j \in \omega$ and therefore $z \in \operatorname{rng}(\bar{\sigma}_{\bar{\tau},\tau})$. So $N_\tau \models \psi[z, y, i, \langle x, p_\tau \rangle]$ and, letting $\bar{z}, \bar{y}, \bar{x}$ be the inverse image of z, y, x under $\bar{\sigma}_{\bar{\tau},\tau}$, we conclude that $\bar{N}_{\bar{\tau}} \models \psi[\bar{z}, \bar{y}, i, \langle \bar{x}, \bar{p}_{\bar{\tau}} \rangle]$, that is, $\bar{h}_{\bar{\tau}}(i, \langle \bar{x}, \bar{p}_{\bar{\tau}} \rangle)$ is defined, its value is $\bar{y}$ and is mapped to y by $\bar{\sigma}_{\bar{\tau},\tau}$.

As a next step we prove that for each $\iota < \iota_\tau$ such that $\tau_\iota \in B^*_\tau$,

$$(9) \qquad \tau_{\iota+1} \cap \tilde{h}_{\tau_{\iota+1}}(\{\xi_\iota^\tau, \alpha_\tau, \tau_{\iota+1}\} \cup \{p_{\tau_{\iota+1}}\}) \not\subseteq \tau^* \text{ for any } \tau^* < \tau_\iota.$$

Assume for a contradiction that this fails. Let $\bar{N}$ be the transitive collapse of $\tilde{h}_{\tau_{\iota+1}}(\{\xi_\iota^\tau, \alpha_\tau, \tau_{\iota+1}\} \cup \{p_{\tau_{\iota+1}}\})$ and $\bar{\sigma} : \bar{N} \to N_{\tau_{\iota+1}}$ be the inverse to the collapsing isomorphism. Let and $(\bar{\tau}, \bar{p}) = \bar{\sigma}^{-1}(p_{\tau_{\iota+1}}, \tau_{\iota+1})$. It follows by standard fine structural considerations that $\bar{N}$ is an acceptable J-structure for the same language as $\bar{N}_{\tau_{\iota+1}}$, the map $\bar{\sigma}$ is $\Sigma_1^{(n)}$-preserving and $\tilde{h}^{n+1}_{\bar{N}}(\bar{\tau} \cup \{\bar{p}\}) = \bar{N}$. The ordinal $\bar{\tau}$ is obviously regular in $\bar{N}$. In fact, $\tilde{h}^{k+1}_{\bar{N}}(\delta \cup \{\bar{p}\})$ is bounded in $\bar{\tau}$ whenever $\delta < \bar{\tau}$ and $k < n$, hence there is no good $\mathbf{\Sigma}_1^{(k)}(\bar{N})$-function singularizing $\bar{\tau}$. To see this notice that for $\delta < \bar{\tau}$ the boundedness of $\tau_{\iota+1} \cap \tilde{h}_{\tau_{\iota+1}}(\bar{\sigma}(\delta) \cup \{p_{\tau_{\iota+1}}\})$ in $\tau_{\iota+1}$ can be expressed as the requirement that for each $i \in \omega$ the $\Sigma_1^{(n)}$-statement

$$(\exists \beta^n)(\forall \zeta^k)(\forall x^k)(\beta^n < \tau_{\iota+1} \;\&\; \varphi(\beta^n, \zeta^k, i, x^k, p_{\tau_{\iota+1}}, \bar{\sigma}(\delta))$$

holds in $N_{\tau_{\iota+1}}$ where $\varphi(\beta^n, \zeta^k, i, x^k, p_{\tau_{\iota+1}}, \bar{\sigma}(\delta))$ is the $\Sigma_1^{(k)}$-statement

$$\Big(\zeta^k = \tilde{h}^{k+1}_{N_{\tau_{\iota+1}}}(i, \langle x^k, p_{\tau_{\iota+1}}) \;\&\; x \in [\bar{\sigma}(\delta)]^{<\omega} \;\&\; \zeta^k < \tau_{\iota+1}\Big) \;\longrightarrow\; \zeta^k < \beta^n$$

and that $\bar{\sigma}$ is sufficiently preserving to transfer these statements down to $\bar{N}$. Let $\tilde{\tau} = \sup(\bar{\sigma}[\bar{\tau}])$. Notice that $\alpha_\tau < \tilde{\tau} < \tau_\iota$ where the first inequality follows from the fact that $\alpha_\tau \in \operatorname{rng}(\bar{\sigma})$ and the second inequality is a consequence of our assumption that (9) fails. Applying the interpolation lemma to $\bar{\sigma} : \bar{N} \to N_{\tau_{\iota+1}}$ we obtain an acceptable J-structure $\tilde{N}$ for the same language as $\bar{N}$ and $N_{\tau_{\iota+1}}$, a $\Sigma_0^{(n)}$-preserving embedding $\tilde{\sigma} : \bar{N} \to \tilde{N}$ cofinal at the n-th level that extends $\bar{\sigma} \restriction \bar{\tau}$ and a $\Sigma_0^{(n)}$-preserving embedding $\sigma' : \tilde{N} \to N_{\tau_{\iota+1}}$ such that $\sigma' \restriction \tilde{\tau} = \mathrm{id}$ and $\sigma' \circ \tilde{\sigma} = \bar{\sigma}$. These preservation properties of $\tilde{\sigma}$ and σ' follow from the fact that $\tilde{h}^{n+1}_{\bar{N}}(\bar{\tau} \cup \{\bar{p}\}) = \bar{N}$; this fact moreover yields that $\tilde{h}^{n+1}_{\tilde{N}}(\tilde{\tau} \cup \{\tilde{p}\}) = \tilde{N}$ where $\tilde{p} = \tilde{\sigma}(\bar{p})$. Since there is no good $\mathbf{\Sigma}_1^{(k)}(\bar{N})$-function singularizing $\bar{\tau}$ for $k < n$ we have $\tilde{\sigma}(\bar{\tau}) = \tilde{\tau}$ hence $\sigma'(\tilde{\tau}) = \bar{\sigma}(\bar{\tau}) = \tau_\iota$; see the proof of Theorem 4.8 above for more details concerning the application of the interpolation lemma. Since $\tilde{\tau} < \tau_\iota$, we can apply Lemma 4.2 to $\tilde{N}, N_{\tau_\iota}, \sigma', \sigma_{\tau_\iota, \tau_{\iota+1}}$ and $N_{\tau_{\iota+1}}$ in place of $N^*, \bar{N}, \sigma^*, \bar{\sigma}$ and N_τ to conclude that $\operatorname{rng}(\sigma') \subseteq \operatorname{rng}(\sigma_{\tau_\iota, \tau_{\iota+1}})$. It follows that $\tilde{h}_{\tau_{\iota+1}}(\{\xi_\iota^\tau\} \cup \{p_{\tau_{\iota+1}}\}) \subseteq \operatorname{rng}(\sigma_{\tau_\iota, \tau_{\iota+1}})$ hence $\tilde{h}_\tau(\{\xi_\iota^\tau\} \cup \{p_\tau\}) = \sigma_{\tau_{\iota+1}, \tau}[\tilde{h}_{\tau_{\iota+1}}(\{\xi_\iota^\tau\} \cup \{p_{\tau_{\iota+1}}\})] \subseteq \operatorname{rng}(\sigma_{\tau_{\iota+1}, \tau} \circ \sigma_{\tau_\iota, \tau_{\iota+1}}) = \operatorname{rng}(\sigma_{\tau_\iota, \tau})$ where the equality on the left follows from (8) and the definition of $\tau_{\iota+1}$. The resulting inclusion contradicts the requirement imposed on ξ_ι^τ by definition and thereby completes the proof of (9).

We are now ready to complete the proof of (a). Let $\bar\tau = \tau_{\bar\iota}$ be as above. If $\beta < \bar\tau$, pick $\iota^* < \bar\iota$ such that $\tau_{\iota^*} \in B^*_\tau \cap \bar\tau$ and $\beta < \tau_{\iota^*}$. Then

$$\begin{aligned}\bar h_{\bar\tau}(\xi^\tau_{\bar\iota} \cup \{\alpha_\tau, \bar\tau\} \cup \{\bar p_{\bar\tau}\}) &\supseteq \bar h_{\bar\tau}(\{\xi^\tau_{\iota^*}, \alpha_\tau, \bar\tau\} \cup \{\bar p_{\bar\tau}\}) \\ &\supseteq \bar\sigma_{\tau_{\iota+1},\bar\tau}[\tilde h_{\tau_{\iota^*+1}}(\{\xi^\tau_{\iota^*}, \alpha_\tau, \tau_{\iota^*+1}\} \cup \{p_{\tau_{\iota^*+1}}\})]\end{aligned}$$

and the last set above has some elements in the interval $[\beta, \bar\tau)$ by (9). The second inclusion above follows by the fact that $\bar\sigma_{\tau_{\iota^*+1},\bar\tau}$ is $\Sigma^{(n)}_0$-preserving, whereas the first one is trivial. This completes the proof of (a).

Now turn to the proof of (b). Let $\bar\tau \in C^*_\tau \cap B^*_\tau$ and $\bar\iota < \iota_\tau$ be such that $\bar\tau = \tau_{\bar\iota}$. Then $B^*_\tau \cap \bar\tau = B^*_\tau$ by Lemma 4.5(a). To prove (b), we show by induction on $\iota < \bar\iota$ that $\bar\tau_\iota = \tau_\iota$ for each $\iota \le \bar\iota$. Since the induction at limit steps ι is trivial and $\bar\tau_0 = \tau_0$ by coherency, the proof can be obviously reduced to showing that $\xi^{\bar\tau}_\iota = \xi^\tau_\iota$ and $\bar\tau_{\iota+1} = \tau_{\iota+1}$ whenever $\iota < \bar\iota$. Without loss of generality we can assume that $\bar\tau_{\iota'} = \tau_{\iota'}$ for all $\iota' \le \iota$ and $\xi^{\bar\tau}_{\iota'} = \xi^\tau_{\iota'}$ for all $\iota' < \iota$. Since $\iota + 1 \le \bar\iota$, from (8) we obtain

$$\tilde h_\tau(\{\xi\} \cup \{p_\tau\}) = \sigma_{\bar\tau,\tau}[\tilde h_{\bar\tau}[(\{\xi\} \cup \{p_\tau\})]$$

whenever $\xi < \xi^\tau_{\bar\iota}$. It follows that $\tilde h_{\bar\tau}(\{\xi\} \cup \{p_{\bar\tau}\})] \subseteq \mathrm{rng}(\bar\sigma_{\tau_\iota,\bar\tau})$ just in case that $\tilde h_\tau(\{\xi\} \cup \{p_\tau\}) \subseteq \mathrm{rng}(\bar\sigma_{\tau_\iota,\tau})$ for each such ξ; here we apply $\sigma_{\bar\tau,\tau}$ to both sides of the first inclusion. Hence ξ^τ_ι is the least ξ with the property $\tilde h_{\bar\tau}(\{\xi\}\cup\{p_{\bar\tau}\})] \subseteq \mathrm{rng}(\bar\sigma_{\tau_\iota,\bar\tau})$, that is, $\xi^\tau_\iota = \xi^{\bar\tau}_\iota$. The same kind of argument yields that $\tau_{\iota+1}$ is the least element τ' of $B^*_{\bar\tau} = B^*_\tau \cap \bar\tau$ such that $\tilde h_{\bar\tau}(\{\xi^{\bar\tau}_\iota\} \cup \{p_{\bar\tau}\}) \subseteq \mathrm{rng}(\sigma_{\tau',\bar\tau})$, so $\bar\tau_{\iota+1} = \tau_{\iota+1}$. This completes the proof of (b).

