CHARACTERIZATION OF $\square_\kappa$ IN CORE MODELS

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Abstract. We present a general construction of a $\square_\kappa$-sequence in Jensen's fine structural extender models. This construction yields a local definition of a canonical $\square_\kappa$-sequence as well as a characterization of those cardinals κ, for which the principle $\square_\kappa$ fails. Such cardinals are called *subcompact* and can be described in terms of elementary embeddings. Our construction is carried out abstractly, making use only of a few fine structural properties of levels of the model, such as solidity and condensation.

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In [32], we announced two main results on the combinatorial principle $\square_\kappa$ in extender models, that is, in fine structural models $\mathbf{L}[E]$ constructed relative to coherent extender sequences. The purpose of the present paper is to give a self-contained proof of the second result, namely to give a characterization of $\square_\kappa$ in Jensen extender models. The current paper should be considered a sequel to [32], where the background material, history, connections with other results and topics and the basic definitions can be found. For the reader's convenience, we nevertheless recall some crucial notions.

Definition (Jensen)**.** *Given a cardinal κ, we say that $\langle C_\alpha;\ \kappa < \alpha < \kappa^+ \ \&\ \lim(\alpha)\rangle$ is a $\square_\kappa$-sequence iff*

a) *C_α is a closed unbounded subset of α;*

b) *$C_{\bar\alpha} = C_\alpha \cap \bar\alpha$ whenever $\bar\alpha \in \lim(C_\alpha)$;*

c) $\mathrm{otp}(C_\alpha) \le \kappa$.

The principle $\square_\kappa$ postulates the existence of a $\square_\kappa$-sequence.

Recall that the property b) of square sequences is called *coherency.* The principle $\square_\kappa$ was originally discovered in $\mathbf{L}$ by Jensen. With later developements, a wide variety of similar principles proved to be useful; we will consider just the following generalization.

Definition (Schimmerling). *Given two cardinals* κ *and* λ*, we say that*

$$\langle \mathcal{C}_\alpha;\ \kappa < \alpha < \kappa^+ \ \&\ \lim(\alpha)\rangle$$

is a $\square_{\kappa,<\lambda}$-sequence *iff*

a) $\mathcal{C}_\alpha$ *is a nonempty family of closed unbounded subset of* α *satisfying* $|\mathcal{C}_\alpha| < \lambda$*;*

b) $C \cap \bar{\alpha} \in \mathcal{C}_{\bar{\alpha}}$ *whenever* $C \in \mathcal{C}_\alpha$ *and* $\bar{\alpha} \in \lim(C)$*;*

c) $\mathrm{otp}(C) \leq \kappa$ *for all* $C \in \mathcal{C}_\alpha$*.*

The principle $\square_{\kappa,<\lambda}$ *postulates the existence of a* $\square_{\kappa,<\lambda}$*-sequence. Also, we write* $\square_{\kappa,\lambda}$ *for* $\square_{\kappa,<\lambda^+}$*.*

Clearly, $\square_{\kappa,1}$ is just Jensen's $\square_\kappa$. It is shown in [2, 4, 5] and [11] that $\square_{\kappa,\lambda}$ does not, in general, follow from $\square_{\kappa,<\bar{\lambda}}$ for $\bar{\lambda} < \lambda$. In extender models, however, exactly the opposite happens. Our ultimate result along these lines shows that the failure of $\square_\kappa$ is equivalent to that of $\square_{\kappa,<\kappa}$ and that both are equivalent to a large cardinal property called *subcompactness.* This property was abstracted by Jensen from Solovay's proof of the failure of square from supercompactness. A very closely related property was considered few years earlier by Burke [3] who proved that $\square_\kappa$ fails whenever

$$S_\kappa = \{x \in \mathcal{P}_\kappa(\kappa^+);\ \mathrm{otp}(x) \text{ is a cardinal}\}$$

is stationary. It follows easily from our main theorem that in $\mathbf{L}[E]$, the stationarity of S_κ is equivalent to the subcompactness of κ.

What makes subcompactness an attractive large cardinal axiom is that, unlike supercompactness, it can be satisfied in extender models of the kind known today. More precisely, if we could prove weak iterability (see preliminaries) of the premice that appear in background certified constructions similar to those in [13, 22] or [37], we would obtain extender models with subcompact cardinals whenever such constructions are carried out in a sufficiently rich universe.

Definition (Jensen). *A cardinal* κ *is* subcompact *iff given any* $A \subset \kappa^+$*, there are* $\mu < \kappa$*,* $\bar{A} \subset \mu^+$ *and an elementary embedding*

$$\sigma : \langle H_{\mu^+}, \bar{A}\rangle \to \langle H_{\kappa^+}, A\rangle$$

with critical point equal to μ.

It is clear that $\sigma(\mu) = \kappa$ whenever σ is as in the previous definition. Notice also that μ is measurable, as it is the critical point of an embedding defined on the entire $\mathcal{P}(\mu)$. (Actually, σ witnesses that μ is superstrong, and even 1-extendible.) Since any large cardinal property of μ that is uniformly first order expressible over H_{μ^+} is carried up to κ, we see that κ is weakly compact, Ramsey, strongly Π^1_n-indescribable, etc. On the other hand, κ need not be measurable: If U is a normal measure of Mitchell order 0 and M is the ultrapower of $\mathbf{V}$ by U then κ is not measurable anymore, but remains subcompact in M. The preservation of subcompactness follows from the fact that M agrees with $\mathbf{V}$ on H_{κ^+} and that each embedding σ witnessing subcompactness is an element of H_{κ^+}.

Theorem 0.1 (Main Theorem)**.** *The following statements are equivalent in any Jensen extender model:*

a) $\square_\kappa$.
b) $\square_{\kappa,<\kappa}$.
c) κ *is* not *subcompact.*
d) *The set of all* $\nu < \kappa^+$ *satisfying* $E_\nu \neq \varnothing$ *is non-stationary in* κ^+.

The heart of the theorem is the implication d) $\longrightarrow$ a), and its proof is based on a generalization of Jensen's original construction of a $\square_\kappa$-sequence in $\mathbf{L}$. His construction uses the so called "condensation properties" of $\mathbf{L}$. An example of a condensation property is Gödel's result that any elementary substructure of J_α has, as its Mostowski collapse, some $J_{\bar\alpha}$. The condensation properties used by Jensen to prove $\square_\kappa$ in $\mathbf{L}$ are more sophisticated. It was realized early on that the kinds of condensation used by Jensen in $\mathbf{L}$ fail in general for $\mathbf{L}[E]$. The bulk of this paper is concerned with overcoming this difficulty. Jensen [17] proved the failure of $\square_\kappa$ under the assumption that κ is subcompact; this proof is a straightforward elaboration on Solovay's proof mentioned above. Later [15] he showed that in $\mathbf{L}[E]$, the failure of d) implies the failure of $\square_\kappa$. His arguments can easily be adapted to yield both b) $\longrightarrow$ c) and c) $\longrightarrow$ d). We give the proofs of both implications at the end of the paper; they are presented with Jensen's permission.

The methods developed in this paper seem to be applicable to a wide variety of combinatorial constructions in extender models. One such application is the construction of morasses [33, 34]; these play a key role in studying cardinal transfer properties of extender models. Another such application is Jensen's global square principle. It postulates the existence of a coherent sequence of closed unbounded sets $C_\alpha \subset \alpha$ defined on all limit ordinals α and satisfying the requirement $\mathrm{otp}(C_\alpha) < \alpha$. Jensen proved that the global square principle is equivalent to the conjunction $(\forall\kappa)\Box_\kappa$ & $\Box^S$ where $\Box^S$ is the global square principle restricted to the class S of all singular *cardinals.* More precisely, $\Box^S$ postulates the existence of a global square sequence $\langle C_\alpha; \alpha \in S\rangle$ with the additional property that $C_\alpha \subset S$ whenever $\mathrm{otp}(C_\alpha) > \omega$. It is shown in [43] that $\Box^S$ holds in any Jensen-type extender model; this result does not require any smallness condition.

The paper is organized as follows. Section 1 contains a detailed description of our notation. In order to make the paper self-contained and because Jensen's fine structure for extender models is not as well known as that of Mitchell-Steel's, we chose to make this description quite comprehensive. Section 2 is devoted to *protomice,* that is, to structures that arise in the course of the construction of a $\Box_\kappa$-sequence and that fail to be premice. In particular, to each protomouse M we assign a unique $\mathbf{L}[E]$-level $N(M)$ and establish the exact relationship between the fine structure of M and that of $N(M)$. We then prove a sort of condensation lemma for protomice. It might happen that some $\mathbf{L}[E]$-levels N can have many associated protomice; the crucial point of the construction is a method for finding a *canonical* protomouse for each such N. We shall develop such a method at the end of Section 2. Section 3 contains the actual construction of a $\Box_\kappa$-sequence. The construction involves a certain amount of non-uniformity. κ^+ is partitioned into two subsets, $\mathcal{S}^0$ and $\mathcal{S}^1$. We construct a $\Box_\kappa$-sequence on either of them independently, and obtain the final $\Box_\kappa$-sequence by merging the two partial sequences. Each of these constructions is treated in a separate subsection. On $\mathcal{S}^0$, we imitate Jensen's $\mathbf{L}$-construction in a more general setting, and without any substantial new ingredients. The only thing we have to verify is that the construction on $\mathcal{S}^0$ will not conflict with that on $\mathcal{S}^1$, more precisely, that $C_\alpha \subset \mathcal{S}^0$ whenever $\alpha \in \mathcal{S}^0$. On $\mathcal{S}^1$, the situation is completely new and the construction deals with protomice rather than with $\mathbf{L}[E]$-levels. There are two main issues we have to deal with. First, showing that the recipe for choosing the canonical protomouse is preserved under the embeddings arising in the course of the construction; these embeddings have very weak preservation properties. Second, as in the case of $\mathcal{S}^0$, we have to verify that our construction will not conflict with that on $\mathcal{S}^0$, in other words, that $C_\alpha \subset \mathcal{S}^1$ whenever $\alpha \in \mathcal{S}^1$. The last subsection of Section 3 contains the proofs of the remaining implications of the main theorem.

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1. Preliminaries

In this section we shall introduce our notation and terminology and recall some background material. The construction will be carried out in Jensen's setting [13]. We shall first explain how our version of the general fine structure theory differs from that of Mitchell-Steel's. The difference is mainly only notational and the reader familiar with the other version will easily be able to find the appropriate translation. We then introduce Jensen premice, which are, in contrast to Mitchell-Steel's premice, based on *delayed* indexing of extenders. This indexing was originally suggested by S. Friedman. A detailed description of both the general fine structure theory as well as Jensen premice can be found in [44]. Finally, we state a list of lemmata which will be used in the course of the construction. We believe that, using the material from this section, the proof of $\square_\kappa$ will be self-contained and understandable to a reader with basic knowledge of the inner model theory.

Fine Structure. We shall recapitulate the general fine structure theory for acceptable J-structures of the form $\langle J^B_\alpha, D\rangle$. The reason for this is that we would like to apply fine structural arguments to a class of structures broader than that of premice. In our notation, J^B_α is the J-structure of height α constructed relative to B, and the language for this structure *includes* a predicate symbol denoting B. The domain of J^B_α is denoted by $|J^B_\alpha|$, but we will be sloppy with the notation whenever there is no danger of confusion and write simply J^B_α instead. For an acceptable structure M we define the *n-th projectum* ϱ^n_M and the *n-th standard (master) code* with respect to some $p \in M$, denoted by $A^{n,p}_M$, in the usual way. The *ultimate projectum* ϱ^ω_M is the ordinal ϱ satisfying $\varrho = \varrho^n_M$ for all but finitely many $n \in \omega$. Setting $H^n_M = |J^B_\varrho|$ where $\varrho = \varrho^n_M$, the structure $M^{n,p} = \langle H^n_M, A^{n,p}_M\rangle$ is called the *n-th reduct* of M with respect to p. The Σ_1-Skolem function for $M^{n,p}$ is denoted by $h^{n,p}_M$. We write h_M instead of $h^{0,\varnothing}_M$. A formula φ is a *Q-formula* iff it is of the form $(\forall u_1)\cdots(\forall u_\ell)(\exists v_1 \supset u_1)(\exists v_\ell \supset u_\ell)\bar\varphi(v_1,\cdots,v_\ell)$ where $\bar\varphi$ is a Σ_1-formula with no free occurences of $u_1,\cdots,u_\ell$. Q-statements are downward preserved by Σ_1-preserving maps, and upward preserved by maps that are Σ_0-preserving and *cofinal.* A map $\sigma : M \to N$ is cofinal just in case that for every $y \in N$ there is some $x \in M$ such that $y \subset \sigma(x)$. Being an acceptable structure is a Q-property, i.e. there is a fixed Q-sentence ψ such that for every transitive structure M closed under forming singletons, M is acceptable iff $M \models \varphi$. The requirement that M is closed under forming singletons is necessary, as this property cannot be expressed in a Q-fashion. To see this, choose any $a \subset \omega$ that is not an element of J_α. Then $\mathrm{id} : J_\alpha \to J_\alpha \cup \{a\}$ is Σ_0-preserving and cofinal map between two transitive structures, but $J_\alpha \cup \{a\}$ is not closed under forming singletons.

In order to avoid talking about reducts, we introduce Jensen's Σ^*-language, whose detailed description can be found in [9] or [44]. This language differs from the usual language for acceptable J-structures in that it has variables of ω many types, say v^i of type i for every $i \in \omega$. The formulae are built in the usual way, but there are restrictions on quantifying. *Bounded* quantifiers $\exists v^i \in v^j$ and $\forall v^i \in v^j$ are allowed only for $j \geq i$ and we can bind a variable of type i in a formula φ only if φ is of sufficiently low complexity. More precisely, formulae containing only bounded quantifiers of type 0 are $\Sigma^{(0)}_0$ and for every i, each $\Sigma^{(i)}_1$ formula is $\Sigma^{(i+1)}_0$ by definition. Propositional connectives preserve the complexity. Finally, if φ is $\Sigma^{(i)}_0$, then so are $(\exists v^i \in v^j)\varphi$ and $(\forall v^i \in v^j)\varphi$, granting that $j \geq i$. A formula φ is $\Sigma^{(i)}_\ell$ iff it consists of an alternating block of *unbounded* quantifiers for variables of type i followed by a $\Sigma^{(i)}_0$-formula, where the above block has length ℓ and its leftmost quatifier is $\exists$. $\Pi^{(i)}_\ell$ formulae are defined dually. A formula φ is Σ^* iff it is

$\Sigma_1^{(i)}$ for all $i \in \omega$. Finally, φ is $Q^{(i)}$ iff it is of the form $(\forall u^i)(\exists v^i \supset u^i)\bar{\varphi}(v^i)$ where $\bar{\varphi}$ is a $\Sigma_1^{(i)}$-formula in which u^i is not free.

Semantically, variables of type i range over the i-th reduct, so binding variables of type i means quantifying over the i-th reduct. That a relation $A \subset M$ is $\Sigma_\ell^{(n)}(M)$ in p, resp. $\mathbf{\Sigma}_\ell^{(n)}(M)$ is defined in the obvious way. It can be shown that

$$\omega\varrho_M^n = \text{the least ordinal } \zeta \text{ such that there is a } \mathbf{\Sigma}_1^{(n)}(M) \text{ subset of } \zeta \text{ which is not an element of } M,$$

and that $A_M^{n,p}$, $h_M^{n,p}$ are uniformly $\Sigma_1^{(n)}(M)$ in p. More precisely, there are fixed $\Sigma_1^{(n)}$ formulae Φ_n and Ψ_n such that given arbitrary M, n and $p \in M$, Φ_n and Ψ_n define $A_M^{n,p}$ and $h_M^{n,p}$ from p, respectively. As we have already mentioned above, the Σ^*-language enables us to talk about the definability over the n-th reduct without making any explicit reference to the reduct. For $\ell > 0$ and an arbitrary p we have: $A \subset H_M^n$ is $\Sigma_\ell^{(n)}(M)$ in p iff A is $\Sigma_\ell(M^{n,p})$; if $\ell = 0$, only the implication $\leftarrow$ holds.

Fix $n \in \omega$. A parameter $p \in [\mathbf{On} \cap M]^{<\omega}$, such that there is a new $\Sigma_1^{(n)}(M)$ in p subset of $\omega\varrho_M^n$ (by "new" we mean a set A such that $A \cap \omega\varrho_M^n$ is not an element of M), is called *good,* and the collection of all good parameters is denoted by P_M^n. $P_M^* = P_M^n$ for the least n satisfying $\varrho_M^n = \varrho_M^\omega$. Let $<^*$ be the canonical well-ordering of $[\mathbf{On} \cap M]^{<\omega}$, where $a <^* b$ means that a is lexicographically less than b when a and b are viewed as descending sequences of ordinals. The $<^*$-least element of P_M^* is called the *standard parameter* of M and denoted by p_M; the $<^*$-least element of P_M^n is called the *standard parameter above the n-th projectum* and denoted by p_M^n.

Given an $n \in \omega$, there is a class of partial $\Sigma_1^{(n)}(M)$ functions called *good functions,* which can be treated the same way as elements of M in many situations. This class comprises all partial $\Sigma_1^{(n)}(M)$-functions with range contained in H_M^n and their compositions. It follows that for every $\Sigma_1^{(n)}(M)$-relation $R(v_1^{i_1}, \ldots, v_\ell^{i_\ell})$ and any list of good $\Sigma_1^{(n)}(M)$-functions $F_1(\vec{z}), \ldots, F_\ell(\vec{z})$, the composition $R(F_1(\vec{z}), \ldots, F_\ell(\vec{z}))$ is again $\Sigma_1^{(n)}(M)$, provided that the ranges of the functions are contained in the domains of these corresponding variables. Morever, the definition of such a composition is uniform and depends only on the definitions of R and F_i. A formula φ is a *functionally absolute* definition for a good function if it defines a good function over every acceptable J-structure. It can be shown that every good function has a functionally absolute definition. An example of a good function is any $h^{n,p}_M$; its functionally absolute definition is uniform in the parameter p. Another example is the $\Sigma^{(n)}_1$ Skolem function $\tilde{h}^{n+1}_M(z,p)$ whose domain is contained in $H^n_M \times M$ and which is defined as follows. For a fixed p, $\tilde{h}^{n+1}_M(-,p)$ is the composition of all $h^{i,p\restriction i}_M$ for $i \leq n$, where the innermost function is $h^{n,p}_M$. For instance, $\tilde{h}^2_M(\langle i,x\rangle,p) = h^{0,\varnothing}_M(i_0, h^{1,p}_M(i_1,x))$ where $i = \langle i_0, i_1\rangle$. Since these functions are universal, each good $\Sigma^{(n)}_1$-function is equal to $\tilde{h}^{n+1}_M(\langle i,-\rangle,-)$ for some $i \in \omega$.

We can now generalize the notion of a Σ_1-hull. A $\Sigma^{(n)}_1$-hull of $X \subset M$ is defined in the obvious way. It follows that the $\Sigma^{(n)}_1$-hull of X is exactly the closure of X under the good $\Sigma^{(n)}_1(M)$-functions and this closure is, in turn, equal to the image of $[X]^{<\omega}$ under $\tilde{h}^{n+1}_M$. We denote this image by $\tilde{h}^{n+1}_M(X)$. More generally, for any $X \subset H^n_M$ and $Y \subset M$ we let

$$\tilde{h}^{n+1}_M(X \cup Y) \stackrel{\text{def}}{=} \{\tilde{h}^{n+1}_M(\langle i,z\rangle,p);\, z \in [X]^{<\omega} \;\&\; p \in [Y]^{<\omega} \;\&\; i \in \omega\}.$$

Recall also that $h^{n,p}_M(X \cup \{q\})$ is the set of all values $h^{n,p}_M(i,\langle x,q\rangle)$ where $i \in \omega$ and $x \in [X]^{<\omega}$ (we assume that $X \cup \{q\} \subset H^n_M$). In most of cases, X and Y will have sufficient closure properties, so that e.g. the images of X and $[X]^{<\omega}$ under the Skolem functions will be the same. If $p \in [\mathbf{On} \cap M]^{<\omega}$ and $\tilde{h}^{n+1}_M(X \cup \{p\}) = M$ (i.e. every $x \in M$ is $\Sigma^{(n)}_1(M)$-definable from p) then p is clearly in P^n_M(diagonalize to get a new subset of the projectum). The collection of all such parameters is denoted by R^n_M and we call them *very good* parameters. R^*_M then has the obvious meaning. M is *n-sound* iff $p^n_M \in R^n_M$ and M is *sound iff* $p_M \in R^*_M$. It can be proved that M is sound iff $P^*_M = R^*_M$ (and similarly for n-soundness). M is *sound above* α iff $M = \tilde{h}^{n+1}_M(\alpha \cup \{p^{n+1}_M\})$ where n is such that $\omega\varrho^{n+1}_M \leq \alpha < \omega\varrho^n_M$.

We shall often work with $\Sigma^{(n)}_\ell$-preserving maps $\pi : M \to N$. Sometimes we will briefly write

$$\pi : M \underset{\Sigma^{(n)}_\ell}{\longrightarrow} N.$$

Note that each such map is automatically $\Sigma^{(i)}_1$ for $i < n$. For any $p \in M$ and $q = \pi(p)$, the restriction $\pi \restriction H^n_M : M^{n,p} \to N^{n,q}$ is Σ_ℓ-preserving. Moreover, the converse holds whenever $p \in R^n_M$, i.e. each such restriction can be lifted to the original map – this fact is known as the *downward extension of embeddings lemma.* Finally, if both $p \in R^n_M$ and $q \in R^n_N$, the map $\pi \restriction H^i_M$, viewed as a map between the corresponding reducts, is in fact $\Sigma^{(i)}_{\ell+n-i}$-preserving for every $i \leq n$. A $\Sigma^{(n)}_{\ell}$-preserving map π is *cofinal* iff π has the above preservation degree and $\pi''H^n_M$ is cofinal in H^n_N. We will often say that π is *cofinal at the n-th level.* If f is a good $\Sigma^{(n)}_1(M)$-function in p and π is $\Sigma^{(n+1)}_0$-preserving, we can define $\pi(f)$ as the unique good $\Sigma^{(n)}_1(N)$-function in $\pi(p)$ with the same functionally absolute definition, as such a function does not depend on the choice of functionally absolute definition for f.

Solidity is a means for characterizing standard parameters. Fix a finite set of ordinals $p \in M$ and any ordinal $\nu \in M$. Let n be such that $\omega\varrho^{n+1}_M \leq \nu < \omega\varrho^n_M$. Let $W^{\nu,p}_M$ be the transitive collapse of $\tilde{h}^{n+1}_M(\nu \cup \{p-(\nu+1)\})$ and σ be the inverse to the collapsing map. $W^{\nu,p}_M$ is called the *standard solidity witness* to ν with respect to M and p and σ is the associated *witness map* . As σ is $\Sigma^{(n)}_1$-preserving, we have

$$M \models \Phi_n(i,\xi_1,\dots,\xi_\ell,q) \quad \text{iff} \quad W^{\nu,p}_M \models \Phi_n(i,\xi_1,\dots,\xi_\ell,\bar{q})$$

for every $\xi_1,\dots,\xi_\ell < \nu$ and $i \in \omega$, where Φ_n is the universal $\Sigma^{(n)}_1$-formula fixed above, $q = p-(\nu+1)$ and $\sigma(\bar{q}) = q$. A *generalized solidity witness* for ν with respect to M and p is a pair $\langle Q, r\rangle$, where $Q \supset \nu$ is an acceptable J-structure, $r \in Q$ is a finite set of ordinals and

$$M \models \Phi_n(i,\xi_1,\dots,\xi_\ell,q) \;\longrightarrow\; Q \models \Phi_n(i,\xi_1,\dots,\xi_\ell,r)$$

for all $\xi_1,\dots,\xi_\ell < \nu$ and $i \in \omega$. Clearly, $W^{\nu,p}_M$ is a generalized solidity witness, but the converse need not hold. On the other hand, $W^{\nu,p}_M \in M$ iff M contains *some* generalized solidity witness for ν with respect to M and p. If $p \in P^*_M$ and for each $\nu \in p$ there is a generalized solidity witness $\langle Q, r\rangle$ with respect to M and p such that $\langle Q, r\rangle$ is in M, then $p = p_M$ (and similarly for p^n_M). We say that M is *solid* iff also the converse holds, i.e. if every $\nu \in p_M$ has a generalized solidity witness with respect to M and p in M (similarly we define *n-solidity*). Note that if M is solid, then p^n_M is a lengthening of p^m_M whenever $m < n$. Thus, if M is solid then M is sound above α iff $M = \tilde{h}^{n+1}_M(\alpha \cup \{p_M\})$ where n is such that $\omega\varrho^{n+1}_M \leq \alpha < \omega\varrho^n_M$.

