A CHARACTERIZATION OF $\square(\kappa^+)$ IN EXTENDER MODELS.

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Abstract. We prove that, in any fine structural extender model with Jensen's λ-indexing, there is a $\square(\kappa^+)$-sequence if and only if there is a pair of stationary subsets of $\kappa^+ \cap \mathsf{cof}(<\kappa)$ without common reflection point of cofinality $<\kappa$ which, in turn, is equivalent to the existence of a family of size $<\kappa$ of stationary subsets of $\kappa^+ \cap \mathsf{cof}(<\kappa)$ without common reflection point of cofinality $<\kappa$. By a result of Burke/Jensen, $\square_\kappa$ fails whenever κ is a subcompact cardinal. Our result shows that in extender models, it is still possible to construct a canonical $\square(\kappa^+)$-sequence where κ is the first subcompact.

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Constructions of canonical coherent sequences have been of interest in inner model theory since their discovery by Jensen [4]. There are at least two reasons why. First, coherent sequences coming from inner models are defined in a uniform way, and differ radically from coherent sequences obtained by forcing. One such notable distinction is that typically they cannot be threaded in any outer model, as long as the length of the sequence remains uncountable from the point of view of the outer model. Second, due to this strong non-threadability property, such sequences can be used in arguments that derive large cardinal strength of various combinatorial situations.

1. Introduction, background and the result

Using a variation on an argument of Solovay, Jensen [6] showed that $\square_\kappa$ fails whenever the set

(1) $$S^*_\kappa = \{x \in [H_{\kappa^+}]^{<\kappa} \mid x \cap \kappa = \mu \;\&\; (x, \in) \simeq (H_{\mu^+}, \in) \text{ for some } \mu < \kappa\}$$

is stationary. Cardinals κ such that S^*_κ is stationary are called subcompact. A seemingly weaker requirement, namely that $S_\kappa = \{x \in [\kappa^+]^{<\kappa} \mid \mathsf{otp}(x) \text{ is a cardinal}\}$, already implies that $\square_\kappa$ fails as is proved in [1], and an argument similar to that used by Jensen described in [2] in the context of Chang Conjecture also gives the failure of $\square_\kappa$ from the stationarity of S_κ. It follows from results in [10] that in an extender model $\mathbf{L}[E]$ the set S_κ is stationary if and only if S^*_κ is. In the same manuscript [6] Jensen shows that κ may be subcompact in an extender model granting that κ is subcompact in $\mathbf{V}$ and the $\mathbf{K}^c$ construction converges. It follows that once we can construct extender models for sufficiently strong large cardinal axioms, these models differ from $\mathbf{L}$ in that $\square_\kappa$ no longer holds in these models at

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all cardinals κ. It is then natural to expect that also constructions of other types of square sequences and other combinatorial objects in extender models may differ in a substantial way from the corresponding constructions in $\mathbf{L}$, in that in extender models certain obstructions may be present which make these constructions break down. Let us note that there are constructions where no obstructions are encountered, as for instance in [12] or [11]. The focus of this paper is however on the the case where obstructions do occur, and the best known example demonstrating this phenomenon is the construction of a $\square_\kappa$ sequence where such obstructions are encountered precisely when S_κ is stationary. The obstruction of this kind relevant in the context of the construction of a $\square_\kappa$ sequence has been isolated in [10], and hinges on the inability to pick a canonical collapsing structure for a local cardinal successor of κ. More precisely, the construction of a canonical $\square_\kappa$ sequence in the model $\mathbf{L}[E]$ begins with assigning a canonical collapsing structure to each ordinal $\tau \in (\kappa, \kappa^+)$ such that κ is the largest cardinal in J^E_τ. Such an ordinal τ is called a local cardinal successor of κ. In $\mathbf{L}$ the canonical collapsing structure for τ is just the right initial segment of $\mathbf{L}$, but in an extender model $\mathbf{L}[E]$ it may be a structure which is not even a premouse, that is, the structure is not an initial segment of any extender model. Such structures are in [10] called protomice, and one of their feature is that they are coherent structures in the sense of [14] whose top extender is not weakly amenable. (See Section 2 for a brief review of coherent structures and protomice.) Also, they code, in a certain canonical way, the complete information about the collapsing segment of $\mathbf{L}[E]$ for τ. The assignment of the canonical collapsing structure thus involves a procedure which for each local cardinal successor τ decides whether the canonical collapsing structure is the initial segment of $\mathbf{L}[E]$ or a protomouse. If the canonical collapsing structure turns out to be a protomouse, one faces the obstacle of specifying which protomouse is the canonical one; one can namely show that there may be many protomice coding the complete information about the collapsing segment of $\mathbf{L}[E]$ for τ in the same canonical way. In [10] it is shown that for each local cardinal successor τ such that E_τ is not an extender one can isolate a small class of well-behaved protomice, and this class contains a protomouse which is in certain canonical sense "smallest". Using this "smallest" protomouse as the canonical collapsing protomouse, it is shown in [10] that the construction of $\square_\kappa$ can be successfully carried out. If E_τ is an extender then the methods of [10] cannot be used to define a collapsing structure for τ in any canonical way, so the construction of a canonical $\square_\kappa$-sequence can be only carried out if E_τ is an extender for only non-stationarily many $\tau < \kappa^+$. On the other hand, it is shown in [10] that letting

$$T_\kappa = \{\alpha < \kappa^+ \mid E_\alpha \neq \varnothing\}, \tag{2}$$

the following holds in the model $\mathbf{L}[E]$.

$$S_\kappa \text{ is stationary} \iff T_\kappa \text{ is stationary} \iff \neg\square_{\kappa,<\kappa} \iff \neg\square_\kappa \tag{3}$$

Hence $\square_\kappa$ must fail in $\mathbf{L}[E]$ whenever T_κ is stationary, and this implication is due to Jensen [6]. This shows that our inability to specify the canonical protomouse for $\tau \in T_\kappa$ is not a shortcoming of the method developed in [10], but a consequence of a more fundamental phenomenon that in this case no canonically defined collapsing structures for stationarily many $\tau \in T_\kappa$ exist. Recall once again that the implications "S_κ is stationary" $\implies \neg\square_{\kappa,<\kappa} \implies \neg\square_\kappa$ are provable in ZFC.

The non-existence of canonical collapsing structures for certain ordinals τ is of course a major issue in pursuing combinatorial constructions. This is because most of these constructions, as known from classical literature on fine structure, heavily rely on canonical collapsing structures. In this paper we show how to overcome this obstacle in a construction of a non-threadable coherent sequence, or in other words a $\square(\kappa^+)$-sequence, in the situation where the set T_κ is stationary. For each $\tau \in T_\kappa$ we assign a structure which is canonically defined, but not necessarily collapsing. We then show that for $\tau \in T_\kappa$ this assignment of canonical structures can be used to construct a $\square(\kappa^+)$-sequence in $\mathbf{L}[E]$. Of course, as we have seen above, it may not be possible to carry out such a construction without a smallness condition, and indeed we provide an argument showing that it is not possible to do so. In [6] Jensen introduced a property he called "quasicompactness", and showed that that analogously as in the case of subcompactness, quasicompact cardinals may exist in $\mathbf{K}^c$ granting that they exist in $\mathbf{V}$ and the $\mathbf{K}^c$-construction converges. Quasicompactness can be characterized in terms of elementary embeddings, and can be viewed as a strong form of 1-extendibility. (Subcompactness can also be characterized in terms of elementary embeddings in a similar fashion as quasicompactness, with the only difference that subcompactness refers to downward reflection, see [6] or [10], whereas quasicompactness refers to upward reflection. However, this characterization of subcompactness will not be relevant in this paper.) Following Jensen [6], a cardinal κ is quasicompact if and only if for every $A \subseteq \kappa^+$ there is $\lambda > \kappa$, $A' \subseteq \lambda^+$, and an elementary/Σ_1-preserving embedding $\pi : (H_{\kappa^+}, A) \to (H_{\lambda^+}, A')$ with critical point κ. Obviously then $\pi(\kappa) = \lambda$. If κ is quasicompact then $\square(\kappa^+)$ fails. This can be seen as follows. Given a coherent sequence $\vec{C} = \langle C_\alpha \mid \alpha < \kappa^+ \rangle$, let this sequence $\vec{C}$ play the role of A (under the obvious coding). If $\pi : (H_{\kappa^+}, \vec{C}) \to (H_{\lambda^+}, \vec{C}')$ is the embedding coming from the quasicompactness of κ and $\nu = \sup(\pi[\kappa^+])$, it is easy to see that $C^* = \pi^{-1}[C'_\nu]$ is a thread through $\vec{C}$, that is, $C^* \cap \alpha = C_\alpha$ for all $\alpha \in \lim(C^*)$. A similar argument shows that if κ is quasicompact then every stationary $R \subseteq \kappa^+ \cap \mathsf{cof}(<\kappa)$ has a reflection point of cofinality $< \kappa$. Namely, if $\pi : (H_{\kappa^+}, R) \to (H_{\lambda^+}, R')$ is the embedding whose existence is guaranteed by the quasicompactness of κ then $\pi[R]$ is stationary in ν, as follows from the stationarity of R and the continuity of π on $\kappa^+ \cap \mathsf{cof}(<\kappa)$. Since $\pi[R] \subseteq R'$, the structure (H_{λ^+}, R') sees that $R' \cap \nu$ is stationary, that this fact is reflected to (H_{κ^+}, R) by the elementarity of π. The same argument shows that, more generally, if κ is quasicompact and $\mathcal{R}$ is a family of stationary subsets of $\kappa^+ \cap \mathsf{cof}(<\kappa)$ such that $\mathsf{card}(\mathcal{R}) < \kappa$ then the sets in $\mathcal{R}$ have a common reflection point of cofinality $< \kappa$.

Given a regular cardinal κ, cardinal $\lambda < \kappa$, and a set of regular cardinals $X \subseteq \kappa$, write $\mathsf{cof}(X) = \{\xi < \kappa \mid \mathsf{cf}(\xi) \in X\}$. Define $\mathsf{Refl}(\kappa, \lambda, X)$ as follows.

(4) $\mathsf{Refl}(\kappa, \lambda, X) \equiv$ "For every family $\mathcal{R}$ of stationary subsets $\kappa \cap \mathsf{cof}(X)$ such that $\mathsf{card}(\mathcal{R}) \leq \lambda$ there is a $\vartheta < \kappa$ such that ϑ is a common reflection point for all sets in $\mathcal{R}$".

Then $\mathsf{Refl}(\kappa, <\lambda, <\mu)$ has the obvious meaning. Although $\mathsf{Refl}(\kappa^+, <\kappa, <\kappa)$ may superficially look weaker than the conclusion of the argument above, in an extender model the two are actually equivalent.

Proposition 1.1. *Assume $\mathbf{V} = \mathbf{L}[E]$ is a fine structural extender model. Let κ be a cardinal such that $\mathsf{Refl}(\kappa^+, <\kappa, <\kappa)$. Then every stationary $S \subseteq \kappa^+ \cap \mathsf{cof}(<\kappa)$ has a reflection point of cofinality $<\kappa$.*

Proof. The proposition has a non-trivial meaning only in the case where κ is regular, and is a consequence of the following fact concerning the canonical square sequences in extender models established in [10].

(5) For any local cardinal successor τ of κ with $\mathsf{cf}(\tau) = \kappa$ the canonical collapsing structure exists, and is an initial segment of $\mathbf{L}[E]$; so in particular the set C_τ on the canonical $\square_\kappa$-sequence is defined.

The well-known argument then shows that $\mathsf{Refl}(\kappa, <\kappa, <\kappa)$ actually implies that for every family $\mathcal{R}$ of stationary subsets of $\kappa^+ \cap \mathsf{cof}(<\kappa)$ there is a common reflection point of cofinality $<\kappa$. Let R^* be the set of all common reflection points for sets in $\mathcal{R}$; it follows from $\mathsf{Refl}(\kappa^+, <\kappa, <\kappa)$ that R^* is stationary. Assume for a contradiction all ordinals in R^* have cofinality κ. Pick $R \in \mathcal{R}$ and notice that the set $\bar{R} = R \cap \bigcup\{C_\tau \mid \tau \in R^*\}$ is stationary. Now C_ξ is defined for each $\xi \in \bar{R}$; this follows from the fact that if C_τ is the on the canonical $\square_\kappa$-sequence in an extender model then C_ξ is defined for all $\xi \in C_\tau$. Hence we can fix $\mathsf{otp}(C_\xi)$ on a stationary $R' \subseteq \bar{R}$. But then R' cannot reflect at any point of uncountable cofinality τ for which C_τ is defined, so in particular at any τ of cofinality κ. By $\mathsf{Refl}(\kappa^+, <\kappa, <\kappa)$ the sets in $(\mathcal{R} - \{R\}) \cup \{R'\}$ have a common reflection point. This point must be of cofinality $<\kappa$, as it is a reflection point of R'. Since $R' \subseteq R$, this point is obviously a common reflection point of all sets in $\mathcal{R}$. $\square$

The above argument also works for $\mathsf{Refl}(\kappa, \lambda, <\kappa)$ in place of $\mathsf{Refl}(\kappa, <\kappa, <\kappa)$, which is of particular interest in the case where λ is finite.