Regarding (c), it suffices to show that the set $Z_\tau = \{\xi^\tau_\iota \mid \iota < \iota_\tau\}$ is bounded in τ. As we have seen that the assignment $\iota \mapsto \xi^\tau_\iota$ is strictly monotonic, we conclude that $\mathrm{otp}(C_\tau) = \mathrm{otp}(Z_\tau) < \tau$. Assume for the contradiction that Z_τ is unbounded in τ. As τ has uncountable cofinality, the intersection $C_\tau \cap \lim(Z_\tau)$ is nonempty, and in fact is closed unbounded in τ. Pick $\iota < \iota_\tau$ with τ_ι is in this intersection. Then $Z_\tau \cap \tau_\iota = \{\tau_{\bar\iota} \mid \bar\iota < \iota\}$ is unbounded in τ_ι by the choice of ι and $\tilde h_\tau(\{\xi\} \cup \{p_\tau\}) \subseteq \mathrm{rng}(\sigma_{\tau_\iota,\tau})$ for all $\xi < \tau_\iota$, as follows directly from the definition of τ_ι. Since τ_ι is primitive recursively closed, we conclude that $\tilde h_\tau(\tau_\iota \cup \{p_\tau\}) \subseteq \mathrm{rng}(\sigma_{\tau_\iota,\tau})$, which implies that $\tau \cap \tilde h_\tau(\tau_\iota \cup \{p_\tau\}) = \tau_\iota > \alpha_\tau$. This contradicts the properties of α_τ.

□(Lemma 4.11)

Remark 4.12. *With some extra effort, we could obtain a stronger conclusion* $[\tau_\iota, \tau_{\iota+1}) \cap \tilde h_{\tau_{\iota+1}}(\{\xi^\tau_\iota, \alpha_\tau, \tau_{\iota+1}\} \cup \{p_{\tau_{\iota+1}}\}) \neq \varnothing$ *instead of (9).*

We we are now ready to give the definition of the global sequence $\langle C_\tau \mid \tau \in \mathcal{S}^0\rangle$. By Lemma 4.6 and Lemma 4.11, if τ has uncountable cofinality then a tail-end of C^*_τ is contained in $\hat B_\tau$; see (4). By Lemma 4.9, $\hat B_\tau$ agrees with B_τ on a tail-end, so a tail-end of C^*_τ is actually contained in $B^*_\tau \supseteq B_\tau$. For each $\tau \in \mathcal{S}^0$ let γ_τ be the least element γ of $C^*_\tau \cup \{\tau\}$ such that $C^*_\tau - \gamma \subseteq B^*_\tau$. We let

$$C_\tau = C^*_\tau - \gamma_\tau. \tag{10}$$

C_τ is obviously a closed subset of τ and is unbounded in τ whenever τ has uncountable cofinality, as follows immediately from the definition of C_τ. It is also obvious that $\mathrm{otp}(C_\tau) \le \mathrm{otp}(C^*_\tau) < \tau$, as follows from Lemma 4.11(c). Regarding the coherency, if $\bar\tau \in C_\tau$ then $\bar\tau \in C^*_\tau \cap B^*_\tau$ hence $C^*_{\bar\tau} = C^*_\tau \cap \bar\tau$ by Lemma 4.11(b), so $\gamma_{\bar\tau} = \gamma_\tau$. It follows that $C_{\bar\tau} = C^*_{\bar\tau} - \gamma_{\bar\tau} = (C^*_\tau \cap \bar\tau) - \gamma_\tau = (C^*_\tau - \gamma_\tau) \cap \bar\tau = C_\tau \cap \bar\tau$. This completes the construction of the global square sequence on $\mathcal{S}^0$.

Remark 4.13. *As mentioned immediately above Definition 4.1, there is a way of defining the sets* B_τ *so that these sets will be closed on a tail-end. Together with* α_τ *in Definition 4.1 we consider the monotonic sequence of ordinals* $\langle\alpha_\tau^i \mid i<\omega\rangle$ *defined inductively by equalities* $\alpha_\tau^0=\alpha_\tau+1$ *and* $\alpha_\tau^{i+1}=\sup(\tau\cap\tilde h_\tau(\alpha_\tau^i\cup\{p_\tau\})$. *It is not hard to see that there is some* $k_\tau\in\omega$ *such that* $\langle\alpha_\tau^i \mid i<k_\tau+1\rangle$ *is strictly increasing and* $\alpha_\tau^i=\tau$ *for all* $i>k_\tau$. *The amendment to the definition of* B_τ *consists in adding requirements to Definition 4.3 that* $k_{\bar\tau}=k_\tau$ *and* $\alpha_{\bar\tau}^{k_\tau}=\alpha_\tau^{k_\tau}$. *This approach to the construction makes certain parts of the proof simpler, but other parts more complicated. In general, the two approaches yield constructions of roughly the same level of complexity.*

We now proceed with the construction of C_τ for $\tau\in\mathcal{S}^1$. Recall the basic notation for objects associated with such τ introduced at the beginning of this section. Since N_τ is a singularizing structure for τ that is exact for τ, Lemma 3.7 guarantees that M_τ is a singularizing structure for τ. Also, $q_\tau=p_{M_\tau}$ if M_τ is a protomouse, as follows from Lemma 3.11, and $q_\tau=d_\tau$ if M_τ is a premouse. Of course, if M_τ is a premouse then M_τ is pluripotent. In either case, M_τ is sound and solid with respect to the language for coherent structures; if M_τ is a premouse, this is equivalent to saying that M_τ is Dodd sound and Dodd solid.

Certain aspects of the construction become simpler for $\tau\in\mathcal{S}^1$. One such aspect involves ordinals α_τ which we used to show that the embeddings between singularizing structures are not cofinal at the level n_τ and which played an important role in the proof that C_τ is closed and has small order type. Recall the definition of Skolem functions $h_\tau^{(\mu_\tau,q_\tau)}$ from the beginning of this section.

Lemma 4.14. *Let* $\tau\in\mathcal{S}^1$ *and* $\alpha<\tau$ *be such that* $\tau\cap h_\tau(\alpha\cup\{q_\tau\})=\alpha$. *Then* $\alpha=0$.

Proof. We first observe that if α satisfies the assumptions of the lemma and $0<\alpha$ then $\tau\cap\tilde h_\tau(\alpha\cup\{p_\tau\})=\alpha$. This is clear if $M_\tau=N_\tau$, so let us focus on the case where M_τ is a protomouse. Notice that if $0<\alpha$ then α is a limit cardinal and $\mu_\tau<\alpha$ so actually $\vartheta_\tau<\alpha$. Given $x\in[\alpha]^{<\omega}$ and $i\in\omega$, assume $\zeta=\tilde h_\tau(i,\langle x,p_\tau\rangle)<\tau$. Then $\{\zeta\}$ is a singleton that is $\Sigma_1^{(n_\tau)}$-definable[3] over N_τ in the parameters x,i and p_τ, so by Lemma 3.7(a) $\{\zeta\}$ is Σ_1-definable over M_τ in the parameters x,i,ϑ_τ and q_τ. It follows that $\zeta=h_\tau(j,\langle x\cup\{\vartheta_\tau\},q\rangle)$ for some $j\in\omega$. Then $\zeta<\alpha$, as $\vartheta_\tau<\alpha$.

We next show that $\alpha_\tau\le\mu_\tau$. If $\omega\varrho_\tau\cap\tilde h_\tau(\alpha_\tau\cup\{p_\tau\})$ is cofinal in $\omega\varrho_\tau$ then $(\alpha_\tau,\varnothing)$ is a divisor for τ. Because the second component in this divisor is $\varnothing$, this divisor is strong, so by maximality of α_τ we have that $(\alpha_\tau,\varnothing)=(\mu_\tau,q_\tau)$ hence $\alpha_\tau=\mu_\tau$. If $\omega\varrho_\tau\cap\tilde h_\tau(\alpha_\tau\cup\{p_\tau\})$ is bounded in $\omega\varrho_\tau$ then $\mu_\tau>\alpha_\tau$ because of (c) in Definition 3.5.

Now assume that $\alpha>0$ satisfies the assumptions of the lemma. By the above results $\alpha_\tau\le\mu_\tau<\alpha$ and $\tau\cap\tilde h_\tau(\alpha\cup\{p_\tau\})=\alpha$, which is impossible. It follows that $\alpha=0$. $\square$(Lemma 4.14)

Following the standard strategy, we next define the sets B_τ for $\tau\in\mathcal{S}^1$.

Definition 4.15. *Let* $\tau\in\mathcal{S}^1$. *The set* B_τ *consists of all* $\bar\tau\in\tau\cap\mathcal{S}^1$ *for which there is a map* $\sigma: M_{\bar\tau}\to M_\tau$ *that is* Σ_0*-preserving with respect to the language of coherent structures and such that:*

(a) $\sigma\restriction\bar\tau=\mathrm{id}\restriction\bar\tau$ *and* $\sigma(\bar\tau)=\tau$.

[3] Recall that N_τ is exact for τ

(b) $\sigma(q_{\bar\tau}) = q_\tau$.
(c) *If* $\beta \in q_\tau$ *then there is a generalized witness* $Q^*_\tau(\beta) = \langle Q_\tau(\beta), t_\tau(\beta)\rangle$ *for* β *with respect to* M_τ *and* q_τ *satisfying* $Q^*_\tau(\beta) \in \mathrm{rng}(\sigma)$.

This definition obviously corresponds to (4) in Definition 4.3. Clauses (1) and (2) do not apply here for obvious reasons and (3) can be omitted because of Lemma 4.14. Since $\tau \in M_\tau$ whenever $\tau \in \mathcal{S}^1$, the second part in (a) always makes sense. The map σ in the above definition is unique and we denote it by $\sigma^*_{\bar\tau,\tau}$, as "$\sigma_{\bar\tau,\tau}$" was already reserved. The following lemma summarizes basic properties of B_τ and is an obvious analogue to Lemma 4.4.

Lemma 4.16. *Let* $\tau^* < \bar\tau$ *be two elements of* B_τ.

(a) $\sigma^*_{\bar\tau,\tau}$ *is not* Σ_1*-preserving, and therefore is not cofinal.*
(b) $\mathrm{rng}(\sigma^*_{\tau^*,\tau}) \subseteq \mathrm{rng}(\sigma^*_{\bar\tau,\tau})$ *and* $\sup(\mathbf{On} \cap \mathrm{rng}(\sigma^*_{\tau^*,\tau}) < \sup(\mathbf{On} \cap \mathrm{rng}(\sigma^*_{\bar\tau,\tau}))$.
(c) $(\sigma^*_{\bar\tau,\tau})^{-1} \circ \sigma^*_{\tau^*,\tau} = \sigma^*_{\tau^*,\bar\tau}$, *so* $\sigma^*_{\tau^*,\bar\tau}$ *exists.*
(d) $\sigma^*_{\tau^*,\bar\tau} : M_{\tau^*} \to M_{\bar\tau}$ *is not cofinal.*
(e) $\sigma^*_{\tau^*,\bar\tau}$ *witnesses Definition 4.15 with* $(\tau^*, \bar\tau)$ *in place of* $(\bar\tau, \tau)$.
(f) $B_\tau \cap \bar\tau = B_{\bar\tau} - \min(B_\tau)$.

Proof. The proof is virtually identical with the proof of Lemma 4.2. One again uses Lemma 4.2 and Lemma 4.14 to prove (a) and (b). □(Lemma 4.16)

We define the set B^*_τ and maps $\sigma^*_{\tau^*,\bar\tau} : M_{\tau^*} \to M_{\bar\tau}$ for $\tau^* \leq \bar\tau$ from $B^*_{\bar\tau} \cup \{\tau\}$ as before.

Lemma 4.17. *Let* $\bar\tau \in B^*_\tau \cup \{\tau\}$, $\tau' < \tau^*$ *be elements of* $B^*_\tau \cap \bar\tau$ *and* $n = n_\tau$. *Then:*

(a) $B^*_{\bar\tau} = B^*_\tau \cap \bar\tau$.
(b) $\sigma^*_{\tau^*,\bar\tau}$ *exists, is* Σ_0*-preserving,* $\sigma^*_{\tau^*,\bar\tau} \restriction \tau^* = \mathrm{id}$ *and* $\sigma^*_{\tau^*,\bar\tau}(\tau^*, q_{\tau^*}) = (\bar\tau, q_{\bar\tau})$.
(c) $\sigma^*_{\tau^*,\bar\tau}$ *is not* Σ_1*-preserving, and therefore is not cofinal.*
(d) $\mathrm{rng}(\sigma^*_{\tau',\bar\tau}) \subseteq \mathrm{rng}(\sigma^*_{\tau^*,\bar\tau})$ *and* $\sup(\mathbf{On} \cap \mathrm{rng}(\sigma^*_{\tau',\bar\tau})) < \sup(\mathbf{On} \cap \mathrm{rng}(\sigma^*_{\tau^*,\bar\tau}))$.
(e) $\sigma^*_{\tau^*,\bar\tau} \circ \sigma^*_{\tau',\tau^*} = \sigma^*_{\tau',\bar\tau}$.