It follows easily that the statement "$\langle Q, r\rangle$ is a generalized solidity witness for ν with respect to M and p" is $\Pi^{(n)}_1$. Hence, the property of being a generalized solidity witness is preserved upward under $\Sigma^{(n)}_1$-maps and downward under $\Sigma^{(n)}_0$-maps. We stress that these preservation properties are typical for generalized solidity witnesses and fail for standard solidity witnesses – this is the point of introducing generalized solidity witnesses. If M is solid and $\pi(p_M) \in P^*_N$ then $\pi(p_M) = p_N$. We shall often use one further criterion for preservation of the standard parameter which we state as a lemma on its own.

Lemma 1.1 ([44]). *Suppose M is solid, $p \in M$ is a finite set of ordinals such that $p \cap \alpha = \varnothing$ and $M = \tilde{h}^{n+1}_M(\alpha \cup \{p\})$ for some $n \in \omega$. If every $\nu \in p$ has a generalized solidity witness with respect to M and p that is an element of M, then $p = p_M - \alpha$.*

A parameter $p \in P^n_M$ is *n-universal* iff $\mathcal{P}(\omega\varrho^n_M) \cap \bar{M} = \mathcal{P}(\omega\varrho^n_M) \cap M$ where $\bar{M}$ is the transitive collapse of $\tilde{h}^n_M(\omega\varrho^n_M \cup \{p\})$; note that $\omega\varrho^n_{\bar{M}} = \omega\varrho^n_M$ in this case. If $n = \omega$ we talk about *universality.* If p^n_M is n-universal, $\bar{M}$ is called the *n-th core* of M and is denoted by $\mathrm{core}_n(M)$; the inverse to the corresponding collapsing map is called the *n-core map.* It follows that p^n_M collapses to $p^n_{\bar{M}}$ and the n-core map is cofinal at the n-th level. If $n = \omega$ we talk about the *core of M,* which we denote by $\mathrm{core}(M)$, and the *core map.* (Here p^ω_M means p_M.) More generally, if $\omega\varrho^{n+1}_M \le \nu < \omega\varrho^n_M$ then the transitive collapse of $\tilde{h}^{n+1}_M(\nu \cup \{p_M\})$ is called the core of M *above* ν and denoted by $\mathrm{core}_\nu(M)$; the inverse to the collapsing map is called the associated *core map.* Thus, if $\nu = \omega\varrho^n_M$ then $\mathrm{core}_\nu(M) = \mathrm{core}_n(M)$.

Our constructions will utilize direct limits of diagrams of the form $\langle M_\iota, \sigma_{\bar{\iota}\iota};\, \bar{\iota} \le \iota < \gamma\rangle$ where each M_ι is acceptable and $\langle \sigma_{\bar{\iota}\iota}\rangle_{\bar{\iota}\le\iota}$ is a system of commuting maps such that for a fixed $n \in \omega$, each $\sigma_{\bar{\iota}\iota} : M_{\bar{\iota}} \to M_\iota$ is $\Sigma^{(n)}_0$-preserving. We are primarily interested in the case where the direct limit of $\langle M_\iota, \sigma_{\bar{\iota}\iota};\, \bar{\iota} \le \iota < \gamma\rangle$ is well-founded; such a direct limit is represented by a unique acceptable structure M together with the *direct limit maps* $\sigma_\iota : M_\iota \to M$ such that each $\sigma_\iota(x)$ is the object represented by the thread $\langle \sigma_{\iota\iota'}(x);\, \iota \le \iota' < \gamma\rangle$. Then $\sigma_\iota \circ \sigma_{\bar{\iota}\iota} = \sigma_{\bar{\iota}}$ and $M = \bigcup\{\mathrm{rng}(\sigma_\iota);\, \iota < \gamma\}$. If, moreover, there are $p_\iota \in P^n_{M_\iota}$ such that $p_\iota = \sigma_{\bar{\iota}\iota}(p_{\bar{\iota}})$ whenever $\bar{\iota} \le \iota < \gamma$, then $p = \sigma_\iota(p_\iota) \in P^n_M$, all maps σ_ι are $\Sigma^{(n)}_0$-preserving and $H^m_M = \bigcup\{\sigma_\iota{}''H^m_{M_\iota};\, \iota < \gamma\}$ whenever $m \le n$. One important property of the direct limit maps σ_ι is that, although they are merely $\Sigma^{(n)}_0$-preserving, they actually downward preserve $\Sigma^{(n)}_1$-statements on a tail-end of γ. To see this, consider a $\Sigma^{(n)}_0$-formula $\varphi(v_0, v_1, \dots, v_\ell)$ and any $a^1, \dots, a^\ell \in M$. If $M \models (\exists z^n)\varphi(z^n, a^1, \dots, a^\ell)$ then there is an $\iota_0 < \gamma$ such that for all $\iota \ge \iota_0$ we have $M_\iota \models (\exists z^n)\varphi(z^n, a^1_\iota, \dots, a^\ell_\iota)$ where $\sigma_\iota(a^h_\iota) = a^h$. To see this, pick a witness $b \in H^n_M$ such that $M \models \varphi(b, a^1, \dots, a^\ell)$ and an ι_0 such that

$\{b, a^1, \ldots, a^\ell\} \subset \mathrm{rng}(\sigma_{\iota_0})$. Then $b_\iota = \sigma_\iota^{-1}(b) \in H^n_{M_\iota}$ and $M_\iota \models \varphi(b_\iota, a^1_\iota, \ldots, a^\ell_\iota)$ whenever $\iota \geq \iota_0$. The downward preservation property we just verified guarantees that if $\zeta = \tilde{h}^{n+1}_M(\xi, p)$ and $(\xi, p) = \sigma_\iota(\xi_\iota, p_\iota)$, then $\zeta_\iota \simeq \tilde{h}^{n+1}_{M_\iota}(\xi_\iota, p_\iota)$ is defined on an tail-end of γ and $\zeta = \sigma_\iota(\zeta)$. The point here is that $\tilde{h}^{n+1}_{M_\iota}$ is good, i.e. it has a functionally absolute definition. Since $\Pi^{(n)}_1$-statements are automatically downward preserved under $\Sigma^{(n)}_0$-preserving embeddings, the above argument actually establishes the downward tail-end preservation for $\Sigma^{(n)}_2$-statements. Thus, if $\psi(u_1, \ldots, u_\ell)$ is a $\Pi^{(n)}_2$-formula and $M_\iota \models \psi(a^1_\iota, \ldots, a^\ell_\iota)$ on an tail-end of γ, then $M \models (a^1, \ldots, a^\ell)$.

Extenders and Coherent Structures. Consider two transitive structures M and N that satisfy a sufficiently large fragment of ZFC. Given a cofinal elementary embedding $\pi : M \to N$ and a pair of ordinals κ, λ such that κ is the critical point of π, $\lambda \in N$ is primitive recursively closed and the Σ_0-hull of $\mathrm{rng}(\pi) \cup \lambda$ is the entire N, the map $F : \mathcal{P}(\kappa) \cap M \to \mathcal{P}(\lambda)$ defined by

$$F(x) = \pi(x) \cap \lambda$$

is the *extender on M at (κ, λ)* derived from π. Then κ is the *critical point* and λ the *length* of F, and we denote them $\mathrm{cr}(F)$ and $\lambda(F)$, respectively. In this situation, N is called the *ultrapower* of M by F and we denote it by $\mathrm{Ult}(M, F)$. Also, π is called the *associated ultrapower map.* The requirement that λ is primitive resursively closed enables us to code finite tuples of ordinals below λ by elements of λ via the Gödel pairing function $\langle \eta, \xi \rangle \mapsto \prec\!\eta, \xi\!\succ$.

The term "ultrapower" here is not chosen accidentally; it is namely the case that N is the transitive isomorph to the term model $\mathbb{D}(M, F)$ resulting from the ultrapower construction. The domain of $\mathbb{D}(M, F)$ consists of all equivalence classes $[\alpha, f]$ where $\alpha < \lambda$, $f \in M$ is a function with domain κ, and the pair $\langle \alpha', f' \rangle$ is in the same equivalence class as $\langle \alpha, f \rangle$ just in case that $\prec\!\alpha, \alpha'\!\succ \in F(\{\prec\!\eta, \eta'\!\succ;\ f(\eta) = f'(\eta')\})$. The predicates of the term model are then defined in the obvious way. We will often write

$$\pi : M \xrightarrow[F]{} N$$

to express that $N = \mathrm{Ult}(M, F)$ and π is the associated ultrapower map.

A closer inspection of the ultrapower construction reveals that in order to establish the basic facts about $\mathrm{Ult}(M,F)$ and π one needs merely a finite list of Σ_0-properties of F (see [44]), so our definition of an extender is easily seen to be equivalent to the traditional one where an extender is presented as a coherent system of measures on κ indexed by finite subsets of λ. Also in our case it will be useful to have such a "local" characterization of extenders, i.e. a characterization that does not make any reference to the ultrapower map. It can be shown that for primitive recursively closed λ, F is an extender on M at (κ,λ) just in case that

$$F : \mathcal{P}(\kappa) \cap M \longrightarrow \mathcal{P}(\lambda)$$

and for every $A_1, \dots, A_\ell, B \in \mathcal{P}(\kappa) \cap M$, if B is primitive recursive in $A_1, \dots, A_\ell$, then $F(B)$ is primitive recursive in $F(A_1), \dots, F(A_\ell)$ by the same definition.

For $\alpha < \lambda$, the measure $\{x \subset \kappa;\, \alpha \in F(x)\}$ is denoted by F_α. Notice that if we want to represent F as a coherent system of measures, these measures can be indexed by ordinals below λ, as λ is primitive recursively closed. Given any primitive recursively closed $\bar\lambda < \lambda$, the "restriction" $F\,|\,\bar\lambda$ of F to $\bar\lambda$ is the unique extender at $(\kappa,\bar\lambda)$ satisfying $\mathrm{dom}(F\,|\,\bar\lambda) = \mathrm{dom}(F)$ and $(F\,|\,\bar\lambda)(x) = F(x) \cap \bar\lambda$.

An extender F at (κ,λ) is *weakly amenable* with respect to M iff the power set of κ remains unchanged in the corresponding ultrapower, which, in turn, is equivalent to the requirement that for every $\alpha < \lambda$ and $R \in M \cap \mathcal{P}(\kappa \times \kappa)$, the set of all $\xi < \kappa$ satisfying $R''\{\xi\} \in F_\alpha$ is an element of M. F is Σ_1-*amenable* with respect to M if each F_α is $\boldsymbol{\Sigma}_1(M)$ for $\alpha < \lambda$, and F is *close* to M iff it has both these properties. The definition of Σ_1-amenability does *not* involve any uniformity, more precisely, for $\alpha \neq \beta$ it allows that F_α is $\Sigma_1(M)$ in p and F_β is $\Sigma_1(M)$ in q where $p \neq q$. Finally, F is *whole* iff $\pi(\kappa) = \lambda$. An ordinal $\bar\lambda < \lambda$ is a *cutpoint* of F iff $F\,|\,\bar\lambda$ is whole.

Recall that, since π is the ultrapower map of M by F, each $x \in N$ is of the form $\pi(f)(\alpha)$ for some $f \in {}^\kappa M \cap M$ and $\alpha < \lambda$. If x is an ordinal less than $\pi(\kappa)$, we can choose f such that $f : \kappa \to \kappa$, and if $x \subset \pi(\kappa)$ in the ultrapower, we can choose f such that $f : \kappa \to \mathcal{P}(\kappa)$. In both cases, f can be recursively encoded into a subset of κ, and this coding is uniform. Also, the same applies to functions f with domain ${}^n\kappa$ for some $n \in \omega$ and range contained in κ or $\mathcal{P}(\kappa)$. Hence, granting that F is whole, we can pretend that we are applying F instead of π to f and shall write $F(f)$ instead of $\pi(f)$. In this notation, for instance, if $f : {}^n\kappa \to \kappa$ and $a \in {}^n\lambda$, we would write $F(f)(a)$ for $\pi(f)(a)$ instead of using the correct, but clumsy $F(f^*)^{**}(a)$ where $f^* = \{\prec\eta_1, \dots, \eta_n, \eta\succ;\ f(\eta_1, \dots, \eta_n) = \eta\}$ and $g^{**} = \{\langle\eta_1, \dots, \eta_n, \eta\rangle;\ \prec\eta_1, \dots, \eta_n, \eta\succ \in g\}$. In general, we will write $g(\xi_1, \dots, \xi_\ell)$ or $g(x)$ instead of $g(\prec\xi_1, \dots, \xi_\ell\succ)$ for any function g whose domain is contained in ordinals and any finite sequence of ordinals $x = \langle\xi_1, \dots, \xi_\ell\rangle$, and $g(\xi_1, \dots, \xi_\ell, y)$ instead of $g(\prec\xi_1, \dots, \xi_\ell, \zeta_1, \dots, \zeta_k\succ)$ where $y = \langle\zeta_1, \dots, \zeta_k\rangle$ is a finite sequence of ordinals. The expressions $F(f)(\alpha_1, \dots, \alpha_n)$, $F(f)(a)$ and $F(f)(\alpha_1, \dots, \alpha_\ell, b)$ then have the obvious meaning. These expressions do make sense even if a, b are finite *sets* of ordinals, as we can naturally view them as finite increasing sequences.

A J-structure M is *coherent* iff M is of the form $\langle J^A_\alpha, F\rangle$ where J^A_α is acceptable, F is a whole extender at (κ, λ) with $\mathrm{dom}(F) = \mathcal{P}(\kappa) \cap J^A_{\bar\alpha}$ for some $\bar\alpha \le \kappa^{+M}$ and $J^A_\alpha = \mathrm{Ult}(J^A_{\bar\alpha}, F)$. Thus, F need not be weakly amenable, but M is always an amenable structure. Clearly, F is weakly amenable iff $\bar\alpha = \kappa^{+M}$ (that is, iff F is an extender on M). If F happens to be weakly amenable, we shall sometimes write "F is a *total* extender on M" to stress this fact. The property of being a coherent structure is a Q-property, i.e. there is a fixed Q-sentence ψ such that for every transitive structure $M = \langle U, F\rangle$ closed under forming singletons, M is a coherent structure iff $M \models \psi$. On the other hand, the weak amenability of F cannot be expressed in a Q-fashion; it is genuinely Π_2. Notice also that the statement "$\bar\lambda$ is a cutpoint of F" is Π_1 over a coherent structure $M = \langle J^A_\alpha, F\rangle$.

Fine Ultrapowers. A *fine ultrapower* of an acceptable J-structure M is obtained by an ultrapower construction which makes use not only of functions that are elements of M, but also of those, which are reasonably *definable* over M. If $k \le \omega$, F is at (κ, λ) and n is such that $\omega\varrho^{n+1}_M \le \kappa < \omega\varrho^n_M$, setting $m = \min\{k, n\}$ we define

$$\Gamma_k(\kappa, M) = \{f : \kappa \to M;\ f \text{ is a good } \mathbf{\Sigma}^{(m-1)}_1(M)\text{-function}\};$$

here we make a convention that good $\mathbf{\Sigma}^{(-1)}_1(M)$-functions are precisely those functions that are elements of M. Notice that for each $k \le \omega$ we have ${}^\kappa M \cap M \subset \Gamma_k(\kappa, M)$ since each $f \in {}^\kappa M \cap M$ has a functionally absolute Σ_0-definition. So $\Gamma_0(\kappa, M) = {}^\kappa M \cap M$. Using functions from $\Gamma_k(\kappa, M)$ we build the term model $\mathbb{D}^k(M, F)$ which, provided that it is well-founded, gives rise to the *k-ultrapower*

$\mathrm{Ult}^k(M,F)$ of M; if $k=\omega$ we talk about the $*$-ultrapower $\mathrm{Ult}^*(M,F)$ and denote the associated term model by $\mathbb{D}^*(M,F)$. We also write $\Gamma(\kappa,M)$ instead of $\Gamma_\omega(\kappa,M)$ in this case. The domain of $\mathbb{D}^k(M,F)$ consists of all equivalence classes $[\alpha,f]$ where $\alpha<\lambda$ and $f\in\Gamma_k(\kappa,M)$. The equivalence relation in question, as well as the predicates of the term model are defined in the same manner as in the case of ordinary ultrapowers (see above); that these definitions make sense follows from the fact that the definability degree of the functions f is at most $n-1$. Notice also that 0-ultrapowers are just ordinary ultrapowers, that is, ultrapowers, where the functions f are elements of M. To indicate that π is the corresponding ultrapower map, we briefly write

$$\pi : M \xrightarrow[F]{k} N, \qquad \text{resp.} \qquad \pi : M \xrightarrow[F]{*} N.$$

Analogously to the case of ordinary ultrapowers, each $x\in N$ is of the form $\pi(f)(\alpha)$ for some $f\in\Gamma_k(\kappa,M)$ and $\alpha<\lambda$.

If $k\geq 0$, we have the Łoś theorem for $\Sigma_0^{(m)}$-formulae, where m is as above. The k-ultrapower map π is $\Sigma_0^{(m)}$-preserving and $\Sigma_2^{(i)}$-preserving for every $i<m$. If $i<n$ then $\pi(\omega\varrho_M^i)=\omega\varrho_N^i$. Moreover, if R_M^m is nonempty, then $\pi(p)\in R_N^m$ for every $p\in R_M^m$ and π is $\Sigma_0^{(m)}$ cofinal, which means that every $x\in H_N^m$ is of the form $\pi(f)(\alpha)$ for some $f\in H_M^m$ and $\alpha<\lambda$. In this case, forming a k-ultrapower of M is equivalent to forming the corresponding (ordinary) ultrapower of the m-th reduct (relative to some $p\in R_M^m$) and to additional lifting the associated ultrapower map to M. For $k=0$, the ultrapower map is always Σ_0-preserving and cofinal. If F is weakly amenable or Σ_1-amenable with respect to M or if $h_M^{n,p}$ maps κ onto κ^{+M} for some p, then the $*$-ultrapower map π is $\Sigma_0^{(n)}$-preserving and cofinal (that is, cofinal at the n-th level); if F is close to M, then π is fully Σ^*-preserving.

The ultrapower construction can be generalized in the sense that the role of an extender is played by a sufficiently preserving embedding; the resulting structure is called a *pseudoultrapower.* (Roughly speaking, this is the process of deriving an extender and taking an ultrapower combined into one.) More precisely, let $\sigma : J_\tau^A \to J_{\tilde\tau}^B$ be Σ_0-preserving and cofinal. Suppose that $M=\langle J_\alpha^A, C\rangle$ and $\tau=\kappa^{+M}$. For k,n,m as above we let $\Gamma_k(\sigma,M)=\Gamma_k(\kappa,M)$; if $k=\omega$ we write briefly $\Gamma(\sigma,M)$. Now the ultrapower construction can be carried out using σ in place of an extender and the corresponding version of the Łoś theorem will hold for the term model $\mathbb{D}^k(M,\sigma)$. If $k=\omega$, we write briefly $\mathbb{D}^*(M,\sigma)$ and if $k=0$, we simply omit the superscript. Granted that the term model is well-founded, we can transitivize it to obtain a structure $N=\langle J^B_\beta, D\rangle$ which is an end-extension of $J^B_{\tilde\tau}$, and a map $\tilde\sigma_k : M\to N$ extending σ. We write $\tilde\sigma$ for $\tilde\sigma_\omega$. The map $\tilde\sigma$, resp. $\tilde\sigma_k$, is called the *canonical extension,* resp. *k-extension,* of σ to M. Letting $\tilde\kappa=\sigma(\kappa)$, we have $\tilde\tau=\tilde\sigma(\tau)=\tilde\kappa^{+N}$. As in the case of extenders, $\tilde\sigma_k$ is always $\Sigma^{(m)}_0$-preserving where m is as above and, if $k=0$, also cofinal; moreover, for $k>0$, this map is $\Sigma^{(i)}_2$-preserving whenever $i<m$. If R^m_M is nonempty, then $\tilde\sigma_k(p)\in R^m_N$ for every $p\in R^m_M$ and $\tilde\sigma_k$ is $\Sigma^{(m)}_0$ cofinal. Furthermore, if $h^{n,p}_M$ maps κ onto τ for some p then $\tilde\sigma$ (note that $k=\omega$ here) is $\Sigma^{(n)}_0$ cofinal; but this is all we can say about preservation properties of these maps in general. As before, each $x\in N$ is of the form $\tilde\sigma_k(f)(\xi)$ for some $f\in\Gamma_k(\sigma,M)$ and $\xi<\tilde\kappa$; if $x\in H^m_N$ and $\tilde\sigma_k$ is cofinal then f can be chosen from H^m_M. Furthermore, there is an obvious relationship between the k-pseudoultrapower of M and the (ordinary) pseudoultrapower of $M^{k,p}$ for $p\in R^k_M$. We shall often use the following criterion on the well-foundedness of pseudoultrapowers (we only state the version useful for us where $k=\omega$). Its proof can be found in [44].

Lemma 1.2 (Interpolation lemma)**.** *Let $\sigma : M\to N'$ be $\Sigma^{(n)}_0$-preserving where M,n,κ,τ are as above (in particular, $\omega\varrho^{n+1}_M\le\kappa<\omega\varrho^n_M$), let $N'=\langle J^{B'}_\beta, D'\rangle$, $\tilde\kappa=\sigma(\kappa)$ and $\tilde\tau=\sup(\sigma''\tau)$. Then the canonical extension $\tilde\sigma : M\to N$ of $\sigma\restriction|J^A_\tau| : J^A_\tau\to J^{B'}_{\tilde\tau}$ exists and there is a unique Σ_0-preserving map $\sigma' : N\to N'$ such that $\sigma=\sigma'\circ\tilde\sigma$ (thus, $\sigma'\restriction|J^{B'}_{\tilde\tau}|=\mathrm{id}$ and $\sigma'(\tilde\tau)=\tilde\kappa^{+N'}$). In fact, σ' is $\Sigma^{(n-1)}_1$-preserving. Moreover, if $\tilde\sigma$ is $\Sigma^{(n)}_0$-preserving and cofinal (i.e. cofinal at the n-th level) then σ' is $\Sigma^{(n)}_0$-preserving.*

Premice and Extender Models. The main objects of our interest are coherent structures $M=\langle J^E_\alpha, E_{\omega\alpha}\rangle$ constructed relative to an extender sequence E. The predicate E is not literally an extender sequence, but a set (class) of triples $\{\langle\nu,x,y\rangle;\ \nu\in\mathbf{On}\ \&\ x,y\subset\mathbf{On}\}$ such that, setting $E_\nu=\{\langle x,y\rangle;\ \langle\nu,x,y\rangle\in E\}$, $E_\nu=\varnothing$ whenever ν is a successor ordinal and, for a limit ordinal $\omega\nu$, $E_{\omega\nu}$ is either the empty set or else an extender *on* J^E_ν, that is, $\mathrm{dom}(E_{\omega\nu})=\mathcal{P}(\mathrm{cr}(E_{\omega\nu}))\cap J^E_\nu$. For $\nu\le\alpha$ we set $M||\nu=\langle J^E_\nu, E_{\omega\nu}\rangle$.

Definition ([13, 44])**.** *A* potential premouse *is an acceptable structure M of the from $\langle J^E_\alpha, E_{\omega\alpha}\rangle$ constructed relative to an extender sequence satisfying:*

a) *If $\nu \le \alpha$ then either $E_{\omega\nu}$ is empty or else $M||\nu$ is a coherent structure and $E_{\omega\nu}$ is weakly amenable with respect to $M||\nu$.*

b) *If $\nu < \alpha$ then $M||\nu$ is sound.*

To understand the exact meaning of b), see the discussion concerning the initial segment condition below. The ordinal α is called the *height* of M; we denote it by $\mathrm{ht}(M)$. To indicate that E is the extender sequence of M, we shall write E^M. It is clear that if $E_{\omega\nu}$ is an extender then $\omega\nu = \nu$. The top extender $E_{\omega\alpha}$ is denoted by E^M_{top}. Potential premice with top extenders are called *active,* those without top extenders are called *passive.* It follows easily from a) that if $N = \mathrm{Ult}(M||\nu, E_\nu)$ then $E^N \restriction \nu = E \restriction \nu$; since ν is a cardinal in N, we also have $E^N_\nu = \varnothing$.