By Jensen's classical result for $\mathbf{L}$ and a general result for extender models in [13], stationary reflection implies the failure of $\square(\theta)$ if θ is inaccessible, as it is equivalent to weak compactness of θ. In general, however, stationary reflection at a regular cardinal θ is consistent with the existence of a $\square(\theta)$-sequence. One simple reason is that for many θ, the consistency strength of stationary reflection at θ is lower than that of the failure of $\square(\theta)$. Stationary reflection of the form $\mathsf{Refl}(\theta, 1, \{\omega\})$ was obtained in [3] via the method of iterative club shooting from a Mahlo cardinal in the case where θ is inaccessible or a successor of a regular cardinal, whereas the consistency strength of the failure of $\square(\theta)$ in these cases is a weakly compact cardinal. One can also show that starting from a Mahlo cardinal θ, if one adds a $\square(\theta)$ sequence using the standard forcing with initial segments, then the Harrington-Shelah construction from [3] over this extension does not add a thread to the generic $\square(\theta)$-sequence. We thus obtain a model where θ is inaccessible or a successor of a regular cardinal, and both $\square(\theta)$ and $\mathsf{Refl}(\theta, 1, \{\omega\})$ hold; see also the discussion in [15] on this topic in a slightly different context. However, the weakest nontrivial amount of simultaneous reflection implies the failure of $\square(\theta)$, as was pointed out to us by Menachem Magidor:

Proposition 1.2. *Let θ be regular and $X \subseteq \kappa$ be a set of regular cardinals. Then $\mathsf{Refl}(\theta, 2, X)$ implies $\neg\square(\theta)$.*

Proof. (Sketch.) Assume $\langle c_\alpha \mid \alpha < \kappa \rangle$ is a $\square(\theta)$-sequence. Given a limit ordinal $\alpha < \kappa$ let

$$\begin{aligned} R_0 &= \{\xi < \kappa \mid \mathsf{cf}(\xi) \in X \;\&\; \alpha \in \lim(C_\xi)\} \\ R_1 &= \{\xi < \kappa \mid \mathsf{cf}(\xi) \in X \;\&\; \alpha \notin \lim(C_\xi)\} \end{aligned}$$

If both R_0, R_1 were stationary then by $\mathsf{Refl}(\kappa, 2, X)$ there would be a common reflection point ϑ for R_0, R_1. So $\lim(c_\vartheta) \cap R_0$ would be nonempty, and by the coherency of the $\langle c_\alpha\rangle$-sequence we conclude that $\alpha \in \lim(c_\vartheta)$. The same reasoning applied to R_1 in place of R_0 yields that $\alpha \notin \lim(c_\vartheta)$, a contradiction.

It follows from the above that letting $c \subseteq \theta$ be defined by

$$\alpha \in c \iff \lim(\alpha)\ \&\ \alpha \in \lim(c_\xi) \text{ for all but non-stationarily many } \xi \in \mathsf{cof}(X),$$

then this definition makes sense, and it is easy to verify that c is a thread through $\langle c_\alpha \mid \alpha < \kappa\rangle$. □

It was pointed out to us by James Cummings that the above proposition actually leads to a characterization of $\square(\theta)$ in terms of simultaneous reflection for pairs.

Proposition 1.3. *Let $\theta > \omega_1$ be regular. The following are equivalent.*

(a) $\square(\theta)$.
(b) *There is a pair R_0, R_1 of disjoint stationary subsets of θ and a coherent sequence $\langle c_\alpha \mid \alpha < \theta\rangle$ such that each c_α is a closed unbounded subset of α, and $c_\alpha \cap R_0 = \varnothing$ or $c_\alpha \cap R_1 = \varnothing$ whenever $\alpha < \theta$.*

Proof. If $\langle c_\alpha \mid \alpha < \theta\rangle$ is as in (b) then it is easy to check that it does not have a thread. On the other hand, if $\langle c_\alpha \mid \alpha < \theta\rangle$ is a $\square(\theta)$-sequence then an argument similar to the proof of Proposition 1.2 provides sets R_0, R_1. □

Thus, whereas the failure of $\mathsf{Refl}(\theta, 2, X)$ guarantees that there is a pair of stationary sets such that each member of some fixed sequence of closed unbounded sets $c_\alpha \subseteq \alpha$ avoids at least one of them, $\square(\theta)$ guarantees that such avoidance can be achieved in a coherent way.

The observations from above yield the following diagram of implications.

$$\kappa \text{ quasicompact} \implies \mathsf{Refl}(\kappa^+, <\kappa, <\kappa) \implies \mathsf{Refl}(\kappa^+, 2, \{\omega\}) \implies \neg\square(\kappa^+) \tag{6}$$

$$\neg\square(\kappa^+), \mathsf{Refl}(\kappa^+, 1, <\kappa) \implies \kappa \text{ subcompact} \tag{7}$$

Concerning (6), the left arrow was justified in the text preceding Proposition 1.1; the middle one is trivial, and the right one is Magidor's result from Proposition 1.2. The implication in (7) holds under the assumption $\mathbf{V} = \mathbf{L}[E]$, and follows from (3) and from the well-known facts that $\neg\square(\kappa^+)$ as well as $\mathsf{Refl}(\kappa)$ imply $\neg\square_\kappa$. Our main theorem adds the converse to the middle and the right arrow in (6), and inserts an arrow between "$\neg\square(\kappa^+)$" and "$\mathsf{Refl}(\kappa^+, 1, <\kappa)$ in (7), granting that we are in an extender model. The theorem is a generalization of the main result in [7].

Theorem 1.4 (Main Theorem). *Assume $\mathbf{V} = \mathbf{L}[E]$ is a fine structural extender model with λ-indexing of extenders. Let κ be an infinite cardinal. The following are equivalent.*

(a) $\square(\kappa^+)$.
(b) $\neg(\mathsf{Refl}(\kappa^+, 2, \{\omega\})$.
(c) $\neg(\mathsf{Refl}(\kappa^+, <\kappa, <\kappa)$.

Moreover, if $\mathcal{R}$ is a family of stationary subsets of κ^+ witnessing (b) or (c) then there is a family $\mathcal{R}'$ of stationary subsets of κ^+ such that every $R' \in \mathcal{R}'$ is contained in some $R \in \mathcal{R}$, and if $\langle C_\alpha \mid \alpha < \kappa^+\rangle$ is the canonical $\square(\kappa^+)$-sequence constructed to witness (a) then for every $\alpha < \kappa^+$ we have $C_\alpha \cap R' = \varnothing$ for some $R' \in \mathcal{R}$.

In view of Propositions 1.2 and 1.3, our main theorem says – and this will be explicitly demonstrated in its proof – that in an extender model, if a pair of stationary subsets of κ^+ can be avoided by a sequence of closed unbounded subsets $\langle c_\alpha \mid \alpha < \kappa^+\rangle$ in the sense that each c_α is disjoint from at least one of these sets, then it can be avoided by a coherent sequence. Moreover, all non-trivial notions of simultaneous stationary set reflection "collapse", that is, they all either hold at κ^+ or fail at κ^+.

The heart of Theorem 1.4 is (c)⇒(a), whose proof comprises Section 3 of this paper. The implication (a)⇒(b) is Magidor's result in Proposition 1.2), and (b)⇒(c) is a triviality. The proof of Theorem 1.4 also yields the following corollary.

Corollary 1.5. *Assume $\mathbf{V} = \mathbf{L}[E]$ is a fine structural extender model with λ-indexing of extenders and κ is an infinite cardinal. If there is a non-reflecting stationary subset of $\kappa^+ \cap \mathsf{cof}(<\kappa)$. Then $\square(\kappa^+)$ holds. Moreover, if R is such a set then there is a stationary $R' \subseteq R$ such that $c_\alpha \cap R' = \varnothing$ for all $\alpha < \kappa^+$ where $\langle c_\alpha \mid \alpha < \kappa^+\rangle$ is the canonical $\square(\kappa^+)$-sequence.*

Theorem 1.4 an outcome of our effort to prove the following stronger variant which asserts that in an extender model, stationary set reflection at κ^+ is equivalent to the failure of $\square(\kappa^+)$. As we have already pointed out above, this is false in general, but it is not unreasonable to conjecture that this is the case in extender models. At inaccessible cardinals this is true by the results in [13]. Also, a somewhat distant analogy to this conjecture is the result from [10] that $\square_\kappa$ is equivalent to $\square_{\kappa,<\kappa}$ in $\mathbf{L}[E]$, that is, variants of weak square that do not follow from GCH cannot be separated in $\mathbf{L}[E]$.

Conjecture 1.6. *Assume $\mathbf{V} = \mathbf{L}[E]$ is a fine structural extender model with λ-indexing of extenders. Then $\mathsf{Refl}(\kappa^+, 1, <\kappa)$, or even $\mathsf{Refl}(\kappa^+, 1, \{\omega\})$ implies the failure of $\square(\kappa^+)$. Equivalently, $\mathsf{Refl}(\kappa^+, 1, \{\omega\})$ is equivalent to $\mathsf{Refl}(\kappa^+, <\kappa, <\kappa)$.*

We would like to make some remarks on the reversibility of implications in (6). First, we believe that the left arrow cannot be reversed, as a little inspection of the argument described above for the failure of $\square(\kappa^+)$ reveals that all one needs to run the argument is an embedding $\pi : (H_{\kappa^+}, \vec{C}) \to (H_\theta, \vec{C}')$ which is not cofinal. A simple reflection argument shows that such a map can be obtained already below the first quasicompact cardinal. However, we do not know if this is the right notion that characterizes the failure of $\square(\kappa^+)$ in the extender model. Concerning the arrow in (7), this one cannot be reversed. This is a consequence of the following proposition.

Proposition 1.7 ([9]). *Assume $\mathbf{V} = \mathbf{L}[E]$ and κ is the first subcompact cardinal, or more generally, κ is a subcompact cardinal such that no $\mu < \kappa$ is subcompact and κ-strong. Then T_κ is a nonreflecting stationary subset of $\kappa^+ \cap \mathsf{cof}(<\kappa)$.*

By the above proposition, $\mathsf{Refl}(\kappa^+, <\kappa)$ fails, and by Theorem 1.4 $\square(\kappa^+)$ holds at the first subcompact. Hence the arrow in (7) cannot be reversed toward any of the two statements. It is proved in [15] that $\neg\square_{\aleph_\omega,<\aleph_\omega}$ is consistent relative to the existence of a cardinal which is measurable and subcompact. In view of this fact, Proposition 1.7 may be understood as an indication that the consistency strength of $\mathsf{Refl}(\aleph_\omega, <\aleph_\omega)$ and $\neg\square(\aleph_{\omega+1})$ is higher than that of the failure of $\neg\square_{\aleph_\omega,<\aleph_\omega}$. We defer the proof of Proposition 1.7 to Section 3 where we will have the sufficient technical background.

Finally some word on the organization of this paper. Section 2 contains the technical background that will be used in the main construction, which in turn will be described in Section 3. The construction will be organized in the way that focuses on the new issues arising at cardinal successors of subcompact cardinals;

obviously this is the heart of the construction. In the course of the construction also tools developed in [10] will be used; however we will summarize the relevant "black boxes" in Section 2, and these will be used in the construction in Section 3 without going into local details. We hope this will make the paper self-contained and comprehensible even without knowledge of the details in [10]. Finally let us note that it is not necessary for our construction that we work with λ-indexing of extenders; everything can be also done with the Mitchell-Steel indexing from [8] as well. However, the advantage of the λ-indexing is that the entire presentation becomes notationally much simpler, thereby allowing to focus on the actual issues in the construction.

2. Technical prerequisites

In this section we summarize facts about the extender model and about constructions of square sequences that will be used in our main construction as black boxes. We omit most of the proofs and details which can be found in [14] and [10].