Proof. Virtually identical to the proof of Lemma 4.5. Note also that (c) can be viewed as a consequence of Lemma 4.14. □(Lemma 4.17)

Notice that $\mu_{\bar\tau} = \mu_\tau$ and $|\,q_{\bar\tau}\,| = |\,q_\tau\,|$ for all $\bar\tau \in B^*_\tau$. Clause (d) in Lemma 4.17 yields the following monotonicity property of the sequence $\langle \vartheta_{\bar\tau} \mid \bar\tau \in B^*_\tau \cup \{\tau\}\rangle$:

$$\tau^* < \bar\tau \;\longrightarrow\; \vartheta_{\tau^*} < \vartheta_{\bar\tau}. \tag{11}$$

This immediately implies that $\mathrm{otp}(B^*_\tau) \leq \vartheta_\tau \leq \mu^+_\tau < \tau$. Unlike the situation in the case where $\tau \in \mathcal{S}^0$ we will be further able to prove that B^*_τ is closed on a tail-end, which will make it possible to define C_τ to be the canonical tail-end of B^*_τ and thereby simplify the construction. There are, of course, new complications in the present situation related to issues with canonical divisors. Similarly as in the previous case we introduce the sets $\hat{B}\tau$.

$\hat{B}_\tau$ is the set of all ordinals $\bar\tau \in \tau \cap \mathcal{S}^1$ satisfying the following.

(a) $N_{\bar\tau}$ has a strong divisor $(\mu_\tau, q_{\bar\tau,\tau})$ such that, letting $M_{\bar\tau,\tau} = N_{\bar\tau}(\mu_\tau, q_{\bar\tau,\tau})$, there is a Σ_0-preserving map $\hat\sigma_{\bar\tau,\tau} : N_{\bar\tau,\tau} \to M_\tau$ satisfying the requirements from Definition 4.15 with $q_{\bar\tau,\tau}$ in place of $q_{\bar\tau}$.
(b) 0 is the only ordinal $\alpha < \bar\tau$ is such that $h_{M_{\bar\tau,\tau}}(\alpha \cup \{q_{\bar\tau,\tau}\}) = \alpha$.

Obviously $B_\tau \subseteq \hat{B}_\tau$. If $\bar\tau \in B_\tau$ then $q_{\bar\tau,\tau} = q_{\bar\tau}$. But $\hat{B}_\tau$ is also allowed to contain elements $\bar\tau$ for which some protomouse associated with $N_{\bar\tau}$, that is not the canonically chosen one, embeds into M_τ. The definition of $\hat{B}_\tau$ makes use of the fact that these protomice are uniquely determined by τ and $\bar\tau$. Our aim is to show that on a tail-end of $\hat{B}_\tau$, these protomice agree with the canonical ones. Notice that $|\, q_{\bar\tau,\tau} \,| = |\, q_\tau \,|$ for all $\bar\tau \in \hat{B}_\tau$.

Lemma 4.18. *If $\tau \in \mathcal{S}^1$ has uncountable cofinality then $\hat{B}_\tau$ is unbounded in τ.*

Proof. We combine arguments from the proof of Lemma 3.10 in [14] and that of Lemma 4.11 above. Given some $\tau' < \tau$, we are again looking for some $\tilde\tau \in \hat{B}_\tau$ such that $\tau' \le \tilde\tau$. By taking elementary hulls and collapsing, we find a countable coherent structure $\bar M$, an elementary map $\sigma : \bar M \to M_\tau$ that has τ', τ, q_τ and some generalized solidity witnesses $Q^*_\tau(\beta)$ for each $\beta \in q_\tau$ in its range. We then let $\bar\tau$ be the σ-preimage of τ and $\tilde\tau = \sup(\tau \cap \sigma''\bar\tau)$; as before $\tilde\tau < \tau$ since τ is not ω-cofinal. Also, $\tilde\tau$ is a limit cardinal in the sense of $\mathbf{L}[E]$. Our aim is to show that $\tilde\tau \in \hat{B}_\tau$.

The interpolation lemma gives us an acceptable structure $\tilde M$ together with Σ_0-preserving maps $\tilde\sigma : \bar M \to \tilde M$ and $\sigma' : \tilde M \to M_\tau$ such that $\sigma' \circ \tilde\sigma = \sigma$. The map $\tilde\sigma$, being a fine pseudoultrapower map, is cofinal, $\sigma' \restriction \tilde\tau = \mathrm{id} \restriction \tilde\tau$ and $\tilde\sigma(\bar\tau) = \tau$. Since $q_\tau \in R_{M_\tau}$ and $\tau = \omega\varrho^1_{M_\tau}$, (all fine structure here is done with respect to the language of coherent structures), the elementarity of σ guarantees that $\bar q \in R_{\bar M}$ and $\omega\varrho^1_{\bar M} = \bar\tau$. The fact that $\tilde\sigma$ is a pseudoultrapower map then yields $\tilde q \stackrel{\text{def}}{=} \tilde\sigma(\bar q) \in R_{\tilde M}$ and $\omega\varrho^1_{\tilde M} = \tilde\tau$. Finally, since $\mathrm{rng}(\sigma) \subseteq \mathrm{rng}(\sigma')$ and $\sigma'(\tilde q) = q_\tau$, the generalized witness $Q^*_\tau(\sigma'(\beta))$ is in the range of σ' for each $\beta \in \tilde q$, and since σ' is Σ_0-preserving, its σ'-preimage is a generalized witness for β with respect to $\tilde M$ and $\tilde q$. By Corollary 1.12.4 in [18], $\tilde q = p_{\tilde M}$, and $\tilde M$ is sound and solid. The fact that $\tilde\sigma$ is cofinal also yields that $\tilde M$ is a coherent structure with $\vartheta_{\tilde M} = \sup(\vartheta_\tau \cap \tilde\sigma[\vartheta_{\bar\tau}])$. The map σ' is Σ_0-preserving but not cofinal, as otherwise $\tilde\tau = \tau \cap h_\tau(\tilde\tau \cup \{p_\tau\})$ which would contradict Lemma 4.14. This verifies the assumptions of the condensation lemma for protomice, Lemma 3.14. We conclude that $\tilde N = N(\tilde M)$ is a proper level of M hence a level of $\mathbf{L}[E]$ and $(\mu, \tilde q)$ is a divisor for $\tilde N$ such that $\tilde M = \tilde N(\mu, \tilde q)$. Since M_τ is a singularizing structure for τ there is some $\delta < \tau$ such that $\tau \cap h_\tau(\delta \cup \{q_\tau\})$ is cofinal in τ. Then $\bar\tau \cap h_{\bar M}(\bar\delta \cup \{\bar q\})$ is cofinal in $\bar\tau$ where $\bar q = \sigma^{-1}(q_\tau)$ and $\bar\delta = \sigma^{-1}[\delta]$. As $\tilde\sigma$ maps $\bar\tau$ cofinally into $\tilde\tau$, we conclude that $\tilde\delta \cap h_{\tilde M}(\delta \cup \{\tilde q\})$ is cofinal in $\tilde\tau$, so $\tilde M$ is a singularizing structure for $\tilde\tau$ with $n(\tilde\tau, \tilde M) = 0$. By the second conclusion in Lemma 3.14 and the above, $\tilde N = N_{\tilde\tau}$ and is exact for $\tilde\tau$.

We next verify that $(\mu_\tau, \tilde q)$ is a strong divisor for $N_{\tilde\tau}$. This can be done using an argument similar to that in the proof of Lemma 3.10 in [14]; however, we present a different argument that avoids the use of Fodor's pressing down lemma. Recall the notion of a closed ordinal from Definition 3.18. The goal is to show:

(12) There are cofinally many $\vartheta^* < \vartheta_\tau$ that are closed in M_τ relative to q_τ.

From (12) we conclude that there are cofinally many ordinals $\vartheta^* < \vartheta_{\bar M}$ that are closed in $\bar M$ relative to $\bar q$. Since $\tilde\sigma$ maps $\vartheta_{\bar M}$ cofinally into $\vartheta_{\tilde M}$, there are cofinally many ordinals $\vartheta^* < \vartheta_{\tilde M}$ that are closed in $\tilde M$ relative to $\tilde q$. But then $\vartheta_{\tilde M}$ itself is closed in $\tilde M$ relative to $\tilde q$, as in any coherent structure M, any limit of ordinals that are closed in M relative to a fixed parameter is itself closed in M relative to the same parameter. By Lemma 3.19, $(\mu_\tau, \tilde q)$ is strong as desired.

To see (12), fix arbitrary $\vartheta_0 < \vartheta_\tau$ larger than μ_τ. Since μ_τ is the largest cardinal in $J^E_{\vartheta_\tau}$, there is a function $g \in J^E_{\vartheta_\tau}$ that maps μ_τ surjectively onto ${}^{\mu_\tau}\mathcal{P}(\mu_\tau) \cap J^E_{\vartheta_0}$. This function can be viewed as a function $g^* : \mu_\tau \to \mathcal{P}(\mu_\tau)$ where

$$g^*(\zeta) = \{\prec\eta, \eta', \eta''\succ \mid \eta'' \in g(\eta)(\prec\zeta, \eta'\succ)\}.$$

Now if $\xi, \xi' < \mu_\tau$ then

$$\begin{aligned} F_\tau(g(\xi))(q_\tau, \xi') \cap \mu_\tau &= \pi_{M_\tau}(g(\xi))(q_\tau, \xi') \cap \mu_\tau \\ &= \pi_{M_\tau}(g)(\xi)(q_\tau, \xi') \cap \mu_\tau \\ &= \{\xi'' < \mu_\tau \mid \prec\xi, \xi', \xi''\succ \in \pi_{M_\tau}(g^*)(q_\tau)\} \\ &= \{\eta'' < \mu_\tau \mid \prec\xi, \xi', \eta''\succ \in F_\tau(g^*)(q_\tau)\}, \end{aligned}$$

and it is clear that $F(g^*)(q_\tau)$ can be replaced by $a^* = F(g^*)(q_\tau) \cap \mu_\tau$ on the bottom line above. Since ϑ_τ is closed in M_τ relative to q_τ the set a^* is an element of $J^E_{\vartheta_\tau}$, so we can pick some $\vartheta_1 < \vartheta_\tau$ larger than ϑ_0 such that $a^* \in J^E_{\vartheta_1}$. Then $F(f)(q_\tau, \xi') \cap \mu_\tau \in J^E_{\vartheta_1}$ whenever $f : \mu_\tau \to \mathcal{P}(\mu_\tau)$ is in $J^E_{\vartheta_0}$ and $\xi' < \mu_\tau$, as $F(f)(q_\tau, \xi') \cap \mu_\tau = \{\xi'' < \mu_\tau \mid \prec\xi, \xi', \xi''\succ \in a^*\}$ where ξ is such that $f = g(\xi)$. Repeating this argument, we inductively construct a strictly increasing sequence $\langle\vartheta_i \mid i \in \omega\rangle$ of ordinals below ϑ_τ such that for every $f : \mu_\tau \to \mathcal{P}(\mu_\tau)$ in $J^E_{\vartheta_i}$ and $\xi' < \mu_\tau$ the set $F(f)(q_\tau, \xi') \cap \mu_\tau$ is in $J^E_{\vartheta_{i+1}}$. Let $\vartheta^* = \sup(\{\vartheta_i \mid i \in \omega\})$. Then $\vartheta^* < \vartheta_\tau$ since ϑ_τ has uncountable cofinality. By construction, ϑ^* is closed in M_τ relative to q_τ.