Definition. *A* premouse *is a potential premouse which satisfies the so-called initial segment condition (ISC):*

$$\textit{If } E^M_\nu \neq \varnothing \textit{ and } \bar\lambda \textit{ is a cutpoint of } E^M_\nu, \textit{ then } E^M_\nu \mid \bar\lambda \in M||\nu.$$

Given a potential premouse M and a $\nu \le \mathrm{ht}(M)$, it follows easily that $\lambda(E^M_\nu)$ is a cardinal in M iff E^M_ν is *superstrong* in M. If $\bar\lambda$ is a cutpoint of E^M_ν then $\bar\lambda$ is a limit cardinal in $M||\nu$. To see this, notice that the canonical map that embeds the coherent structure associated with $E^M_\nu \mid \bar\lambda$ into $M||\nu$ is Σ_0-preserving and cofinal. Granting that M is a premouse, $E^M_\nu \mid \bar\lambda$ is a superstrong extender in $M||\nu$. Thus, the existence of cutpoints is a very strong large cardinal axiom under the initial segment condition.

Extender models are models of the form $\mathbf{L}[E]$ constructed relative to an extender sequence, whose each proper initial segment is a premouse which is *weakly iterable* in the sense specified below.

We shall consider normal k-iterations of (potential) premice where k-ultrapowers are taken at successor steps as defined in [13, 44]. Let us briefly recall the notation. If $\Im$ is a normal iteration then $T^\Im$ is the tree structure indicating where the iteration maps live, $\nu_i = \nu^\Im_i$ are the iteration indices and $D^\Im$ is the set of indices i for which $E^{M_i}_{\nu_i} \neq \varnothing$. We also set $\kappa_i \simeq \mathrm{cr}(E^{M_i}_{\nu_i})$ and $\lambda_i \simeq \lambda(E^{M_i}_{\nu_i})$. Given an $i \in D^\Im$, the T-predecessor of $i+1$ is the least $\xi \in D^\Im$ such that $\kappa_i < \lambda_\xi$. *Iterations* are linear concatenations of normal iterations. At limit steps of an iteration we choose a cofinal well-founded branch through $T^{\Im}$; a function $\mathbb{S}$ which picks such branches is called *iteration strategy.* We say that M is (θ,μ,k)-$\mathbb{S}$-iterable, resp. (θ,μ)-$\mathbb{S}$-iterable if every k-iteration, resp. $*$-iteration of M according to $\mathbb{S}$ of length less than θ which is a concatenation of less than μ normal iterations can be continued. M is (θ,μ)*-iterable* if there is an iteration strategy (i.e. a $*$-iteration strategy) $\mathbb{S}$ such that M is (θ,μ)-$\mathbb{S}$-iterable; k-iterability is defined analogously. A premouse M is *weakly iterable* iff every countable premouse elementarily embeddable into M is (ω_1+1,ω_1)-iterable. This amount of iterability suffices for developing the basic fine structure theory of extender models along the lines of [13, 44].

The initial segment condition is imposed on premice in order to guarantee the termination of the comparison process. It would be therefore desirable that it is preserved under embeddings arising in condensation arguments and ultrapower constructions. Unfortunately, the initial segment condition need not be downward preserved under maps that are merely Σ_1-preserving. To overcome this issue, we divide the class of active premice into subclasses according to the distribution of cutpoints of the top extender, and add a predicate for the cutback of $E^M_{\rm top}$ to its largest cutpoint, provided that such a cutpoint exists. Obviously, this procedure involves expanding the language of premice. Originally, the language for premice contained merely the symbols $\dot\in, \dot E$ and $\dot F$ naming the membership relation, extender sequence and the top extender, respectively. The expanded language for premice also contains a predicate symbol for the cutback of the top extender to its largest cutpoint. As there is no danger of confusion, we shall be a bit sloppy with the notation and shall rarely notationally distinguish between the predicate symbol and the corresponding predicate.

Let us now turn to a more detailed description of the expanded language for premice. Given an active potential premouse M, we let $\lambda_M=\lambda(E^M_{\rm top})$ and C_M be the set of all cutpoints of $E^M_{\rm top}$. We say that M is a *type A* premouse iff $C_M=\varnothing$, *type B* premouse iff C_M is nonempty, but bounded in λ_M and *type C* premouse iff C_M is unbounded in λ_M. For type B premice we we set $\lambda^*_M=\max(C_M)$ and $F^*_M=E^M_{\rm top}\restriction\lambda^*_M$. The standard way of treating type B premice (see e.g. [15]) is to augment the language by adding a constant symbol $\dot F^*$ whose interpretation is the object

F^*_M. This has a consequence that, for type B premice, the standard codes, Skolem functions, reducts and the standard parameter of M are to be computed relative to F^*_M. Note that the notion of soundness is also affected by this amendment; this should be kept in mind when dealing with clause b) in the definition of potential premouse. Of course, the projecta will be the same no matter which of the two languages we choose. The point of adding the symbol for F^*_M is to guarantee that all embeddings of type B premice into potential premice have F^*_M in their ranges, and thus preserve the initial segment condition. In order to simplify the notation, it will be more convenient to add the constant symbol $\dot{\gamma}$ denoting the *index* γ_M of F^*_M instead of adding $\dot{F}^*$ itself. This does make sense, as it follows by an easy coiteration argument (see [46]) that F^*_M is on the E^M-sequence. Clearly, γ_M is definable from F^*_M in a Σ_1-fashion over M and vice versa, so all fine structural characteristics of M remain unchanged if the computations are relative to γ_M instead of F^*_M. Switching from the language with $\dot{F}^*$ to that with $\dot{\gamma}$ is therefore a purely cosmetic issue. This completes the discussion of type B premice. In order to obtain a uniform terminology, we also add $\dot{\gamma}$ to the language in the case of type A, type C and passive premice, but we always interpret it as $\varnothing$. This trivial amendment to the language does not bring anything new, of course, but will help us to simplify formulations in many situations.

Since most of our arguments will be sensitive to the language we work with, it is convenient to introduce the following terminology. The *language for coherent structures* is the language containing the symbols $\dot{\in}$ denoting the membership relation, $\dot{E}$ denoting the extender sequence the structure is built from and $\dot{F}$ denoting the top extender of the structure. The *language for premice* will also contain the additional symbol $\dot{\gamma}$ intended to denote the restriction of the top extender to its largest cutpoint. Clearly, it does not make any difference whether we treat type A, type C and passive premice in the language for coherent structures or that for premice, and we will usually choose the language for coherent structures (which is actually more natural).

Let us fix a convention that whenever we work with *premice,* the language in question will always be the language for premice, unless *explicitly stated otherwise.* For instance, the standard parameters and the Skolem functions $\tilde{h}^n_M$ and $h^{n,p}_M$ are all computed in the language for premice. Similarly, when we say "$\sigma : M \to M'$ is $\Sigma_\ell^{(n)}$-preserving" where M and M' are two premice, we always mean the $\Sigma_\ell^{(n)}$-preservation with respect to the language for premice. In some cases, however, we will ignore this convention and write explicitly that the language in question is the language for premice; the intention here is not to confuse the reader, but rather to stress which language we work with. Notice that if $\sigma : M \to M'$ is Σ_0-preserving with respect to the language for premice then M and M' are of the same type. On the other hand, if we consider preservation properties with respect to the language for coherent structures, it is, of course, possible that the premice in question are of distinct types.

The following two lemmata summarize basic facts concerning preservation of the initial segment condition. We recall that passive premousehood is a Q-condition and active potential premousehood a Π_2-condition, both in the language for coherent structures. Also, being a cutpoint is a Π_1-condition in the language for coherent structures. For any type C premouse M we have $\omega\varrho_M^1 = \lambda_M$ and $p_M \subset \lambda_M$, i.e. $p_M^1 = \varnothing$. That $\omega\varrho_M^1 \leq \lambda_M$ is immediate. To see the converse, we show that given any $\alpha < \lambda_M$ and any A that is $\Sigma_1(M)$ in some $p \in [\mathbf{On}]^{<\omega}$, the intersection $A \cap \alpha$ is in M. Choose $\tilde\lambda \in C_M$ large enough that $\alpha \leq \tilde\lambda$ and $p \in \mathrm{rng}(\sigma)$; here $\sigma : \tilde M \to M$ is the canonical embedding of the coherent structure $\tilde M$ associated with $E^M_{\mathrm{top}} \mid \tilde\lambda$ into M. The map σ is Σ_1-preserving in the language for coherent structures. Letting $\tilde A$ be the unique set that is $\Sigma_1(\tilde M)$ in $\sigma^{-1}(p)$ by the same definition as A, the fact that $\sigma \restriction \tilde\lambda = \mathrm{id}$ guarantees that $A \cap \alpha = \tilde A \cap \alpha$. By the initial segment condition, $\tilde M \in M$, so $\tilde A \cap \alpha \in M$ as well.

Lemma 1.3. *Let $M = \langle J_\nu^E, F\rangle$ be an active premouse, $\bar M$ be a potential premouse of the same type as M and let $\sigma : \bar M \to M$ be such that one of the following holds:*

a) *M is of type A or B and σ is Σ_1-preserving;*
b) *M is of type C and σ is $\Sigma_0^{(1)}$-preserving or Σ_2-preserving.*

Then $\bar M$ is a premouse.

For transitive collapses of definable hulls we obtain the following analogue of the above lemma.

Lemma 1.4. *Let M be and active premouse and $\sigma : \bar{M} \to M$ be an uncollapsing embedding associated with a $\Sigma_1^{(n)}$-hull over M. Assume that one of the following holds:*

a) *$n \geq 0$ and M is of type A or B;*

b) *$n > 0$ and M is of type C.*

Then $\bar{M}$ is a premouse of the same type as M; moreover, $\sigma(\gamma_M) = \gamma_{M'}$ whenever they are of type B.

Consistently with our convention, a) and b) in the above lemmata talk about preservation properties of σ with respect to the language for premice. Recall again that in the case of type A or type C premice, there is no real difference between the two languages.

Lemma 1.5. *Let $M = \langle J_\nu^E, F\rangle$ be an active premouse and let $\sigma : M \to M'$ be such that one of the following holds:*

a) *M is of type A or B and σ is Σ_2-preserving with respect to the language for coherent structures;*

b) *M is of type C and σ is $Q^{(1)}$-preserving with respect to the language for coherent structures and $\omega\varrho^1_{M'} = \lambda_{M'}$;*

c) *σ is Σ_0-preserving with respect to the language for coherent structures and cofinal, and maps λ_M cofinally into $\sigma(\lambda_M)$.*

Then M' is a premouse of the same type as M; moreover, $\sigma(\gamma_M) = \gamma_{M'}$ whenever they are of type B.

Steel's $\mathbf{K}^c$-construction [37] can be carried out in the above setting and provides us with the extender model $\mathbf{K}^c$ whose initial segments are weakly iterable, which enables us to give a fine structural analysis of the model. Moreover, granting that the set-theoretic universe is sufficiently rich, we can get a Woodin cardinal (see [13] and [14]) or even many Woodin cardinals in $\mathbf{K}^c$. [1] describes the largest known $\mathbf{K}^c$-model of this kind which can be constructed by present methods; it goes up to an externally measurable cardinal λ that is a limit of Woodin cardinals and cardinals strong up to λ. With a different, carefully-chosen indexing of extenders, one can get even further, slightly beyond a strong limit of Woodin cardinals. The next lemma summarizes all fine structural properties of extender models which will be needed for the construction of square sequences. See [44] for the proof.

Lemma 1.6 (Condensation lemma, [13, 44])**.** *Let $\bar{M}$ and M be premice of the same type where M is a level of $\mathbf{L}[E]$ and let $\sigma : \bar{M} \to M$ be an embedding which is both cardinal preserving and $\Sigma_0^{(n)}$-preserving and such that $\sigma \restriction \omega\varrho_{\bar{M}}^{n+1} = \mathrm{id}$. Then $\bar{M}$ is solid and p_M is k-universal for every $k \in \omega$. Furthermore, if $\bar{M}$ is sound above $\nu = \mathrm{cr}(\sigma)$ then one of the following holds:*

a) *$\bar{M} = M$ and $\sigma = \mathrm{id}$;*

b) *$\bar{M}$ is a proper initial segment of M;*

c) *$\bar{M} = \mathrm{Ult}^*(M||\eta, E_\alpha^M)$ where $\nu \leq \eta < \mathrm{ht}(M)$, $\alpha \leq \omega\eta$ and $\nu = \kappa^{+M||\eta}$ where $\kappa = \mathrm{cr}(E_\alpha^M)$; moreover, E_α^M has a single generator κ;*

d) *$\bar{M}$ is a proper initial segment of $\mathrm{Ult}(M, E_\nu^M)$.*

The assumption that σ is cardinal preserving is superfluous unless $n = 0$, in which case it cannot be omitted. It might namely happen that $\bar{M}$, although being of the form $\langle J_{\bar{\alpha}}^E, \varnothing\rangle$ where $E = E^M$, is distinct from $M||\bar{\alpha}$, and so is not literally an initial segment of M. By inspecting all cases that might occur in the condensation lemma, one can easily verify that this is the only way the condensation lemma can fail if σ fails to be cardinal preserving. Obviously, $E_{\bar{\alpha}} \neq \varnothing$ whenever this happens.

It follows that the ordinal η in c) is in strictly less than κ^{+M} and that $M||\eta$ projects to κ. Clause d) implies that E_ν^M is a superstrong extender, as $\lambda(E_\nu^M)$, being the cardinal predecessor of ν in $\bar{M}$, is a cardinal in M. In fact, it is easy to see that the set of all $\bar{\nu} < \lambda(E_\nu^M)^{+M}$ such that both

$$E_{\bar{\nu}}^M \text{ is a superstrong extender and } \lambda(E_{\bar{\nu}}^{\bar{M}}) = \lambda(E_\nu^M)$$

is stationary in $\bar{M}$. Indeed, if $c \in \bar{M}$ is closed and unbounded in ν then $\sigma(c)$ is closed in M and has ν as its limit point, so $\nu \in \sigma(c)$. Thus, ν witnesses the statement "there is a $\zeta \in \sigma(c)$ such that E_ζ^M is superstrong and $\lambda(E_\zeta^M) = \lambda(E_\nu^M)$" in M. But the acceptability of M enables us to phrase this statement inside $J^E_{\lambda(E_\nu^M)^{+M}} = \sigma(J_\nu^{\bar{E}})$ (here $\bar{E} = E^{\bar{M}}$), that is, in a Σ_0-manner over M. It follows that c contains some index of a superstrong extender whose length is precisely $\lambda(E_\nu^M)$.

Dodd Parameters. The last ingredient needed in our main construction is some information concerning fine structural properties of the Dodd parameters of certain extenders. A detailed account of this topic can be found [46] where a more general situation is treated. For a sound mouse $M = \langle J_\nu^E, F\rangle$ satisfying $\mu = \mathrm{cr}(F) <$

$\omega\varrho_M^1 < \lambda_M$, we inductively define a descending sequence of ordinals δ_i as follows:

$\delta'_i =$ the least δ such that every ordinal less than λ_M is of the form $F(f)(\delta_0, \ldots, \delta_{i-1}, \xi_1, \ldots \xi_m)$ for some $m \in \omega$, $\xi_1, \ldots, \xi_m < \delta$ and $f : \mu \to \mu$ in M.

If δ'_i is a successor ordinal, we set $\delta_i = \delta'_i - 1$. As δ_i is descending, after finitely many steps we reach an m such that δ'_m is limit. Then δ'_m is called the *Dodd* projectum and denoted by δ_M, and $d_M = \{\delta_0, \ldots, \delta_{m-1}\}$ is the *Dodd parameter* of M. This is the traditional definition of the Dodd projectum and Dodd parameter. Notice that M must be a type A or type B premouse, as $\omega\varrho_M^1 < \lambda_M$. If M is of type A, then $d_M = p_M$ by an easy computation. If M is of type B, one can show that $d_M = p_M \cup e_M$ where e_M is defined below. In either case, the Dodd projectum is equal to $\omega\varrho_M^1$. Let us now turn to our official definitions, which will be more convenient for our purposes.

Definition ([46]). *Let M be an active premouse such that $\mathrm{cr}(E_{\mathrm{top}}^M) < \omega\varrho_M^1$. The parameter d_M is the $<^*$-least finite set of ordinals d such that $M = h_M^*(\omega\varrho_M^1 \cup \{d\})$, if defined. Here h_M^* is the canonical Σ_1-Skolem function for M computed in the language for coherent structures.*

The above definition is formulated for arbitrary active premouse M; if M has a very good parameter, this definition is equivalent to the traditional definition. Clearly, $d_M \cap \omega\varrho_M^1 = \varnothing$. Although it is obvious, let us stress again that the use of h_M^* in the above definition is vital; if we used h_M instead, d_M would be just the least very good parameter from R_M^1. This is actually the case whenever M is of type A or C, so the notion of Dodd parameter brings something new only for type B premice.

Dodd solidity witnesses are just solidity witnesses with respect to the language for coherent structures. Thus, given an ordinal $\nu \geq \omega\varrho_M^1$ and a parameter p in $[\mathbf{On} \cap M]^{<\omega}$, the standard Dodd solidity witness ${}^*W_M^{\nu,p}$ for ν with respect to p and M is the transitive collapse of the hull $h_M^*(\nu \cup \{p - (\nu+1)\})$. The reader will now easily formulate the definition of a generalized Dodd solidity witness. Obviously, all general facts about solidity witnesses apply to Dodd solidity witnesses. In particular, ${}^*W_M^{\nu,p} \in M$ just in case that *some* generalized Dodd solidity witness with respect to ν and M is an element of M. A premouse M is *Dodd solid* iff d_M is defined and ${}^*W_M^{\nu,d_M} \in M$ for every $\nu \in d_M$. Notice that if M is Dodd solid then d_M is nothing but the standard parameter of M with respect to the language of coherent structures.

Definition. *Let M be a type B potential premouse. The $<^*$-least finite set of ordinals e such that $\gamma_M \in h^*_M(\omega\varrho^1_M \cup \{p_M \cup e\})$ is denoted by e_M.*

Obviously, e_M is always defined and $e_M \subset \gamma_M + 1$. We have already mentioned above that in all relevant cases, the Dodd parameter d_M can be computed from the standard parameter of M and the set of ordinals e_M. It is of a great importance that the conversion between the Dodd parameter and the standard parameter is Σ_1 and sufficiently uniform, which guarantees that it is preserved under embeddings with relatively weak preservation properties. Let us now formulate everything rigorously. The following two lemmata summarize the main results concerning Dodd parameters we will rely on.

Lemma 1.7 ([46])**.** *Let M be an $\mathbf{L}[E]$-level of type B satisfying $\mathrm{cr}(E^M_{\mathrm{top}}) < \omega\varrho^1_M$. Then d_M is defined and:*

a) $d_M = (p_M - \omega\varrho^1_M) \cup e_M$.
b) *M is Dodd solid.*

Lemma 1.8 ([46])**.** *Let M be a type B premouse with $\mathrm{cr}(E^M_{\mathrm{top}}) < \omega\varrho^1_M$ and such that there is a Σ_0-preserving embedding $\sigma : M \to N$ into an $\mathbf{L}[E]$-level N satisfying $\sigma \restriction \omega\varrho^1_M = \mathrm{id}$. Assume that d is a parameter satisfying:*

- $d \cap \omega\varrho^1_M = \varnothing$.
- $M = h^*_M(\omega\varrho^1_M \cup \{d\})$.
- *Every $\nu \in d$ has a generalized Dodd solidity witness with respect to d and M, that is an element of M.*

Then

a) $d = d_M$;
b) $d_M = (p_M - \omega\varrho^1_M) \cup e_M$;
c) *M is sound above $\omega\varrho^1_M$.*

The preservation properties of σ in the above lemma are, of course, formulated with respect to the language of premice. Thus, granting that M is active, M and N must be of the same type.

2. Extender Fragments

The construction of a $\square_\kappa$ sequence involves interpolation arguments which, in turn, give rise to embeddings with low preservation degree. A typical example is an embedding $\sigma : \bar{M} \to M$ where M is an active $\mathbf{L}[E]$-level, σ is Σ_0-preserving but *not* cofinal and $\mathrm{cr}(\sigma) > \mathrm{cr}(E^M_{\mathrm{top}})^{+M}$. The top predicate $\bar{F}$ of $\bar{M}$ is then an extender which is *not* weakly amenable with respect to $\bar{M}$. In other words, some subsets of $\mathrm{cr}(\bar{F})$ which are in $\bar{M}$ are not measured by $\bar{F}$. This means that $\bar{M}$ is not even a potential premouse and our task is to find a correspondence between $\bar{M}$ and some suitable level of $\mathbf{L}[E]$. Studying connections between structures like $\bar{M}$ and levels of $\mathbf{L}[E]$ is the main issue addressed in the current section. Structures like $\bar{M}$ are called *protomice* and their top extenders can be viewed as "fragments" of total extenders. It will turn out that the fine ultrapower of the longest possible initial segment of $\bar{M}$ by $\bar{F}$ is a level of $\mathbf{L}[E]$ and we will discuss the relationship between the definability over $\bar{M}$ and its corresponding $\mathbf{L}[E]$-level. It is clear that each protomouse is associated with at most one such level, but the converse is false in general, and our next task will be identifying the *canonical* protomice for collapsing $\mathbf{L}[E]$-levels of "local" successor cardinals.

The reader should pay particular attention to the definitions of divisor and strong divisor to come, as they will play important roles in identifying canonical protomice. These two notions were used already in the authors' earlier papers on square principles in extender models but with different terminology. Before launching into the technicalities, let us offer a bit of motivating discussion. In order to explain the main ideas quickly, it will be necessary for us to be slightly inaccurate, but only in this paragraph. As one familiar with Jensen's proof of $\square_\kappa$ in $\mathbf{L}$ would expect, we must consider ordinals $\kappa < \tau \leq \alpha$ where κ is the largest cardinal in J^E_τ and $\mathbf{L}[E]\,||\,\alpha$ is the collapsing $\mathbf{L}[E]$-level for τ, that is, τ is a cardinal in $\mathbf{L}[E]\,||\,\alpha$ and $\mathbf{L}[E]\,||\,\alpha$ projects to κ. Let us now focus on the case where κ is the *first* projectum of the collapsing level M for τ and $\mathrm{cr}(E^M_{\mathrm{top}}) < \kappa$. As indicated in the previous paragraph, it is this case that prevents a naive adaptation of Jensen's method because the corresponding interpolation arguments give rise to protomice. There are, however, canonical club subsets of τ of order type ϑ where $\mu = \mathrm{cr}(E^M_{\mathrm{top}})$ and $\vartheta = \mu^{+\mathbf{L}[E]} \leq \kappa$. One such club set consists of all ordinals of the form $\bar{\tau} = \kappa^{+N}$

where $N = \mathrm{Ult}(M \,||\, \zeta, F)$, $\mu < \zeta < \vartheta$ and F is the restriction of E^M_{top} to sets in $M \,||\, \zeta$ and coordinates from $\kappa \cup d_M$. We may make additional assumptions about ζ, for example, that $M \,||\, \zeta$ projects to μ. This allows us to read off that N projects to κ and $p_N = \pi(p_{M \,||\, \zeta}) \cup q$ where $\pi : M \,||\, \zeta \to N$ is the ultrapower map and q is the parameter represented by the identity function on coordinate d_M. It is also true that $\max(q) < \pi(\mu) \leq \min(\pi(p_{M \,||\, \zeta}))$. Moreover, F is the extender derived from the Mostowski collapse of the hull of $\mu \cup \pi(p_{M \,||\, \zeta})$ taken inside N. Identifying F in this way over N (but without reference to E^M_{top}!) is key to our proof of the coherency of our ultimate square sequence. Such an F will turn out to be the top extender of some protomouse arising in one of the interpolation arguments in the course of the main construction. We call the pair (μ, q) a *divisor* of N since it has the data needed to identify F. The challenge is in knowing which divisor to use; what we have said so far does not seem to be enough to guarantee a unique choice. A certain additional closure condition on the ordinals ζ will ensure that (μ, q) is a so-called *strong divisor* of N. This will make possible a unique choice of divisor.