We first recall that initial levels of $\mathbf{L}[E]$, and in fact all premice we are considering here are coherent structures in the sense of [14], that is, they are of the form $N = \langle J^E_\nu, F\rangle$ where E is the extender sequence of the model N, the predicate F is an extender in functional representation, and, letting $\kappa = \mathsf{cr}(F)$ and τ be the largest ordinal $\leq \kappa^{+N}$ such that F measures all sets in $\mathcal{P}(\kappa) \cap J^E_\tau$, we have $J^E_\nu = \mathsf{Ult}(J^E_\tau, F)$. Such structure is always amenable, and F is weakly amenable if and only if $\tau = \kappa^{+N}$. So if N is a premouse then $\tau = \kappa^{+N}$. But our construction will also make use of coherent structures of special kind whose top extender is not weakly amenable; following [10] these are called protomice.

Our fine structural notation is taken from [14]. In particular, ϱ^n_M is the n-th projectum of the structure M, ϱ^ω_M is the ultimate projectum of M, p^n_M is the n-standard parameter of M, p_M is the full standard parameter of M, and $\tilde{h}^{n+1}_M$ is the uniform $\Sigma^{(n)}_1$-Skolem function for M. We write $\tilde{h}^{n+1}_M(X \cup \{p\})$ to denote the $\Sigma^{(n)}_1$-Skolem hull of $X \cup \{p\}$, that is, the set of all values $\tilde{h}^{n+1}_M(i, \langle x, p\rangle)$ where $i \in \omega$ and $x \in X$. Recall also three basic facts that will be used throughout the argument. First, ϱ^{n+1}_M is the least ordinal ϱ such that there is some set of ordinals A which is $\Sigma^{(n)}_1$-definable over M in the parameter p^{n+1}_M such that $A \cap \omega\varrho \notin M$. Second, M is n-sound means that $\tilde{h}^n_M(\omega\varrho^n_M \cup \{p^n_M\}) = M$; more generally, given an ordinal ν we say that M is sound above ν if and only if $\tilde{h}^{n+1}_M(\nu \cup \{p^{n+1}_M\}) = M$ where n is such that $\omega\varrho^{n+1}_M \leq \nu < \omega\varrho^n_M$. Third, a parameter $p \in M$ is n-universal just in the case where the transitive collapse of $\tilde{h}^n_M(\omega\varrho^n_M \cup \{p\})$ contains all sets in $\mathcal{P}(\omega\varrho^n_M) \cap M$.

If M is an acceptable structure and F is an M-extender, $\mathsf{Ult}(M, F)$ denotes the coarse ultrapower of M by F and $\mathsf{Ult}^*(M, F)$ denotes the fine structural ultrapower of M by F. In the case of the fine structural ultrapower, the functions representing the objects in the ultrapower are all good $\boldsymbol{\Sigma}^{(n)}_1(M)$-definable functions with domain $\mathsf{cr}(F)$ where n is such that $\omega\varrho^{n+1}_M > \mathsf{cr}(F)$. For details see [14].

We also briefly review our terminology concerning solidity; the details can be found in [14], Section 1.12. Given a parameter $p \in [\mathbf{On}]^{<\omega} \cap M$, the standard solidity witness $W^{\nu,p}_M$ for an ordinal ν with respect to p and M is the transitive collapse of the hull $\tilde{h}^{n+1}_M(\nu \cup \{p - (\nu+1)\})$ where n is such that $\omega\varrho^{n+1}_M \leq \nu < \omega\varrho^n_M$. A generalized solidity witness for ν with respect to p and M is any pair (Q, r) where Q is an acceptable structure and $r \subseteq Q$ is a finite set of ordinals of the same size as $p-(\nu+1)$, such that for every $\Sigma_1^{(n)}$-formula $\varphi(u,v_1,\dots,v_\ell)$ and every tuple of ordinals $\xi_1,\dots,\xi_\ell<\nu$ if $M\models\varphi(p-(\nu+1),\xi_1,\dots,\xi_\ell)$ then $Q\models\varphi(r,\xi_1,\dots,\xi_\ell)$. The standard witness $W_M^{\nu,p}$ can be viewed as a generalized witness (W,r) where r is the image of $p-(\nu+1)$ under the collapsing isomorphism. The point of the notion of generalized witness is that it can be characterized by a $\Pi_1^{(n)}$ formula, whereas the standard witness not. It is easy to see that the standard witness for ν with respect to p and M can be constructed from any generalized witness (Q,r) for ν with respect to p and M inside any acceptable structure which has (Q,r) as an element. Recall also that an acceptable structure is solid if and only if $W_M^{\nu,p}\in M$ whenever $\nu\in p_M$, which is equivalent to saying that some generalized witness (Q,r) for ν with respect to p and M is an element of M whenever $\nu\in p_M$.

One of our main tools is the condensation lemma for premice embeddable into levels of $\mathbf{L}[E]$; this is a simplified variant of a more general condensation lemma in [14].

Lemma 2.1 (Condensation Lemma)**.** *Let $\sigma:\bar M\to M$ be a $\Sigma_0^{(n)}$-preserving embedding where M is an initial segment of $\mathbf{L}[E]$ and $\bar M$ is a premouse. Let E be the extender sequence of $\mathbf{L}[E]$, so $E^M=(E\restriction(\mathbf{On}\cap M)+1)$.*

If

(i) $\sigma\restriction\nu=\mathsf{id}$ *where* $\omega\varrho_{\bar M}^{n+1}\le\nu$.

Then $\bar M$ is sound, solid, and its standard parameter is universal.

If additionally

(ii) *$\bar M$ is sound above ν.*

Then one of the following holds.

(a) *$\bar M$ is a proper initial segment of $\mathbf{L}[E]$.*
(b) *$\nu=\mathsf{cr}(\sigma)$, $E_\nu\neq\varnothing$ and $\bar M$ is a proper initial segment of* $\mathsf{Ult}(M,E_\nu)$.

The following consequence of the condensation lemma will also be useful. It can be proved by applying Lemma 2.1 to levels of $\bar M$ that project to $\nu^{+\bar M}$.

Lemma 2.2. *Assume $\sigma:\bar M\to M$ is a Σ_0-preserving embedding where M is an initial segment of $\mathbf{L}[E]$ and $\bar M$ is a premouse. Let $\nu=\mathsf{cr}(\sigma)$ and $\tau=\nu^{+\bar M}$. Then one of the following holds.*

(a) *$E_\tau=\varnothing$, in which case $E^{\bar M}\restriction\tau=E\restriction\tau$.*
(b) *$E_\tau\neq\varnothing$, in which case $E\restriction\tau=E^{M'}\restriction\tau$ where $M'=\mathsf{Ult}(M,E_\nu)$.*

In the following we briefly review the notion of divisor. The details can be found in [10]. First of all we recall that we will make use of two languages. The language for coherent structures is the language of set theory enriched with symbols $\dot E,\dot F$ where $\dot E$ denotes the extender sequence of the structure and $\dot F$ the top extender. The language of premice contains additionally a constant symbol for the index of the extender obtained by cutting the top extender at its largest cutpoint in the case where the largest cutpoint exists. Since the language of premice is richer, one may compute fine structural characteristics of premice in the language of coherent structures. This will not change the projecta, but may change the standard parameter and other characteristics. We thus make the following conventions. All fine structural characteristics of premice are computed in the language of premice. Also the notion of a $\Sigma_k^{(n)}$-preserving embedding between premice is understood to be with respect to the language of premice. On the other hand, all fine structural characteristics of protomice will be computed in the language of coherent structures. Protomice have been mentioned above, and we briefly mention them also below. Again, the notion of a Σ_k-preserving map between protomice will also be considered in the language of coherent structures. Whenever the situation is ambiguous, we will explicitly mention which language is used.

Let N be an initial segment of the extender model $\mathbf{L}[E]$ we work in such that N is a collapsing level of $\mathbf{L}[E]$ for $\tau = \kappa^{+N}$. Here recall that κ is the cardinal for which we aim to construct a $\square(\kappa^+)$-sequence. Given an active premouse N, we let h^*_N denote the uniform Σ_1-Skolem function computed in the language for coherent structures. The premouse N is called pluripotent if and only if N is active, its top extender has critical point strictly below κ, and $\omega\varrho^1_N = \kappa$. If N is pluripotent, the Dodd parameter d_N of N is the first standard parameter for N computed in the language for coherent structures. Let n be the least integer such that $\omega\varrho^{n+1}_N = \kappa$.

We say that (μ, q) is a divisor for N just in case one of the following holds.

(a) N is not pluripotent. In this case, letting $r = p_N - q$ and $X = \tilde{h}^{n+1}_N(\mu \cup \{r\})$, we have $X \cap (\max(q)+1) = \mu$ and $X \cap \omega\varrho^n_N$ is cofinal in $\omega\varrho^n_N$.
(b) N is pluripotent. In this case, letting $r = d_N - q$ and $X = h^*_N(\mu \cup \{r\})$, we have $X \cap (\max(q)+1) = \mu$ and X is cofinal in N.

Denote the transitive collapse of the structure X from the definition of a divisor by $N^*(\mu, q)$ and the inverse of the Mostowski collapsing isomorphism by $\pi(\mu, q)$. Let $\lambda = \pi(\mu, q)(\mu)$ and $\nu = \lambda^{+N}$. The protomouse associated with N and (μ, q) is the coherent structure (J^E_ν, F) where $E = E^N$ and F is the $N^*(\mu, q)$-extender at (μ, λ) derived from the map $\pi(\mu, q)$. We denote this protomouse by $N(\mu, q)$. The top extender F of $N(\mu, q)$ is not weakly amenable. It is proved in [10] that $N = \mathsf{Ult}^*(N^*(\mu, q), F)$ and $\pi(\mu, q)$ is the associated ultrapower embedding.

A divisor (μ, q) for N is called strong just in case one of the following holds.

(a) N is not pluripotent. In this case we require that $x \cap \mu \in N^*(\mu, q)$ whenever $x \in \tilde{h}^{n+1}_N(\mu \cup \{p_n\})$.
(b) N is pluripotent. In this case we require that $x \cap \mu \in N^*(\mu, q)$ whenever $x \in h^*_N(\mu \cup \{d_N\})$.

It is proved in [10] that if N admits a strong divisor then among all strong divisors there is one with largest μ. This μ is still strictly smaller than κ, and "top" portion q of such divisor is shortest possible. Moreover, this divisor is unique, and is called the canonical divisor for N. We denote it by (μ_N, q_N). Additionally we define the notion of canonical divisor for pluripotent premice N which do not admit a strong divisor to be the pair $(\mu_N, q_N) = (\mathsf{cr}(E^N_{\mathrm{top}}), d_N)$. This pair may not even be a divisor, but it is convenient to make this definition, as it unifies the notation and terminology. If N admits a strong divisor we define the canonical protomouse for N to be the protomouse $N(\mu_N, q_N)$. For pluripotent N which does not admit a strong divisor we then let $N(\mu_N, q_N) = N$ and call $N(\mu_N, q_N)$ the canonical protomouse for N. This is the only case where the canonical protomouse is identical with the premouse it comes from, and again we introduce this notion in order to unify our notation and terminology.

3. The construction

Throughout this section we will work in an extender model $W = \mathbf{L}[E]$. The following lemma is the key for the choice of the canonical structure for elements of T_κ. As already indicated in the introduction, these structures will not be collapsing, but rather will provide a witnesses that the sets in $\mathcal{R}$ do not have a common reflection point.

Lemma 3.1. *Assume $\mathcal{R} \subseteq \mathcal{P}(\kappa^+ \cap \mathsf{cof}(<\kappa))$ is a a family of size $<\kappa$ consisting of stationary sets such that no $\nu \in \kappa \cap \mathsf{cof}(<\kappa)$ is a common reflection point for all sets in $\mathcal{R}$. Then*

$$K(\mathcal{R}) = \{\xi < \kappa^+ \mid E_\xi \neq \varnothing \text{ and } R \cap \xi \text{ is stationary in } \mathsf{Ult}(W, E_\xi) \text{ for all } R \in \mathcal{R}\}$$

is nonstationary in W.

Proof. Assume not. Let $\mathcal{R}$ be the $<_E$-least family as in the statement of the lemma such that $K(\mathcal{R})$ is stationary. Let $\theta >> \kappa$ be regular and X be an elementary substructure of J^E_θ of size κ such that $\kappa + 1 \subseteq X$ and $\nu = X \cap \kappa^+ \in K(\mathcal{R})$. Let M be the transitive collapse of X and $\sigma : M \to J^E_\theta$ be the inverse of the Mostowski collapsing isomorphism. Then $\mathcal{R} \in X$, $\mathcal{R} \subseteq X$, and $\sigma(R \cap \nu) = R$ whenever $R \in \mathcal{R}$. In fact, letting $\bar{\mathcal{R}} = \sigma^{-1}(\mathcal{R})$, we have $\bar{\mathcal{R}} = \{R \cap \nu \mid R \in \mathcal{R}\}$. By Lemma 2.2, $E' \restriction \tau = E^M \restriction \tau$ where $\mathbf{L}[E'] = \mathsf{Ult}(W, E_\nu)$ and $\tau = \nu^{+W'}$ where we write W' for $\mathbf{L}[E']$. This along with the assumptions of Lemma 3.1 tells us that all sets in $\bar{\mathcal{R}}$ are stationary in W'. Also, from the point of view of M, the sets in $\bar{\mathcal{R}}$ do not have a common nonreflection point, as follows from the elementarity of σ. Since M and W' agree up to ν and ν is a cardinal both in M and W', the sets in $\mathcal{R} \cap \nu$ do not have a common reflection point from the point of view of W'.