It remains to prove that $\tilde{\tau}$ satisfies (b) in the definition of $\hat{B}_\tau$. Assume $0 < \alpha < \tilde{\tau}$. Let $\bar{\alpha} = \sigma^{-1}[\alpha]$. By Lemma 4.14, there is some finite $x \subseteq \alpha$ and some $i \in \omega$ such that $\alpha \leq h_\tau(i, \langle x, q_\tau\rangle) < \tau$ and since σ is fully elementary, there is some finite $\bar{x} \subseteq \bar{\alpha}$ satisfying $\bar{\alpha} \leq h_{\bar{M}}(i, \langle\bar{x}, \bar{q}\rangle) < \bar{\tau}$. Since $\tilde{\sigma} \restriction \bar{\tau} = \sigma \restriction \bar{\tau}$, we conclude that $\alpha \leq \tilde{\sigma}(\bar{\alpha}) \leq h_{\tilde{M}}(i, \langle\tilde{\sigma}(\bar{x}), \tilde{q}\rangle) < \tilde{\tau}$. Obviously $\tilde{\sigma}(\bar{x}) = \sigma(\bar{x}) \subseteq \alpha$. □(Lemma 4.18)

Lemma 4.19. *Assume $\tau \in \mathcal{S}^1$. Then $\hat{B}_\tau$ is closed.*

Proof. We assume without loss of generality that $\hat{B}_\tau$ is unbounded in τ. Given a limit point $\tilde{\tau} < \tau$ of $\hat{B}$, we prove that $\tilde{\tau} \in \hat{B}_\tau$. Following the strategy from the case where $\tau \in \mathcal{S}^0$, we form the direct limit of the diagram

$$\langle M_{\bar{\tau},\tau}, \sigma^*_{\bar{\tau},\bar{\tau}'} \mid \bar{\tau} < \bar{\tau}' \;\&\; \bar{\tau}, \bar{\tau}' \in \hat{B}_\tau \cap \tilde{\tau}\rangle.$$

We obtain the transitivized direct limit structure $\tilde{M}$ together with the direct limit maps $\tilde{\sigma}_{\bar{\tau}} : M_{\bar{\tau},\tau} \to \tilde{M}$ and the canonical embedding $\tilde{\sigma} : \tilde{M} \to M_\tau$ such that $\tilde{\sigma} \circ \tilde{\sigma}_{\bar{\tau}} = \sigma^*_{\bar{\tau},\tau}$ for all $\bar{\tau} \in \hat{B}_\tau \cap \tilde{\tau}$. All these maps are Σ_0-preserving, so $\tilde{M}$ is a coherent structure, as this is a Π_2-property that is true about all structures in the direct limit diagram. Also $\tilde{\sigma}_{\bar{\tau}} \restriction \bar{\tau} = \mathrm{id} \restriction \bar{\tau}$, $\tilde{\sigma}_{\bar{\tau}}(\bar{\tau}) = \tilde{\tau}$, $\tilde{\sigma} \restriction \tilde{\tau} = \mathrm{id} \restriction \tilde{\tau}$ and $\tilde{\sigma}(\tilde{\tau}) = \tau$. We let $\tilde{q} = \tilde{\sigma}_{\bar{\tau}}(q_{\bar{\tau},\tau})$; this object is again independent of the choice of $\bar{\tau} \in \hat{B}_\tau \cap \tilde{\tau}$. Our intention is to show that $\tilde{\tau}$ is a singular cardinal in $\mathbf{L}[E]$, $\tilde{M} = N_{\tilde{\tau}}(\mu_\tau, \tilde{q})$ and $\tilde{\sigma}$ witnesses that $\tilde{\tau} \in \hat{B}_\tau$.

Since $M_{\bar{\tau},\tau}$ is sound whenever $\bar{\tau} \in \hat{B}_\tau \cap \tilde{\tau}$, the Σ_0-elementarity of the maps $\tilde{\sigma}_\tau$ guarantees that $h_{\tilde{M}}(\tilde{\tau} \cup \{\tilde{q}\}) = \tilde{M}$, and so $\omega\varrho^1_{\tilde{M}} = \tilde{\tau}$ and $\tilde{q} \in R_{\tilde{M}}$. Since $\mathrm{rng}(\tilde{\sigma}) \supseteq \mathrm{rng}(\hat{\sigma}_{\bar{\tau},\tau})$ and $\mathrm{rng}(\hat{\sigma}_{\bar{\tau},\tau})$ contains a generalized witness $Q^*(\beta)$ for β with respect to M_τ and q_τ whenever $\beta \in q_\tau$, any such witness is in the range $\tilde{\sigma}$. By the Σ_0-elementarity of $\tilde{\sigma}$, the object $\tilde{\sigma}^{-1}(Q^*(\tilde{\sigma}(\tilde{\beta})))$ is a generalized witness for $\tilde{\beta}$ whenever $\tilde{\beta} \in \tilde{q}$. By Corollary 1.12.4 in [18], $\tilde{q} = p_{\tilde{M}}$ and the structure $\tilde{M}$ is solid and sound. Finally, the map $\tilde{\sigma}$ is not Σ_1-preserving, and therefore not cofinal. Otherwise $h_\tau(\tilde{\tau} \cup \{q_\tau\}) = \tilde{\sigma}[h_{\tilde{M}}(\tilde{\tau} \cup \{\tilde{q}\})]$, so $\tau \cap h_\tau(\tilde{\tau} \cup \{q_\tau\}) = \tilde{\tau}$, contradicting Lemma 4.14. This verifies the assumptions of Lemma 3.14. It follows that $\tilde{N} = N(\tilde{M})$ is a level of N_τ hence a level of $\mathbf{L}[E]$ and $\tilde{M} = \tilde{N}(\mu_\tau, \tilde{q})$.

As a next step we prove that $\tilde{M}$ is a singularizing structure for $\tilde{\tau}$ with $n(\tilde{\tau}, \tilde{M}) = 0$. Then again by Lemma 3.14, $\tilde{N} = N_{\tilde{\tau}}$ and is exact for $\tilde{\tau}$. Let $\bar{\tau} \in \hat{B}_\tau \cap \tilde{\tau}$, $\zeta_{\bar{\tau}} = \sup(\tilde{\sigma}_{\bar{\tau}}[\mathbf{On} \cap M_{\bar{\tau},\tau}])$ and $\tilde{M}_{\bar{\tau}} = \langle S^E_{\zeta_{\bar{\tau}}}, \tilde{F} \cap S^E_{\zeta_{\bar{\tau}}} \rangle$ where $\tilde{F}$ is the top extender of $\tilde{M}$. Then $\tilde{\sigma}_{\bar{\tau}}$ when viewed as a map from $M_{\bar{\tau},\tau}$ to $\tilde{M}_{\bar{\tau}}$ is cofinal hence Σ_1-preserving. By Lemma 4.14, $\bar{\tau}$ is the least ordinal τ' smaller than $\tilde{\tau}$ satisfying the condition $\tilde{\tau} \cap h_{\tilde{M}_{\bar{\tau}}}(\tau' \cup \{\tilde{q}\}) = \tau'$. This observation enables us to show that the map $\vartheta_{\bar{\tau}} \mapsto \bar{\tau}$ is a singularizing function for $\tilde{\tau}$ that is Σ_1-definable over $\tilde{M}$. Notice that $\zeta_{\bar{\tau}}$ is fully determined by $\vartheta_{\bar{\tau}}$ as the least ordinal ζ such that $\tilde{F}(x) \in S^E_\zeta$ for all $x \in \mathcal{P}(\mu_\tau) \cap J^E_{\vartheta_{\bar{\tau}}}$. Thus, letting $\psi(w, u, v)$ be the Σ_1-formula that constitutes a functionally absolute definition of a Σ_1-Skolem function, the map $\vartheta_{\bar{\tau}} \mapsto \bar{\tau}$ can be defined as follows: $\xi = \bar{\tau}$ just in case that $\tilde{M}$ satisfies the Σ_1-statement

$(\exists \zeta, X, G, M, h)$
- $X = S^E_\zeta \ \&\ G = F \cap X \ \&\ M = \langle X, G \rangle \ \&\ \xi < \tilde{\tau}$
- $(\forall \bar{\zeta} < \zeta)(\forall \bar{X} \in X)(\exists x \in X)(\exists y \in X)(\langle x, y \rangle \in G \ \&\ y \notin \bar{X})$
- $h = \{\langle y, i, x \rangle \in X \times \omega \times X \mid M \models \psi[y, i, x]\}$
- $(\forall x \in [\xi]^{<\omega})(\forall i \in \omega)(\forall \eta < \tilde{\tau})\eta = h(i, \langle x, \tilde{q} \rangle) \to \eta < \xi)$
- $(\forall \xi' < \xi)(\exists x' \in [\xi']^{<\omega})(\exists i' \in \omega)(\exists \eta' < \tilde{\tau})[\eta' = h(i', \langle x', \tilde{q} \rangle) \ \&\ \eta \geq \xi')]$.

We now finish the proof that $\tilde{\tau}$ satisfies (a) in the definition of $\hat{B}_\tau$ and $\tilde{\sigma} = \sigma_{\tilde{\tau},\tau}$ it remains to verify that $(\mu_\tau, \tilde{q})$ is a strong divisor for $N_{\tilde{\tau}}$. By Lemma 3.19, this is equivalent to showing that $\vartheta_{\tilde{M}}$ is closed in $\tilde{M}$ relative to $\tilde{q}$. We first observe that $\vartheta_{N_{\bar{\tau},\tau}}$ is closed in $\tilde{M}$ relative to $\tilde{q}$ whenever $\bar{\tau} \in \hat{B}_\tau \cap \tilde{\tau}$. Pick a map $f : \mu_\tau \to \mathcal{P}(\mu_\tau)$ from $J^E_{\vartheta_{N_{\bar{\tau},\tau}}}$ and an ordinal $\xi < \mu_\tau$. Letting $\tilde{F}$ and $F_{\bar{\tau},\tau}$ be the top extender of $\tilde{M}$ and $M_{\bar{\tau},\tau}$ respectively, the facts that $\tilde{\sigma}_{\bar{\tau}}$ is Σ_0-preserving and $\mu_\tau < \mathrm{cr}(\tilde{\sigma}_{\bar{\tau}})$ guarantee that $\tilde{F}(f)(\tilde{q}, \xi) \cap \mu_\tau = F_{\bar{\tau},\tau}(f)(q_{\bar{\tau},\tau}, \xi) \cap \mu_\tau$. The set on the right side of this equality is an element of $J^E_{\vartheta_{N_{\bar{\tau},\tau}}}$, as $(\mu_\tau, q_{\bar{\tau},\tau})$ is a strong divisor for $N_{\bar{\tau},\tau}$. Finally since the property of being strong relative to a fixed parameter is closed under limits and $\vartheta_{\tilde{M}} = \sup\{\vartheta_{N_{\bar{\tau},\tau}} \mid \bar{\tau} \in \hat{B}_\tau \cap \tilde{\tau}\}$, we conclude that $\vartheta_{\tilde{M}}$ is closed relative to $\tilde{q}$ in $\tilde{M}$.

To complete the proof that $\tilde{\tau} \in \hat{B}_\tau$ we have to verify that $\tilde{M}$ and $\tilde{q}$ satisfy (b) in the definition of $\hat{B}_\tau$. Assume there is an ordinal α such that $0 < \alpha < \tilde{\tau}$ and $\tilde{\tau} \cap h_{\tilde{M}}(\alpha \cup \{\tilde{q}\}) = \alpha$. Let $\bar{\tau} \in \hat{B}_\tau \cap \tilde{\tau}$ be such that $\alpha < \bar{\tau}$. Pick a finite $x \subseteq \alpha$ and an $i \in \omega$. If $\zeta = h_{M_{\bar{\tau},\tau}}(i, \langle x, q_{\bar{\tau},\tau} \rangle)$ is defined and $\zeta < \bar{\tau}$ then using the fact that $\tilde{\sigma}_{\bar{\tau}}$ is Σ_0-preserving we obtain $h_{\tilde{M}}(i, \langle x, \tilde{q} \rangle) = \zeta < \tilde{\tau}$. By our assumption on α then $\zeta < \alpha$. This means that $\bar{\tau} \cap h_{M_{\bar{\tau},\tau}}(\alpha \cup \{q_{\bar{\tau},\tau}\}) = \alpha$, which contradicts the fact that $\bar{\tau} \in \hat{B}_\tau$. $\square$(Lemma 4.19)

The last essential lemma in the treatment of the case $\tau \in \mathcal{S}^1$ is an analogue to Lemma 4.9.