From now on we shall work *in* $\mathbf{L}[E]$ and fix a cardinal κ for which we intend to construct a $\square_\kappa$-sequence. Throughout this section we assume that

- $\tau < \kappa^+$ is such that κ is the largest cardinal in J^E_τ.
- N is the *collapsing level* for τ. More precisely, N is a level of $\mathbf{L}[E]$, τ is a cardinal in N and $\omega\varrho^\omega_N = \kappa$;
- $n = n(N)$ is such that $\omega\varrho^{n+1}_N \leq \kappa < \omega\varrho^n_N$.

Premice and Protomice. Given a coherent structure $M = \langle J^E_\alpha, F \rangle$, let $\vartheta(M)$ be the least ϑ such that $\mathrm{dom}(F) \subset J^E_\vartheta$. Then $\mu = \mathrm{cr}(F)$ is the largest cardinal in $J^E_{\vartheta(M)}$. If $\vartheta(M) < \kappa^{+M}$, we define $N^*(M)$ to be the collapsing level of $\vartheta(M)$ in M.

Definition. *A* protomouse *is a coherent structure* $M = \langle J^E_\alpha, F \rangle$ *such that* $N(M) = \mathrm{Ult}^*(N^*(M), F)$ *is a level of* $\mathbf{L}[E]$. M *is a* collapsing protomouse *for* τ *iff* $N(M)$ *is the collapsing* $\mathbf{L}[E]$*-level for* τ.

Definition. *A pair* (μ, q) *is a* divisor *of* N *iff there is an ordinal* $\lambda = \lambda_N(\mu, q)$ *such that, setting* $r = p_N - q$, *the following holds:*

a) $\mu \leq \kappa < \lambda < \omega\varrho^n_N$;
b) $q = p_N \cap \lambda$;
c) $\tilde{h}^{n+1}_N(\mu \cup \{r\}) \cap \omega\varrho^n_N$ *is cofinal in* $\omega\varrho^n_N$;

d) $\lambda = \min\Big(\mathbf{On} \cap \tilde{h}^{n+1}_N(\mu \cup \{r\}) - \mu\Big)$.

In the next couple of paragraphs, we shall assume that (μ, q) is a divisor of N and that $r = p_N - q$. So $r = p_N - \lambda$.

Let N^* be the transitive collapse of $\tilde{h}^{n+1}_N(\mu \cup \{r\})$ and $\pi : N^* \to N$ be the associated uncollapsing map; π is clearly $\Sigma^{(n)}_0$-preserving and cofinal, $\mu = \mathrm{cr}(\pi)$ (so μ is regular in N^*) and $\lambda = \pi(\mu)$. Notice also that $\lambda > \tau$, as otherwise κ, being the largest cardinal in J^E_τ, would also be in $\tilde{h}^{n+1}_N(\mu \cup \{r\}) - \mu$, thus contradicting clause d) above. Furthermore,

(1) $\qquad \mu$ is both inaccessible in N^* and a limit cardinal in $\mathbf{L}[E]$.

Consequently, λ is inaccessible in N. To see (1), it suffices to show that μ is a limit cardinal in N^*; the rest follows from the preservation properties of the map π and acceptability. Pick a $\bar{\mu} < \mu$; then $N \models (\exists\theta < \lambda)(\bar{\mu} < \theta \;\&\; \theta$ is a cardinal in $J^E_\lambda)$, as is witnessed by τ. Since this statement is downward preserved under π, choosing $\bar{\mu}$ arbitrarily large below μ yields that J^E_μ doesn't have a largest cardinal. So μ is a limit cardinal in N^*, as N^* is acceptable.

From the fact that π is an embedding arising from collapsing of a $\Sigma^{(n)}_1$-hull in N and Lemma 1.4 we obtain

(2) $\qquad N^*$ is a premouse of the same type as N.

This is immediate whenever N is passive or active of type A or B. If N is of type C, we have to verify that $n > 0$. However, if $n = 0$ then $\lambda_N = \kappa$, so there are no cardinals in N strictly between κ and $\omega\varrho^0_N = \mathbf{On} \cap N$. Since $\lambda(\mu, q)$ is a cardinal in N, we have a contradiction with a) in the definition of divisor.

Set $p^* = \pi^{-1}(r)$. Then $N^* = \tilde{h}^{n+1}_{N^*}(\mu \cup \{p^*\})$, which together with (1) yields

(3) $$\omega\varrho^\omega_{N^*} = \omega\varrho^{n+1}_{N^*} = \mu < \omega\varrho^n_{N^*} \quad \text{and} \quad p^* \in R^n_{N^*} = R^*_{N^*}.$$

Let F be the extender at (μ, λ) derived from π, i.e.

(4) $$F = \pi \restriction (\mathcal{P}(\mu) \cap N^*).$$

Lemma 2.1. $\pi : N^* \xrightarrow[F]{*} N$.

Proof. Define an embedding $\sigma : \mathbb{D}^*(N^*, F) \to N$ by $\sigma([\alpha, f]) = \pi(f)(\alpha)$ for $\alpha < \lambda$ and $f \in \Gamma(\mu, N^*)$. The map σ is Σ_0-preserving by the Łoś theorem, so $\mathbb{D}^*(N^*, F)$

is well-founded, and we identify it with its transitive isomorph $\bar{N}$. Let $\bar{\pi}$ be the associated ultrapower map. Then $\pi = \sigma \circ \bar{\pi}$ and $\bar{\pi}$ is $\Sigma_0^{(n)}$-preserving and cofinal, as N^* has a very good parameter, namely p^*. From the cofinality of π, using the Łoś theorem for $\Sigma_0^{(n)}$-formulae we infer that σ is $\Sigma_0^{(n)}$-preserving. Finally, the cofinality of π guarantees that σ is in fact $\Sigma_0^{(n)}$-preserving and cofinal.

Notice that $\kappa \cup \{p_N\} \subset \mathrm{rng}(\sigma)$, as $\sigma(\bar{\pi}(p^*)) = r = p_N - \lambda$ and σ is the identity on λ. By the discussion in the previous paragraph, σ is $\Sigma_1^{(n)}$-preserving, so $N = \tilde{h}_N^{n+1}(\kappa \cup \{p_N\}) \subset \mathrm{rng}(\sigma)$. This proves that σ is an isomorphism. It follows that $\bar{N} = N$, $\sigma = \mathrm{id}$ and $\bar{\pi} = \pi$. $\square$(Lemma 2.1)

Lemma 2.2. *N^* is sound.*

Proof. For active N^* we can assume w.l.o.g. that $\lambda_{N^*} > \mu$, as otherwise the conclusion is immediate. Let $\bar{N}^*$ be the core of N^* and $\bar{\sigma}^*$ the core map. We will show that $\bar{N}^* = N^*$ and $\bar{\sigma}^* = \mathrm{id}$. The proof of (2) with $\bar{N}^*$ and $\pi \circ \bar{\sigma}^*$ in place of N^* and π yields that $\bar{N}^*$ is a premouse. Let $\vartheta = \mu^{+N^*}$. We observe:

(5) $$E^{\bar{N}^*} \restriction \vartheta = E^{N^*} \restriction \vartheta = E \restriction \vartheta \quad \text{and} \quad \vartheta = \mu^{+\bar{N}^*}.$$

To see that $E^{N^*} \restriction \vartheta = E \restriction \vartheta$, notice that there are cofinally many ζ in ϑ such that $N^* || \zeta$ projects to μ. Given any such ζ, the map $\pi \restriction N^* || \zeta : N^* || \zeta \to N || \pi(\zeta)$ is fully elementary, so the assumptions of the condensation lemma are met, as $\pi \restriction \mu = \mathrm{id}$ and $N^* || \zeta$ is sound. From the possibilities given by the condensation lemma, we can easily rule out c) and d), as these require $\mathrm{cr}(\pi)$ to be a successor cardinal in N^*. In order to rule out a), pick an $a \subset \mu$ which is in N^* but not in $N^* || \zeta$, and observe that $a = \pi(a) \cap \mu \in N$. It follows that $N^* || \zeta$ is a proper initial segment of N. As $\zeta < \vartheta$ can be chosen arbitrarily large, we have the desired conclusion.

Set $\bar{\vartheta} = \mu^{+\bar{N}^*}$. The argument from the previous paragraph with $\bar{N}^*$ and $\pi \circ \bar{\sigma}^*$ in place of N^* and π yields that $E^{\bar{N}^*} \restriction \bar{\vartheta} = E \restriction \bar{\vartheta}$. So the sequences $E^{\bar{N}^*} \restriction \bar{\vartheta}$ and $E^{N^*} \restriction \vartheta$ are compatible in the sense that one of them is an initial segment of the other.

The preservation properties of $\bar{\sigma}^*$ obviously yield the inclusion $\mathcal{P}(\mu) \cap \bar{N}^* || \bar{\vartheta} \subset \mathcal{P}(\mu) \cap N^* || \vartheta$. The universality of p_{N^*}, guaranteed by the condensation lemma, implies that the converse also holds, so $\bar{\vartheta} = \vartheta$. $\square$(5)

It follows that F measures all subsets of μ which are in $\bar{N}^*$, so we can attempt to apply F to $\bar{N}^*$. Define an embedding $\sigma : \mathbb{D}(\bar{N}^*, F) \to N$ by

$$\sigma([\alpha, f]) = \pi(\bar{\sigma}^*(f))(\alpha)$$

for $\alpha < \lambda$ and $f \in \Gamma(\mu, \bar{N}^*)$. Given a Σ_0-formula φ together with ordinals $\alpha_0, \ldots, \alpha_\ell < \lambda$ and $f_0, \ldots, f_\ell \in \Gamma(\mu, \bar{N}^*)$, the Łoś theorem yields

$$\mathbb{D}(\bar{N}^*, F) \models \varphi([\alpha_0, f_0], \ldots [\alpha_\ell, f_\ell]) \quad \text{iff} \quad \prec\alpha_0, \ldots, \alpha_\ell\succ \in F(u)$$

where

$$\begin{aligned} u &= \{\prec\eta_0, \ldots, \eta_\ell\succ \in \mu\,;\ \bar{N}^* \models \varphi(f_0(\eta_0), \ldots, f_\ell(\eta_\ell))\} \\ &= \{\prec\eta_0, \ldots, \eta_\ell\succ \in \mu\,;\ N^* \models \varphi(\bar{\sigma}^*(f_0)(\eta_0), \ldots, \bar{\sigma}^*(f_\ell)(\eta_\ell))\}. \end{aligned}$$

Since F is an extender derived from π, the right side of the above equivalence is equivalent to $N \models \varphi(\pi(\bar{\sigma}^*(f_0))(\alpha_0), \ldots, \pi(\bar{\sigma}^*(f_\ell))(\alpha_\ell))$. This proves that σ is structure preserving. Consequently, $\mathbb{D}^*(\bar{N}^*, F)$ is well-founded, so we can identify it with its transitive isomorph $\bar{N}$. Also, let $\bar{\pi}$ be the associated ultrapower map. From the definition of $\bar{\sigma}^*$ we easily get

$$\sigma \circ \bar{\pi} = \pi \circ \bar{\sigma}^*. \tag{6}$$

The map $\bar{\pi}$ is $\Sigma_0^{(n)}$-preservning and cofinal; this follows from the fact that $\bar{N}^*$ has a very good parameter, namely the $\bar{\sigma}^*$-preimage of p_{N^*}. The argument from the previous paragraph now goes through for $\Sigma_0^{(n)}$-formulae; of course, this time we have to use the corresponding version of the Łoś theorem. Consequently, σ is $\Sigma_0^{(n)}$-preserving. The universality of p_{N^*} implies that $\bar{\sigma}^*$ is $\Sigma_0^{(n)}$-preserving and cofinal, which together with the cofinality of π and (6) yields that σ is $\Sigma_0^{(n)}$-preserving and cofinal as well.

We next conclude that $\bar{N}$ is a premouse of the same type as $\bar{N}^*$. (Obviously, the case of passive $\bar{N}^*$ is not an issue.) This time we make use of the upward preservation properties of $\bar{\pi}$. First assume that $\bar{N}^*$ is of type A or B. If $\omega\varrho^1_{\bar{N}^*} > \mu$, we appply a) of Lemma 1.5, otherwise we use clause c). That all conditions are met follows from the properties of fine ultrapowers. Now assume that $\bar{N}^*$ is of type C. Notice that the assumption $\mu < \lambda_{\bar{N}}$ we made at the beginning implies $\mu < \lambda_{\bar{N}^*}$, so we automatically have $\omega\varrho^1_{\bar{N}^*} > \mu$. The properties of fine ultrapowers guarantee that

$\bar\pi$ is $Q^{(1)}$-preserving (this requires to discuss the cases $n=1$ and $n>1$ separately), so we can apply clause c) of Lemma 1.5.

Set $\bar p^* = p_{\bar N^*}$. The universality of p_{N^*} guarantees that that $\bar\sigma^*(\bar p^*) = p_{N^*}$ and that $\bar N^*$ is sound. We also have the solidity of $\bar N^*$, which follows from the condensation lemma and the fact that $\pi \circ \bar\sigma^* : \bar N^* \to N$ is $\Sigma_0^{(n)}$-preserving. Using the preservation properties of $\bar\pi$ we infer that $\bar N = \tilde h_{\bar N}^{n+1}(\lambda \cup \{\bar\pi(\bar p^*\})$ and for every $\alpha \in \bar p^*$, the structure $\bar\pi(W_{\bar N^*}^{\alpha,\bar p^*})$ is a generalized witness to $\bar\pi(\alpha)$ with respect to $\bar\pi(\bar p^*)$ and $\bar N$. This yields the inequality $\omega\varrho_{\bar N}^{n+1} \le \lambda$ and, together with Lemma 1.1, also the identity $\bar\pi(\bar p^*) = p_{\bar N} - \lambda$ and the soundness of $\bar N$ above λ.

Let ν be the cardinal successor of λ in N. The agreement in (5) guarantees that

$$\nu = \lambda^{+\bar N} \quad \text{and} \quad E^{\bar N} \restriction \nu = E \restriction \nu. \tag{7}$$

We now apply the condensation lemma to $\bar N, N$ and σ. Notice that neither of b), c) and d) is the case: Otherwise we would have $\bar N \in N||\nu$, as follows from the fact, given by the discussion in the previous paragraph, that $\bar N$ can be encoded into a $\mathbf{\Sigma}_1^{(n)}(\bar N)$ subset of λ. But then ν would be collapsed in N, contradicting (7). Thus, $\bar N = N$ and $\sigma = \mathrm{id}$. As a consequence we obtain that $\bar N^*$ is the transitive collapse of $\tilde h_{\bar N}^{n+1}(\mu \cup \{p_{\bar N} - \lambda\}) = \tilde h_N^{n+1}(\mu \cup \{p_N - \lambda\})$ and $\bar\pi$ that is the associated uncollapsing map. Hence $\bar N^* = N^*$ and $\bar\pi = \pi$. Finally, (6) yields that $\bar\sigma^* = \mathrm{id}$, so N^* is sound. □(Lemma 2.2)

Lemma 2.3. *N^* is a proper initial segment of N, and thus the collapsing level for $\vartheta = \mu^{+N^*}$. Moreover, setting $\nu = \pi(\vartheta)$, the structure $M = \langle J_\nu^E, F\rangle$ is a protomouse, $N(M) = N$ and $\vartheta(M) = \vartheta$ (so $N^*(M) = N^*$).*

Proof. The fact that π is $\Sigma_0^{(n)}$-preserving together with (3) and Lemma 2.2 guarantee that we can apply the condensation lemma to $\pi : N^* \to N$. Clauses c) and d) can be ruled out immediately, as they would imply that μ is not a limit cardinal in N^*, which contradicts (1). Clause a) is obviously false for $\mu < \kappa$. For $\mu = \kappa$ notice that $q \neq \varnothing$, as $\kappa \notin \mathrm{rng}(\pi)$. Then $\mathrm{rng}(\pi)$ is contained in $\mathrm{rng}(\sigma_\beta)$, where $\beta = \max(q)$ and σ_β is the witness map for β. This means that N^*, being sound, can be encoded into a $\mathbf{\Sigma}_1^{(n)}(W_N^\beta)$ subset of μ. Thus, $N^* \in N$, which rules out a) also in this case. It follows that b) must hold, i.e. N^* must be a proper initial segment of N. The rest of the proof is obvious. □(Lemma 2.3)

From now on we are fixing the notation: Unless specified otherwise, the symbols $\mu, \vartheta, \lambda, \kappa, q, r$ and N^* will have the same meaning as in the previous lemmata.

There is a class of $\mathbf{L}[E]$-levels which resemble protomice in many respects. For instance, they look very much like protomice with the only difference that their top extenders are total. Also, we will treat them the same way as protomice. We shall see later that the reason why we have to introduce protomice is precisely the existence of such levels.

Definition. *The* $\mathbf{L}[E]$*-level* N *is* pluripotent *iff* N *is active,* $\mathrm{cr}(E^N_{\mathrm{top}}) < \kappa$ *and* $\omega\varrho^1_N = \kappa$*.*

In the next subsection, we will develop fine structure theory for protomice. The language of protomice will consist of the membership predicate, predicate for the extender sequence and that for the top predicate. In other words, it will be the language for coherent structures. Protomice typically arise in interpolation arguments where the target structure is a pluripotent $\mathbf{L}[E]$-level. In order to give a uniform definition of square sequences, we will therefore need to do certain fine structural computations for type B levels in the language for coherent structures, i.e. without referring to γ_N. Equivalently, we will need some information about the Dodd parameter. (The Dodd projectum is κ in this case, as is explained in the preliminaries.)

A pluripotent N is of type C exactly when $\lambda_N = \kappa$. This means that κ is Σ_1-definable over N, so N cannot have any divisors. If N is of type A then $d_M = p_M$. For N of type B, there are two ways how to determine divisors: We can use either the standard parameter p_N and the language for premice, or else the Dodd parameter d_N and the language for coherent structures. The interpolation arguments that are applied in the construction of square sequences will force us to use both, which might badly injure the uniformity of the definition of these sequences. Luckily, the two ways of determining the divisor turn out to be equivalent: The divisors determined by Dodd parameter exactly agree with those determined by the standard parameter with the only exception: The "divisor" (μ, d_N) does not have any counterpart when using the standard parameter instead of the Dodd parameter. We shall prove this in the next series of lemmata. Before stating these lemmata notice that for pluripotent

N, the requirements

$$\begin{aligned}&\lambda_N \cap h^*_N(\mu \cup \{d_N - q\}) \text{ contains an ordinal larger than } \mu\\ &\lambda_{N^*} > \mu\end{aligned} \tag{8}$$

are equivalent where N^* is the transitive collapse of $h^*_N(\mu \cup \{d_N - q\})$.

Lemma 2.4. *Let N be pluripotent of type B and let (μ, q) be such that a) – d) of the definition of divisor hold with h^*_N in place of $\tilde{h}^{n+1}_N$ and d_N in place of p_N (note that $n = 0$). Suppose further that $\lambda_{N^*} > \mu$ (see (8)). Then $\gamma_N \in h^*_N(\mu \cup \{d_N - q\})$, so the transitive collapse N^* of this hull is a type B premouse.*

Proof. Let $\pi : N^* \to N$ be the uncollapsing map associated with the above hull. Then $\omega\varrho^\omega_{N^*} = \omega\varrho^1_{N^*} = \mu$ and $p^* = \pi^{-1}(r)$ is a very good parameter for N^* relative to the language for coherent structures. This is just enough to make the proof of Lemma 2.1 go through. Thus, $N = \mathrm{Ult}(N^*, F)$ where F is the extender derived from π and π is the ultrapower map. We stress once again that, by the fact that the first projectum drops to μ, this ultrapower is *coarse* (that is, it uses only functions that are *elements* of N^*).

We first show that the largest cutpoint λ^*_N of E^N_{top} is in the range of π. Recall that $\lambda_{N^*} > \mu$. Notice also that $\bar{\mu} \stackrel{\mathrm{def}}{=} \mathrm{cr}(E^N_{\mathrm{top}}) < \mu$, as $\bar{\mu} \in \mathrm{rng}(\pi)$. The embedding π, being a coarse ultrapower map, is continuous at all ordinals of N^*-cofinality other than μ. In particular, π is continuous at λ_{N^*}, so π maps λ_{N^*} cofinally into λ_N. The potential premouse N^* cannot be of type C, as otherwise $\pi''C_{N^*}$ would be a set of cutpoints of E^N_{top} cofinal in λ_N. This is impossible, as N is of type B. If N^* were of type A, there would be unboundedly many ordinals of the form $E^{N^*}_{\mathrm{top}}(f)(\bar{\mu})$ in λ_{N^*} where $f : \bar{\mu} \to \bar{\mu}$ is an element of N^*. By the preservation properties of π, we would have unboundedly many ordinals of the form $E^N_{\mathrm{top}}(f)(\bar{\mu})$ in λ_N where $f : \bar{\mu} \to \bar{\mu}$ ranges over functions in N (note that $\mathcal{P}(\bar{\mu}) \cap N^* = \mathcal{P}(\bar{\mu}) \cap N$). In other words, N would be of type A, which is impossible as well. So N^* is of type B and $\pi(\lambda^*_{N^*})$ is a cutpoint of N. The same argument we used to rule out the possibility that N^* is of type A then shows that $\pi(\lambda^*_{N^*}) = \lambda^*_N$. Now if $\lambda^*_{N^*} < \mu$, then also $\gamma_N < \mu$ and it is easy to see that $\gamma_{N^*} = \gamma_N$. From now on suppose that $\lambda^*_{N^*} \geq \mu$. Thus, $\lambda^*_N \geq \lambda \stackrel{\mathrm{def}}{=} \pi(\mu)$.

By the initial segment condition, $E^N_{\mathrm{top}} \,|\, \lambda^*_N = E^N_{\gamma_N}$, so $\mathrm{Ult}(J^E_{\bar{\vartheta}}, E^N_{\mathrm{top}} \,|\, \lambda^*_N) = J^E_{\gamma_N}$ where $\bar{\vartheta} = \bar{\mu}^{+N} = \bar{\mu}^{+N^*}$. Let σ be the associated ultrapower map. Let further

$J^{E^*}_{\gamma^*} = \mathrm{Ult}(J^E_{\bar\vartheta}, E^{N^*}_{\mathrm{top}} \,|\, \lambda^*_{N^*})$ with the associated ultrapower map σ^*. Our next task is to show:

(9) $J^{E^*}_{\gamma^*}$ is an initial segment of N^*, that is, $E^* = E^{N^*} \restriction \gamma^*$.

Given a $\zeta < \gamma^*$ such that $Q_\zeta = J^{E^*}_{\gamma^*} \,||\, \zeta$ projects to $\lambda^*_{N^*}$, we show that Q_ζ is an initial segment of N^*. Since there are unboundedly many such ordinals ζ in γ^*, this proves (9). A standard approach here would be to observe that Q_ζ can be embedded into a proper initial segment of N^* and then apply the condensation lemma. However, we restrict ourselves to make use of Lemma 1.6 instead of the full condensation lemma, so we can only consider embeddings whose target structures are $\mathbf{L}[E]$-levels. For this reason we have to do some extra work.