As a next step we show that in the sense of W', the set $\bar{\mathcal{R}}$ is the $<_{E'}$-least family $\mathcal{R}'$ of subsets of $\nu \cap \mathsf{cof}(<\kappa)$ such that $\mathsf{card}(\mathcal{R}') < \kappa$ and the sets in $\mathcal{R}'$ do not have a common reflection point of cofinality $<\kappa$. Assume $\mathcal{R}' \in W'$ is a family of stationary subsets of $\nu \cap \mathsf{cof}(<\kappa)$ such that $\mathsf{card}^{W'}(\mathcal{R}') < \kappa$, the sets in $\mathcal{R}'$ have no common reflection point of cofinality $<\kappa$, and $\mathcal{R}' <_{E'} \bar{\mathcal{R}}$. We have seen above that $E^M \restriction \tau = E' \restriction \tau$. Since $\bar{\mathcal{R}} \in M \,||\, \tau$ and $\mathcal{R}' <_{E'} \bar{\mathcal{R}}$, necessarily $\mathcal{R}' \in M$, $\mathcal{R}' <_{E^M} \bar{\mathcal{R}}$, and

$M \models$ "$\mathcal{R}'$ is a family of stationary subsets of $\nu \cap \mathsf{cof}(<\kappa)$ such that $\mathsf{card}(\mathcal{R}') < \kappa$ and the sets in $\mathcal{R}'$ have no common reflection point of cofinality $<\kappa$".

By the elementarity of σ, in the sense of W we have $\sigma(\mathcal{R}') <_E \mathcal{R}$, $\mathsf{card}^W(\sigma(\mathcal{R}')) < \kappa$, all sets in $\sigma(\mathcal{R}')$ are stationary subsets of $\kappa^+ \cap \mathsf{cof}(<\kappa)$, and the family $\sigma(\mathcal{R}')$ does not have a common reflection point of cofinality $<\kappa$. As $\mathcal{R}$ was chosen $<_E$-minimal with these properties such that $K(\mathcal{R})$ is stationary, the set $K_{\sigma(\mathcal{R}')}$ must be nonstationary, so there is a closed unbounded $C \subseteq \kappa^+$ such that $C \cap K_{\sigma(\mathcal{R}')} = \varnothing$. The $<_E$-least such C, being definable in W' from the parameter $\sigma(\mathcal{R}')$, is in $\mathrm{rng}(\sigma)$. Hence $\nu = \kappa^+ \cap \mathrm{rng}(\sigma)$ is an element of this set C. It follows that $\nu \notin K_{\sigma(\mathcal{R}')}$, so $\mathcal{R}' = \{R \cap \nu \mid R \in \sigma(R')\}$ must contain a set R' that is nonstationary in W'. This contradicts our choice of $\mathcal{R}'$, and thereby completes the proof of the $<_{E'}$-minimality of $\bar{\mathcal{R}}$.

Let $\pi : W \to W'$ be the ultrapower embedding associated with $\mathsf{Ult}(W, E_\nu)$. Let $\mu = \mathsf{cr}(\pi) = \mathsf{cr}(E_\nu)$ and $\vartheta = \mu^{+W}$. Then $\pi(\mu) = \kappa$, $\pi(\vartheta) = \nu$, and π maps ϑ

cofinally into ν. By the previous paragraph, the family $\bar{\mathcal{R}}$ is definable in W' from the parameter $\nu \in \operatorname{rng}(\pi)$, so $\bar{\mathcal{R}} \in \operatorname{rng}(\pi)$. Let $\mathcal{R}^* = \pi^{-1}(\bar{\mathcal{R}})$. By the elementarity of π, the set $\mathcal{R}^*$ is a family of size $< \mu$ of stationary subsets of $\vartheta \cap \mathsf{cof}(< \mu)$ in the sense of W. Since $\mathcal{R}^*$ is of size $< \mu$, we have $\bar{\mathcal{R}} = \pi(\mathcal{R}^*) = \pi[\mathcal{R}^*]$. Since π is continuous at ordinals of cofinality $< \mu$, maps ϑ cofinally into ν and $\pi \in W$, for every $R \in \mathcal{R}^*$ the set $\pi[R]$ is a stationary (in the sense of W) subset of ν, and so is $\pi(R) \cap \nu \supseteq \pi[R]$. Since $\bar{\mathcal{R}} = \pi[\mathcal{R}^*]$, we see that ν is a common reflection point of cofinality $< \kappa$ for all sets in $\bar{\mathcal{R}}$. This is a contradiction. $\square$

Based on the above lemma, we can identify the canonical structures assigned to local cardinal successors. The construction in [10] is based on the fact that T_κ is non-stationary. In this situation there is a closed unbounded subset $\mathcal{S} \subseteq \kappa^+$ of local cardinal successors disjoint from T_κ, and the construction of a $\square_\kappa$-sequence is carried out on this set. The ordinals in $\mathcal{S}$ are split into two disjoint classes $\mathcal{S}^0, \mathcal{S}^1$ depending on whether the canonical collapsing structure is chosen to be the collapsing level of $\mathbf{L}[E]$ or the canonical collapsing protomouse. On each of these classes the construction is carried out separately, and it is shown that the two constructions do not interfere. In our current situation we follow the same scenario, and also the same notation whenever possible. However this time we split our closed unbounded set of local cardinal successors – which we again denote by $\mathcal{S}$, but whose intersection with T_κ will be now nonempty – into three disjoint classes $\mathcal{S}^0, \mathcal{S}^1, \mathcal{S}^2$, carry out the construction on each of these classes separately, and show that the three constructions do not interfere. The constructions on $\mathcal{S}^0$ and $\mathcal{S}^1$ will be identical with those in [10] whereas the construction on $\mathcal{S}^2$ will be new and will cover the ordinals in T_κ. The slight difference between the current situation and the situation in [10] is that the sets $\mathcal{S}^0$ and $\mathcal{S}^1$ will be smaller (in the sense of inclusion) than the sets denoted by the same symbols in [10], as some of the ordinals which in [10] are elements of $\mathcal{S}^0 \cup \mathcal{S}^1$ will move to $\mathcal{S}^2$ in our current construction.

Recall that $W = \mathbf{L}[E]$ is our extender model. Fix a family $\mathcal{R}$ of size $< \kappa$ consisting of stationary sets $R \subseteq \kappa^+ \cap \mathsf{cof}(< \kappa)$ with no common reflection point of cofinality $< \kappa$, and a closed unbounded set $C(\mathcal{R}) \subseteq \kappa^+$ such that $C(\mathcal{R}) \cap K(\mathcal{R}) = \varnothing$. The existence of $C(\mathcal{R})$ is guaranteed by Lemma 3.1. Since $\mathcal{R}$ is a family of subsets of κ^+ if size $< \kappa$, it can be encoded into a single subset of κ^+, hence $\mathcal{R} \in J^E_{\kappa^{++}}$. The following definition thus makes sense.

$$\mathcal{S} = \{\tau \in C(\mathcal{R}) \mid \kappa^+ \cap \tilde{h}^1_{W\,||\,\kappa^{++}}(\tau \cup \{\mathcal{R}\}) = \tau\}. \tag{8}$$

Obviously $\mathcal{S}$ is a closed unbounded subset of κ^+ whose elements are local cardinal successors of κ, and in fact J^E_τ is an elementary substructure of $J^E_{\kappa^+}$ whenever $\tau \in \mathcal{S}$. For $\tau \in \mathcal{S}$ define

$$\mathcal{R}_\tau = \{R \cap \tau \mid R \in \mathcal{R}\}. \tag{9}$$

We are going to split $\mathcal{S}$ into disjoint sets $\mathcal{S}^0, \mathcal{S}^1, \mathcal{S}^2$. The following lemma will help to streamline this procedure. Recall also that throughout the rest of the construction W^τ will denote the ultrapower of W by E_τ and E^τ its extender sequence, that is,

$$W^\tau = \mathbf{L}[E^\tau] = \mathsf{Ult}(W, E_\tau) \tag{10}$$

whenever $\tau \in \mathcal{S} \cap T_\kappa$, that is, whenever $\tau \in \mathcal{S}$ indexes an extender.

Lemma 3.2. *If $\tau \in \mathcal{S} - T_\kappa$ then there is some $\zeta \in \mathbf{On}$ such that $\tau = \kappa^{+W\,||\,\zeta}$, $\mathcal{R}_\tau \in W\,||\,\zeta$, and R is a stationary subset of τ in $W\,||\,\zeta$ whenever $R \in \mathcal{R}_\tau$.*

Similarly, if $\tau \in \mathcal{S} \cap T_\kappa$ then there is some $\zeta \in \mathbf{On}$ such that $\tau = \kappa^{+W^\tau\,||\,\zeta}$, $\mathcal{R}_\tau \in W^\tau\,||\,\zeta$, and R is a stationary subset of τ in $W^\tau\,||\,\zeta$ whenever $R \in \mathcal{R}_\tau$.

Proof. Let $\sigma : \bar{M} \to M$ be the inverse of the Mostowski collapsing isomorphism coming from collapsing $\tilde{h}^1_{W\,||\,\kappa^{++}}(\tau \cup \{\mathcal{R}\})$. The map σ is Σ_1-preserving, and since $W\,||\,\kappa^+ = J^E_{\kappa^+}$ is a passive premouse, so is $\bar{M}$. Obviously $\tau = \mathsf{cr}(\sigma) = \kappa^{+\bar{M}}$. Since $\mathcal{R} \in \mathrm{rng}(\sigma)$ the preimage $\sigma^{-1}(\mathcal{R}_\tau)$ is defined, and since the size of $\mathcal{R}$ is $< \kappa$, is equal to $\mathcal{R}_\tau$. As the notion of being stationary is Π_1, it is downward transferred under σ. So R is stationary in $\bar{M}$ whenever $R \in \mathcal{R}_\tau$. Let $\zeta = \mathsf{ht}(\bar{M})$. By Lemma 2.2, $E^{\bar{M}} \restriction \omega\zeta = E \restriction \omega\zeta$ if $\tau \notin T_\kappa$, and $E^{\bar{M}} \restriction \omega\zeta = E^\tau \restriction \omega\zeta$ if $\tau \in T_\kappa$. Since τ is the largest cardinal in $\bar{M}$, this completes the proof. $\square$

Lemma 3.2 says that for $\tau \in \mathcal{S} - T_\kappa$, the ordinal τ is not yet collapsed earlier than the family $\mathcal{R}_\tau$ enters the $\mathbf{L}[E]$-hierarchy, and a witness that the sets $\mathcal{R}_\tau$ have no common reflection point if cofinality $< \kappa$ does not enter the $\mathbf{L}[E]$-hierarchy earlier either. For $\tau \in \mathcal{S} \cap T_\kappa$ the lemma gives a similar conclusion with W^τ in place of W, but the collapsing of τ is in this case not an issue, as τ is a cardinal in W^τ.

For each $\tau \in \mathcal{S}$ we now identify the level of W or of W^τ which will be used to define the canonical structure associated with τ in the definition of a $\square(\kappa^+)$-sequence. Given $\tau \in \mathcal{S}$,

(11) N_τ is the longest initial segment N of
  (a) W if $\tau \in \mathcal{S} - T_\kappa$
  (b) W^τ if $\tau \in \mathcal{S} \cap T_\kappa$
such that $\tau = \kappa^{+N}$, $\mathcal{R}_\tau \in N$, and all sets in $\mathcal{R}_\tau$ are stationary in the sense of N.