Lemma 4.20. *Assume $\tau \in \mathcal{S}^1$ and $\hat{B}_\tau$ is unbounded in τ. Then $\hat{B}_\tau$ agrees with B_τ on a tail-end. That is, there is some $\hat{\tau} < \tau$ such that $\hat{B}_\tau - \hat{\tau} = B_\tau - \hat{\tau}$.*

Proof. We have to show that $(\mu_\tau, q_{\bar{\tau},\tau}) = (\mu_{\bar{\tau}}, q_{\bar{\tau}})$ for all $\bar{\tau}$ in some tail-end of $\hat{B}_\tau$. So suppose for a contradiction that this fails. Then for cofinally many $\bar{\tau} \in \hat{B}_\tau$, the strong divisor $(\mu_\tau, q_{\bar\tau,\tau})$ fails to be the canonical one. Letting $\hat{B}'_\tau$ be the set of all such $\bar\tau$, it follows from Lemma 3.21 that $\mu_\tau < \mu_{\bar\tau}$ and $q_{\bar\tau}$ is a (not necessarily proper) bottom segment of $q_{\bar\tau,\tau}$ whenever $\bar\tau \in \hat{B}'_\tau$. As in the proof of Lemma 4.9, we find a strictly monotonic sequence $\langle \tau_\iota \mid \iota < \gamma\rangle$ cofinal in τ such that the ordinals τ_ι are all in $\hat{B}'_\tau$, the sequence $\langle \mu_{\tau_\iota} \mid \iota < \gamma\rangle$ is (not necessarily strictly) monotonic and q_{τ_ι} are of the same size for all $\iota < \gamma$; say this size is m. Let $\mu = \sup\{\mu_\iota \mid \iota < \gamma\}$ and q be the bottom segment of q_τ of size m. Obviously $\mu_\tau < \mu \le \tau$ and $\hat\sigma_{\tau_\iota,\tau}(q_{\tau_\iota}) = q$ for all $\iota < \gamma$.

We first verify that (μ, q) is a divisor for N_τ whenever $\mu < \tau$. This is a straightforward application of Lemma 3.10; an argument of this kind can also be found in the proof of Lemma 3.12 in [14]. To make this text self-contained, we give the argument. Let $\beta = \max(q)$ where $\max(\varnothing) = \tau$. If $\xi < \mu$ and $f : \mu_\tau \to \mu_\tau$ are such that $\zeta \stackrel{\text{def}}{=} F_\tau(f)(q_\tau - q, \xi) \le \beta$, pick $\iota < \gamma$ large enough that $\xi < \mu_{\tau_\iota}$ and $f \in J^E_{\vartheta_{\tau_\iota,\tau}}$ where $\vartheta_{\tau_\iota,\tau} = \vartheta(M_{\tau_\iota,\tau})$. Letting $F_{\tau_\iota,\tau}$ be the extender $F_{N_{\tau_\iota,\tau}}(\mu_\tau, q_{\tau_\iota,\tau})$, the fact that $\hat\sigma_{\tau_\iota,\tau}$ is Σ_0-preserving implies $\zeta_{\tau_\iota} \stackrel{\text{def}}{=} F_{\tau_\iota,\tau}(f)(q_{\tau_\iota,\tau} - q_{\tau_\iota}, \xi) \le \beta_{\tau_\iota}$ where $\beta_{\tau_\iota} = \max(q_{\tau_\iota})$ and $\max(\varnothing) = \tau_\iota$. Since $(\mu_{\tau_\iota}, q_{\tau_\iota})$ is a divisor for N_{τ_ι}, we have $\zeta_{\tau_\iota} < \mu_{\tau_\iota}$, so $\zeta = \hat\sigma_{\tau_\iota,\tau}(\zeta_{\tau_\iota}) = \zeta_{\tau_\iota} < \mu$.

We next observe that $q \ne \varnothing$. Assuming the contrary, we have $q_{\tau_\iota} = \varnothing$ for some/any $\iota < \gamma$. By the definition of a divisor, $\tau_\iota \cap \tilde{h}_{\tau_\iota}(\mu_{\tau_\iota} \cup \{q_{\tau_\iota}\}) = \mu_\iota$. Since $\mu_{\tau_\iota} > \vartheta_{\tau_\iota,\tau}$, Lemma 3.7(b) yields $\tau \cap h_{M_{\tau_\iota,\tau}}(\mu_{\tau_\iota} \cup \{q_{\tau_\iota,\tau}\}) = \mu_{\tau_\iota}$, which contradicts clause (b) in the definition of $\hat{B}_\tau$.

We are now heading towards the final contradiction. Notice that if $\mu < \tau$ then (μ, q) is a divisor for N_τ that is not strong. Using this for $\mu < \tau$ and the fact that $\tilde{h}_\tau(\mu \cup \{p_\tau\}) = N_\tau$ if $\mu = \tau$ we show that $(\mu_{\tau_\iota}, q_{\tau_\iota})$ cannot be strong after all, which yields the desired contradiction. The proof makes use in a crucial way of the following claim that describes how to transfer witnesses between N_τ and $N_{\tau_\iota,\tau}$. A version of this claim was also used in [14], proof of Lemma 3.12.

Claim. Let $\mu \le \beta < \min(q_\tau - q)$ where $\min(\varnothing) = \lambda_\tau$. Let further $s_\tau = p_\tau - q$ if N_τ is either passive or active with $\lambda_{N^*_\tau} > \mu_\tau$ and $s_\tau = d_\tau - q$ if N_τ is active with $\lambda_{N^*_\tau} = \mu_\tau$. Similarly, for $\iota < \gamma$ let $s_{\tau_\iota} = p_{\tau_\iota} - q_{\tau_\iota}$ if N_{τ_ι} is either passive or active with $\lambda_{N^*_{\tau_\iota,\tau}} > \mu_\tau$ and $s_{\tau_\iota} = d_{\tau_\iota} - q_{\tau_\iota}$ if N_{τ_ι} is active with $\lambda_{N^*_{\tau_\iota,\tau}} = \mu_\tau$; here $N^*_{\tau_\iota,\tau}$ denotes the structure $N^*_{\tau_\iota}(\mu_\tau, q_{\tau_\iota,\tau})$. Assume that $Q^*_\tau(\beta) \in J^E_{\lambda_\tau}$ and $x, x' \in [\mu]^{<\omega}$ and $i, i' \in \omega$ are objects satisfying one of the following hypotheses.

(a) N_τ is either passive or active with $\lambda_{N^*_\tau} > \mu_\tau$, $Q^*_\tau(\beta) = \tilde{h}_\tau(i, \langle x, p_\tau\rangle)$ is a generalized n_τ-witness for β with respect to N_τ and s_τ and $\beta = \tilde{h}_\tau(i', \langle x', p_\tau\rangle)$.

(b) N_τ is active with $\lambda_{N^*_\tau} = \mu_\tau$ or $N_\tau = M_\tau$, $Q^*_\tau(\beta) = h^*_\tau(i, \langle x, d_\tau\rangle)$ is a generalized Dodd witness for β with respect to N_τ and s_τ and $\beta = h^*_\tau(i', \langle x', d_\tau\rangle)$.

Granting these assumptions there is an ι^* such that if $\iota^* \le \iota < \gamma$ then $\beta, Q^*_\tau(\beta) \in \operatorname{rng}(\hat\sigma_{\tau_\iota,\tau})$ and there are $y, y' \in [\mu_{\tau_\iota}]^{<\omega}$ and $i, i' \in \omega$ such that the following is true. Let β_{τ_ι} and $Q^*_{\tau_\iota}$ be the inverse image of β and $Q^*_\tau(\beta)$ under $\hat\sigma_{\tau_\iota,\tau}$, respectively.

(c) If N_{τ_ι} is either passive or active with $\lambda_{N^*_{\tau_\iota,\tau}} > \mu_\tau$ then $Q^*_{\tau_\iota} = \tilde{h}_{\tau_\iota}(i, \langle y, p_{\tau_\iota}\rangle)$ is a generalized n_{τ_ι}-witness for β_{τ_ι} with respect to N_{τ_ι} and s_{τ_ι} and $\beta_{\tau_\iota} = \tilde{h}_{\tau_\iota}(i', \langle y', p_{\tau_\iota}\rangle)$.

(d) If N_{τ_ι} is active with $\lambda_{N^*_{\tau_\iota,\tau}} = \mu_\tau$ then $Q^*_{\tau_\iota} = h^*_{\tau_\iota}(i, \langle y, d_{\tau_\iota}\rangle)$ a generalized Dodd witness for β_{τ_ι} with respect to N_{τ_ι} and s_{τ_ι} and $\beta_{\tau_\iota} = h^*_{\tau_\iota}(i', \langle y', d_{\tau_\iota}\rangle)$.

Proof. For N_τ we have to consider three options: (i) N_τ is either passive or active with $\lambda_{N^*_\tau} > \mu_\tau$, (ii) N_τ is active with $\lambda_{N^*_\tau} = \mu_\tau$ and $N_\tau = M_\tau$. For N_{τ_ι} we only have two options: (iv) N_{τ_ι} is either passive or active with $\lambda_{N^*_{\tau_\iota,\tau}} > \mu_\tau$ and (v) N_{τ_ι} is active and $\lambda_{N^*_{\tau_\iota,\tau}} = \mu_\tau$. The option $N_{\tau_\iota} = M_{\tau_\iota}$ does not apply as N_{τ_ι} cannot be a pluripotent premouse; this is a consequence of the fact that $\hat\sigma_{\tau_\iota,\tau}$ fails to be cofinal. Thus, we have altogether six cases. As an example we treat the case where N_τ is as in (i) and N_{τ_ι} is as in (v). The proofs in the remaining cases are similar.

By Lemma 3.12, the standard witness $W^{0,\beta,q_\tau-q}_{M_\tau}$ is Σ_0-definable from $Q^*_\tau(\beta),\beta$ and ϑ_τ inside $J^E_{\lambda_\tau}$. Moreover, this witness can be encoded into a set $w \subseteq \beta$ in a canonical way so that w is Σ_0-definable from w inside any transitive admissible structure containing the respective objects. It follows that w is of the form $\tilde h_\tau(k,\langle z,p_\tau\rangle)$ for some $k \in \omega$ where $z = x \cup x' \cup \{\vartheta_\tau\}$. By Lemma 3.8, there are functions $f,f' \in J^E_{\vartheta_\tau}$ such that $w = F_\tau(f)(q_\tau,z)$ and $\beta = F(f')(q_\tau,x')$. Fix some ι^* such that $z \subset \mu_{\tau_{\iota^*}}$ and $f,f' \in J^E_{\vartheta_{\tau_{\iota^*}}}$. Such an ι^* exists, as the ordinals μ_{τ_ι} and $\vartheta_{\tau_\iota,\tau}$ monotonically converge to μ and ϑ_τ, respectively. Now consider any $\iota \geq \iota^*$. Letting $w_\iota = F_{\tau_\iota,\tau}(f)(q_{\tau_\iota,\tau},z)$ and $\beta_{\tau_\iota} = F_{\tau_\iota,\tau}(f')(q_{\tau_\iota,\tau},x')$ where $F_{\tau_\iota,\tau}$ denotes the extender $F_{N_{\tau_\iota,\tau}}(\mu_\tau,q_{\tau_\iota,\tau})$, obviously $\hat\sigma_{\tau_\iota,\tau}(\beta_{\tau_\iota}) = \beta$ and $\hat\sigma_{\tau_\iota,\tau}(w_\iota) = w$, so $\hat\sigma_{\tau_\iota,\tau}(Q'_{\tau_\iota}) = W^{\beta,q_\tau-q}_{M_\tau}$. The last equality here follows from the uniformity of the coding used to produce w. Since $\hat\sigma_{\tau_\iota,\tau}$ is Σ_0-preserving, Q'_{τ_ι} is a generalized witness for β_{τ_ι} with respect to $M_{\tau_\iota,\tau}$ and $q_{\tau_\iota,\tau} - q_{\tau_\iota}$. By Lemma 3.8, $w_\iota = h^*_{\tau_\iota}(i_2,\langle z \cup \{\xi\}, q_{\tau_\iota,\tau}\rangle)$ and $\beta_{\tau_\iota} = h^*_{\tau_\iota}(i',\langle x' \cup \{\xi'\}, q_{\tau_\iota,\tau}\rangle)$ for some $\xi,\xi' < \mu_\tau$ and $i_2,i' \in \omega$ ($r = \varnothing$ here where r is the parameter as in Lemma 3.8). Due to the uniformity of the coding, Q'_{τ_ι} is lightface definable from w_ι inside N_{τ_ι}, so there is an $i_1 \in \omega$ such that $Q'_{\tau_\iota} = h^*_{\tau_\iota}(i_1,\langle z \cup \{\xi\}, q_{\tau_\iota,\tau}\rangle)$. Applying Lemma 3.13, the standard witness $W^{\beta_{\tau_\iota},q_{\tau_\iota,\tau}-q_{\tau_\iota}}_{M_{\tau_\iota,\tau}}$ is Σ_0-definable in N_{τ_ι} from $Q'_{\tau_\iota},\beta_{\tau_\iota}$ and $\vartheta_{\tau_\iota,\tau}$, so by Lemma 3.11(e), the standard Dodd witness ${}^*W^{\beta_{\tau_\iota},q_{\tau_\iota,\tau}-q_{\tau_\iota}}_{N_{\tau_\iota}}$ is Σ_0-definable in N_τ from the same parameters. Since $q_{\tau_\iota,\tau} = d_{\tau_\iota}$ in this case, we have $d_{\tau_\iota,\tau} - q_{\tau_\iota} = s_{\tau_\iota}$. Letting $y = z \cup \{\xi,\xi',\vartheta_{\tau_\iota,\tau}\}$ and $y' = x' \cup \{\xi'\}$, obviously $y,y' \in [\mu_{\tau_\iota}]^{<\omega}$ and there is an $i \in \omega$ such that

$$\begin{aligned} {}^*W^{\beta_{\tau_\iota},s_{\tau_\iota}}_{N_{\tau_\iota}} &= h^*_{\tau_\iota}(i,\langle y, d_{\tau_\iota}\rangle) \\ \beta_{\tau_\iota} &= h^*_{\tau_\iota}(i',\langle y', d_{\tau_\iota}\rangle), \end{aligned}$$

which is in fact a little bit more than we intended to prove. □(Claim)