Let $\tilde\sigma^*$ be the ultrapower map associated with $\mathrm{Ult}(J^E_{\bar\vartheta}, E^{N^*}_{\mathrm{top}})$. Pick a $\vartheta' < \bar\vartheta$ such that $\bar\mu$ is the largest cardinal in $J^E_{\vartheta'}$, the embedding σ^* maps ϑ' cofinally into $\sigma^*(\vartheta') \stackrel{\mathrm{def}}{=} \nu'$ and $\zeta < \nu'$. This is possible, since there are unboundely many ordinals ϑ' in $\bar\vartheta$ satisfying the former two requirements (obviously, σ^* is continuous at any ordinal of N^*-cofinality strictly smaller than μ). It is also easy to see that $\tilde\sigma^*$ maps ϑ' cofinally into $\tilde\sigma^*(\vartheta') \stackrel{\mathrm{def}}{=} \tilde\nu'$. We know that N^* is an amenable structure, so $\tilde F^* = E^{N^*}_{\mathrm{top}} \restriction (\mathcal{P}(\bar\mu) \cap J^E_{\vartheta'}) \in N^*$. Let $F^* = \tilde F^* \,|\, \lambda^*_{N^*}$. A standard argument shows that our choice of ϑ' guarantees that $\tilde\sigma^* \restriction J^E_{\vartheta'}$ and $\sigma^* \restriction J^E_{\vartheta'}$ are the ultrapower maps associated with $\mathrm{Ult}(J^E_{\vartheta'}, \tilde F^*)$ and $\mathrm{Ult}(J^E_{\vartheta'}, F^*)$, respectively. Define an embedding $\bar\sigma^* : J^{E^*}_{\nu'} \to J^{E^{N^*}}_{\tilde\nu'}$ in the usual way, i.e. $\bar\sigma^*(\sigma^*(f)(\alpha)) = \tilde\sigma^*(f)(\alpha)$ for $f \in J^E_{\vartheta'}$ with $\mathrm{dom}(f) = \bar\mu$ and $\alpha < \lambda^*_{N^*}$. Then $\bar\sigma^*$ is Σ_0-preserving and cofinal, has critical point $\lambda^*_{N^*}$ and $\bar\sigma^* \in N^*$. By applying π we obtain that $\pi(\bar\sigma^*) : \pi(J^{E^*}_{\nu'}) \to J^E_{\pi(\tilde\nu')}$ is Σ_0-preserving and cofinal and has critical point λ^*_N. Letting $\tilde Q_\zeta = N^* \,||\, \bar\sigma^*(\zeta)$, we see that $\pi(\bar\sigma^*) \restriction \pi(Q_\zeta) : \pi(Q_\zeta) \to \pi(\tilde Q_\zeta)$ is fully elementary and also the remaining assumptions of the condensation lemma are met. Exactly as in the proof of (5) we infer that $\pi(Q_\zeta) = N \,||\, \pi(\zeta)$. So Q_ζ is an initial segment of N^* by the preservation properties of π. This completes the proof of (9).

A standard verification yields that $\pi^* : \sigma^*(f)(\alpha) \mapsto \sigma(f)(\pi(\alpha))$ is an embedding of $J^E_{\gamma^*}$ into $J^E_{\gamma_N}$ which is Σ_0-preserving and cofinal, and which agrees with π up to $\lambda^*_{N^*}$. We improve this by further showing that, in fact, π^* and π agree up to γ^*, which has an immediate consequence that

$$\pi \text{ maps } \gamma^* \text{ cofinally into } \gamma_N. \tag{10}$$

By the basic properties of ultrapowers, each $\zeta < \gamma^*$ can be written in the form $\mathrm{otp}(\sigma(f)(\alpha))$ where $f : \bar\mu \to \mathcal{P}(\bar\mu)$ is in N^* and $\alpha < \lambda^*_{N^*}$. According to our convention introduced in the preliminaries (see the subsection on extenders and coherent structures), we can think of f as the set of all Gödel pairs $\prec\eta,\xi\succ$ satisfying $\xi \in f(\eta)$ and write $E^{N^*}_{\mathrm{top}} \mid \lambda^*_{N^*}$ instead of σ^*. We obtain:

$$\begin{aligned}\pi(\zeta) &= \pi(\mathrm{otp}[(E^{N^*}_{\mathrm{top}} \mid \lambda^*_{N^*})(f)(\alpha)]) = \pi(\mathrm{otp}[(E^{N^*}_{\mathrm{top}}(f) \cap \lambda^*_{N^*})(\alpha)]) = \\ &= \mathrm{otp}[(E^{N}_{\mathrm{top}}(f) \cap \lambda^*_{N})(\pi(\alpha))] = \mathrm{otp}[(E^{N}_{\mathrm{top}} \mid \lambda_{N})(f)(\pi(\alpha))] = \pi^*(\zeta).\end{aligned}$$

Since $\lambda^*_{N^*}$ is the largest cardinal in $J^{E^*}_{\gamma^*}$, (9) guarantees that $\gamma^* \le (\lambda^*_{N^*})^{+N^*}$. We show that γ^* is actually strictly smaller than $(\lambda^*_{N^*})^{+N^*}$. Recall that $\gamma^* > \mu$. If γ^* were the cardinal successor of $\lambda^*_{N^*}$ in N^* then π, being a coarse ultrapower embedding, would map γ^* cofinally into the N-cardinal $\pi(\gamma^*)$. But (10) tells us that the cofinal image of γ^* under π is γ_N, and this is certainly not a cardinal in N, as it indexes an extender on E.

Finally we are ready to show that $\gamma_N \in \mathrm{rng}(\pi)$. Assume for a contradiction that this is not the case. Then $\pi(\gamma^*) > \gamma_N$, as follows from (10). For the rest of the proof, let $\bar Q$ be the collapsing level for γ^* in N^* and $Q = \pi(\bar Q)$. Notice that Q is an initial segment of N, and thus an $\mathbf{L}[E]$-level. So the preservation properties of π guarantee that $\bar Q$ is a premouse that is sound and solid. By our assumptions, $\bar Q$ projects to $\lambda^*_{N^*}$ and $\gamma^* = (\lambda^*_{N^*})^{+\bar Q}$. The interpolation lemma guarantees that the canonical extension $\tilde\pi : \bar Q \to \tilde Q$ of $\pi \restriction J^{E^{N^*}}_{\gamma^*} : J^{E^{N^*}}_{\gamma^*} \to J^{E}_{\gamma_N}$ to $\bar Q$ exists and is $\Sigma^{(m)}_0$-preserving and cofinal where m is such that $\omega\varrho^{m+1}_{\bar Q} \le \lambda^*_{N^*} < \omega\varrho^{m}_{\bar Q}$. The cofinality of $\tilde\pi$ follows from the soundness of $\bar Q$. Moreover, letting $\tilde p \overset{\mathrm{def}}{=} \tilde\pi(p_{\bar Q})$, standard fine structural computations yield:

- $\omega\varrho^{m+1}_{\tilde Q} \le \lambda^*_N < \omega\varrho^{m}_{\tilde Q}$;
- $\tilde h^{m+1}_{\tilde Q}(\lambda^*_N \cup \{\tilde p\}) = \tilde Q$;
- $\tilde\pi(W^{\bar\beta,p_{\bar Q}}_{\bar Q})$ is a generalized solidity witness for $\beta = \pi(\bar\beta)$ with respect to $\tilde Q$ and $\tilde p$ whenever $\beta \in \tilde p$, and this witness is an element of $\tilde Q$.

Finally, there is a $\Sigma_0^{(m)}$-preserving map $\pi' : \tilde{Q} \to Q$ which is the identity on γ_N and $\pi'(\gamma_N) = \pi(\gamma^*)$.

In order to be able to use the condensation lemma, we must verify that $\tilde{Q}$ is a premouse. This is certainly true if $\bar{Q}$ is passive or $\omega\varrho^1_{\bar{Q}} > \lambda^*_{N^*}$, as in this case we can directly apply a) or b) of Lemma 1.5. The map $\tilde{\pi}$ is easily seen to be Σ_2-preserving and, if $\bar{Q}$ is of type C, also $Q^{(1)}$-preserving; in the latter case we also have $\omega\varrho^1_{\tilde{Q}} = \lambda_{\tilde{Q}}$ (this, actually requires a discussion of two cases, namely $\omega\varrho^2_{\bar{Q}} = \lambda^*_{N^*}$ and $\omega\varrho^2_{\bar{Q}} > \lambda^*_{N^*}$). Now suppose that $\bar{Q}$ is active and its first projectum drops to $\lambda^*_{N^*}$. Let $\bar{\kappa}$ and $\bar{\tau}$ be the critical point of $E^{\bar{Q}}_{\mathrm{top}}$ and its $\bar{Q}$-successor, respectively. If $\bar{\tau} > \lambda^*_{N^*}$, then $\tilde{\pi}$, being a coarse pseudoultrapower embedding, maps $\bar{\tau}$ cofinally into $\tilde{\pi}(\bar{\tau})$. If $\bar{\tau} < \lambda^*_{N^*}$, then $\bar{\tau}$ is in fact a successor cardinal in N^*, as $\lambda^*_{N^*}$ is a limit cardinal in N^*. So π, being a coarse ultrapower embedding, maps $\bar{\tau}$ cofinally to $\pi(\bar{\tau})$ and the same is true with $\tilde{\pi}$ in place of π, as $\tilde{\pi}$ is an extension of π. In either case, the top extender of $\tilde{Q}$ is weakly amenable, so $\tilde{Q}$ is a potential premouse. The cofinality of $\pi''\bar{\tau}$ in $\pi(\bar{\tau})$ also guarantees that $\lambda_{\bar{Q}} > \lambda^*_{N^*}$, as otherwise $\mathrm{ht}(\bar{Q}) = \gamma^*$ and $\pi(\gamma^*)$ would be the cofinal image of γ^* under π. By (10), the latter is nothing other than γ_N, so $\gamma_N \in \mathrm{rng}(\pi)$, which contradicts our initial assumption. Hence $\lambda_{\bar{Q}} > \lambda^*_{N^*}$, and the premousehood of $\tilde{Q}$ then follows from c) of Lemma 1.5.

At last we can apply the condensation lemma to $\pi' : \tilde{Q} \to Q$. Here we will substantially use the fact that $\gamma_N \in \tilde{Q}$ and the consequence $\gamma_N = \mathrm{cr}(\pi')$. To see this, it suffices to show that $\gamma^* \in \bar{Q}$. But $\mathrm{ht}(\bar{Q}) = \gamma^*$ is only possible if $\bar{Q}$ is active, since $J^{E^*}_{\gamma^*}$ is a ZFC^--model. Then $\lambda(\bar{Q}) = \lambda^*_{N^*}$, and the argument from the end of the previous paragraph yields a contradiction. Option a) in the condensation lemma is impossible, as $\lambda_N^{+\tilde{Q}} = \gamma_N < \pi(\gamma^*) = \lambda_N^{+Q}$. Option b) fails, as γ_N indexes an extender in Q but, being a $\tilde{Q}$-cardinal, cannot index any extender in $\tilde{Q}$. Option c) is false, as $\tilde{Q}$ is sound above λ_N. Finally, option d) cannot occur as $\tilde{Q}$ projects to λ_N, while γ_N is a cardinal in $\mathrm{Ult}(Q, E_{\gamma_N})$. This yields the desired contradiction and completes the proof of the lemma. □(Lemma 2.4)

Lemma 2.5. *Let N be pluripotent of type B and let (μ, q) be as in the previous lemma. Assume further that $\lambda_{N^*} > \mu$. Then $q = p_N \cap \lambda$. Moreover, e_N and γ_N are $\Sigma_1(N)$-definable one from the other with respect to the language for coherent* *structures, and using only parameters from $\mu \cup \{r\}$ where $r = p_N - \lambda$. It follows that (μ, q) is a divisor.*

On the other hand, if N is pluripotent of type B and (μ, q) is a divisor of N such that $\lambda_{N^*} > \mu$, then $e_N \in \text{rng}(\pi)$ where $\pi = \pi_N(\mu, q)$. Moreover, we have the same conclusion regarding the relationship between e_N and γ_N as before.

Proof. We only give a proof of the former; the latter follows by a similar and, in fact, simpler argument. It follows from the previous lemma that γ_M is in the range of π. If $\gamma_N < \kappa$ then $\gamma_N < \mu$, since no ordinals from $[\mu, \kappa)$ are in the range of π. So there is nothing to prove, as $d_N = p_N$ and $e_N = \varnothing$. From now on assume $\gamma_N \geq \kappa$; the argument we have just given yields $\gamma_N \geq \lambda$.

Let $\bar{\mu}$ be, as in the proof of the previous lemma, the common critical point of E^N_{top} and $E^{N^*}_{\text{top}}$ and let $e' = e_N - \lambda$. From Lemma 1.7 we obtain $r \cup e' = d_N - \lambda$, and from the previous lemma we have $\gamma_N \in h^*_N(\mu \cup \{r, e'\})$. If $e_N = \varnothing$, the conclusions of the current lemma follow immediately. So suppose that e_N is nonempty. We now borrow Lemma 2.14 from the next subsection. We found it more natural to place the lemma there, although it states a general fact about pluripotent $\mathbf{L}[E]$-levels which depends merely on elementary properties of premice. As γ_N is in $h^*_N(\mu \cup \{r, e'\})$, there is an $f : \bar{\mu} \to \bar{\mu}$ such that $\gamma_N = E^N_{\text{top}}(f)(r, e', \xi)$ for some $\xi < \mu$. From the fact that $r \subset p_N$ and e_N is the $\leq^*$-least d with the property that $\gamma_N = E^N_{\text{top}}(f)(p_N, d, \xi)$ for some f as above and $\xi < \kappa$ we immediately obtain $e_N \leq^* e'$. Hence $e_N = e'$, as $e' \subset e_N$. This shows that $q = p_N \cap \lambda$. Also, it is clear that γ_N is $\Sigma_1(N)$-definable from parameters in $\mu \cup \{r, e_N\}$.

We now show that $e_N \in h^*_N(\mu \cup \{r, \gamma_N\})$. Let e be the $\leq^*$-least finite set of ordinals in $h^*_N(\mu \cup \{r, \gamma_N\})$ such that $\gamma_N = E^N_{\text{top}}(f)(r, e, \xi)$ for some $f : \bar{\mu} \to \bar{\mu}$ and $\xi < \mu$. If $e_N \neq e$, then $e_N <^* e$, so there is a $g : \bar{\mu} \to \bar{\mu}$ such that e_N witnesses the statement

$$(\exists x, \eta)[x <^* d \ \&\ \eta < \lambda' \ \&\ \gamma_N = E^N_{\text{top}}(g)(r, x, \eta)]$$

holds in N where $\lambda' = \min(h^*_N(\mu \cup \{r, \gamma_N\}) - \mu)$. To see that η can be bounded by λ', note that $\lambda' \geq \lambda > \kappa$, as $h^*_N(\mu \cup \{r, \gamma_N\}) \subset h^*_N(\mu \cup \{r, e_N\})$ and η can be certainly bounded by κ. Notice also that $g \in h^*_N(\mu \cup \{r, \gamma_N\})$. The above statement, being Σ_1, is then witnessed by some e' and η from $h^*_N(\mu \cup \{r, \gamma_N\})$. Obviously, such an η is smaller than μ. So e was not chosen minimally after all. Contradiction.

The argument just given shows that

$$h_N(\mu \cup \{r\}) = h^*_N(\mu \cup \{r, \gamma_N\}) = h^*_N(\mu \cup \{r, e_N\}),$$

so q is a divisor of N. □(Lemma 2.5)

It only remains to discuss the case where $\lambda_{N^*} = \mu$; it is clear that $q = d_N$ in this case. By the proof of (1), μ is inaccessible in $\mathbf{L}[E]$ both in Lemma 2.3 as well as in the following lemma. Furthermore, if π is the associated uncollapsing map and F the derived extender, then $N = \mathrm{Ult}^*(N^*, F) = \mathrm{Ult}(N^*, F)$ and π turns out to be the associated ultrapower map, as follows from the proof of Lemma 2.1.

Lemma 2.6. *Let N be pluripotent and $\mu \leq \kappa$ be such that*

a) *$h^*_N(\mu)$ is cofinal in N;*
b) $\lambda \overset{\text{def}}{=} \min(h^*_N(\mu) - \mu) = \lambda_N$.

If N^* is the transitive collapse of $h^*_N(\mu)$, then $N^* = N\,||\,\vartheta$ where $\vartheta = \mathrm{ht}(N^*) = \mu^{+N^*}$. Moreover, we can define a protomouse $M(N)$ similarly as in Lemma 2.3.

Proof. As before, $\mu = \mathrm{cr}(\pi)$ where π is as above. We observe that μ is a cutpoint of E^N_{top}. This follows from the equality $E^N_{\mathrm{top}}(f)(\xi) = E^{N^*}_{\mathrm{top}}(f)(\xi) < \mu$ for $f : \bar{\mu} \to \bar{\mu}$ and $\xi < \mu$ which, in turn, is a consequence of the fact that $\mu = \mathrm{cr}(\pi)$. So N^* is a proper initial segment of N by the initial segment condition. Clearly, $\mu^{+N^*} = \mathrm{ht}(N^*)$. Notice also that N^* can be of type A or C even if N is of type B in this case. □(Lemma 2.6)

The discussion in the previous three lemmata guarantees that the following definition is consistent with our earlier definiton of divisor.

Definition. *Given a pluripotent N, a pair (μ, q) is a* divisor *of N iff a) – d) of our earlier definition hold with h^*_N and d_N in place of $\tilde{h}^{n+1}_N$ and p_N, respectively.*

The protomouse associated with the divisor (μ, q) of N is denoted by $N(\mu, q)$ and its top extender by $F_N(\mu, q)$. The further auxiliary parameters associated with N and (μ, q) are denoted in the obvious way, i.e. we write $\vartheta_N(\mu, q)$ for $\vartheta(N(\mu, q))$, $\pi_N(\mu, q)$ for the associated ultrapower map π, $\lambda_N(\mu, q)$ for $\pi(\mu) = \lambda(F)$, $r(\mu, q)$ for $r = p_N - q = \pi(p_{N^*})$, $\nu_N(\mu, q)$ for $\pi(\vartheta_N(\mu, q))$, etc. If N is clear from the context we omit the subscript; this applies also to the notation which will be introduced later.

Let us finally stress that for an active $\mathbf{L}[E]$-level N we have the equivalence

$$\lambda_{N^*(\mu,q)} > \mu \quad \text{iff} \quad \lambda_N(\mu,q) < \lambda_N, \tag{11}$$

which is obvious by (8). Notice this statement provides us with a nontrivial piece of information only if N is pluripotent. Also, $\lambda_N(\mu,q) = \lambda_N$ implies that N is pluripotent.

Definability: Premice versus Protomice. For our fixed local successor τ, the associated collapsing level N is the object of our primary interest. As we have already indicated, some arguments will require us to replace this collapsing level by a protomouse, say M. What will be important for us later is the ability to talk about N in the language of the structure M and vice versa. We now develop a uniform procedure which provides such a translation. Also, our later applications will require that such a translation procedure between M and N works even in those cases where N is not necessarily an $\mathbf{L}[E]$-level. Thus, unless specified otherwise, throughout this subsection we shall assume:

A) $M = \langle J^E_\nu, F\rangle$ is a coherent structure such that J^E_ν is a premouse all of whose proper initial segments are solid and F is an extender at μ,λ;

B) $\vartheta < \mu^{+M}$, μ is the largest cardinal in J^E_ϑ, $\mathrm{dom}(F) = \mathcal{P}(\mu)\cap J^E_\vartheta$ and N^* is the collapsing level for ϑ in M;

C) $\omega\varrho^{n+1}_{N^*} \le \mu < \omega\varrho^n_{N^*}$;

D) $\pi : N^* \xrightarrow[F]{*} N$.

This is certainly true of any $\mathbf{L}[E]$-level N and $N(\mu,q)$ where (μ,q) is a divisor of N. Notice also that N^*, being a proper initial segment of J^E_ν, is sound and solid. This means that π is $\Sigma^{(n)}_0$-preserving and cofinal. Setting $r = \pi(p_{N^*})$, we obtain $N = \tilde{h}^{n+1}_N(\lambda\cup\{r\})$, so $\omega\varrho^{n+1}_N \le \lambda$. Also, each $\alpha\in r$ has a generalized solidity witness in N with respect to r and N, namely $\pi(W^{\bar\alpha,p_{N^*}}_{N^*})$ where $\bar\alpha$ is the π-preimage of α. By Lemma 1.1, $r = p_N - \lambda$ whenever N is solid, in particular, whenever N is an $\mathbf{L}[E]$-level. Recall that if N is active, then N^* and N are of the same type whenever $\lambda_N(\mu,q) < \lambda_N$.

Lemma 2.7. *Assume N is either passive or else $\lambda_N(\mu,q)<\lambda_N$. Let $\varphi(v_0,\dots,v_\ell)$ be a $\Sigma^{(n)}_1$-formula in the language for premice. There is a Σ_1-formula $\varphi^*(v_0,\dots,v_\ell)$ in the language for coherent structures such that for every tuple $x_1,\dots,x_\ell\in J^E_\lambda$ we* *have*

$$N \models \varphi(r, x_1, \ldots, x_\ell) \quad \text{iff} \quad M \models \varphi^*(\vartheta, x_1, \ldots, x_\ell)$$

If N is active with $\lambda_N(\mu, q) = \lambda_N$ then an analogous conslusion holds for any Σ_1-formula $\varphi(v_1, \ldots, v_\ell)$ in the language for coherent structures: There is a Σ_1-formula $\varphi^*(v_0, \ldots, v_\ell)$ in the language for coherent structures such that for every tuple $x_1, \ldots, x_\ell \in J^E_\lambda$ we have

$$N \models \varphi(x_1, \ldots, x_\ell) \quad \text{iff} \quad M \models \varphi^*(\vartheta, x_1, \ldots, x_\ell)$$

Moreover, the formula φ^* depends only on φ and is independent of M. In fact, φ^* effectively computable from φ.

Proof. If N is as in the first part of the lemma and active then N^* and N are of the same type, so it does make sense to use the language for premice, as the constant symbol $\dot{\gamma}$ is interpreted in the same way in both N^* and N. In the second part of the lemma, however, the two structures might be of different types; this explains why we demand that φ is a formula in the language for coherent structures.

We give a proof of the first part of the lemma where N is either passive or $\lambda_N(\mu, q) < \lambda_N$, as the proof of the second part is similar. Furthermore, we will focus on the special case where each x_i is an ordinal less than λ, and write α_i instead of x_i. The same argument proves the lemma in its full generality if we additionally employ the fact that each $x \in J^E_\lambda$ is $\Sigma_1(J^E_\lambda)$-definable from an ordinal $\alpha < \lambda$ as the α-th element of J^E_λ in the ordering $<_E$.

Suppose φ is of the form $(\exists z^n)\psi(z^n, v_0, \ldots, v_\ell)$ where ψ is $\Sigma^{(n)}_0$. Then $N \models \varphi(r, \alpha_1, \ldots, \alpha_\ell)$ iff $(\exists u \in H^n_{N^*})[N \models (\exists z^n \in \pi(u))\psi(z^n, r, \alpha_1, \ldots, \alpha_\ell)]$. Using the Łoś theorem, the latter can be expressed in a Σ_1-fashion over M as

$$(\exists Q, p^*, \mu, a, y, u, m)\varphi^*_0(Q, p^*, \mu, a, y, u, m, \vartheta, \alpha_1, \ldots, \alpha_\ell)$$

where $\varphi^*_0(Q, p^*, \mu, a, y, u, m, \vartheta, \alpha_1, \ldots, \alpha_\ell)$ is the conjunction of the following statements:

- Q is an initial segment of M and $m \in \omega$
- $\omega\vartheta = \mu^{+Q}$, $\omega\varrho^{m+1}_Q = \mu < \omega\varrho^m_Q$, $p^* = p_Q$ and $u \in H^m_Q$
- $a = \{\prec\eta_1, \ldots, \eta_\ell\succ \in \mu;\ Q \models (\exists z \in u)\psi(z, p^*, \eta_1, \ldots \eta_\ell)\}$
- $y = F(a)$ and $\prec\alpha_1, \ldots \alpha_\ell\succ \in y$

It is obvious that the conversion from φ to φ^* is effective and independent on M.

$\square$(Lemma 2.7)

The same proof yields in fact a "boldface" version of the above lemma, but the conversion from φ to φ^* is then *not* uniform. As any parameter $c \in N$ is of the form $\pi(f)(\alpha)$ for some $\alpha < \lambda$ and $f \in \Gamma(\mu, N^*)$, the image $\pi(f)$ is a good $\Sigma_1^{(n-1)}(N)$-function in a parameter from $\mathrm{rng}(\pi)$. Then φ^* will clearly depend on the functionally absolute definition of f and both α as well as the above mentioned parameter from $\mathrm{rng}(\pi)$ must also be considered.