It follows from the definition of N_τ that either τ is collapsed via a $\mathbf{\Sigma}^{(n)}_1(N_\tau)$-definable function for some $n \in \omega$, in which case $\omega\varrho^{n+1}_{N_\tau} = \kappa$, or else τ is a $\mathbf{\Sigma}^*$-cardinal over N_τ, in which case there is a closed unbounded set $d \subseteq \tau$ such that $d \cap R = \varnothing$ for some $R \in \mathcal{R}_\tau$ and d is $\mathbf{\Sigma}^{(n)}_1$-definable over N_τ for some $n \in \omega$. In the latter case the set d witnesses that $\omega\varrho^{n+1}_{N_\tau} = \tau$, and since τ is a $\mathbf{\Sigma}^*$-cardinal over N_τ we actually have $\omega\varrho^{n+1}_{N_\tau} = \omega\varrho^{\omega}_{N_\tau} = \tau$. By the soundness of N_τ we may without loss of generality assume that the set d is $\Sigma^{(n)}_1(N_\tau)$-definable in parameters p_{N_τ} and some $\xi_d < \nu$. We will use the observation just made to split $\mathcal{S}$ into $\mathcal{S}^0, \mathcal{S}^1$ and $\mathcal{S}^2$. There are of course other options to consider here, and the one we are going to use may seem too crude, but it turns out to work, whereas the other ones seem to have substantial deficiencies when it comes to verification that the definitions of our $\square(\kappa^+)$-sequence on the sets $\mathcal{S}^i$ do not interfere. For $\tau \in \mathcal{S}$ we can now define basic fine structural characteristics; see Section 2 for the notation.

- n_τ is the least $n \in \omega$ such that exactly one of the following holds:
  (i) $\omega\varrho^{n+1}_{N_\tau} = \kappa$.
  (ii) $\omega\varrho^{\omega}_{N_\tau} = \tau$ and there is a $\mathbf{\Sigma}^{(n)}_1(N_\tau)$-definable closed unbounded $d \subseteq \tau$ that is disjoint from some $R \in \mathcal{R}_\tau$.
- $\rho_\tau = \rho^{n_\tau}_{N_\tau}$.
- $p_\tau = p_{N_\tau}$.
- $\tilde{h}_\tau = \tilde{h}^{n_\tau+1}_{N_\tau}$.

It follows immediately from the definition of N_τ that

(12) $$R \text{ is stationary in } N_\tau \text{ whenever } R \in \mathcal{R}_\tau.$$

We can now define splitting of $\mathcal{S}$ into $\mathcal{S}^0, \mathcal{S}^1$ and $\mathcal{S}^2$. For information about strong divisors see again Section 2, or for more details, the paper [10]. Notice that by our definition of n_τ,

(13) $$\omega\varrho_{N_\tau}^{n_\tau+1} = \omega\varrho_{N_\tau}^{\omega}, \text{and this value is either } \kappa \text{ or } \tau.$$

Given $\tau \in \mathcal{S}$, we let

(14)

$\tau \in \mathcal{S}^2$ iff $\omega\varrho_{N_\tau}^{n_\tau+1} = \tau$

$\tau \in \mathcal{S}^1$ iff $\omega\varrho_{N_\tau}^{n_\tau+1} = \kappa$, and N_τ is pluripotent or N_τ admits a strong divisor

$\tau \in \mathcal{S}^0$ iff $\omega\varrho_{N_\tau}^{n_\tau+1} = \kappa$, and N_τ does not admit a strong divisor

The following important observation is an immediate consequence of our definition of $\mathcal{S}$ combined with the fact that τ is a cardinal in $\mathsf{Ult}(\mathbf{L}[E], E_\tau)$ whenever $\tau \in T_\kappa$.

(15) $$T_\kappa \subseteq \mathcal{S}^2.$$

For $\tau \in \mathcal{S}^1$ we define the following characteristics.

- (μ_τ, q_τ) is the canonical divisor for N_τ.
- $M_\tau = N_\tau(\mu_\tau, q_\tau)$.

Next we define the canonical structures for $\tau \in \mathcal{S}$ as follows. For $\tau \in \mathcal{S}^0 \cup \mathcal{S}^1$ the canonical structure for τ will be a collapsing structure for τ, whereas for $\tau \in \mathcal{S}^2$ the canonical structure will definably destroy the stationarity of some set in $\mathcal{R}_\tau$, and τ will remain a cardinal definably over its canonical structure.

(a) If $\tau \in \mathcal{S}^0 \cup \mathcal{S}^2$ then the canonical structure for τ is N_τ.
(b) If $\tau \in \mathcal{S}^1$ then the canonical structure for τ is M_τ.

For $\tau \in \mathcal{S}$ we can now define the sets B_τ which are the first approximations to the $\square(\kappa^+)$-sequence; this is analogous to the situation in [10].

(A) $\tau \in \mathcal{S}^0$. We let $\bar\tau \in B_\tau$ precisely when $\bar\tau \in \mathcal{S}^0$ and the following conditions are satisfied.
  (a) $n_{\bar\tau} = n_\tau$.
  (b) There is a $\Sigma_0^{(n_\tau)}$-preserving map (relative to the language of premice!) $\sigma : N_{\bar\tau} \to N_\tau$ such that:
    (i) $\mathsf{cr}(\sigma) = \bar\tau$ and $\sigma(\bar\tau) = \tau$.
    (ii) $\sigma(p_{\bar\tau}) = p_\tau$.
    (iii) $\sigma(\mathcal{R}_{\bar\tau}) = \mathcal{R}_\tau$.
    (iv) For every $\beta \in p_\tau$ there is some generalized witness (Q, t) with respect to p_τ and N_τ such that $(Q, t) \in \mathrm{rng}(\sigma)$.

(B) $\tau \in \mathcal{S}^1$. We let $\bar\tau \in B_\tau$ precisely when $\bar\tau \in \mathcal{S}^1$ and there is a Σ_0-preserving map (relative to the language of coherent structures !) $\sigma : M_{\bar\tau} \to M_\tau$ such that:
  (i) $\mathsf{cr}(\sigma) = \bar\tau$ and $\sigma(\bar\tau) = \tau$.
  (ii) $\sigma(p_{\bar\tau}) = p_\tau$.
  (iii) $\sigma(\mathcal{R}_{\bar\tau}) = \mathcal{R}_\tau$.
  (iv) For every $\beta \in p_\tau$ there is some generalized witness (Q, t) with respect to p_τ and N_τ such that $(Q, t) \in \mathrm{rng}(\sigma)$.

(C) $\tau \in \mathcal{S}^2$. We let $\bar{\tau} \in B_\tau$ precisely when $\bar{\tau} \in \mathcal{S}^2$ and the following conditions are satisfied.
  (a) $n_{\bar{\tau}} = n_\tau$.
  (b) There is a Σ^*-preserving map (relative to the language of premice!) $\sigma : N_{\bar{\tau}} \to N_\tau$ such that:
    (i) $\mathsf{cr}(\sigma) = \bar{\tau}$ and $\sigma(\bar{\tau}) = \tau$.
    (ii) $\sigma(p_{\bar{\tau}}) = p_\tau$.
    (iii) $\sigma(\mathcal{R}_{\bar{\tau}}) = \mathcal{R}_\tau$.
    (iv) For every $\beta \in p_\tau$ there is some generalized witness (Q, t) with respect to p_τ and N_τ such that $(Q, t) \in \mathrm{rng}(\sigma)$.
    (v) There is a $\Sigma_1^{(n_\tau)}$-formula $\psi(u, v_0, v_1)$ and a parameter $q \in \mathrm{rng}(\sigma)$ such that $d = \{\xi < \tau \mid N_\tau \models \psi(\xi, p_\tau, q)\}$ is a closed unbounded subset of τ disjoint from some $R \in \mathcal{R}_\tau \cap \mathrm{rng}(\sigma)$.

The requirements in (A) and (B) are almost precisely the same as in the corresponding cases in [10] with the only exception that in the current situation we additionally require $\sigma(\mathcal{R}_{\bar{\tau}}) = \mathcal{R}_\tau$ in the requirement (ii) on σ. We stress that the preservation degree in (C) is required to be Σ^* and not $\Sigma_0^{(n_\tau)}$. Due to high preservation degree of the map σ in (C), clauses (C)(a) and (C)(b)(iv),(v) are redundant; we nevertheless decided to include them in the definition in order to keep the uniformity of the definition, transparency of the proof of the main theorem, and also since dropping them is not going to save any significant amount of work.

The following observation will be used to prove that the coherent sequence we will define does not have a thread.

(16) If $\tau \in \mathcal{S}^2$ and R is as in (C)(b)(v) then $B_\tau \cap R = \varnothing$.

To see this, let $\bar{\tau} \in B_\tau$, $\sigma : N_{\bar{\tau}} \to N_\tau$ be as in (C)(b), and $\psi(u, v_0, v_1)$ and q be as in (C)(b)(v). Given $\zeta < \bar{\tau}$, the statement "there is a $\xi \in d$ such that $\xi > \zeta$" can be expressed as $(\exists \xi^{n+1})(\zeta < \xi^{n+1} < \tau \;\&\; \psi(\xi^{n+1}, p_\tau, q))$, so it is a $\Sigma_1^{(n+1)}$-statement. Since $q, p_\tau \in \mathrm{rng}(\sigma)$ and $\mathrm{rng}(\sigma)$ is a Σ^*-elementary substructure of N_τ, some such ξ^{n+1} is in $\mathrm{rng}(\sigma) \cap \tau = \bar{\tau}$. As such ξ^{n+1} can be found for every $\zeta < \bar{\tau}$, it follows that $\bar{\tau} \in \lim(d)$, so $\bar{\tau} \in d$ by the requirement that d is closed. But then $\bar{\tau} \notin R$.

Given $\bar{\tau} < \tau$ in $\mathcal{S}$, there is at most one map σ with the required preservation degree satisfying (i) and (ii) in (A) – (C) above; this follows from the fact that the domain structure of σ is sound. So the following definition makes sense.

(17) $\sigma_{\bar{\tau},\tau}$ = the unique map σ with the corresponding preservation degree from the canonical collapsing structure for $\bar{\tau}$ to the canonical collapsing structure for τ satisfying (i) and (ii) in (A) –(C).

Furthermore, if $\tau^* < \bar{\tau}$ are in B_τ then $\mathrm{rng}(\sigma_{\tau^*,\tau}) \subseteq \mathrm{rng}(\sigma_{\bar{\tau},\tau})$. In the cases (A) and (B) this, along with the fact that $\sigma_{\bar{\tau},\tau}$ is not cofinal at the corresponding level, is proved in [10], and requires some work. In the case (C) this is an easy consequence of the requirement that $\sigma_{\bar{\tau},\tau}$ must be Σ^*-preserving. Based on these facts, one can define

(18) If $\tau \in \mathcal{S}$ and $\tau^* < \bar{\tau}$ are in B_τ we let $\sigma_{\tau^*,\bar{\tau}} = \sigma_{\bar{\tau},\tau}^{-1} \circ \sigma_{\tau^*,\tau}$,

and establish the following facts:

(a) The maps $\sigma_{\tau^*,\bar\tau}$ are $\Sigma_0^{(n_\tau)}$-preserving and not cofinal at the n_τ-th level in the case (A), Σ_0-preserving and not cofinal in the case (B), and Σ^*-preserving in the case (C).
(b) $\sigma_{\tau^*,\tau'} = \sigma_{\tau^*,\bar\tau} \circ \sigma_{\bar\tau,\tau'}$ whenever $\tau^* \le \bar\tau \le \tau'$ are in $B_\tau \cup \{\tau\}$.

Lemma 3.3. *Let $\tau \in \lim(\mathcal{S})$ be of uncountable cofinality. Then B_τ is an unbounded subset of τ, and is closed in τ on a tail-end, that is, $B_\tau - \bar\tau$ is closed in τ for some $\bar\tau < \tau$.*

Proof. First we discuss the case where $\tau \in \mathcal{S}^2$, and return to the other two cases later. Let $n = n_\tau$, $\psi(u, v_0, v_1)$ be a $\Sigma_1^{(n)}$-formula, and $q \in N_\tau$ be such that

$$d = \{\xi < \tau \mid N_\tau \models \psi(\xi, p_\tau, q)\}$$

is a closed unbounded subset of τ disjoint from some set $R \in \mathcal{R}_\tau$. We first prove that B_τ is unbounded in τ. This is done by the standard interpolation argument; the basic form of the Interpolation Lemma can be found in [14], Lemma 3.6.10. We will recall the relevant facts here. Let $\bar\tau < \tau$; we construct some $\tilde\tau \in B_\tau$ such that $\tilde\tau > \bar\tau$. Let θ be large regular, and X be a countable elementary substructure of H_θ such that $\tau, \bar\tau, q, R, \mathcal{R} \in X$. Let H be the transitive collapse of X and $\sigma : H \to H_\theta$ be the inverse of the Mostowski collapsing isomorphism. By elementarity, $\kappa, p_\tau, (Q_{\tau,\beta}, t_{\tau,\beta})$ are in $\mathrm{rng}(\sigma)$ where $(Q_{\tau,\beta}, t_{\tau,\beta}) \in N_\tau$ is some generalized witness for $\beta \in p_\tau$ with respect to p_τ and N_τ. Let $N^*, \kappa^*, \tau^*, p^*, q^*, \mathcal{R}^*, R^*, (Q^*_{\beta^*}, t^*_{\beta^*})$ be the preimage of $N, \kappa, \tau, p, q, \mathcal{R}_\tau, R, (Q_{\tau,\beta}, t_{\tau,\beta})$ under σ. By elementarity, N^* is a sound premouse with standard parameter p^* such that $\omega\varrho^{n+1}_{N^*} = \omega\varrho^\omega_{N^*} = \tau^* = \kappa^{+N^*}$, $(Q^*_{\beta^*}, t^*_{\beta^*})$ is a generalized witness for $\beta^* = \sigma^{-1}(\beta)$ with respect to p^* and N^*, $\mathcal{R}^*$ is a family of stationary subsets of $\tau^* \cap \mathsf{cof}(<\kappa^*)$ in N^* and has cardinality $<\kappa^*$ in N^*, and $R^* \in \mathcal{R}^*$ is such that $d^* \cap R^* = \varnothing$ where $d^* = \{\xi < \tau^* \mid N^* \models \psi(\xi, p^*, q^*)\}$. Notice that $\sigma(d^*) = d$ and d^* is a closed unbounded subset of τ^*.