Returning to the proof of the lemma, we are now ready to reach the final contradiction by showing that for sufficiently large $\iota < \gamma$, the canonical divisor $(\mu_{\tau_\iota},q_{\tau_\iota})$ cannot be strong after all. Here we apply the above Claim.

For τ we have to consider five possibilities: (i) N_τ is either passive or active with $\lambda_{N^*_\tau} > \mu_\tau$, (ii) N_τ is active with $\lambda_{N^*_\tau} = \mu_\tau$ and (iii) $N_\tau = M_\tau$. If (i) applies and N_τ is active then $\lambda_{N^*_\tau(\mu,q)} > \mu$, as follows directly by inspecting the associated hulls. If (ii) applies than N_τ is active and either of the two options $\lambda_{N^*_\tau(\mu,q)} = \mu$ and $\lambda_{N^*_\tau(\mu,q)} > \mu$ must be considered. The same is true if (iii) applies, that is, we again have to consider two options. Hence five options have to be considered for τ. For τ_ι we only have to consider two cases: either $\lambda_{N^*_{\tau_\iota}} > \mu_{\tau_\iota}$ or else $\lambda_{N^*_{\tau_\iota}} = \mu_{\tau_\iota}$, as M_{τ_ι} cannot be a premouse. This makes altogether ten cases; all these cases are treated in a similar way. As an example we treat the case where N_τ is active, $\lambda_{N^*_\tau} = \mu_\tau$, $\lambda_{N^*_\tau(\mu,q)} > \mu$, N_{τ_ι} is active with $\lambda_{N^*_{\tau_\iota}} = \mu_{\tau_\iota}$, hence $\lambda_{N^*_{\tau_\iota,\tau}} = \mu_\tau$.

Let $s_\tau = d_\tau - q$. If $\mu = \tau$, let $\beta = \max(q)$ and $Q^*_\tau(\beta) = {}^*W^{\beta,s_\tau}_{N_\tau}$. We have seen above that $q \neq \varnothing$, so β exists. By the Dodd solidity of N_τ, the witness $Q^*_\tau(\beta)$ is an element of N_τ and by acceptability, $Q^*_\tau(\beta) \in J^E_{\lambda_\tau}$. Since N_τ is Dodd sound, we can find $x, x' \in [\mu]^{<\omega}$ and $i, i' \in \omega$ such that $Q^*_\tau(\beta) = h^*_\tau(i, \langle x, d_\tau\rangle)$ and $\beta = h^*_\tau(i', \langle x', d_\tau\rangle)$. Now assume that $\mu < \tau$. Let $r_\tau = p_\tau - q$. Since (μ, q) is a divisor for N_τ and $\mu > \mu_\tau$, this divisor cannot be strong. Letting $r' = {\pi'}^{-1}(r_\tau)$ where $\pi' : N'_\tau(\mu) \to N_\tau$ is the inverse to the Mostowski collapsing isomorphism associated with the hull $\tilde{h}_\tau(\mu \cup \{p_\tau\})$ (see Definition 3.15), the parameter r' is a proper top segment of $p_{N'_\tau(\mu)}$. Notice that $N'_\tau(\mu)$ is a premouse; this follows from the fact that π' is Σ_1-preserving[4] with respect to the language of premice. Since $\omega\varrho^1_{N'_\tau(\mu)} = \mu$ and $\pi' \restriction N'_\tau(\mu) = \mathrm{id}$, the first part of the condensation lemma (Lemma 3.2) yields that $N'_\tau(\mu)$ is solid. From all of this we conclude that there is some β' such that $\mu \leq \beta < {\pi'}^{-1}(\lambda_{N_\tau}(\mu, q))$ and $W^{0,\beta',r'}_{N'_\tau(\mu)} \in N'_\tau(\mu)$. Let $\beta = \pi'(\beta')$. Since π' is Σ_1-preserving, $\pi'(W^{0,\beta',r'}_{N'_\tau(\mu)})$ is a generalized witness for β with respect to N_τ and r_τ and this generalized witness is essentially a subset of $\beta < \lambda_{N_\tau}(\mu, q)$, modulo the natural uniform coding. Hence this witness is an element of $J^E_{\lambda_\tau}$. Notice that $\mathrm{rng}(\pi') = \tilde{h}_\tau(\mu \cup \{p_\tau\}) = h^*_\tau(\mu \cup \{d_\tau\})$, as follows by Lemma 3.6. Hence $\pi'(W^{0,\beta',r'}_{N'_\tau(\mu)}) = h^*_\tau(i_1, \langle x_1, d_\tau\rangle)$ and $\beta = h^*_\tau(i', \langle x', d_\tau\rangle)$ for suitably chosen $x_1, x' \in [\mu]^{<\omega}$ and $i_1, i' \in \omega$. By Lemma 3.12, $W^{0,\beta,r}_{N_\tau}$ is Σ_0-definable from $\pi'(W^{0,\beta',r'}_{N'_\tau(\mu)})$, β and ϑ_τ inside $J^E_{\lambda_\tau}$, so there are $x \in [\mu]^{<\omega}$ and $i \in \omega$ satisfying $W^{0,\beta,r}_{N_\tau} = h^*_\tau(i, \langle x, p_\tau\rangle)$. By (2), $W^{0,\beta,r}_{N_\tau} = {}^*W^{\beta,s}_{N_\tau}$, so we can let $Q^*(\beta) = {}^*W^{\beta,s}_{N_\tau}$. To summarize, in either case $\mu = \tau$ or $\mu < \tau$ we proved that the hypothesis (b) in the above Claim is met.

Applying the Claim, we obtain some $\iota < \gamma$ and $Q^*_{\tau_\iota} \in N_{\tau_\iota}$ such that conclusion in the Claim is satisfied. Since we are treating the case where N_{τ_ι} is active with $\lambda_{\tau_\iota,\tau} = \mu_\tau$, conclusion (d) applies. It follows that $Q^*_{\tau_\iota}$ is a generalized witness for $\beta_{\tau_\iota} = \hat{\sigma}^{-1}_{\tau_\iota,\tau}(\beta)$ with respect to N_{τ_ι} and $s_{\tau_\iota} = d_{\tau_\iota} - q_{\tau_\iota}$. Moreover, both $Q^*_{\tau_\iota}$ and β_{τ_ι} are in the hull $h^*_\tau(\mu_{\tau_\iota} \cup \{d_{\tau_\iota}\})$. Arguing as in the previous paragraph, we conclude that the standard Dodd witness ${}^*W^{\beta_{\tau_\iota},s_{\tau_\iota}}_{N_{\tau_\iota}}$ is in the same hull. Let $N'_{\tau_\iota}(\mu_{\tau_\iota})$ be the transitive collapse of $h^*_\tau(\mu_{\tau_\iota} \cup \{d_{\tau_\iota}\})$, $\pi'_{\tau_\iota} : N'_{\tau_\iota}(\mu_{\tau_\iota}) \to N_{\tau_\iota}$ be the inverse to the collapsing isomorphism and $\beta'_{\tau_\iota}, Q'_{\tau_\iota}$ be the inverse image of $\beta_{\tau_\iota}, {}^*W^{\beta_{\tau_\iota},s_{\tau_\iota}}_{N_{\tau_\iota}}$ under π', respectively. Then Q'_{τ_ι} is a generalized Dodd witness for β'_{τ_ι} with respect to N'_{τ_ι} and s'_{τ_ι} where β'_{τ_ι} and s'_{τ_ι} are the π'_{τ_ι}-preimages of β_{τ_ι} and s_{τ_ι}, respectively. Since this generalized Dodd witness is an element of $N'_{\tau_\iota}(\mu_{\tau_\iota})$, also the standard Dodd witness ${}^*W^{\beta'_{\tau_\iota},s'_{\tau_\iota}}_{N'_{\tau_\iota}(\mu_{\tau_\iota})}$ is in $N'_{\tau_\iota}(\mu_{\tau_\iota})$; denote this standard Dodd witness by ${}^*W$. Since $\mu_{\tau_\iota} \leq \beta'_{\tau_\iota}$, we conclude that $E^{N'_{\tau_\iota}(\mu_{\tau_\iota})}_{\mathrm{top}} \restriction \mu_{\tau_\iota} = E^{{}^*W}_{\mathrm{top}} \restriction \mu_{\tau_\iota} \in N'_{\tau_\iota}(\mu_{\tau_\iota})$, so by Lemma 3.16(e,f) the divisor $(\mu_{\tau_\iota}, q_{\tau_\iota})$ cannot be strong. □(Lemma 4.20)

The discussion of various other cases is simpler than that treated in the above proof, as it is not always necessary to extract standard witnesses from generalized ones. Indeed, the Claim in the previous lemma is formulated for generalized witnesses. The reason we had to look at standard witnesses in the above proof was that we needed to convert a solidity witness into a Dodd solidity witness in order to meet the assumptions of the Claim. One might suggest that this conversion should

[4]Recall that $n_\tau = 0$ in our case

also be included in the Claim; however, it is not clear how to do this in an elegant way at the level of generality the Claim is formulated, as the conversion between solidity witnesses and Dodd solidity witnesses was granted by Lemma 3.6 and (2).

We are now ready to define C_τ for $\tau \in \mathcal{S}^1$:

$$\begin{aligned} \gamma_\tau &= \text{the least } \gamma \in B_\tau \cup \{\tau\} \text{ such that } B_\tau - \gamma \text{ is closed.} \\ C_\tau &= B_\tau - \gamma_\tau . \end{aligned}$$

Then each $C_\tau \subseteq \mathcal{S}^1$, the same argument as in the proof of (10) shows that the sets C_τ are coherent, Lemma 4.18, Lemma 4.19 and Lemma 4.20 imply that C_τ is unbounded whenever τ has uncountable cofinality and $\mathrm{otp}(C_\tau) \leq \mathrm{otp}(B_\tau) \leq \vartheta_\tau$, as was pointed out immediately below (11). This completes the construction of C_τ for $\tau \in \mathcal{S}^1$.