Lemma 2.8. *Let $\varphi(v, v_0, \dots, v_\ell)$ be as in Lemma 2.7 and $c \in N$. There is a Σ_1-formula $\varphi^*(v, v_0, \dots, v_\ell)$ in the language for coherent structures and a $c^* \in J^E_\lambda$ such that for every $\alpha_1, \dots, \alpha_\ell$,*

$$N \models \varphi(c, r, \alpha_1, \dots, \alpha_\ell) \quad \textit{iff} \quad M \models \varphi^*(c^*, \vartheta, \alpha_1, \dots, \alpha_\ell)$$

if N is as in the first part of Lemma 2.7, and

$$N \models \varphi(c, \alpha_1, \dots, \alpha_\ell) \quad \textit{iff} \quad M \models \varphi^*(c^*, \vartheta, \alpha_1, \dots, \alpha_\ell)$$

if N is as in the second part of Lemma 2.7. $\square$(Lemma 2.8)

Corollary 2.9. *Given a finite set of ordinals $q \subset \lambda$, the following conversions hold between N and M. In particular, these conversions hold between N and $N(\mu, q)$ whenever N is an $\mathbf{L}[E]$-level and (μ, q) a divisor thereof. In the two clauses below, A is computed in the language for premice if N is either passive or else $\lambda_N(\mu, q) < \lambda_N$, and in the language for coherent structures otherwise. A^* is computed in the language for coherent structures.*

a) *To every A that is $\Sigma_1^{(n)}(N)$ in the parameter $r \cup q$ there is an A^* that is $\Sigma_1(M)$ in the parameters q and ϑ such that $A \cap \lambda = A^* \cap \lambda$.*

b) *To every $A \in \mathbf{\Sigma}_1^{(n)}(N)$ there is an A^* in $\mathbf{\Sigma}_1(M)$ such that $A \cap \lambda = A^* \cap \lambda$.*

$\square$(Corollary 2.9)

The following three lemmata provide us with the dual conversions.

Lemma 2.10. *Let $\varphi(v_1, \dots, v_\ell)$ be a Σ_1-formula in the language for coherent structures.*

Assuming that N is either passive or else $\lambda_N(\mu, q) < \lambda_N$, there is a $\Sigma_1^{(n)}$-formula ${}^*\varphi(v, v', v_0, \dots, v_\ell)$ in the language for premice and an ordinal $\xi_0 < \mu$ such that for *every* $x_1, \ldots, x_\ell \in M$ *we have*

$$M \models \varphi(x_1, \ldots, x_\ell) \quad \textit{iff} \quad N \models {}^*\varphi(r, \xi_0, \mu, x_1, \ldots, x_\ell).$$

If N *is active with* $\lambda_N(\mu, q) = \lambda_N$, *there is a* Σ_1*-formula* ${}^*\varphi(v_0, \ldots, v_\ell)$ *in the language for coherent structures such that for every* $x_1, \ldots, x_\ell \in M$ *we have*

$$M \models \varphi(x_1, \ldots, x_\ell) \quad \textit{iff} \quad N \models {}^*\varphi(\mu, x_1, \ldots, x_\ell).$$

As before, the formula ${}^*\varphi$ *depends only on* φ *and is independent of* M *and, in fact, can be effectively computed from* φ.

Proof. Since φ is a Σ_1-formula, $M \models \varphi(x_1, \ldots, x_\ell)$ iff there is a $\zeta < \omega\nu$ such that $\langle S^E_\zeta, F \cap S^E_\zeta \rangle \models \varphi(x_1, \ldots, x_\ell)$. Fixing an ordinal $\xi_0 < \mu$ such that $\tilde{h}^{n+1}_N(\xi_0, r) = \lambda$, the latter is expressible in a $\Sigma^{(n)}_1$-fashion over N in the parameters r, μ, ξ_0 as

$$(\exists \zeta^n, \lambda^n, \eta^n_1, \eta^n_2, f^n, G^n, Q^n)^*\varphi_0(\zeta^n, \lambda^n, \eta^n_1, \eta^n_2, f^n, G^n, Q^n, r, \xi_0, \mu, x_1, \ldots, x_\ell)$$

where ${}^*\varphi_0(\zeta^n, \lambda^n, \eta^n_1, \eta^n_2, f^n, G^n, Q^n, r, \xi_0, \mu, x_1, \ldots, x_\ell)$ is the conjunction of the following statements:

- $\eta^n_1, \eta^n_2 < \mu$ and $\zeta^n < (\lambda^n)^{+N}$
- $\zeta^n = \tilde{h}^{n+1}_N(\eta^n_1, r)$, $f^n = \tilde{h}^{n+1}_N(\eta^n_2, r)$ and $\lambda^n = \tilde{h}^{n+1}_N(\xi_0, r)$
- $f^n : \lambda^n \xrightarrow{\text{onto}} \mathcal{P}(\lambda^n) \cap S^E_{\zeta^n}$
- $G^n = \{\langle f^n(\eta) \cap \mu, f^n(\eta) \rangle \,;\, \eta < \mu\}$ and $Q^n = \langle S^E_{\zeta^n}, G^n \rangle$
- $Q^n \models \varphi(x_1, \ldots, x_\ell)$

It is clear that the conversion from φ to ${}^*\varphi$ is effective and does not depend on M.
$\square$(Lemma 2.10)

An argument similar to that above yields a version of the previous lemma where the pair of parameters ξ_0, μ is replaced by ϑ; the conversion from φ to ${}^*\varphi$ is again effective.

Lemma 2.11. *Suppose* $\varphi(v_1, \ldots, v_\ell)$ *is a* Σ_1*-formula in the language for coherent structrures.*

Assuming that N *is either passive or else* $\lambda_N(\mu, q) < \lambda_N$, *there is a* $\Sigma^{(n)}_1$*-formula* ${}^*\varphi(v, v_0, \ldots, v_\ell)$ *such that for every* $x_1, \ldots, x_\ell \in M$,

$$M \models \varphi(x_1, \ldots, x_\ell) \quad \textit{iff} \quad N \models {}^*\varphi(r, \vartheta, x_1, \ldots, x_\ell),$$

and the obvious reformulation is true if N is active with $\lambda_N(\mu,q)=\lambda_N$.

As before, the formula ${}^*\varphi$ depends only on φ and is independent of M and, in fact, can be effectively computed from φ. □(Lemma 2.11)

Corollary 2.12. *Given any $c\in M$, the following conversions hold between M and N. In particular, these conversions hold for N and $N(\mu,q)$ where N is an arbitrary $\mathbf{L}[E]$-level and (μ,q) is a divisor thereof. The situation here is dual to that in Corollary 2.9, so ${}^*A$ is computed in the language for premice if N is either passive or else $\lambda_N(\mu,q)<\lambda_N$, and in the language for coherent structures otherwise. A is computed in the language for coherent structures.*

a) *To every A from $\Sigma_1(M)$ in c there is an ${}^*A$ in $\Sigma_1^{(n)}(N)$ in c,r,ϑ such that $A\cap\lambda={}^*A\cap\lambda$.*

b) *To every $A\in\mathbf{\Sigma}_1(M)$ there is an ${}^*A\in\mathbf{\Sigma}_1^{(n)}(N)$ such that $A\cap\lambda={}^*A\cap\lambda$.*

□(Corollary 2.12)

When dealing with *singletons* definable over N, there is another way of switching between the languages of N and M. Such a conversion is then *not* uniform in the above sense, but will be useful for proving that certain properties of protomice are preserved under embeddings with weak preservation properties. The formulations of the next lemmata make use of the convention we made at the end of the section on extenders and coherent structures in preliminaries. Thus, if f is a function defined on ordinals, we will write $f(a)$ even if a is a finite set or sequence of ordinals, as such objects are recursively encodable into ordinals.

Lemma 2.13. *Let $b\subset\lambda$ be finite, $\zeta<\lambda$ and $y\subset\lambda$.*

a) *If ζ, resp. y, is $\Sigma_1^{(n)}(N)$-definable from r and b, then there is an $f:\mu\to\mu$, resp. $f:\mu\to\mathcal{P}(\mu)$ in N^* such that $\zeta=F(f)(b)$, resp. $y=F(f)(b)$.*

b) *If $\zeta=F(f)(b)$, resp. $y=F(f)(b)$ where $f:\mu\to\mu$, resp. $f:\mu\to\mathcal{P}(\mu)$ is in N^*, then ζ, resp. y is $\Sigma_1^{(n)}(N)$-definable from r,b and some $\xi<\mu$.*

As usual, here we consider $\Sigma_1^{(n)}(N)$-definability with respect to the language for premice if $\lambda_N(\mu,q)<\lambda_N$, and with respect to the language for coherent structures if $\lambda_N(\mu,q)=\lambda_N$.

Proof. We give a proof for ζ. To see a), suppose ζ is the unique object such that $N\models(\exists z^n)\psi(z^n,\zeta,r,b)$ where ψ is a $\Sigma_0^{(n)}$-formula. Fix a $\delta^*<\omega\varrho^n_{N^*}$ large enough such that, setting $\delta = \pi(\delta^*)$, there is a $z \in S^E_\delta$ witnessing this existential statement; such a δ^* exists since π is cofinal at the n-th level. Define a partial map $f : \mu \to \mu$ as follows:

$$f(x) \simeq \text{ the unique } \xi < \mu \text{ satisfying } N \models (\exists z^n \in S^E_{\delta^*})\psi(z^n, \xi, p_{N^*}, x)$$

Then f, being a $\mathbf{\Sigma}^{(n)}_0(N)$ subset of $\mu < \omega\varrho^n_{N^*}$, is an element of N^*. Applying π, it follows that $\pi(f)(x)$, if defined, is the unique ordinal $\xi < \lambda$ such that we have $(\exists z^n \in J^E_\delta)\psi(z^n, \xi, r, x)$ in N. But for $x = b$ we know that $\pi(f)(x)$ *is* defined, so $F(f)(b) = \pi(f)(b) = \zeta$. Obviously, f can be turned into a total function on μ by setting $f(x) = 0$ whenever $f(x)$ is undefined by the above definition.

To see b), suppose $\zeta = F(f)(b)$ for f, b as above; since N^* is sound, there is a $\xi < \mu$ such that $f = \tilde{h}^{n+1}_{N^*}(\xi, p_{N^*})$. The preservation properties of π, yield $\pi(f) = \tilde{h}^{n+1}_N(\xi, r)$, so ζ can be defined in a $\Sigma^{(n)}_1$-fashion over N as follows:

$$(\exists g^n)(g^n = \tilde{h}^{n+1}_N(\xi, r) \;\&\; \zeta = g^n(b)).$$

□(Lemma 2.13)

The following is a version of the previous lemma that will be useful when dealing with pluripotent $\mathbf{L}[E]$-levels N.

Lemma 2.14. *Let $N = \langle J^E_\nu, F\rangle$ be a coherent structure such that F is at μ, λ and* $\mathrm{dom}(F) = \mathcal{P}(\mu) \cap N$. *Suppose further that $b \subset \lambda$ is finite, $\zeta < \lambda$ and $y \subset \lambda$.*

a) *If ζ, resp. y is $\Sigma_1(N)$-definable from b then there is an $f : \mu \to \mu$, resp. $f : \mu \to \mathcal{P}(\mu)$ in N such that $\zeta = F(f)(b, \mu)$, resp. $y = F(f)(b, \mu)$.*

b) *If $\zeta = F(f)(b, \mu)$, resp. $y = F(f)(b, \mu)$) for an $f \in N$ such that $f : \mu \to \mu$, resp. $f : \mu \to \mathcal{P}(\mu)$, then ζ, resp. y is $\Sigma_1(N)$-definable from b and f.*

Both a) and b) concerns $\Sigma_1(N)$-definability in the language for coherent structures.

In particular, a) and b) is true of any pluripotent $\mathbf{L}[E]$-level N.

Proof. Note first that b) follows immediately: The fact that $\zeta = F(f)(b, \mu)$ can be expressed in a Σ_1-fashion in b over N as

$$(\exists w)(\langle f, w\rangle \in F \;\&\; \zeta = w(\prec b, \mu \succ)).$$

The ordinal μ can be elimitated from this definition, as it is Σ_1-definable over N without parameters.

To see the converse a), let ζ be the unique object such that $N \models (\exists z)\psi(z,\zeta,b)$ for a Σ_0-formula ψ. Let π be the ultrapower map associated with $\mathrm{Ult}(J^E_\vartheta, F)$, where $\vartheta = \mu^{+N}$. Pick a $\delta^* < \vartheta$ large enough such that μ is the largest cardinal in $J^E_{\delta^*}$ and the above formula still holds in $\tilde{N} = \langle J^E_\delta, \tilde{F}\rangle$, where $\delta = \pi(\delta^*)$ and $\tilde{F} = F \cap J^E_\delta$. Fix an enumeration $g : \mu \xrightarrow{\text{onto}} \mathcal{P}(\mu) \cap J^E_{\delta^*}$, which is itself in N. For convenience suppose that π maps δ^* cofinally to δ; this holds whenever $\mathrm{cf}^N(\delta^*) \neq \mu$ (thus, there are stationarily many – in the sense of J^E_ν – such ordinals δ^* below ϑ). Setting

$$g^*(\eta) = \langle\langle g(\bar\eta), g(\bar\eta) \cap \eta\rangle \,;\, \bar\eta < \eta\rangle$$

and $\tilde{g}(\eta) = \langle J^E_{\delta^*}, g^*(\eta)\rangle$ for $\eta < \mu$, it is easy to verify that $\pi(\tilde{g})(\mu) = \tilde{N}$. Define a partial map $f : \mu \to \mu$ by

$$f(\prec x, \eta \succ) \simeq \text{ the unique } \xi < \mu \text{ such that } \tilde{g}(\eta) \models (\exists z)\psi(z,\xi,x);$$

then $\pi(f)(x,\mu)$ is the unique ordinal $\xi < \lambda$ such that $\pi(\tilde{g})(\mu) = \tilde{N}$ satisfies $(\exists z)\psi(z,\xi,x)$, provided that such a ξ exists. As before, we know that for $x = b$ such a ξ exists, so $F(f)(b,\mu) = \zeta$. $\square$(Lemma 2.14)

The previous lemmata give us a method for expressing the statement "(μ', q') is a divisor of N" over $N(\mu,q)$ whenever $\mu' \geq \mu$ and q' is a bottom segment of q. The important point here is that over $N(\mu,q)$ we use the language for coherent structures, and the above statement is uniformly expressible in this language, no matter what kind of premouse N is or whether $\lambda_N(\mu,q) < \lambda_N$.

Lemma 2.15. *Let (μ,q) be a divisor of N where N is the collapsing $\mathbf{L}[E]$-level for $\tau < \kappa^+$. Let further $\mu \leq \mu' \leq \kappa$ and $q' = q \cap \bar\lambda$ for some $\bar\lambda \leq \max(q)+1$ where $\max(\varnothing) = \kappa + 1$ by definition. Set $r' = q - q'$ and $F = F(\mu,q)$. Then (μ', q') is a divisor of N iff for every $\xi < \mu'$ and $f : \mu \to \mu$ we have*

$$\zeta = F(f)(r',\xi) \;\&\; \zeta \leq \max(q') \;\longrightarrow\; \zeta < \mu' \tag{12}$$

The same is also true for pluripotent N and $\mu' > \mu$ with $F = E^N_{\mathrm{top}}$, $\mu = \mathrm{cr}(F)$ and q a bottom part of d_N.

Proof. Let (μ,q) be a divisor of N where N is the collapsing $\mathbf{L}[E]$-level for τ and let $F = F_N(\mu,q)$. Let us first discuss the case where N is either passive or else $\lambda_N(\mu,q) < \lambda_N$. Granting (12), we show that (μ',q') is a divisor of N. Let $\zeta \leq \max(q')$ be such that $\zeta = \tilde{h}^{n+1}_N(\xi, r \cup r')$ for some $\xi < \mu'$. By Lemma 2.13,

$\zeta = F(f)(r',\xi)$ for some $\xi < \mu \leq \mu'$. So $\zeta < \mu'$ by (12). This tells us that $\tilde{h}_N^{n+1}(\mu' \cup \{r,r'\}) \cap (\max(q')+1) = \mu'$, which verifies a), b) and d) of the definition of divisor. To see that also c) holds, notice that $\tilde{h}_N^{n+1}(\mu' \cup \{r,r'\}) \supset \tilde{h}_N^{n+1}(\mu \cup \{r\})$ and (μ,q) is a divisor.

Now assume that (μ',q') is a divisor of N. If $\zeta = F(f)(r',\xi)$ for some function $f : \mu \to \mu$ and an ordinal $\xi < \mu'$, then, again by Lemma 2.13, there is a $\xi' < \mu$ such that $\zeta = \tilde{h}_N^{n+1}(\xi, \{\xi', r', r\})$, so $\zeta \in \tilde{h}_N^{n+1}(\mu' \cup \{r',r\})$. As (μ',q') is a divisor and $\zeta \leq \max(q')$, we have $\zeta < \mu'$.

If N is active with $\lambda_N(\mu,q) = \lambda_N$ (recall that N is automatically pluripotent in this case) then the same argument goes through, but we have to work in the language for coherent structures. Here we also use the consequence of Lemmata 2.4 and 2.5 that, granting that N is pluripotent and $h_N^*(\mu' \cup \{d_N - q'\})$, the pair (μ',q') is a divisor for N computed in the language for coherent structures just in case that (μ',q') is a divisor for N computed in the language for premice.

Finally consider the case where N is pluripotent, $F = E_{\text{top}}^N$ and $\mu = \text{cr}(F)$. We can again imitate the above argument, this time using Lemma 2.14 in place of Lemma 2.13. Notice that μ occurs as a parameter in $F(f)(r',\xi,\mu)$. This is, however, harmless, as any candidate (μ',q') for a divisor must satisfy the restriction $\mu' > \mu$. Indeed, the ordinal μ, being the critical point of the top predicate F, is automatically in the range of $\pi_N(\mu',q')$. $\square$(Lemma 2.15)

Protomice, Fine Structure and Condensation. Using the lemmata from the previous subsection we determine the relationship between the fine structural parameters of $N(\mu,q)$ from those of N. Moreover, we establish a sort of condensation lemma for protomice. We first turn to general facts about coherent structures. For our later purposes, it will be useful to introduce "solidity" witnesses in the following weaker sense. To understand the next definition, we recommend to recall the facts about solidity witnesses from preliminaries.

Definition. *Let $M = \langle J_\nu^E, F\rangle$ be a coherent structure satisfying A) – D) from the previous subsection and let N be the premouse associated with M. Given a finite $s \subset \lambda$ and an α such that $\vartheta \leq \alpha < \lambda$, we define the* standard witness $W_M^{\alpha,s}$ *for α with respect to M and s to be the transitive collapse of $h_M(\alpha \cup \{s\})$ and, similarly, the* standard witness $W_N^{\alpha,r\cup s}$ *for α with respect to N and $s \cup r$ to be the transitive* *collapse of* $\tilde{h}_N^{n+1}(\alpha \cup \{r \cup s\})$. *The canonical witness maps are then defined in the obvious way. More generally, a* generalized witness *for* α *with respect to* M *and* s *is a pair* $\langle Q, t\rangle$, *where* Q *is an acceptable* J*-structure and* $t \subset Q$ *a finite set of ordinals such that given any* $\xi_1, \ldots, \xi_\ell < \alpha$,

$$M \models \Phi(i, \xi_1, \ldots, \xi_\ell, s) \;\longrightarrow\; Q \models \Phi(i, \xi_1, \ldots, \xi_\ell, t)$$

where Φ *is the universal* Σ_1*-formula. A* generalized witness *for* α *with respect to* N *and* $r \cup s$ *is a pair* $\langle Q, t\rangle$, *where* Q *is an acceptable* J*-structure and* $t \subset Q$ *a finite set of ordinals such that given any* $\xi_1, \ldots, \xi_\ell < \alpha$,

$$N \models \Phi(i, \xi_1, \ldots, \xi_\ell, r \cup s) \;\longrightarrow\; Q \models \Phi(i, \xi_1, \ldots, \xi_\ell, t)$$

where Φ *is the universal* $\Sigma_1^{(n)}$*-formula.*

The main point of this definiton is that generalized witnesses in this weaker sense are related merely to $\Sigma_1^{(n)}$-satisfaction even if $\alpha < \omega\varrho_N^{n+1}$ (and correspondingly for M), so they omit too much information to be able to say anything about the solidity of N, resp. M. The reason why we introduce them is that they might become genuine solidity witnesses after collapsing certain Skolem hulls in the original structure, which will be utilized in our later constructions. We again stress that this notion is defined *exclusively* for the present context, i.e. for coherent structures satisfying A) – D) as above, and should be considered as an ad hoc notion introduced for our convenience.

Lemma 2.16. *Let* $M = \langle J_\nu^E, F\rangle$ *be a coherent structure satisfying A) – D) from the previous subsection. Then*

a) $\omega\varrho_M^1 = \omega\varrho_N^{n+1}$.

Denote this common value by $\omega\varrho$. *Granting that* $\mu < \omega\varrho$, *the following holds:*

b) $p_M^1 = p_N^{n+1} \cap \lambda$.

c) M *is* 1*-sound iff* N *is* $(n+1)$*-sound.*

d) *Let* s *be a finite subset of* λ *and* $\vartheta \leq \alpha < \lambda$. *Then* $W_N^{\alpha, s\cup r} = \mathrm{Ult}^*(N^*, G)$ *where* G *is the top extender of* $W_M^{\alpha,s}$. *Moreover, the associated ultrapower embedding is precisely the uncollapsing map associated with the* $\Sigma_1^{(n)}(W_N^\alpha)$*-hull of* $\mu \cup \{r \cup s - (\alpha+1)\}$.

e) M *is* 1*-solid iff* N *is* $(n+1)$*-solid.*

Proof. Note first that neither $\omega\varrho^1_M$ nor $\omega\varrho^{n+1}_N$ is larger than λ. In the following we show that a) – c) are direct consequences of Corollaries 2.9 and 2.12.

If A is a $\Sigma_1(M)$-relation in p^1_M then there is a $\Sigma^{(n)}_1(N)$-relation ${}^*\!A$ in p^1_M, r and ϑ such that ${}^*\!A$ agrees with A up to $\omega\varrho^1_M$. Choose A such that $A \cap \omega\varrho^1_M \notin M$. Then ${}^*\!A \cap \omega\varrho^1_M$ is not a member of J^E_ν and therefore of N. This follows from acceptability and the fact that ν is a cardinal in N. Thus, $\omega\varrho^{n+1}_N \leq \omega\varrho^1_M$. The dual argument using Corollary 2.9 yields the converse, which proves a).

From now on suppose that $\mu < \omega\varrho$. The ordinal $\omega\varrho$, being smaller than λ, is a cardinal in both M and N. By our assumption on ϑ we have $\vartheta < \omega\varrho$. Given A as above, there is a $\Sigma^{(n)}_1(N)$ relation ${}^*\!A$ such that $A(\eta) \longleftrightarrow {}^*\!A(p^1_M, r, \vartheta, \eta)$ whenever $\eta < \lambda$. (Here we use Lemma 2.11.) From ${}^*\!A$ we obtain a new subset of $\omega\varrho$ which $\Sigma^{(n)}_1(N)$ in $p^1_M \cup r$, so $r \cup (p^{n+1}_N \cap \lambda) = p^{n+1}_N \leq^* r \cup p^1_M$ and, consequently, $p^{n+1}_N \cap \lambda \leq^* p^1_M$. As before, the dual argument yields the converse, which proves b).

If M is 1-sound, then every $\xi < \lambda$ is $\Sigma_1(M)$-definable from $p^1_M = p^{n+1}_N \cap \lambda$ and a parameter less than $\omega\varrho$. Thus, every $\xi < \lambda$ is $\Sigma^{(n)}_1(N)$ definable from $r \cup (p^{n+1}_N \cap \lambda) = p^{n+1}_N$ and parameters less then $\omega\varrho$, in other words, $N = \tilde{h}^{n+1}_N(\omega\varrho \cup \{p^{n+1}_N\})$. Thus, $p^{n+1}_N \in R^{n+1}_N$, i.e. N is $(n+1)$-sound. The converse follows again by the dual argument. This proves c).

Next we prove e) from d). To see that the 1-solidity of M implies the $(n+1)$-solidity of N, notice that W^α_M, being the transitive collapse of $h_M(\alpha \cup \{p_M\})$, can be encoded into a $\mathbf{\Sigma}_1(M)$ subset a of α. Such an a is in J^E_ν by acceptability and W^α_M can be reconstructed from a inside J^E_ν. But then also W^α_N is in J^E_ν by d). For the converse use again the dual argument.