Since τ has uncountable cardinality, $X = \mathrm{rng}(\sigma)$ is countable, and $\bar\tau \in X$, letting $\tilde\tau = \sup(\sigma[\tau^*])$ we see that $\bar\tau < \tilde\tau < \tau$. We now apply the Interpolation lemma to the map $\sigma \restriction N^* : N^* \to N_\tau$. We obtain a transitive structure $\tilde N$ and maps $\tilde\sigma : N^* \to \tilde N$ and $\sigma' : \tilde N \to N_\tau$ with the following properties.

(a) $\tilde\sigma$ is the fine ultrapower (in the language of [14] it is a "pseudoultrapower") of N^* by $\sigma \restriction (N^* \,||\, \tau^*)$. Since $\tau = \omega\varrho^\omega_{N^*}$, the Łoś theorem holds for all Σ^*-formulae, so in particular the map $\tilde\sigma$ is Σ^*-preserving and $\omega\varrho^\omega_{\tilde N} \ge \tilde\tau$. (See [14], Section 3.6)
(b) $\mathsf{cr}(\sigma') = \tilde\tau$ and $\sigma'(\tilde\tau) = \tau$.
(c) The map σ' is Σ^*-preserving; this follows from the fact that the Łoś theorem in (a) holds for all Σ^*-formulae and that σ' is the factor map between σ and $\tilde\sigma$.
(d) $\tilde h^{n+1}_{\tilde N}(\tilde\tau \cup \{\tilde p\}) = \tilde N$ where $\tilde p = \tilde\sigma(p^*)$; this follows from the soundness of N^* and the properties of the ultrapower embedding. In particular, we see that $\omega\varrho^{n+1}_{\tilde N} \le \tilde\tau$. Combined with (a), we conclude that $\omega^{n+1}_{\tilde N} = \omega\varrho^\omega_{\tilde N} = \tilde\tau$.
(e) $\tilde p = p_{\tilde N}$. This follows from (d) combined with the fact that $\tilde\sigma((Q^*_{\beta^*}, t^*_{\beta^*}))$ is a generalized witness for $\tilde\sigma(\beta^*)$ with respect to $\tilde p$ and $\tilde N$, which in turn is a consequence of the preservation properties of $\tilde\sigma$.

It follows from the above that $\tilde{N}$ is a premouse (as $\tilde{\sigma}$ is a Σ^*-preserving ultrapower embedding), $\tilde{p} = p_{\tilde{N}}$, as follows from (d), (e) and Lemma 1.12.5 in [14], and

$$\tilde{h}^{n+1}_{\tilde{N}}(\omega\varrho^{\omega}_{\tilde{N}} \cup \{p_{\tilde{N}}\}) = \tilde{h}^{n+1}_{\tilde{N}}(\tilde{\tau} \cup \{p_{\tilde{N}}\}) = \tilde{N},$$

that is, $\tilde{N}$ is fully sound. Since $\omega\varrho^{n+1}_{\tilde{N}} = \tilde{\tau}$, we can apply the condensation lemma (Lemma 2.1) to conclude that either $E_{\tilde{\tau}} = \varnothing$ and $\tilde{N}$ is an initial segment of W, or $E_{\tilde{\tau}}$ is an extender and $\tilde{N}$ is an initial segment of $W^{\tilde{\tau}} = \mathsf{Ult}(W, E_{\tilde{\tau}})$, see (10). Since $\mathcal{R}_\tau \in \mathrm{rng}(\sigma')$, $\mathsf{cr}(\sigma') = \tilde{\tau}$, $\sigma'(\tilde{\tau}) = \tau$ and $\mathcal{R}_\tau$ has cardinality smaller than κ in N_τ, it is easy to see that $(\sigma')^{-1}(\mathcal{R}_\tau) = \{R' \cap \tilde{\tau} \mid R' \in \mathcal{R}_\tau\} = \mathcal{R}_{\tilde{\tau}}$.

To see that $\tilde{N} = N_{\tilde{\tau}}$ and $\sigma' = \sigma_{\tilde{\tau},\tau}$ it suffices to verify that, letting $\tilde{q} = \tilde{\sigma}(\bar{q}) = (\sigma')^{-1}(q)$, the set

$$\tilde{d} = \{\xi < \kappa \mid \tilde{N} \models \psi(\xi, p_{\tilde{N}}, \tilde{q})\}$$

is a closed unbounded subset of $\tilde{\tau}$ disjoint from $\tilde{R} = \tilde{\sigma}(R^*)$, and that $n = n_{\tilde{\tau}}$, that is, $d' \cap R' \neq \varnothing$ whenever d' is a closed unbounded subset of $\tilde{\tau}$ that is $\mathbf{\Sigma}^{(k)}_1(\tilde{N})$-definable for some $k < n$. Concerning the set $\tilde{d}$, notice that the statements:

$$(\forall\zeta^{n+1})(\exists\xi^n)(\zeta^{n+1} < \xi^n < \tau \;\&\; \psi(\xi^n, p_N, q)) \tag{19}$$

$$(\forall\zeta^{n+1})[(\forall\xi^n < \zeta^{n+1})(\exists\eta^n < \zeta^{n+1})\psi(\eta^n, p_N, q)] \;\longrightarrow\; \psi(\zeta^{n+1}, p_N, q) \tag{20}$$

$$(\exists\xi^{n+1})(\psi(\xi^{n+1}, p_N, q) \;\&\; \xi^{n+1} \in R) \tag{21}$$

express the facts that the set d is an unbounded subset of τ, the set d is a closed subset of τ, and $d \cap R \neq \varnothing$, respectively. All these statements have complexity $\Sigma^{(n+1)}_1$. Since σ' is Σ^*-preserving and $p_N, q, R \in \mathrm{rng}(\sigma')$, it is possible to pull the first two statements and the negation of the last statement via σ' back to $\tilde{N}$. The first two statements will then express over $\tilde{N}$ that $\tilde{d}$ is a closed unbounded subset of $\tilde{\tau}$, and the negation of last statement will express that $\tilde{d} \cap (\sigma')^{-1}(R) = \varnothing$. Clearly $(\sigma')^{-1}(R) \in \mathcal{R}_{\tilde{\tau}}$. Now to see that $n = n_{\tilde{\tau}}$, given $k < n$, the conjunction of formulae (19) and (20) above with k in place of n and v_1 in place of q is a formula with the only free variable v_1 and the parameter p_N expressing the fact that the set $d(v_1) = \{\xi < \tau \mid N_\tau \models \psi(\xi, p_N, v_1)\}$ is a closed unbounded subset of τ. Denote this formula by $\psi'(v_1)$. It has been mentioned above that by the soundness of the structures we work with, it is sufficient to consider the parameter q be an ordinal smaller than τ, as we can substitute the uniform $\Sigma^{(n)}_1$-Skolem function for v_1 in $\psi'(v_1)$ (see the discussion below (11)). Let $\psi^*(v)$ be the formula $\psi'(\tilde{h}^{n+1}((v)^2_0, \langle((v)^2_1, p_N)))$. Then $\psi^*(v)$ is a $\Sigma^{(n)}_1$-formula in the parameter p_N, and the statement

$$(\forall\eta^{n+1})[\psi^*(\eta^{n+1}) \;\longrightarrow\; (\exists\xi^n < \tau)(\psi(\xi^n, p_N, \eta^{n+1}) \;\&\; \xi^n \in R')$$

expresses that the intersection $d(\eta^{n+1}) \cap R'$ is nonempty whenever $d(\eta^{n+1})$ is a closed unbounded subset of τ. Since this is true in N_τ for every $R' \in \mathcal{R}_\tau$ and the above statement is $\Pi^{(n+1)}_1$, it can be pulled back to $\tilde{N}$ via the Σ^*-preserving map σ'. And since this can be done for ever formula $\psi(u, v_0, v_1)$ that is $\Sigma^{(k)}_1$ for some $k < n$, this proves that $d' \cap \tilde{R}$ is nonempty whenever d' is a closed unbounded subset of $\tilde{\tau}$ that is $\mathbf{\Sigma}^{(k)}_1(\tilde{N})$-definable for some $k < n$ and $\tilde{R} \in \mathcal{R}_{\tilde{\tau}}$. At this point we have verified that $\tilde{\tau}$ meets all requirements in (C); notice that, although we did not spell this explicitly, our argument also verifies that $\tilde{\tau} \in \mathcal{S}^2$. This completes the proof that B_τ is unbounded in τ.

Next we prove that B_τ is closed. So if $\tau \in \mathcal{S}^2$ we have actually a stronger conclusion than in the formulation of the lemma. The argument here is again standard. Given $\bar\tau < \tau$ a limit point of B_τ, form the direct limit $\bar N$ of the diagram $\langle N_{\tau^*}, \sigma_{\tau^*,\tau'} \mid \tau^* < \tau' < \bar\tau\rangle$; let $\sigma_{\tau^*} : N_{\tau^*} \to \bar N$ be the direct limit maps and $\bar\sigma : \bar N \to N_\tau$ be the canonical embedding of the direct limit into N_τ. Here $\bar N$ is taken transitive, which can obviously be done, as it is well-founded. As the maps $\sigma_{\tau^*,\tau'}$ are all Σ^*-preserving, standard computations with direct limits yield that the maps σ_{τ^*} and $\bar\sigma$ are Σ^*-preserving as well. We need to verify that $\omega\varrho^{n+1}_{\bar N} = \omega\varrho^{\omega}_{\bar N} = \bar\tau$, which involves a similar kind of argument. Let $\bar p = \sigma_{\tau^*}(p_{\tau^*})$; this value is obviously independent of the choice of τ^*. By the soundness of the structures N_{τ^*} we see that $\tilde h^{n+1}_{N_{\tau^*}}(\tau^* \cup \{p_{\tau^*}\}) = N_{\tau^*}$ whenever $\tau^* \in \bar\tau \cap B_\tau$. Since $\bar N = \bigcup\{\mathrm{rng}(\sigma_{\tau^*}) \mid \tau^* \in \bar\tau \cap B_\tau\}$ and we already know that the maps σ_{τ^*} are sufficiently preserving, we have $\tilde h^{n+1}_{\bar N}(\bar\tau \cup \{\bar p\}) = \bar N$. This verifies the inequality $\omega\varrho^{n+1}_{\bar N} \le \bar\tau$. To see $\ge$, assume $\varphi(u,v)$ is a $\Sigma^{(n)}_1$-formula, $r \in \bar N$, and $\alpha < \tau$. If $\tau^* \in \bar\tau \cap B_\tau$ is larger than α and r_{τ^*} is the preimage of r under σ_{τ^*} then $a = \{\xi < \alpha \mid N_{\tau^*} \models \varphi(\xi, r_{\tau^*})\} \in N_{\tau^*}$; hence $a = \sigma_{\tau^*}(a) \in \bar N$. By the preservation properties of σ_{τ^*}, we obtain $a = \{\xi < \alpha \mid \bar N \models \varphi(\xi, r)\}$, which completes of the verification of $\ge$. The same argument can be run for $n' > n$ to show that actually $\omega\varrho^{\omega}_{\bar\tau} = \tau$. At this point we see that $\bar\tau \notin \mathcal{S}^0 \cup \mathcal{S}^1$, as $\omega\varrho^{\omega}_{N_{\tau'}} = \kappa$ whenever $\tau' \in \mathcal{S}^0 \cup \mathcal{S}^1$. We have also seen that $\tilde h^{n+1}_{\bar N}(\bar\tau \cup \{\bar p\}) = \bar N$; this along with the fact that for each $\beta \in \bar p$ there is a generalized witness with respect to $\bar p$ and $\bar N$ which is an element of $\bar N$, namely $\sigma_{\tau^*}((Q_{\tau^*,\beta^*}, t_{\tau^*,\beta^*}))$ where $\beta^* = \sigma^{-1}_{\tau^*}(\beta)$, makes it possible to argue exactly as in the proof of unboundedness of B_τ, and conclude, relying on Lemma 1.12.5 in [14] that $\bar p = p_{\bar N}$ and $\bar N$ is fully sound. We can now apply Lemma 2.1 to run the same kind of condensation argument as in the proof of unboundedness of B_τ and conclude that either $E_{\bar\tau} = \varnothing$ in which case $\bar N$ is an initial segment of W, or else $E_{\bar\tau}$ indexes an extender in W in which case $\bar N$ is an initial segment of $W^{\bar\tau}$.