So far we have constructed a $\Box^{\mathcal{S}}$-sequence $\langle C_\tau \mid \tau \in \mathcal{S}\rangle$. To complete of the proof of Theorem 1.2, we present a procedure that produces, given a class $A \subseteq \mathcal{S}$, a class $A' \subseteq A$ such that $A' \cap \kappa$ is stationary in κ whenever $A \cap \kappa$ is, and a $\Box^{\mathcal{S}}$-sequence $\langle C^{A'}_\tau \mid \tau \in \mathcal{S}\rangle$ such that $\lim(C^{A'}_\tau) \cap A' = \varnothing$ for all $\tau \in \mathcal{S}$. We begin with the construction of the class A'. We will follow Jensen's idea from [4]; see also [3]. Let $A \subseteq \mathcal{S}$ be given. The class $A' \subseteq A$ comprises all $\tau \in A$ for which there is an $\mathbf{L}[E]$-level $P = J^E_\beta$ and a parameter $a \in P$ such that

$$(13) \qquad \begin{array}{ll} \text{(a)} & P \models \mathsf{ZFC}^-,\ \tau \text{ is a largest cardinal in } P \text{ and is regular in } P. \\ \text{(b)} & X \cap \tau \notin A \text{ whenever } X \prec P \text{ is such that } a \in X \text{ and } X \cap \tau \in \tau. \end{array}$$

Here $X \prec Z$ means that X is a fully elementary substructure of P. A' is clearly a well-defined class; we verify that A' is as required.

Lemma 4.21. *Assume κ is an inaccessible cardinal and $A \cap \kappa$ is stationary in κ. Then A' is stationary in κ.*

Proof. Let $C \subseteq \kappa$ be closed unbounded in κ. By acceptability, $C \in J^E_{\kappa^+}$. Since $A \cap \kappa$ is stationary in κ there is some $X \prec J^E_{\kappa^+}$ with $C \in X$ and $X \cap \kappa \in A$. From now on assume that $\tau = X \cap \kappa \in A$ is the least possible. Let $J^{\bar{E}}_{\bar\beta}$ be the transitive collapse of X and $\sigma : J^{\bar{E}}_{\bar\beta} \to J^E_{\kappa^+}$ be the inverse to the Mostowski collapsing isomorphism. Then τ is the largest cardinal in $J^{\bar{E}}_{\bar\beta}$, the map σ is fully elementary, $\tau = \mathrm{cr}(\sigma)$, $\sigma(\tau) = \kappa$ and $\sigma(C \cap \tau) = C$. Clearly $\tau \in C$, as τ is a limit point of C and C is closed. We prove that $\tau \in A'$.

Obviously $J^{\bar{E}}_{\bar\beta} \models \mathsf{ZFC}^-$. Since $J^{\bar{E}}_\tau$ has no largest cardinal, a straightforward application of the Condensation Lemma (that is, Lemma 3.2) to $\sigma \restriction Q$ where Q is an arbitrarily large level of $J^{\bar{E}}_{\bar\beta}$ projecting to τ yields that all such levels Q are $\mathbf{L}[E]$-levels, hence $\bar{E} = E \restriction \bar\beta$. It follows that $J^E_{\bar\beta}$ is an $\mathbf{L}[E]$-level in which τ is regular. Now if $Y \prec J^E_{\bar\beta}$ is such that $C \cap \tau \in Y$ and $Y \cap \tau \in \tau$ then necessarily $\bar\tau = Y \cap \tau \notin A$, as $\sigma[Y] \prec J^E_{\kappa^+}$ is such that $C \in \sigma[Y]$ and $\kappa \cap \sigma[Y] = \bar\tau < \tau$. Thus, it suffices to set $P = J^{\bar{E}}_{\bar\beta}$ and $a = C \cap \tau$ in (13) for the current value of τ. $\Box$(Lemma 4.21)

We next have to modify our global square sequence slightly. Recall the definition of λ_τ and ϑ_τ from the beginning of this section. We let

$$\begin{aligned} \mathcal{S}^1_* &= \{\tau \in \mathcal{S}^1 \mid J^E_{\vartheta_\tau} \models \mathsf{ZFC}^-\} \\ \mathcal{S}^0_* &= \mathcal{S} - \mathcal{S}^1_*. \end{aligned}$$

This way $\mathcal{S}^0_*$ becomes larger than $\mathcal{S}^0$, as it may contain cardinals τ for which (μ_τ, q_τ) is defined, but for such τ the premouse N_τ is passive and λ_τ is the largest cardinal in N_τ. To see this notice that if λ_τ is not the largest cardinal in N_τ then $\vartheta_\tau \in N^*_\tau$ hence $J^E_{\vartheta_\tau} \models \mathsf{ZFC}^-$. If λ_τ is the largest cardinal in N_τ and N_τ is active then $J^E_{\nu_\tau} \models \mathsf{ZFC}^-$ and the ultrapower map $\pi : J^E_{\vartheta_\tau} \to J^E_{\nu_\tau}$ associated with $E^{M_\tau}_{\text{top}}$ is Σ_0-preserving and cofinal, so the standard argument then yields that π is fully elementary and $J^E_{\vartheta_\tau} \models \mathsf{ZFC}^-$.

For $\tau \in \mathcal{S}^0$ we let C_τ be defined as before. For $\tau \in \mathcal{S}^0_* - \mathcal{S}^0$ the canonical divisor (μ_τ, q_τ) exists; we define the set C_τ similarly as for elements of $\mathcal{S}^0$. We first let

B_τ be the set of all $\bar\tau \in \tau \cap \mathcal{S}^0_* - \mathcal{S}^0$ such that:
(i) $\bar\tau$ satisfies all requirements in Definition 4.3.
(ii) $\mu_{\bar\tau} = \mu_\tau$ and $|\, q_{\bar\tau}\,| = |\, q_\tau\,|$.

We then define B^*_τ exactly as in the case where $\tau \in \mathcal{S}^0$; we also define $\hat B_\tau$ analogously to the definition for $\tau \in \mathcal{S}^1$; here of course we disregard clause (b) in that definition. An argument using (12) in a way similar to the proof of Lemma 4.18 shows that $\hat B_\tau$ is unbounded in τ whenever τ has uncountable cofinality. As in the proof of Lemma 4.19 we show that $\hat B_\tau$ is closed in τ; here of course we work with structures N_τ instead of M_τ. Finally, as in Lemma 4.9 we show that $\hat B_\tau$ agrees with B_τ on a tail-end. We can let $C_\tau = B^*_\tau$ in this case; it follows exactly as in the case $\tau \in \mathcal{S}^1$ above that $\operatorname{otp}(C_\tau) \le \vartheta_\tau$. Notice that the construction yields $C_\tau \subseteq \mathcal{S}^0_* - \mathcal{S}^0$ whenever $\tau \in \mathcal{S}^0_* - \mathcal{S}^0$.

Lemma 4.22. *Let $\tau \in \mathcal{S}^0_*$ and $\bar\tau \in \lim(C_\tau) \cap A'$. Then there is some $\tau^* \in C_\tau \cap \bar\tau$ such that $\tau' \notin A$ whenever $\tau' \in C_\tau \cap (\tau^*, \bar\tau)$.*

Proof. Let (P, a) be a pair witnessing that $\bar\tau \in A'$. Since $N_{\bar\tau}$ is a singularizing structure for $\bar\tau$, τ is regular in P and P is an $\mathbf{L}[E]$-level, we conclude that P is an initial segment of J^E_β where $N_{\bar\tau} = \langle J^E_\beta, E_{\omega\beta}\rangle$.

First assume that $P \in N_{\bar\tau}$. Let $\tau^* \in C_\tau \cap \bar\tau$ be such that $P, a \in \operatorname{rng}(\sigma_{\tau^*,\bar\tau})$. For any $\tau' \in C_\tau \cap (\tau^*, \bar\tau)$ let P' be the preimage of P under $\sigma_{\tau',\bar\tau}$ and $X = \sigma_{\tau',\bar\tau}[P']$. Then $X \prec P$ and $a \in X$, so $\tau' = X \cap \bar\tau \notin A$.

Now consider the case where $P \notin N_{\bar\tau}$, that is, $P = J^E_\beta$. Since $P \models \mathsf{ZFC}^=$ and $N_{\bar\tau}$ is a singularizing structure for $\bar\tau$, necessarily $\tau \in \mathcal{S}^0$ and $E^{N_{\bar\tau}}_{\text{top}} \neq \varnothing$. As $\bar\tau$ is the largest cardinal in P, we have $\lambda(E^{N_{\bar\tau}}_{\text{top}}) = \bar\tau$, so $\mu = \operatorname{cr}(E^{N_{\bar\tau}}_{\text{top}}) < \bar\tau$. Since $\tau \in \mathcal{S}^0$ and the construction of C_τ guarantees that $C_\tau \subseteq \mathcal{S}^0$ we conclude that $N_{\tau'}$ is a premouse with $\operatorname{cr}(E^{N_{\tau'}}_{\text{top}}) = \mu < \tau'$ and actually $\mu^+ < \tau'$ whenever $\tau' \in C_\tau$; notice that $n_\tau > 0$ in this case as N_τ is not pluripotent, so all maps $\sigma_{\tau',\tau''}$ for $\tau' \le \tau''$ in C_τ are Σ_1-preserving. As a consequence of all of the above we have that $\operatorname{dom}(E^{N_{\tau'}}_{\text{top}}) = \mathcal{P}(\mu) = \operatorname{dom}(E^{N_{\bar\tau}}_{\text{top}})$ and $E^{N_{\bar\tau}}_{\text{top}}(x) = \sigma_{\tau',\bar\tau}(E^{N_{\tau'}}_{\text{top}}(x))$ whenever $\tau' \in C_\tau \cap \bar\tau$. Letting $\beta' = \operatorname{ht}(N_{\tau'})$ and $\vartheta = \mu^+$, we have $J^E_{\beta'} = \operatorname{Ult}(J^E_\vartheta, E^{N_{\tau'}}_{\text{top}})$ and $J^E_\beta = \operatorname{Ult}(J^E_\vartheta, E^{N_{\bar\tau}}_{\text{top}})$. It follows that $\sigma_{\tau',\bar\tau} : \pi'(f)(\alpha) \mapsto \pi(f)(\alpha)$ where π', π are the corresponding ultrapower maps, so $\sigma_{\tau',\bar\tau}$ is fully elementary when viewed as a map from $J^E_{\beta'}$ to J^E_β and $X = \mathrm{rng}(\sigma_{\tau',\bar\tau}) \prec J^E_\beta$. We can now proceed as above to show that $\tau' \notin A$ whenever $a \in \mathrm{rng}(\sigma_{\tau',\bar\tau})$. $\square$(Lemma 4.20)

For each $\tau \in \mathcal{S}^0 \cap A'$ let

$$\delta_\tau = \text{the least ordinal } \delta \in C_\tau \text{ such that } (\delta,\tau) \cap A = \varnothing.$$

and

$$C^{A'}_\tau = C_\tau - \bigcup\{(\delta_{\bar\tau},\bar\tau) \mid \bar\tau \in \lim(C_\tau) \cap A'\} \tag{14}$$

It follows immediately from the definition that each $C^{A'}_\tau$ is a closed subset of τ and $\mathrm{otp}(C^{A'}_\tau) \le \mathrm{otp}(C_\tau) < \tau$. Because the sequence $\langle C_\tau \rangle_\tau$ is coherent, it follows easily that $\langle C^{A'}_\tau \rangle_\tau$ is also coherent; this is similar to the verification that the sets defined in (10) are coherent. To complete the verification that $\langle C^{A'}_\tau \rangle_\tau$ is a $\square^{\mathcal{S}^0_*}$-sequence we show that $C^{A'}_\tau$ is unbounded in τ whenever C_τ is. This is clear if $\lim(C_\tau) \cap A'$ is bounded in τ, so assume without loss of generality that $\lim(C_\tau) \cap A'$ is unbounded in τ. We will rely on the following immediate consequence of Lemma 4.20.

$$\tau^*, \bar\tau \in \lim(C_\tau) \cap A' \ \&\ \tau^* < \bar\tau \implies \tau^* < \delta_{\bar\tau} \tag{15}$$

It follows that $C^{A'}_\tau$ is unbounded in τ, as it contains $\lim(C_\tau) \cap A'$.

Finally notice that any limit point $\bar\tau$ of $C^{A'}_\tau$ is a limit point of C_τ, but $\bar\tau$ cannot be an element of A' since all ordinals from $\lim(C_\tau) \cap A'$ are successor points of $C^{A'}_\tau$. This shows that $A' \cap \lim(C^{A'}_\tau) = \varnothing$ and thereby completes the proof of Theorem 1.2 in the case where $\tau \in \mathcal{S}^0_*$.