It remains to prove d). Let $\bar\sigma : W^{\alpha,s}_M \to M$ be the canonical witness map, $\bar\lambda = \lambda(G)$ and $\bar\nu = \mathrm{ht}(W^{\alpha,s}_M)$. As $\bar\sigma$ is Σ_1-preserving, $\mathrm{dom}(G) = \mathcal{P}(\mu) \cap J^E_\vartheta$, so G can be applied to N^* (notice that N^* is an initial segment of $W^{\alpha,s}_M$). Let W be the $*$-ultrapower of N^* by G and $\bar\pi : N^* \to W$ the associated ultrapower map. This ultrapower exists, as $\sigma : \bar\pi(f)(\xi) \mapsto \pi(f)(\bar\sigma(\xi))$ is a structure preserving embedding of $\mathbb{D}^*(N^*, G)$ to N; here $\bar\pi$ is viewed as a map into $\mathbb{D}^*$, f ranges over $\Gamma(\mu, N^*)$ and $\xi < \bar\lambda$. (To verify this, use Łos' theorem for Σ_0 formulae.) By the remarks at the beginning of the previous subsection, $\bar\pi$ is $\Sigma^{(n)}_0$-preserving and cofinal, and $W = \tilde{h}^{n+1}_W(\lambda \cup \{\bar r\})$ where $\bar r = \bar\pi(p_{N^*})$.

Using the cofinality of both $\bar\pi$ and π together with the Łoś theorem for $\Sigma_0^{(n)}$-formulae we show that σ is $\Sigma_0^{(n)}$-preserving and cofinal. Also, $\pi = \sigma \circ \bar\pi$ and σ agrees with $\bar\sigma$ up to $\bar\nu$. It follows that $\sigma(\bar r) = r$, $\sigma \restriction \alpha = \mathrm{id}$ and, letting $\bar s$ be the $\bar\sigma$-preimage of $s - (\alpha+1)$, also $\sigma(\bar s) = \bar\sigma(\bar s) = s - (\alpha+1)$. As each $\zeta < \lambda$ is $\Sigma_1(W_M^{\alpha,s})$-definable from $\bar s$ and parameters below α, Lemma 2.10 combined with the observation at the end of the previous paragraph allow us to conclude that $W = \tilde h_W^{n+1}(\alpha \cup \{\bar r \cup \bar s\})$, so $\tilde h_N^{n+1}(\alpha \cup \{r \cup s - (\alpha+1)\}) = \mathrm{rng}(\sigma)$. But this means that $W = W_N^{\alpha, r\cup s}$ and σ is the associated witness map. $\square$(Lemma 2.16)

The next corollary summarizes the properties of protomice $N(\mu,q)$ associated with $\mathbf{L}[E]$-levels. Let us explain the notation used in d) below. Let N be an $\mathbf{L}[E]$-level, $\alpha < \lambda_N(\mu,q)$ and W_N^{α,p_N} be the standard witness. Let further $\sigma_\alpha : W_N^{\alpha,p_N} \to N$ be the associated canonical witness map. Setting $(r_\alpha, q_\alpha, \lambda_\alpha) = \sigma_\alpha^{-1}(r,q,\lambda)$ where $\lambda = \lambda_N(\mu,q)$ and $\kappa_\alpha = (\sigma^{-1})''\kappa$, the pair (μ, q_α) satisfies a) – d) in the definition of divisor with W_N^{α,p_N} in place of N and $(r_\alpha, q_\alpha, \kappa_\alpha, \lambda_\alpha)$ in place of (r,q,κ,λ). Then $W_N^{\alpha,p_N}(\mu,q)$ is the coherent structure $\langle J_{\nu_\alpha}^{E_\alpha}, G\rangle$ where E_α is the extender sequence of W_N^{α,p_N}, ν_α is the cardinal successor of λ_α in W_N^{α,p_N} and G is the extender derived from the uncollapsing embedding associated with $\tilde h_W^{n+1}(\mu \cup \{\bar r\})$ (here $W = W_N^{\alpha,p_N}$). Intuitively, $W_N^{\alpha,p_N}(\mu,q)$ is the protomouse associated with W_N^{α,p_N} and (μ, q_α).

Corollary 2.17. *Given a divisor (μ,q) of N,*

a) *$\omega\varrho^1_{N(\mu,q)} = \kappa$ is the ultimate projectum of $N(\mu,q)$.*

Granting that $\mu < \kappa$, the following holds:

b) *$p_{N(\mu,q)} = p_N \cap \lambda$ if N is not pluripotent, and $p_{N(\mu,q)} = d_N \cap \lambda$ otherwise.*

c) *$N(\mu,q)$ is sound and solid.*

d) *If N is not pluripotent and $p \overset{\mathrm{def}}{=} p_{N(\mu,q)}$, then for every $\alpha \in p$ we have $W^{\alpha,p}_{N(\mu,q)} = W_N^{\alpha,p_N}(\mu,q)$.*

If N is pluripotent then the same holds with W_N^{α,d_N} in place of W_N^{α,p_N}; this time everything is computed in the language for coherent structrues.

$\square$(Corollary 2.17)

It follows from our previous results that in b), the latter condition says something new only if $\lambda_N(\mu,q) = \lambda_N$. The next lemma gives a more detailed information about the definability of *standard* witnesses over N. We will need such a detailed information only in the proof of Lemma 3.12, so the reader can skip over this lemma now and return to it when necessary.

Lemma 2.18. *Let (μ, q) be a divisor of N such that $q \neq d_N$ if N is pluripotent. Assume that $\langle Q, t\rangle \in N$ is a generalized witness for β with respect to N and $r \cup s$ where $\mu \leq \beta < \lambda$ and $s \subset \lambda$. Then $W_N^{\beta, r\cup s} \in N$ is Σ_0-definable inside N from $\langle Q, t\rangle$, β and $\vartheta = \vartheta_N(\mu, q)$. By d) of Lemma 2.16, $W^{\beta,s}_{N(\mu,q)}$ is also Σ_0-definable inside N from $\langle Q, t\rangle$, β and ϑ.*

If M is a coherent structure, $\vartheta(M) \leq \beta < \lambda_M$ and $\langle Q, t\rangle$ is a generalized witness for β with respect to M and some $s \subset \lambda_M$, then $W_M^{\beta,s}$ is Σ_0-definable inside M from $\langle Q, t\rangle, \beta$ and $\vartheta(M)$.

Proof. The proof is a refinement of the standard argument which shows that if some generalized solidity witness for α with respect to N and p is in N then so is the standard solidity witness. We shall focus on the proof of the first part of the lemma. The proof of the second part is similar and, in fact, much easier, so we leave it to the reader. Set $p = r \cup s$. Let $W = W_N^{\beta,p}$, let $\sigma : W \to N$ be the canonical witness map and let $\sigma(\bar{p}) = p$. We can define a $\Sigma_0^{(n)}$-preserving embedding $\tilde{\sigma} : W \to Q$ sending $\bar{p}$ to t which is the identity on β by

$$\tilde{h}_W^{n+1}(\xi, \bar{p}) \mapsto \tilde{h}_Q^{n+1}(\xi, t),$$

as every element of W can be written in the form on the left hand side. Given a $\Sigma_0^{(n)}$-formula $\varphi(v_1, \ldots, v_\ell)$, each of the following statements is a consequence of the previous one:

$$\begin{aligned} W &\models \varphi(\tilde{h}_W^{n+1}(\xi_1, \bar{p}), \ldots, \tilde{h}_W^{n+1}(\xi_\ell, \bar{p})) \\ N &\models \varphi(\tilde{h}_N^{n+1}(\xi_1, p), \ldots, \tilde{h}_N^{n+1}(\xi_\ell, p)) \\ Q &\models \varphi(\tilde{h}_Q^{n+1}(\xi, t), \ldots, \tilde{h}_Q^{n+1}(\xi_\ell, t)). \end{aligned}$$

As substituting $\tilde{h}^{n+1}$ into a $\Sigma_0^{(n)}$-formula yields a $\Sigma_1^{(n)}$-statement, the first implication follows from the fact that σ is $\Sigma_1^{(n)}$-preserving, and the second implication follows from the definition of a generalized witness. It is now easy to verify that $\tilde{\sigma}$ is well-defined and $\Sigma_0^{(n)}$-preserving, as well as that $\tilde{\sigma} \restriction \beta$ is the identity and $\tilde{\sigma}(\bar{p}) = t$.

Let

$$\omega\tilde{\varrho} = \sup(\tilde{\sigma}''\omega\varrho_W^n).$$

Then

$$\mathrm{rng}(\tilde{\sigma}) = \{y \in Q;\ (\exists z \in J_{\tilde{\varrho}}^{E^Q})(\exists \xi < \beta)Q \models \psi(z, y, \xi, t)\}$$

where $(\exists z^n)\psi(z^n, y, x, u)$ is the functionally absolute $\Sigma_1^{(n)}$-definition of $\tilde{h}^{n+1}$ and ψ is a $\Sigma_0^{(n)}$-formula. Thus, $\mathrm{rng}(\tilde{\sigma})$ is an element of N which is Σ_0-definable from Q, t, β and $\tilde{\varrho}$. Clearly, this applies also to W, which is the transitive collapse of $\mathrm{rng}(\tilde{\sigma})$.

It remains to eliminate $\tilde{\varrho}$ form the definition of W. Here we use the fact that (μ, q) is a divisor of N. If $\pi = \pi_N(\mu, q)$ and $N^* = N^*(\mu, q)$, then π maps $\omega\varrho_{N^*}^n$ cofinally into $\omega\varrho_N^n$. But $\mathrm{rng}(\pi) = \tilde{h}_N^{n+1}(\mu \cup \{r\}) \subset \mathrm{rng}(\sigma)$, so letting $\bar{r}$ be the top segment of $\bar{p}$ of length $|r|$, we obtain:

- If $\xi < \mu$, then $\tilde{h}_W^{n+1}(\xi, \bar{r})$ is defined iff $\tilde{h}_{N^*}^{n+1}(\xi, p_{N^*})$ is defined.
- $Y = \tilde{h}_W^{n+1}(\mu \cup \{\bar{r}\})$ is cofinal in $\omega\varrho_W^n$, so $\tilde{\sigma}''Y \cap \omega\varrho_Q^n$ is cofinal in $\omega\tilde{\varrho}$.

Now it follows easily that

$$\omega\tilde{\varrho} = \sup\{\zeta < \omega\varrho_Q^n;\ (\exists \xi < \mu)(\tilde{h}_{N^*}^{n+1}(\xi, p_{N^*}) \text{ is defined and } \zeta = \tilde{h}_Q^{n+1}(\xi, \bar{t}))\}$$

where $\bar{t}$ is the top part of t of length $|r| = |p_{N^*}|$. The objects ϱ_Q^n and $\tilde{h}_Q^{n+1}$ are Σ_0-definable from Q; the objects $N^*, p_{N^*}, \tilde{h}_{N^*}^{n+1}$ and μ are Σ_0-definable from ϑ and $\bar{t}$ is Σ_0-definable from t and $|p_{N^*}|$. Thus, Q, t and ϑ are *all* parameters we need in the definition of $\tilde{\varrho}$. $\square$(Lemma 2.18)

We are now ready to state a condensation lemma for protomice. Recall again that all fine structural parameters of protomice are computed in the language for coherent structues.

Lemma 2.19. *Let $\bar{M}$ be a coherent structure and let $\sigma : \bar{M} \to N(\mu, q)$. Assume further that the following conditions are met:*

a) $\omega\varrho_{\bar{M}}^1 = \kappa$ *and $\bar{M}$ is sound and solid;*

b) σ *is a Σ_0-preserving map which is the identity on $\bar{\tau} = \kappa^{+\bar{M}}$ and such that $\sigma(\bar{\tau}) = \tau$ and $\sigma(p_{\bar{M}}) = q$;*

c) $\mu < \kappa$.

Then $\bar{M} = \bar{N}(\mu, p_{\bar{M}})$ where $\bar{N}$ is the collapsing level for $\bar{\tau}$.

Proof. Let $\bar{F}, \bar{\lambda}, \bar{\nu}, \bar{\vartheta}, \bar{N}^*, \bar{n}$ have the obvious meaning. We can assume w.l.o.g. that σ is not cofinal; otherwise σ would be Σ_1-preserving and, by soundness of $N(\mu, q)$, we would have $\bar{M} = N(\mu, q)$ and $\sigma = \mathrm{id}$. Our assumption implies that $\bar{\vartheta} < \vartheta$ and $\bar{N}^*$ is a proper initial segment of N^*. Also, the non-cofinality of σ guarantees that $\bar{\tau} < \tau$, as otherwise τ could be definably collapsed over $\langle J^E_{\tilde{\nu}}, F \cap J^E_{\tilde{\nu}} \rangle \in N$ where $F = F(\mu, q)$ and $\tilde{\nu} = \sup(\sigma''\bar{\nu})$. Thus, $\bar{\tau} = \mathrm{cr}(\sigma)$.

Let $\bar{N} = \mathrm{Ult}^*(\bar{N}^*, \bar{F})$ and $\bar{\pi} : \bar{N}^* \to \bar{N}$ be the associated ultrapower map. Let further $\pi = \pi(\mu, p_{\bar{M}})$. Using the methods from the proof of Lemma 2.16 d) we show that this ultrapower exists, $\bar{\pi}$ is $\Sigma_0^{(\bar{n})}$-preserving and cofinal and, letting $\tilde{\sigma} : \bar{N} \to \pi(\bar{N}^*)$ be the map defined by

$$\tilde{\sigma}(\bar{\pi}(f)(\alpha)) = \pi(f)(\sigma(\alpha))$$

for $f \in \Gamma(\mu, \bar{N}^*)$ and $\alpha < \lambda_M$, $\tilde{\sigma}$ is $\Sigma_0^{(\bar{n})}$-preserving and cofinal (here we apply the cofinality of $\bar{\pi}$ at the $\bar{n}$-th level together with the Łoś theorem for $\Sigma_0^{(\bar{n})}$-formulae) and $\tilde{\sigma} \restriction \lambda_{\bar{M}} = \sigma \restriction \lambda_{\bar{M}}$. Hence $\bar{\tau} = \mathrm{cr}(\tilde{\sigma})$.

Assume first that $\bar{N}^*$ is either passive or else $\lambda_{\bar{N}^*} > \mu$. In this case the conclusions of the previous paragraph apply to the language for premice. Similarly as in the case of the pseudoultrapower in the proof of Lemma 2.4 we can apply Lemma 1.5 and infer that $\bar{N}$ is a premouse of the same type as $\bar{N}^*$. Furthermore, $\omega\varrho^{\bar{n}+1}_{\bar{N}} = \kappa < \omega\varrho^{\bar{n}}_{\bar{N}}$, $p_{\bar{N}} = \bar{\pi}(p_{\bar{N}^*}) \cup p_{\bar{M}}$ and $\bar{N}$ is sound, as follows from Lemma 2.16. This verifies the assumptions of the condensation lemma. A simple discussion of the four cases given by the condensation lemma will give us the conclusion that $\bar{N}$ is a proper initial segment of $\pi(\bar{N}^*)$. Thus $\bar{N}$ is an $\mathbf{L}[E]$-level, as this is certainly true of $\pi(\bar{N}^*)$. It follows that $\bar{M} = \bar{N}(\mu, p_{\bar{M}})$. Here are more details as to the application of the condensation lemma. Option a) is impossible, as $\bar{\tau} < \tau$ is the cardinal successor of κ in $\bar{N}$. Option c) contradicts the soundness of $\bar{N}$, and option d) contradicts the fact that $\bar{N}$ projects to κ while $\bar{\tau}$ is a cardinal in $\mathrm{Ult}(\pi(\bar{N}^*), E_{\bar{\tau}})$.

From now on suppose that $\bar{N}^*$ is active with $\lambda_{\bar{N}^*} = \mu$. Then $\bar{N}$ is a potential premouse which need not be of the same type as $\bar{N}^*$. Now we have to be more careful when applying Lemma 2.16; the conclusions of this lemma apply to the language for coherent structures in this case. Notice first that $\omega\varrho^1_{\bar{N}} = \kappa < \lambda_{\bar{N}}$, as $h^*_{\bar{M}}(\kappa \cup \{p_{\bar{M}}\}) = \bar{N}$. It follows that $\bar{N}$ cannot be of type C.

Let $\tilde\lambda = \sup(\tilde\sigma''\lambda_{\bar N})$. If $\tilde\lambda < \lambda(\tilde F)$ where $\tilde F$ is the top extender of $\pi(\bar N^*)$ then $\tilde\lambda$ is a cutpoint of $\tilde F$. To see this, pick any monotonic $f:\bar\mu\to\bar\mu$ that is an element of $\pi(\bar N)$ and any $\alpha<\tilde\lambda$; here $\bar\mu = \mathrm{cr}(\tilde F) = \mathrm{cr}(E^{\bar N^*}_{\mathrm{top}})$. It suffices to consider just monotonic maps, as any $f':\bar\mu\to\bar\mu$ from $\pi(\bar N^*)$ is dominated by such a map. Let $\bar\alpha<\lambda_{\bar N}$ be such that $\tilde\sigma(\bar\alpha)\ge\alpha$. Notice that $f\in\bar N$ and, letting $\bar\zeta = E^{\bar N}_{\mathrm{top}}(f)(\bar\alpha)$, we have $\bar\zeta<\lambda_{\bar N}$ and

$$\tilde F(f)(\alpha)\le\tilde F(f)(\tilde\sigma(\bar\alpha)) = \tilde\sigma(E^{\bar N}_{\mathrm{top}}(f)(\bar\alpha)) = \tilde\sigma(\bar\zeta)<\tilde\lambda.$$

If $\tilde\lambda=\lambda(\tilde F)$ let $N'=\pi(\bar N^*)$ and $\tilde\sigma'=\tilde\sigma$. Otherwise let $N'=\pi(\bar N^*)\,||\,\tilde\nu$ where $\tilde\nu$ is the index of $\tilde F\,|\,\tilde\lambda$. If $\sigma':N'\to\pi(\bar N^*)$ is the canonical embedding (i.e. $\sigma':E_{\tilde\nu}(f)(\alpha)\mapsto\pi(f)(\alpha)$ for $f:\bar\mu\to\bar\mu$ in $\bar N^*$ and $\alpha<\tilde\lambda$) then $\mathrm{cr}(\sigma')=\tilde\lambda$ and it is easy to see that $\mathrm{rng}(\tilde\sigma)\subset\mathrm{rng}(\sigma')$. Define $\tilde\sigma'$ by $\tilde\sigma'=(\sigma')^{-1}\circ\tilde\sigma$. In either case, $\tilde\sigma':\bar N\to N'$ is Σ_0-preserving and cofinal with respect to the language for coherent structures, and maps $\lambda_{\bar N}$ cofinally into $\lambda_{N'}$. We will use this to show that $\bar N$ is a potential premouse of the same type as N' and $\tilde\sigma'$ has the above preservation properties with respect to the language for premice. Since we already know that N' cannot be of type C, so assume w.l.o.g. that N' is of type B; the proof is even easier for N' of type A.

Recall that $\lambda^*_{\bar N}$ is the largest cutpoint of $E^{\bar N}_{\mathrm{top}}$. We show that $\tilde\sigma'(\lambda^*_{\bar N})$ is the largest cutpoint of $E^{N'}_{\mathrm{top}}$. Since being a cutpoint of the top extender is a Π_1-property in the language for coherent structures, it suffices to show that there are no cutpoints of $E^{N'}_{\mathrm{top}}$ above $\tilde\sigma'(\lambda^*_{\bar N})$. If $\lambda'>\tilde\sigma'(\lambda^*_{N'})$ were a cutpoint of $E^{N'}_{\mathrm{top}}$, we would have $\lambda_{\bar N}<\bar\lambda<\lambda_{\bar N}$ where $\bar\lambda$ is the pointwise preimage of λ' under $\tilde\sigma'$; the inequality on the right follows from the cofinality of $\tilde\sigma'$. Now we obtain a contradiction by showing that $\bar\lambda$ is a cutpoint of $\bar N$: If $f:\bar\mu\to\bar\mu$ is in $\bar N$ and $\alpha<\bar\lambda$ then $\tilde\sigma'(E^{\bar N}_{\mathrm{top}}(f)(\alpha)) = E^{N'}_{\mathrm{top}}(f)(\tilde\sigma'(\alpha))<\lambda'$, so $E^{\bar N}_{\mathrm{top}}(f)(\alpha))<\bar\lambda$.

Working in the language for coherent structures, from Lemma 2.16 we obtained that $h^*_{\bar N}(\kappa\cup\{p_{\bar M}\})=\bar N$ and $W^{\alpha,p_{\bar M}}_{\bar N}\in\bar N$ whenever $\alpha\in p_{\bar M}$; recall that $p_{\bar M}=\sigma^{-1}(q)$. The former obviously guarantees that $\bar\alpha\overset{\mathrm{def}}{=}\max(p_{\bar M})\ge\lambda^*_{\bar N}$, and the latter implies that the top extender G of $W^{\alpha,p_{\bar M}}_{\bar N}$ is an element of $\bar N$. Since $E^{\bar N}_{\mathrm{top}}\,|\,\lambda^*_{\bar N} = G\,|\,\lambda^*_{\bar N}$, we see that $\bar N$ satisfies the initial segment condition. So $\bar N$ is a premouse. Now it is easy to verify that $\tilde\sigma'(E^{\bar N}_{\mathrm{top}}\,|\,\lambda^*_{\bar N}) = E^{N'}_{\mathrm{top}}\,|\,\lambda^*_{N'}$, which implies that the preservation properties of $\tilde\sigma'$ transfer to the language of premice.

The rest of the argument is then similar to that in the previous case where $\lambda_{\bar{N}^*} > \mu$. We intend to apply the condensation lemma, this time to $\tilde{\sigma}' : \bar{N} \to N'$. This is possible, as N' is an $\mathbf{L}[E]$-level, $\mathrm{cr}(\tilde{\sigma}') = \mathrm{cr}(\sigma) = \bar{\tau}$, $\omega\varrho^1_{\bar{N}} \leq \tau$ and, as we have seen above, $\tilde{\sigma}'$ has the required preservation properties. It only remains to verify that $\bar{N}$ is sound with respect to the language for premice. But this follows immediately from the two properties of $p_{\bar{M}}$ applied in the previous paragraph and from Lemma 1.8 The discussion of the four cases in the condensation lemma is then the same as in the previous case. So we conclude that $\bar{N}$ is the collapsing $\mathbf{L}[E]$-level for $\bar{\tau}$. Since this time we work in the language for coherent structures, we have $p_{\bar{M}} = d_{\bar{N}}$. It is now clear from the definition following Lemma 2.6 that $\bar{M} = \bar{N}(\mu, p_{\bar{M}})$. $\Box$(Lemma 2.19)

We shall also need the version of this condensation lemma with pluripotent $\mathbf{L}[E]$-levels as target structures.

Lemma 2.20. *Let $\bar{M}$ be a coherent structure and let $\sigma : \bar{M} \to N$ where N is pluripotent. Assume further that the following conditions are met:*

a) $\omega\varrho^1_{\bar{M}} = \kappa$ *and $\bar{M}$ is sound and solid;*
b) σ *is a Σ_0-preserving map which is the identity on $\bar{\tau} = \kappa^{+\bar{M}}$ and such that $\sigma(\bar{\tau}) = \tau$ and $\sigma(p_{\bar{M}}) = d_N$;*
c) $\mu < \kappa$.

Then $\bar{M} = \bar{N}(\mu, p_{\bar{M}})$ where $\bar{N}$ is the collapsing level for $\bar{\tau}$.

Proof. The proof is virtually the same as that of the previous lemma.

$\Box$(Lemma 2.20)

Strong Divisors. The construction of a $\Box_\kappa$-sequence in $\mathbf{L}$ is based on the fact that to every local successor one can assign a *uniquely* determined collapsing structure, namely the collapsing level, and that this assignment is preserved under interpolation arguments. The circumstances in $\mathbf{L}[E]$ are complicated by the fact that premousehood of pluripotent $\mathbf{L}[E]$-levels is *not* preserved under such arguments. More preciely, such arguments give rise to collapsing protomice. One way of dealing with this issue is incorporating protomice in the construction. This means that to every local successor we have to decide whether the collapsing structure used in the construction will be an $\mathbf{L}[E]$-level of some protomouse associated with it. As there can be a large collection of protomice associated with the same collapsing level, have to find a *canonical* way of choosing such a protomouse.