What remains is to show that $\bar N = N_{\bar\tau}$ and $n = n_{\bar\tau}$, that is, there is a closed unbounded set $\bar d$ of $\bar\tau$ that avoids some set in $\mathcal{R}_{\bar\tau}$ such that $\bar d$ is $\mathbf{\Sigma}^{(n)}_1(\bar N)$, and n is the least with this property. First, as $\mathcal{R}_\tau \in \mathrm{rng}(\sigma_{\tau^*,\tau})$ whenever $\tau^* \in \bar\tau \cap B_\tau$, using the preservation properties of $\bar\sigma$ and the fact that $\mathrm{rng}(\sigma_{\tau^*,\tau}) \subseteq \mathrm{rng}(\bar\sigma)$ it follows as before that $\bar\sigma^{-1}(\mathcal{R}_\tau) = \mathcal{R}_{\bar\tau} = \{R' \cap \bar\tau \mid R' \in \mathcal{R}_\tau\}$. Similarly, given $\tau^* \in \bar\tau \cap B_\tau$ there is some $\Sigma^{(n)}_1$-formula $\psi(u, v_0, v_1)$ and some $q \in \mathrm{rng}(\sigma_{\tau^*,\tau})$ such that

$$d = \{\xi < \tau \mid N_\tau \models \psi(\xi, p_\tau, q)\}$$

is a closed unbounded subset of τ disjoint from some $R \in \mathcal{R}_\tau$; this is given by the requirement (C)(b)(v). Now relying on the preservation properties of $\bar\sigma$, using the formulae (19) – (21) we can argue exactly as above in the proof of unboundedness of B_τ to prove that, letting $\bar q = \bar\sigma^{-1}(q)$, the set $\bar d = \{\xi < \bar\tau \mid \bar N \models \psi(\xi, p_{\bar N}, \bar q)\}$ is a closed unbounded subset of $\bar\tau$ disjoint from $R \cap \bar\tau \in \mathcal{R}_{\bar\tau}$, and all closed unbounded subsets of $\bar\tau$ that are $\mathbf{\Sigma}^{(k)}_1(\bar N)$ for some $k < n$ meet all sets in $\mathcal{R}_{\bar\tau}$. This completes the proof that $\bar\tau \in B_\tau$, and thereby also the proof that B_τ is a closed subset of τ. Thus, this also completes the proof of Lemma 3.3 in the case where $\tau \in \mathcal{S}^2$.

It remains to discuss the case where $\tau \in \mathcal{S}^0 \cup \mathcal{S}^1$. This case is essentially taken care of by Lemma 3.5 in [10]. To see why, we first note that the proof in this case is based on the same strategy as the one we did above for $\tau \in \mathcal{S}^2$. Say $\tau \in \mathcal{S}^0$. This means that the unboundedness of B_τ is verified using an interpolation argument which, given $\bar{\tau} < \tau$ produces some $\tilde{\tau} \in (\bar{\tau}, \tau)$ along with a structure $\tilde{N}$ and a map $\sigma' : \tilde{N} \to N_\tau$. In the course of the proof one eventually shows that $\tilde{N} = N_{\tilde{\tau}}$ and $\sigma' = \sigma_{\tilde{\tau},\tau}$ and, granting that $\tilde{\tau}$ is sufficiently large, also that $\tilde{\tau} \in \mathcal{S}^0$ which then implies that $\tilde{\tau} \in B_\tau$. The tail-end-closedness of B_τ is proved by computing the direct limit $\bar{N}$ of the diagram $\langle N_{\tau^*}, \sigma_{\tau^*,\tau'} \mid \tau^* < \tau' \text{ in } \bar{\tau} \cap B_\tau \rangle$ along with the map $\bar{\sigma} : \bar{N} \to N_\tau$ whenever $\bar{\tau} \in \lim(B_\tau)$. Again, one eventually shows that $\bar{N} = N_{\bar{\tau}}$ and $\bar{\sigma} = \sigma_{\bar{\tau},\tau}$ and, granting that $\bar{\tau}$ is sufficiently large, also that $\bar{\tau} \in \mathcal{S}^0$ which implies that $\bar{\tau} \in B_\tau$. By the assumption $\tau \in \mathcal{S}^0$, we have $\omega\varrho^{n_\tau+1}_{N_\tau} = \kappa$. Then the interpolation argument yields $\omega\varrho^{n_\tau+1}_{\tilde{N}} = \kappa$, and from the properties of the direct limits we obtain $\omega\varrho^{n_\tau+1}_{\bar{N}} = \kappa$. It follows that $\tilde{\tau}, \bar{\tau} \notin \mathcal{S}^2$. Similarly, if $\tau \in \mathcal{S}^1$ we work with canonical protomice (see the definition of the canonical protomouse M_τ below (15)) instead of initial segments of extender models, and again use the interpolation argument for the proof of unboundedness and the direct limit argument for the proof of closedness. As $\tau \in \mathcal{S}^1$ we have $\omega\varrho^{n_\tau+1}_{N_\tau} = \kappa$, and it follows by the results in [10], Section 2.3 that $\omega\varrho^1_{M_\tau} = \kappa$. This time the two constructions used above yield protomice in place of premice $\tilde{N}$ and $\bar{N}$; we denote these protomice by $\tilde{M}$ and $\bar{M}$, respectively. By the two constructions, the first projecta of these protomice are equal to κ, $\tilde{M}$ is a collapsing protomouse for $\tilde{\tau}$, and $\bar{M}$ is a collapsing protomouse for $\bar{\tau}$. Letting $\tilde{N}$ be the premouse reconstructed from $\tilde{M}$ and $\bar{N}$ be the premouse from $\bar{M}$ (see Section 2 in this paper for a brief explanation, and Section 2 in [10] for details), by [10], Section 2.3 we then conclude that $\omega\varrho^\omega_{\tilde{N}} = \kappa$ and $\tilde{N}$ is a collapsing premouse for $\tilde{\tau}$, and $\omega\varrho^\omega_{\bar{N}} = \kappa$ and $\bar{N}$ is a collapsing premouse for $\bar{\tau}$. We thus arrived at the conclusion that $\tau \in \mathcal{S}^0 \cup \mathcal{S}^1$ implies $\tilde{\tau}, \bar{\tau} \in \mathcal{S}^0 \cup \mathcal{S}^1$. This makes it possible to run the proof of Lemma 3.5 in [10] which shows that $\tilde{N} = N_{\tilde{\tau}}$ and $\bar{N} = N_{\bar{\tau}}$. The same proof moreover shows that $\tilde{\tau}, \bar{\tau} \in \mathcal{S}^i$ whenever $\tau \in \mathcal{S}^i$, granting that $\tilde{\tau}, \bar{\tau}$ are sufficiently large (as has been already indicated above). This completes the discussion in the case where $\tau \in \mathcal{S}^0 \cup \mathcal{S}^1$, and thereby also the proof of Lemma 3.3. □

Analogously to the situation in [10], the sets B_τ are "almost" coherent in the sense of the following lemma. The lack of full coherency is caused by clauses (iv) and (v) in the definition of B_τ which require certain unspecified objects to be in $\text{rng}(\sigma_{\bar{\tau},\tau})$.

Lemma 3.4. *Let $\bar{\tau} \in B_\tau$. Then $B_\tau \cap \bar{\tau} = B_{\bar{\tau}} - \min(B_\tau)$.*

Proof. Given $\tau^* \in [\min(B_\tau), \bar{\tau})$ we want to see that $\tau^* \in B_{\bar{\tau}}$ if and only if $\tau^* \in B_\tau$. Assuming $\tau^* \in B_{\bar{\tau}}$ this amounts to showing that $\sigma_{\tau^*,\tau} = \sigma_{\bar{\tau},\tau} \circ \sigma_{\tau^*,\bar{\tau}}$ satisfies the requirements (A), (B) or (C) to guarantee that $\tau^* \in B_\tau$. Conversely, assuming $\tau^* \in B_\tau$ this amounts to showing that $\sigma_{\tau^*,\bar{\tau}} = (\sigma_{\bar{\tau},\tau})^{-1} \circ \sigma_{\tau^*,\tau}$ meets these requirements. For $\tau \in \mathcal{S}^0 \cup \mathcal{S}^1$ this is proved in [10]. For $\tau \in \mathcal{S}^2$ the proof is similar, but involves dealing with the additional clause (C)(b)(v). Notice also that the verification of (a) is a triviality, so we may assume $n_\tau = n_{\bar{\tau}} = n_{\tau^*}$, and denote this number by n. The verification of the remaining clauses in (C) is actually much easier than in the cases (A) and (B) where $\tau \in \mathcal{S}^0 \cup \mathcal{S}^1$, due to strong preservation properties of the maps $\sigma_{\tau^*,\tau'}$ in the case $\tau \in \mathcal{S}^2$. The verification of (v) uses the formulae in (19) – (21) in the straightforward way similar to that in the proof of Lemma 3.3. By (C)(b)(v) for τ^* and $\bar{\tau}$, there is a $\Sigma^{(n)}_1$-formula $\psi(u, v_0, v_1)$ which defines a closed unbounded subset of $\bar{\tau}$ over $N_{\bar{\tau}}$ in the parameters $p_{\bar{\tau}}$ and

$q \in \operatorname{rng}(\sigma_{\tau^*,\bar\tau})$ that is disjoint from some $R \in \mathcal{R}_{\bar\tau}$ such that $R \in \operatorname{rng}(\sigma_{\tau^*,\bar\tau})$. The same formula then defines, from p_τ and $\sigma_{\bar\tau,\tau}(q) \in \operatorname{rng}(\sigma_{\tau^*,\tau})$ over N_τ, a closed unbounded subset of τ disjoint from $\sigma_{\bar\tau,\tau}(R) \in \operatorname{rng}(\sigma_{\tau^*,\tau})$. Conversely, Assuming $\tau^* \in B_\tau$, let $\hat\tau = \min(B_\tau)$, $\psi(u, v_0, v_1)$ be a $\Sigma_1^{(n)}$-formula, and $q, R \in \operatorname{rng}(\sigma_{\hat\tau,\tau})$ be such that $R \in \mathcal{R}_\tau$ and ψ defines a closed unbounded subset of τ over N_τ that is disjoint from R. Again, the existence of such ψ, q, R is guaranteed by (C)(b)(v) for $\hat\tau$ and τ. Letting $(\bar q, \bar R) = \sigma^{-1}_{\bar\tau,\tau}(q, R)$, we see that $\bar q, \bar R \in \operatorname{rng}(\sigma_{\tau^*,\bar\tau})$, $\bar R \in \mathcal{R}_{\bar\tau}$ and ψ defines a closed unbounded subset of $\bar\tau$ over $N_{\bar\tau}$ in parameters $p_{\bar\tau}, \bar q$ that disjoint from $\bar R$. □

We are now ready to define a coherent approximation to our $\square(\kappa^+)$-sequence. Let

$$\begin{aligned} \tau_0 &= \tau \\ \tau_{i+1} &= \min(B_{\tau_i}) \\ \ell_\tau &= \text{the least } i \in \omega \text{ such that } B_{\tau_i} = \varnothing. \end{aligned}$$

and

$$B^*_\tau = B_{\tau_0} \cup \cdots \cup B_{\tau_{\ell-1}}. \tag{22}$$

Lemma 3.5. *Let $\tau \in \mathcal{S}$ and $\bar\tau \in B^*_\tau$. Then $B^*_{\bar\tau} = B^*_\tau \cap \bar\tau$.*

Proof. This is a straightforward verification based on Lemma 3.4, and is identical to the proof of Lemma 3.4 in [10]. □

Now for $\tau \in \mathcal{S}$ let β_τ be the least $\beta \in B_\tau \cup \{\tau\}$ such that $B_\tau - \beta_\tau$ is closed, and define

$$C^*_\tau = B_\tau - \beta_\tau. \tag{23}$$

Lemma 3.6. *The sequence $\langle C^*_\tau \mid \tau \in \mathcal{S}\rangle$ is a coherent sequence of closed sets, C^*_τ is unbounded in τ whenever τ has uncountable cofinality, and and $C^*_\tau \cap R = \varnothing$ for some $R \in \mathcal{R}$ whenever $\tau \in \mathcal{S}^2$. Moreover, $\langle C^*_\tau \mid \tau \in \mathcal{S}\rangle$ has no thread.*

Proof. The coherency of $\langle C^*_\tau \mid \tau \in \mathcal{S}\rangle$ follows easily from the coherency of the sequence $\langle B^*_\tau \mid \tau \in \mathcal{S}\rangle$. It is also clear that the remaining properties except for the non-existence of a thread follow immediately from the conclusions established above.