We next turn to the case where $\tau \in \mathcal{S}^1_*$. We need the analogue to Lemma 4.20, but in order to prove it we have to refine our sets C_τ. We let

$$C'_\tau = \{\bar\tau \in C_\tau \mid \vartheta_{\bar\tau} \prec \vartheta_\tau\}$$

We first verify that $\langle C'_\tau \rangle_\tau$ is a $\square^{\mathcal{S}^1_*}$-sequence. Using the coherency of the sequence $\langle C_\tau \rangle_\tau$ and elementary properties of $\prec$ it is easy to see that the sequence $\langle C'_\tau \rangle_\tau$ is coherent; again, this is similar to the situation in (10). Since ϑ_{τ^*} converges to $\vartheta_{\bar\tau}$ as τ^* converges to τ, we see that $J^E_{\vartheta_{\bar\tau}} \prec J^E_{\vartheta_\tau}$ whenever $\bar\tau$ is a limit point of C'_τ, so $\bar\tau \in C'_\tau$. This shows that C'_τ is closed. It remains to show that C'_τ is unbounded in τ whenever τ has uncountable cofinality.

Lemma 4.23. *Let $\tau \in \mathcal{S}^1_*$ be of uncountable cofinality. Then the set*

$$\hat{B}'_\tau = \{\bar\tau \in \hat{B}_\tau \mid J^E_{\vartheta_{\bar\tau,\tau}} \prec J^E_{\vartheta_\tau}\}$$

is unbounded in τ.

Proof. It is sufficient to show that in the proof of Lemma 4.18, the interpolant $\tilde M$ emerging from the construction satisfies $J^E_{\tilde\vartheta} \prec J^E_{\vartheta_\tau}$ where $\tilde\vartheta = \vartheta_{\tilde M}$. We follow the notation from that proof. Recall that $\sigma : \bar M \to M_\tau$ is fully elementary where $\bar M$ is countable. We have $\bar\mu, \bar\vartheta \in \bar M$ such that $\sigma(\bar\mu,\bar\vartheta) = (\mu_\tau,\vartheta_\tau)$. We have seen that $\tilde\vartheta = \sup(\sigma[\bar\vartheta])$. Let $\bar E = E^{\bar M}$.

Assume first that $M_\tau = N_\tau$; in this case $\vartheta_\tau = \mu^+_\tau < \mathrm{ht}(M_\tau)$. Then $\bar\vartheta = \bar\mu^{+\bar M} < \mathrm{ht}(\bar M)$, so we can apply the interpolation lemma to obtain an interpolant $\tilde M$ extending $J^E_{\tilde\vartheta}$ and Σ_0-preserving maps $\sigma_0 : \bar M \to \tilde M$ and $\sigma_1 : \tilde M \to M_\tau$ such that $\sigma_1 \restriction \tilde\vartheta = \mathrm{id}$ and $\sigma_1(\tilde\vartheta) = \vartheta_\tau$. Here $\tilde M$ is the ultrapower of $\bar M$ by $\sigma \restriction J^E_{\bar\vartheta}$ and σ_0

is the associated ultrapower map. It follows from the above properties of σ_1 that $J^E_{\tilde\vartheta} \prec J^E_{\vartheta_\tau}$.

From now on assume that M_τ is a protomouse. If $\lambda_{N^*_\tau(\mu_\tau,q_\tau)} > \mu_\tau$ then N^*_τ is a proper extension of $J^E_{\vartheta_\tau}$ and $\vartheta_\tau = \mu_\tau^{+N^*_\tau}$ where $\bar N^*$ be the inverse image of N^*_τ under σ. Then $\bar N^*$ is the collapsing level of $\bar N$ for $\bar\vartheta$ and $J^{\bar E}_{\bar\vartheta}$ is a proper initial segment of $\bar N^*$. Hence we can use the interpolation lemma to obtain an interpolant $\bar N'$ end-extending $J^E_{\tilde\vartheta}$ and Σ_0-preserving embeddings $\sigma_0 : \bar N^* \to \bar N'$ and $\sigma_1 : \bar N' \to N^*_\tau$ such $\tilde\vartheta = \mathrm{cr}(\sigma_1)$ and $\sigma_1(\tilde\vartheta) = \vartheta_\tau$. (For the present purpose we do not even need to use a fine structural ultrapower in the construction of the interpolant; it is sufficient to construct σ_0 as the ultrapower map associated with the coarse ultrapower $\mathrm{Ult}(\bar N^*, \sigma \restriction J^{\bar E}_{\bar\vartheta})$.) Then obviously $\sigma_1 \restriction J^E_{\tilde\vartheta} : J^E_{\tilde\vartheta} \to J^E_{\vartheta_\tau}$ is fully elementary, so $J^E_{\tilde\vartheta} \prec J^E_{\vartheta_\tau}$.

Finally consider the case where $\lambda_{N^*_\tau}(\mu_\tau, q_\tau) = \mu_\tau$ and M_τ is a protomouse. Then $\vartheta_\tau < \mu_\tau^+$, so $N^*_\tau = \langle J^E_{\vartheta_\tau}, E_{\vartheta_\tau}\rangle$. Let $\mu_1 = \mathrm{cr}(E_{\vartheta_\tau})$ and $\vartheta_1 = \mu_1^+$; then $\mu_1 < \mu_\tau = \lambda(E_{\vartheta_\tau})$. Letting $\bar\mu_1, \bar\vartheta_1$ be the inverse images of μ_1, ϑ_1 under σ and $\tilde\vartheta_1 = \sup(\sigma[\bar\vartheta_1])$, we see that $\bar\vartheta_1 = \bar\mu_1^{+\bar M}$ and $J^{\bar E}_{\bar\vartheta_1}$ is a proper initial segment of $\bar M$, so we can apply the interpolation lemma to obtain an interpolant M' and Σ_0-preserving maps $\sigma_0 : \bar M \to M'$ and $\sigma_1 : M' \to M_\tau$ such that σ_0 is cofinal, $\sigma_0(\bar\vartheta_1) = \tilde\vartheta_1$, $\mathrm{cr}(\sigma_1) = \tilde\vartheta_1$ and $\sigma_1(\tilde\vartheta_1) = \vartheta_1$. As above we conclude that $J^E_{\tilde\vartheta_1} \prec J^E_{\vartheta_1}$. Since σ_0 maps $\bar\vartheta_1$ cofinally into $\tilde\vartheta_1$ and $\sigma_0 \restriction \bar\vartheta_1 = \mathrm{id}$, the restriction $\sigma_0 \restriction \bar M \,||\, \bar\vartheta$ maps the coherent structure $\bar M \,||\, \bar\vartheta$ cofinally into the coherent structure $M' \,||\, \vartheta'$ where $\vartheta' = \sigma_0(\bar\vartheta)$, so $\sup(\sigma_1[\vartheta']) = \tilde\vartheta$. Now notice that $\sigma_1 : M' \,||\, \vartheta' \to M^*$ is cofinal where $M^* = \langle J^E_{\tilde\vartheta}, E_{\vartheta_\tau} \cap J^E_{\tilde\vartheta}\rangle$, so M^* is a coherent structure and $\vartheta_{M^*} = \tilde\vartheta_1$. Since $J^E_{\tilde\vartheta_1} \prec J^E_{\vartheta_\tau}$, it follows by a straightforward verification that the map $\sigma^* : J^E_{\tilde\vartheta} \to J^E_{\vartheta_\tau}$ defined by $\sigma^* : \tilde\pi(f)(\alpha) \mapsto \pi(f)(\alpha)$ for $f \in J^E_{\tilde\vartheta}$ and $\alpha < \mu_\tau$ is fully elementary.
□(Lemma 4.23)

We can now prove the analogue to Lemma 4.22.

Lemma 4.24. *Let $\tau \in \mathcal{S}^1_*$ and $\bar\tau \in \lim(C'_\tau) \cap A'$. Let (P, a) be a pair witnessing that $\bar\tau \in A'$. Then there is some $\tau^* \in C'_\tau \cap \bar\tau$ such that $\tau' \notin A$ whenever $\tau' \in C'_\tau \cap (\tau^*, \bar\tau)$.*

Proof. If $P \in M_{\bar\tau}$ we argue exactly as in the proof of Lemma 4.22. Otherwise $P = J^E_{\nu_{\bar\tau}}$. For every $\tau' \in C'_\tau \cap \bar\tau$ we have $J^E_{\vartheta_{\tau'}} \prec J^E_{\vartheta_{\bar\tau}}$, so $\sigma_{\tau',\bar\tau} : J^E_{\nu_{\tau'}} \to J^E_{\nu_{\bar\tau}} = P$ is fully elementary as $\sigma_{\tau',\bar\tau} : F_{\tau'}(f)(\alpha) \mapsto F_{\bar\tau}(f)(\sigma_{\tau',\bar\tau}(\alpha))$ for $f \in J^E_{\vartheta_{\tau'}}$ and $\alpha < \lambda_{\tau'}$. So we can let $\tau^* \in C'_\tau \cap \bar\tau$ be such that $a \in \mathrm{rng}(\sigma_{\tau^*,\bar\tau})$ and again proceed as in the proof of Lemma 4.22.
□(Lemma 4.23)

With Lemma 4.24 in hand we can define $C^{A'}_\tau$ for $\tau \in \mathcal{S}^1_*$ similarly as in the previous case. We let

$$\delta_\tau = \text{the least ordinal } \delta \in C'_\tau \text{ such that } (\delta, \tau) \cap A = \varnothing$$

and

$$C^{A'}_\tau = C'_\tau - \bigcup\{(\delta_{\bar\tau}, \bar\tau) \mid \bar\tau \in \lim(C'_\tau) \cap A'\}.$$

The verification that $\langle C^{A'}_\tau \mid \tau \in \mathcal{S}^1_*\rangle$ is a $\square^{\mathcal{S}^1_*}(A')$-sequence is then the same as before.

References

[1] Burke, D., Generic embeddings and the failure of box, Proceedings of American Mathematical Society 123(9) (1995), 2867 – 2871

[2] Cox, S., A covering lemma with applications, Ph. D. thesis, Irvine, 2009

[3] Devlin, K. J., *Constructibility,* Springer, 1984

[4] Jensen, R. B., The fine structure of the constructible hierarchy, Annals of Mathematical Logic 4 (1972), 229-308

[5] Jensen, R. B., More on $\square_\kappa$ below $0^\P$ in $\mathbf{K}$, handwritten notes, Oxford, 1994

[6] Jensen, R. B., A new fine structure for higher core models, handwritten notes, Berlin, 1997

[7] Jensen, R. B. and Zeman, M., Smooth categories and global square, Annals of Pure and Applied Logic 102(1-2) (2000), 101 – 138

[8] Koepke, P. and Welch, P., Global square and mutual stationarity at $\aleph_n$'s, to appear

[9] Kypriotakis, K. and Zeman, M., Fine structural nonthreadable square sequences at successors cardinals, in preparation

[10] Mitchell, W. J. and Steel, J. R., *Fine Structure and Iteration Trees,* Lecture Notes in Logic 3, Springer, Berlin, 1994

[11] Neeman, I. and Schimmerling, E., Hierarchies of forcing axioms I, Journal of Symbolic Logic 73 (2008) 343 - 362

[12] Neeman, I., Hierarchies of forcing axioms II, J. of Symbolic Logic 73 (2008), 522 - 542

[13] Schimmerling, E. and Zeman, M., Square in core models, Square in core models, Bulletin of Symbolic Logic 7(3) (2001), 305 – 314

[14] Schimmerling, E. and Zeman, M., Characterization of $\square_\kappa$ in core models, Journal of Mathematical Logic 4(1) (2004), 1 – 72

[15] Steel, J. R., Outilne of Inner Model Theory, *Handbook of Set Theory,* Foreman and Kanamori eds., Springer, to appear

[16] P.D. Welch, D. Phil. dissertation, Oxford 1979

[17] Wylie, D. J., Ph. D. dissertation, MIT, 1990

[18] Zeman, M., *Inner Models and Large Cardinals,* de Gruyter Series in Logic and its Applications 5, de Gruyter (2002)

[19] Zeman, M., Dodd parameters and λ-indexing of extenders, Journal of Mathematical Logic 4(1) (2004), 73 – 108

[20] Zeman, M., Ph. D. dissertation, HU Berlin, 1997

[21] Zeman, M., More structural global square sequences, accepted for publication in Archive for Mathematical Logic

Department of Mathematics, University of California at Irvine, Irvine, CA 92697

E-mail address: mzeman@math.uci.edu