Assume for a contradiction C is a thread for $\langle C^*_\tau \mid \tau \in \mathcal{S}\rangle$. We first observe that $\mathcal{S}^0 \cup \mathcal{S}^1$ must be non-stationary. If $\mathcal{S}^0$ were stationary so would be $\mathcal{S}^0 \cap \lim(C)$. Now if $\bar\tau < \tau$ are in $\mathcal{S}^0 \cap \lim(C)$ then $C^*_{\bar\tau} = C \cap \bar\tau$ is a limit point of $C \cap \tau = C^*_\tau$, so $\bar\tau \in C^*_\tau$ as C^*_τ is closed. Then by the definition of C^*_τ, we see that $n_{\bar\tau} = n_\tau$ and we have the unique $\Sigma_0^{(n_\tau)}$-preserving map $\sigma_{\bar\tau,\tau} : N_{\bar\tau} \to N_\tau$ such that $\sigma_{\bar\tau,\tau}$ has critical point $\bar\tau$, and $\sigma_{\bar\tau,\tau}(\bar\tau, p_{\bar\tau}) = (\tau, p_\tau)$. It follows that $n_\tau = n_{\bar\tau}$ whenever $\bar\tau, \tau \in \mathcal{S}^0 \cap \lim(C)$, and we denote this number by n. So we have a commutative diagram $\langle C^*_\tau, \sigma_{\bar\tau,\tau} \mid \bar\tau < \tau \text{ in } \mathcal{S}^0 \cap \lim(C)\rangle$ where all maps $\sigma_{\bar\tau,\tau}$ are $\Sigma_1^{(n)}$-preserving, and $\sigma_{\bar\tau,\tau}(\bar\tau, p_{\bar\tau}) = (\tau, p_\tau)$. The direct limit of this diagram is obviously well-founded, and letting N be its transitive collapse and $\sigma_\tau : N_\tau \to N$ be the direct limit maps, it is straightforward to see that all maps σ_τ are $\Sigma_0^{(n)}$-preserving, $\mathsf{cr}(\sigma_\tau) = \tau$, $\sigma_\tau(\tau) = \kappa^+$, and the value $p = \sigma_\tau(p_\tau)$ does not depend on the choice of τ. Using all of this and the fact that $N_\tau = \tilde h^{n+1}_{N_\tau}(\kappa \cup \{p_\tau\})$, we conclude that $\tilde h^{n+1}_N(\kappa \cup \{p\}) = N$. So in particular $\tilde h^{n+1}_N$ collapses κ^+ to κ. As $\tilde h^{n+1}_N \in W$, this is a contradiction. If

$\mathcal{S}^1$ were stationary so would be $\mathcal{S}^1 \cap \lim(C)$, and we could proceed as above, but this time we would have a commutative diagram of protomice and Σ_0-preserving maps between them, say $\langle M_\tau, \sigma_{\bar\tau,\tau} \mid \bar\tau < \tau \text{ in } \mathcal{S}^1\rangle$. Now we could argue similarly as above that κ^+ is collapsed, but notice we an also argue as follows. Letting μ be the common critical point of the top extenders F_τ of M_τ and ϑ_τ be the largest $\vartheta < \mu^+$ such that F_τ measures all subsets of μ in J^E_ϑ, we have $\bar\tau < \tau \implies \vartheta_{\bar\tau} < \vartheta_\tau$, so if τ is the μ^+-th element of $\mathcal{S}^1 \cap \lim(C)$ then $\vartheta_\tau = \mu^+$. This is a contradiction, as F_τ is not a total extender on M_τ.

Now that we know $\mathcal{S}^0 \cup \mathcal{S}^1$ is non-stationary, we may choose our thread C so that $C \subseteq \mathcal{S}^2$. By the definition of C^*_τ in this case and (16), for each $\tau \in \mathcal{S}^2$ there is some $R_\tau \in \mathcal{R}$ such that $C^*_\tau \cap R_\tau$ is empty. Since $\mathsf{card}(\mathcal{R}) < \kappa$, we can fix an $R \in \mathcal{R}$ such that $R = R_\tau$ for stationarily many $\tau \in \mathcal{S}^2$, hence there will be stationarily many such τ in $\mathcal{S}^2 \cap \lim(C)$. For each such τ we have $C \cap \tau \cap R = C^*_\tau \cap R = \varnothing$. Since there are arbitrarily large such τ, we conclude that $C \cap R$ is empty, contradicting the fact that R is stationary. This completes the proof of the implication (c)$\Rightarrow$ (a) in Theorem 1.4.

To see that we can find a family $\mathcal{R}'$ and a sequence $\langle C_\tau \mid \tau \in \mathcal{S}\rangle$ as in the statement of Theorem 1.4, notice that, by construction, the conclusion holds with $\mathcal{R}$ in place of $\mathcal{R}'$ and C^*_τ in place of C_τ whenever $\tau \in \mathcal{S}^2$. To extend the result to $\tau \in \mathcal{S}^0 \cup \mathcal{S}^1$, we perform the standard construction that does the job in the case we have a $\square_\kappa$-sequence. This can be done, as our $\square(\kappa^+)$-sequence can be turned into a $\square_\kappa$-sequence on $\mathcal{S}^0 \cup \mathcal{S}^1$. In fact, if $\tau \in \mathcal{S}^1$ and μ is the critical point of the top extender of M_τ then it is not hard to see that $\mathsf{otp}(C^*_\tau) \le \mu^+ \le \kappa$; see [10] for details. So it is sufficient to focus on $\tau \in \mathcal{S}^0$. Here the construction is essentially the same as Jensen's original construction in $\mathbf{L}$. For $\tau \in \mathcal{S}^0$ we define sequences of ordinals τ_ι, ξ_ι as follows.

$$
\begin{aligned}
\tau_0 &= \min(C^*_\tau) \\
\xi_\iota &= \text{the least } \xi \text{ such that } \tilde h_\tau(\{\xi\} \cup \{p_\tau\}) \subsetneq \mathrm{rng}(\sigma_{\tau_\iota,\tau}) \\
\tau_{\iota+1} &= \text{the least } \bar\tau \in C^*_\tau \cup \{\tau\} \text{ such that } \tilde h_\tau(\{\xi_\iota\} \cup \{p_\tau\}) \subseteq \mathrm{rng}(\sigma_{\bar\tau,\tau}) \\
\tau_\iota &= \sup_{\bar\iota<\iota} \tau_{\bar\iota} \quad \text{for limit } \iota \\
\iota_\tau &= \text{the least } \iota \text{ such that } \tau_\iota = \tau.
\end{aligned}
$$

Letting

$$
C'_\tau = \begin{cases} C^*_\tau & \text{if } \tau \in \mathcal{S}^1 \cup \mathcal{S}^2 \\ \{\tau_\iota \mid \iota < \iota_\tau\} & \text{if } \tau \in \mathcal{S}^0, \end{cases} \tag{24}
$$

the sequence

$$\langle C'_\tau \mid \tau \in \mathcal{S}\rangle$$

is a $\square(\kappa^+)$-sequence such that $\langle C'_\tau \mid \tau \in \mathcal{S}^0 \cup \mathcal{S}^1\rangle$ is a $\square_\kappa$-sequence. This is proved in [10]. More precisely, the argument in [10] shows that $\langle C'_\tau \mid \tau \in \mathcal{S}^0\rangle$ is a $\square_\kappa$-sequence on $\mathcal{S}^0$, which is just enough for the above conclusion. To complete the proof of Theorem 1.4, for each $R \in \mathcal{R}$ let

$$
R' = \begin{cases} R \cap \mathcal{S}^2 & \text{if } R \cap (\mathcal{S}^0 \cup \mathcal{S}^1) \text{ is nonstationary, and} \\ (R \cap \mathcal{S}^2) \cup \tilde R & \text{where } \tilde R \subseteq R \cap (\mathcal{S}^0 \cup \mathcal{S}^1) \text{ is stationary such that} \\ & \mathsf{otp}(C'_\tau) \text{ is constant on } \tilde R, \text{ otherwise.} \end{cases}
$$

The set $\tilde{R}$ is obtained by the pigeonhole argument, using the fact that $\langle C'_\tau \mid \tau \in \mathcal{S}\rangle$ is a $\square_\kappa$-sequence on $\mathcal{S}^0 \cup \mathcal{S}^1$, or more precisely, that $\mathsf{otp}(C'_\tau) \leq \kappa$ for all $\tau \in \mathcal{S}^0 \cup \mathcal{S}^1$. By the definition of R' and the coherency of the sequence $\langle C'_\tau \mid \tau \in \mathcal{S}\rangle$, the intersection $R' \cap \lim(C'_\tau)$ can contain at most one ordinal. Thus, letting

$$\begin{aligned}\gamma(R',\tau) &\simeq \text{the unique } \gamma \in R' \cap \lim(C'_\tau) \text{ if this intersection is nonempty,}\\ \gamma_\tau &= \begin{cases}\text{the least possible value of } \gamma(R',\tau)+1 \text{ where } R \in \mathcal{R}\\ 0 \text{ if } \gamma(R',\tau) \text{ is undefined for all } R \in \mathcal{R},\end{cases}\end{aligned}$$

and

$$C_\tau = \begin{cases}\lim(C_\tau) - (\gamma_\tau + 1) & \text{if this set is unbounded in } \tau\\ \text{some random } \omega\text{-sequence cofinal in} & \text{otherwise,}\\ \tau \text{ consisting of successor ordinals} & \end{cases}$$

the sequence $\langle C_\tau \mid \tau \in \mathcal{S}\rangle$ along with the family $\mathcal{R}' = \{R' \mid R \in \mathcal{R}\}$ are as in the conclusion of Theorem 1.4. $\square$

The argument at the end of the proof of Theorem 1.4 can also be used to show that no thread through $\langle C^*_\tau \mid \tau < \kappa^+\rangle$ exists in $\mathbf{V}$, granting that the model W is sufficiently "full", that is, contains all relevant premice that exist in $\mathbf{V}$, and that the inner model theory can be developed at the level of subcompact/quasicompact cardinals in a way that makes it possible to lift the arguments from the current inner model theory. Given a thread $C \in \mathbf{V}$ through $\langle C^*_\tau \mid \alpha < \kappa^+\rangle$, we proceed as in the above proof and construct the corresponding direct limit. Notice that we may without loss of generality assume that $C \subseteq \mathcal{S}^i$ for some $i \in \{0,1,2\}$. Denote this limit N if $\tau \in \mathcal{S}^0 \cup \mathcal{S}^2$ and M if $\tau \in \mathcal{S}^1$. Using the "fullness" of W and the appropriate generalization of the inner model theory, we then argue that $N, M \in W$. Having this, if $\tau \in \mathcal{S}^0 \cup \mathcal{S}^1$ we arrive at a contradiction as before, arguing that κ^{+W} is collapsed in W. If $\tau \in \mathcal{S}^2$, we observe that the thread C is contained in the set $C^* = \{\tau < \kappa^{+W} \mid \kappa^{+W} \cap \tilde{h}^{n+1}_N(\tau \cup \{p_N\}) = \tau\}$, and $C^* \subseteq \mathcal{S}^2$, as $\sigma_\tau(\mathcal{R}_\tau) = \mathcal{R}$ whenever $\tau \in C, C^*$ where $\sigma_\tau : N_\tau \to N$ or $\sigma_\tau : M_\tau \to M$ are the corresponding direct limit maps. It is then easy to verify that C^* is also a thread through $\langle C^*_\tau \mid \tau < \kappa^+\rangle$. As C^* is obviously an element of W, we then arrive at a contradiction exactly as in the proof of Theorem 1.4.

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