Given a fixed q, elementary properties of Skolem hulls imply that the set

$$\mathcal{D}_q = \mathcal{D}_q(N) = \{\mu < \kappa\,;\, (\mu, q) \text{ is a divisor of } N\}$$

is closed in κ and $\lambda(\mu_1, q) \geq \lambda(\mu_2, q)$ whenever $\mu_1 < \mu_2$. Thus, one might attempt to choose the divisor with the shortest possible q and largest possible μ, and show that such divisors are preserved in interpolation arguments. This works in some situations, as Lemma 2.15 enables us to express, uniformly over $N(\mu, q)$, that there are no divisors $(\mu', q') \neq (\mu, q)$ with $\mu \leq \mu'$ and q' a bottom segment of q. In fact, one can give a proof of $\square_\kappa$ in Jensen extender models below a measurable Woodin cardinal based on this choice of divisor [45]. The smallness condition is used to rule out the possibility that too many $\mathbf{L}[E]$-levels will have a divisor (μ', q') "overlapping" the chosen (μ, q) in the sense that q' is a bottom segment of q and $\mu' < \mu$. Notice that Lemma 2.15 will not help here. For the choice of a canonical divisor that avoids the use of any unnecessary smallness condition we need a more sophisticated tool. The following is an elaboration on a notion from [25].

Definition. *A divisor (μ, q) of N is* strong *iff for every $\xi < \mu$ and every x of the form $\tilde{h}_N^{n+1}(\xi, p_N)$ we have $x \cap \mu \in N^*(\mu, q)$. If N is pluripotent, we define the notion of strong divisor in the same way, but with h_N^* and d_N in place of $\tilde{h}_N^{n+1}$ and p_N, respectively.*

By acceptability, (μ, q) is strong iff $x \cap \mu \in J^E_{\vartheta(\mu,q)}$ for every x as above. Equivalently,

$$(13) \qquad (\mu, q) \text{ is strong} \quad \text{iff} \quad \mathcal{P}(\mu) \cap N^*(\mu, q) = \mathcal{P}(\mu) \cap N'(\mu)$$

where $N'(\mu)$ is the transitive collapse associated with $\tilde{h}_N^{n+1}(\mu \cup \{p_N\})$. If N is pluripotent, we define $N'(\mu)$ in the same way, but with h_N^* and d_N in place of $\tilde{h}_N^{n+1}$ and p_N, respectively. Notice also that $E^{N'} \restriction \mu^{+N'} = E \restriction \mu^{+N'}$ by the proof of (5).

Translating facts about fine structural parameters between N and $N(\mu, q)$ requires that $\mu < \kappa$. In particular, it is not clear how to translate facts concerning the soudness and standard parameters if $\mu = \kappa$. For this reason, all lemmata in the previous subsection require the assumption $\mu < \kappa$. The next lemma tells us that this assumption is automatically satisfied if (μ, q) is a strong divisor.

Lemma 2.21. *Suppose* (κ, q) *is a divisor. Then* (κ, q) *is not strong.*

Proof. By Lemmata 2.3 and 2.6, $N^*(\mu, q)$ is a *proper* initial segment of N projecting to κ. Thus, there is a subset of κ in N which is not in $N^*(\mu, q)$. By the soundness, resp. Dodd soundness of N we have $N'(\mu) = N$, so (μ, q) cannot be strong by (13). □(Lemma 2.21)

The next couple of lemmata will enable us to prove preservation of strong divisors under maps with weak preservation properties that arise in interpolation arguments. The first of them provides us with a characterization of strong divisors.

Lemma 2.22. *Let* (μ, q) *be a divisor of* N. *If* N *is pluripotent, assume moreover that* $\lambda_N(\mu, q) < \lambda_N$. *The following are equivalent:*

a) (μ, q) *is strong;*
b) $N^*(\mu, q) = \mathrm{core}(N'(\mu))$*;*
c) $|p_{N'(\mu)}| = |p_{N^*(\mu,q)}| = |r|$ *(recall* $r = p_N - q$*).*

Proof. If $\mu = \kappa$ then a) – c) are obviously false, so we can assume w.l.o.g. that $\mu < \kappa$. Let $N^* = N^*(\mu, q)$ and $N' = N'(\mu)$. Also, let $\pi' : N' \to N$ be the associated uncollapsing map and $\pi^* = \pi'^{-1} \circ \pi : N^* \to N'$. Both π' and π^* are $\Sigma_0^{(n)}$-preserving and cofinal, and N^* and N' have the same $(n+1)$-st projectum μ. The map π' is sufficiently preserving to guarantee that N' is a premouse; if N is pluripotent then γ_N is in the range of π by Lemma 2.4. By the condensation lemma, N' is solid and its standard parameter is universal.

Setting $p' = \pi^*(p_{N^*})$, the parameter p' is a top segment of $p_{N'}$. More precisely,

$$p' = p_{N'} - \min(p'). \tag{14}$$

This follows from Lemma 1.1; just observe that $N' = \tilde{h}^{n+1}_{N'}(\min(p') \cup \{p'\})$ and that for every $\alpha \in p_{N^*}$, the structure $\pi^*(W^{\alpha}_{N^*})$ is a generalized witness for $\pi^*(\alpha)$ with respect to N and p'.

To see that a) implies b), notice first that $p' \in P^{n+1}_{N'}$. This follows by the standard argument: Pick an A that is $\Sigma_1^{(n)}(N^*)$ in p_{N^*} and such that $A \cap \mu \notin N^*$. Letting A' be $\Sigma_1^{(n)}(N')$ in the parameter p' by the same definition, we have $A' \cap \mu = A \cap \mu$, so $A' \cap \mu \notin N'$ by (13). Hence $p' = p_{N'}$ by (14) and the $\leq^*$-minimality of $p_{N'}$.

Clause c) is an immediate consequence of b), so it remains to derive a) from c). By (14) we know that $p_{N'}$ is a lengthening of p'; clause then c) implies that they actually agree. So N^* is the transitive collapse of $h^{n+1}_{N'}(\mu \cup \{p_{N'}\})$ and a) follows from the universality of $p_{N'}$ and (13). $\square$(Lemma 2.22)

Again, we will need the version of the previous lemma for pluripotent N with $\lambda_N(\mu, q) = \lambda_N$. As we already know, this is possible only if $q = d_N$. Notice also that μ must be a cutpoint of E^N_{top}, so N must be of type B.

Lemma 2.23. *Assume N is pluripotent and (μ, d_N) is a divisor of N such that $\lambda_N(\mu, d_N) = \lambda_N$, the following are equivalent:*

a) (μ, d_N) *is strong;*
b) $E^{N'(\mu)}_{\mathrm{top}} \,|\, \mu \notin N'(\mu)$.

Proof. Let $\bar\gamma$ be such that $E_{\bar\gamma} = E^N_{\mathrm{top}} \,|\, \mu$ and let $N^* = N^*(\mu, d_N)$. Then $E_{\bar\gamma} = E^{N^*}_{\mathrm{top}} = E^{N'(\mu)}_{\mathrm{top}} \,|\, \mu$. Recall also that $N'(\mu)$ agrees with N up to $\mu^{+N'(\mu)}$ and that $N^*, N'(\mu)$ and N all agree up to $\bar\gamma$. More precisely, $E^{N^*} \restriction \bar\gamma = E^{N'(\mu)} \restriction \bar\gamma = E^N \restriction \bar\gamma$.

Since $\bar\gamma = \mathrm{ht}(N^*)$, the agreement between N^* and $N'(\mu)$ together with (13) guarantee that (μ, d_N) is strong just in case that $\bar\gamma$ is the cardinal successor of μ in $N'(\mu)$. If $\bar\gamma = \mu^{+N'(\mu)}$ then $E^{N'(\mu)}_{\mathrm{top}} \,|\, \mu$ cannot be an element of $N'(\mu)$, as it collapses $\bar\gamma$. If $\bar\gamma \neq \mu^{+N'(\mu)}$ then $E_{\bar\gamma} = E^{N'(\mu)}_{\bar\gamma} \in N'(\mu)$ by the agreement between $N'(\mu)$ and N. $\square$(Lemma 2.23)

Notice also that in the situation from the previous lemma, $\langle \langle J^E_{\bar\gamma}, E_{\bar\gamma} \rangle, \varnothing \rangle$ is the standard witness for μ with respect to $N'(\mu)$ and $\varnothing$. This terminology is consistent with that introduced at the beginning of the subsection on fine structure for protomice. We will make use of this fact in the proof of Lemma 3.12.

The strongness of a divisor (μ, q) can also be characterized over the associated protomouse $N(\mu, q)$; this fact will play an important role in our main construction.

Definition. *Let M be a coherent structure with the top extender F and $\mu = \mathrm{cr}(F)$. We say that an ordinal $\vartheta \leq \vartheta(M)$ is* closed *in M* relative *to $p \in [\lambda_M]^{<\omega}$ just in case that $F(f)(p, \xi) \cap \mu \in J^{E^M}_\vartheta$ for every $f : \mu \to \mathcal{P}(\mu)$ from $J^{E^M}_\vartheta$ and every $\xi < \mu$.*

If M is a protomouse, we say that $\vartheta \leq \vartheta(M)$ is closed *in M iff ϑ is closed in M relative to p_M.*

If M is a pluripotent $\mathbf{L}[E]$-level, we say that $\vartheta \leq \vartheta(M)$ is closed *in M iff ϑ is closed in M relative to d_M.*

Notice that if $\vartheta \le \vartheta(M)$ is a limit of ordinals that are closed in M relative to q then ϑ itself is closed in M relative to q. Thus, the set of all $\vartheta \le \vartheta(M)$ which are closed in M relative to q is closed. Notice also that if M is a pluripotent $\mathbf{L}[E]$-level then $\vartheta(M)$ is trivially closed in M.

Lemma 2.24. *Let (μ, q) be a divisor of N. Then*

$$\vartheta(\mu, q) \text{ is closed in } N(\mu, q) \quad \text{iff} \quad (\mu, q) \text{ is a strong divisor of } N.$$

Proof. Given a divisor (μ, q) of N, Lemma 2.13 guarantees that for each $x \subset \lambda$ we have $x \in \tilde{h}_N^{n+1}(\mu \cup \{p_N\})$ (resp. $x \in h_N^*(\mu \cup \{d_N\})$ whenever N is pluripotent with $\lambda_N(\mu, q) = \lambda_N$) just in case that x is of the form $F(f)(q, \xi)$ for some $\xi < \mu$ and $f : \mu \to \mathcal{P}(\mu)$ from $\mathrm{dom}(F)$; here $F = F_N(\mu, q)$. So (μ, q) is strong iff $x \cap \mu \in N^*(\mu, q)$ for each such x which, in turn is equivalent to the requirement that $F(f)(q, \xi) \cap \mu \in J^E_{\vartheta(\mu,q)}$ for each f and ξ as above. □(Lemma 2.24)

For a fixed q, we define

$$\mathcal{D}^*_q = \mathcal{D}^*_q(N) = \{\mu < \kappa\,;\, (\mu, q) \text{ is a strong divisor of } N\}.$$

Lemma 2.25. *$\mathcal{D}^*_q$ is closed and bounded in κ.*

Proof. It suffices to prove that $\mathcal{D}^*_q$ is closed; its boundedness then follows immediately from Lemma 2.21. Given a limit point μ of $\mathcal{D}^*_q$, the pair (μ, q) is clearly a divisor of N. Suppose for a contradiction that (μ, q) is not strong. As usual, the proof splits into two cases, depending on whether N is pluripotent and $\lambda_N = \lambda_N(\mu, q)$, or not.

Let us first focus on the case where N is either not pluripotent or else $\lambda_N(\mu, q) < \lambda_N$. Let $\pi = \pi_N(\mu, q)$, $\pi' : N'(\mu) \to N$ be the uncollapsing embedding associated with $N'(\mu)$ and $\pi^* = (\pi')^{-1} \circ \pi$. Let further r' and p' be the π'-preimage of r and p_N, respectively. Then both $N^*(\mu, q)$ and $N'(\mu)$ are premice; this follows from Lemma 2.4 for pluripotent N of type B. Also, γ_N is in the ranges of both π and π' in this case. As in the proof of Lemma 2.22 we observe that $p_{N'(\mu)}$ is a lengthening of r'.

Granting that (μ, q) is not strong, $p_{N'(\mu)}$ is a *proper* lengthening of r'. Here we use the previous lemma. Let $\alpha = \max(p_{N'(\mu)} - r')$. As $N'(\mu)$ is solid (this follows from the condensation lemma), $W \in N'(\mu)$ where W is the standard solidity witness for

α with respect to $N'(\mu)$ and $p_{N'(\mu)}$. Let r'_α be the preimage of r' under the canonical witness map $\sigma'_\alpha : W \to N'(\mu)$. Pick $\xi < \mu$ such that $\langle \alpha, W, r'_\alpha \rangle = \tilde{h}^{n+1}_{N'(\mu)}(\xi, p')$; this is possible since $p' \in R^{n+1}_{N'(\mu)}$. Pick further a strong divisor $(\bar{\mu}, q)$ for N such that $\xi < \bar{\mu} < \mu$. Let $\bar{\pi}' : N'(\bar{\mu}) \to N$ be the uncollapsing map associated with $N'(\bar{\mu})$ and $\bar{p}' = (\bar{\pi}')^{-1}(p_N)$. Notice that $\sigma \stackrel{\text{def}}{=} (\pi')^{-1} \circ \bar{\pi}'$ is a $\Sigma^{(n)}_1$-preserving embedding of $N'(\bar{\mu})$ into $N'(\mu)$ which sends $\bar{p}'$ to p'. By our choice of $\bar{\mu}$, the range of σ contains α, W as well as r'_α. Let $\varphi(x_1, \cdots, x_\ell)$ be a $\Sigma^{(n)}_1$-formula. Since W is the standard solidity witness for α with respect to $N'(\mu)$ and $p_{N'(\mu)}$, the premouse $N'(\mu)$ satisfies the $\Pi^{(n)}_2$-statement

$$(\forall \xi_1 \cdots \xi_\ell < \alpha)\Big(\varphi(\xi_1 \cdots \xi_\ell, r') \longleftrightarrow W \models \varphi(\xi_1 \cdots \xi_\ell, r'_\alpha)\Big).$$

Since $\Pi^{(n)}_2$-statements are downward preserved under σ, the same is true of $\bar{\alpha}$, $\bar{r}'$, $\bar{Q}$ and $\bar{r}'_\alpha$ in $N'(\bar{\mu})$ where $\bar{\alpha}$, $\bar{Q}$ and $\bar{r}'_{\bar{\alpha}}$ is the σ-preimage of α, W and p'_α, respectively. Thus, if A is any set of ordinals which is $\Sigma^{(n)}_1(N'(\bar{\mu}))$-definable in the parameter $\bar{r}'$ then $A \cap \bar{\mu}$, being $\Sigma^{(n)}_1(\bar{Q})$-definable in $\bar{r}'_\alpha$, is an element of $N'(\bar{\mu})$. It follows that $\bar{r}' \neq p_{N'(\bar{\mu})}$. By Lemma 2.22, $(\bar{\mu}, q)$ cannot be strong after all, a contradiction.

Now consider the case where N is pluripotent and $\lambda_N(\mu, q) = \lambda_N$. Notice that $\lambda_N(\bar{\mu}, d_N) = \lambda_N$ whenever $(\bar{\mu}, d_N)$ is a divisor of N with $\bar{\mu} \leq \mu$. Assuming that (μ, d_N) is not strong, $E^{N'(\mu)}_{\text{top}} \,|\, \mu = E^{N'(\mu)}_\gamma$ for a suitable γ. Here we simply apply Lemma 2.23. Let $(\gamma', \mu') = \pi'(\gamma, \mu)$ where $\pi' : N'(\mu) \to N$ is the uncollapsing map associated with $N'(\mu)$. Then $E^N_{\text{top}} \,|\, \mu' = E_{\gamma'}$ and $\gamma = h^*_N(\xi, d_N)$ for some $\xi < \mu$. Pick a strong divisor $(\bar{\mu}, d_N)$ such that $\xi < \bar{\mu} < \mu$. Then both γ' and μ' are in the range of $\bar{\pi}' : N'(\bar{\mu}) \to N$, the uncollapsing map associated with $N'(\bar{\mu})$. Since $\bar{\pi}'$ is the identity on μ', necessarily $\mu' = \bar{\pi}'(\hat{\mu})$ for some $\hat{\mu} \geq \bar{\mu}$. So $E^{N'(\bar{\mu})}_{\text{top}} \,|\, \hat{\mu} = E^{N'(\bar{\mu})}_{\hat{\gamma}}$ where $\bar{\pi}'(\hat{\gamma}) = \gamma'$ and, consequently, $E^{N'(\bar{\mu})}_{\text{top}} \,|\, \bar{\mu} \in N'(\bar{\mu})$. Again by Lemma 2.23, $(\bar{\mu}, d_N)$ cannot be strong, which yields the desired contradiction. $\square$(Lemma 2.25)

The next lemma rules out that a strong divisor can be overlapped by *any* other divisor, and thus solves the issue of overlapping discussed in the introduction to this subsection.

Lemma 2.26. *Let (μ, q) be a strong divisor of N and $\bar{q}$ be a* proper *bottom segment of q. Then there is no $\bar{\mu} \leq \mu$ such that $(\bar{\mu}, \bar{q})$ is a divisor of N.*

Proof. Suppose the contrary; let this be witnessed by $(\bar{\mu}, \bar{q})$. Let $\bar{r} = p_N - \bar{q}$; clearly $\bar{r}$ is a proper lengthening of r and $\alpha = \max(q)$ is an element of $\bar{r} - \bar{q}$. If N is pluripotent and $\lambda_N(\mu, q) = \lambda_N$, then γ_N is in the range of $\pi_N(\bar{\mu}, \bar{q})$ by Lemma 2.4, so it must also be in the larger $\mathrm{rng}(\pi')$ where $\pi' : N'(\mu) \to N$ is the associated map. It follows that $N'(\mu)$ is a premouse and, consequently, $E_{\mathrm{top}}^{N'(\mu)} \mid \mu \in N'(\mu)$. By Lemma 2.23, this contradicts the strongness of (μ, q).

Let us now focus on the remaining case where N is either not pluripotent or else $\lambda_N(\mu, q) < \lambda_N$. As in the proofs of the previous lemmata we observe that $\tilde{h}_N^{n+1}(\bar{\mu} \cup \{\bar{r}\}) = \mathrm{rng}(\pi_N(\bar{\mu}, \bar{q}))$ contains a generalized witness $\langle Q, t\rangle$ for α with respect to N and p_N, so $\langle Q, t\rangle$ must be in the range of the larger embedding $\pi' : N'(\mu) \to N$. Notice that $\langle Q, t\rangle$ is a generalized solidity witness for α with respect to N and $r = p_N - q$. Then $\langle Q', t'\rangle \stackrel{\text{def}}{=} (\pi')^{-1}(\langle Q, t\rangle)$ is a generalized witness for $(\pi')^{-1}(\alpha) \geq \mu$ with respect to $N'(\mu)$ and $r' \stackrel{\text{def}}{=} (\pi')^{-1}(r)$. As in the proof of the previous lemma we conclude that $p_{N'(\mu)}$ must be a proper lengthening of r'. Again by Lemma 2.23, this contradicts the strongness of (μ, q). □(Lemma 2.26)

We are now ready to define the notion of canonical protomouse associated with N. By Lemma 2.25, each $\mathcal{D}_q^*(N)$ has a largest element, provided that it is nonempty. The protomouse we are looking for is that associated with the shortest possible q and the maximal element of $\mathcal{D}_q^*(N)$. The previous lemma implies that this divisor is in fact fully determined by the requirement that μ is largest possible: Indeed, if (μ, q) is a strong divisor with the largest possible μ then no divisor $(\mu', q') \neq (\mu, q)$ with $q' \supset q$ and $\mu' \geq \mu$ can be strong. On the other hand, if q' is a proper segment of q, then (μ', q') can only be a divisor if $\mu' > \mu$, and none of such divisors can be strong by our choice of μ. We thus set:

$$\begin{aligned} &q(N) \simeq \text{ the shortest } q \text{ such that } \mathcal{D}_q^*(N) \neq \varnothing \\ &\mu(N) \simeq \max(\mathcal{D}_q^*(N)). \end{aligned} \tag{15}$$

If N is pluripotent and does not admit strong divisors, i.e. $(\mu(N), q(N))$ as in (15) is undefined, we set

$$\begin{aligned} &(\mu(N), q(N)) = (\mu, d_N) \quad \text{where} \quad \mu = \mathrm{cr}(E_{\mathrm{top}}^N) \\ &N(\mu(N), q(N)) = N \end{aligned} \tag{16}$$

In view of the remarks preceding Lemma 2.24, we can consider (μ, d_N) a strong divisor of N.

3. The Square Sequence

With the technical means developed in the previous sections, we are ready to begin with the construction of a $\square_\kappa$-sequence. As it is typical for constructions of square sequences in fine structural models, we shall actually construct a so-called $\square'_\kappa$-sequence indexed by ordinals from a certain c.u.b. subset of κ^+. Such a sequence can easily be turned into a $\square_\kappa$-sequence in the usual sense using merely combinatorial manipulations (see [6]). Recall that we work in $\mathbf{L}[E]$.

Theorem 3.1. *Let $\mathcal{S}$ be a c.u.b. subset of κ^+ such that all $\tau \in \mathcal{S}$ satisfy:*

- *κ is the largest cardinal in J^E_τ.*
- *J^E_τ is a fully elementary substructure of $J^E_{\kappa^+}$.*
- *$E_\tau = \varnothing$.*

Then there is a sequence

$$\mathcal{C} = \langle C_\tau;\, \tau \in \mathcal{S}\rangle$$

satisfying the following requirements:

- *$C_\tau \subset \mathcal{S} \cap \tau$ is closed.*
- *C_τ is unbounded in τ whenever τ is a limit point of $\mathcal{S}$ and $\mathrm{cf}(\tau) > \omega$.*
- *$C_\tau \cap \bar\tau = C_{\bar\tau}$ whenever $\bar\tau \in C_\tau$; this property is called* coherency.
- $\mathrm{otp}(C_\tau) \leq \kappa$

Moreover, each C_τ is uniformly definable over $N_\tau \cup \{N_\tau\}$ where N_τ is the collapsing $\mathbf{L}[E]$-level for τ.

The requirement that $E_\tau = \varnothing$ holds on a c.u.b. set, or equivalently, that only non-stationarily many $\tau < \kappa^+$ can index an extender, constitutes a smallness condition on the extender sequence E, and at the same time an upper bound at which our methods break down. However, $\square_\kappa$ turns out to fail in $\mathbf{L}[E]$ if we move beyond this bound, so our methods are in fact optimal. We shall return to these matters in detail in the last subsection which summarizes the proof of the main theorem; for now just notice that if $E_\tau \neq \varnothing$ for stationarily many $\tau < \kappa^+$, then E_τ is a superstrong extender with $\lambda(E_\tau) = \kappa$.

Defining the Square Sequence. This subsection is devoted to the notation, which is quite massive. We split the set $\mathcal{S}$ into two disjoint parts $\mathcal{S}^0$ and $\mathcal{S}^1$ depending on whether we can imitate Jensen's construction of square in $\mathbf{L}$ in the more general $\mathbf{L}[E]$-context or not. It will turn out that a modification of the same construction that employs protomice in place of $\mathbf{L}[E]$-levels will provide us with a square sequence on $\mathcal{S}^1$.

To each $\tau \in \mathcal{S}$ we assign several auxiliary objects. Our notation builds on that developed in the previous section, recall also the introductory settings there.

- N_τ is the collapsing level for τ.
- $n_\tau = n(N_\tau)$ is the unique n such that $\omega\varrho^{n+1}_{N_\tau} = \kappa < \omega\varrho^n_{N_\tau}$.
- $p_\tau = p_{N_\tau}$.
- If N_τ is pluripotent, then $d_\tau = d_{N_\tau}$.
- $\varrho_\tau = \varrho^{n_\tau}_{N_\tau}$ and $\mathcal{H}_\tau = H^{n_\tau}_{N_\tau}$ (that is, $\mathcal{H}_\tau = H^{N_\tau}(\omega\varrho^{n_\tau}) = |J^E_{\varrho^{n_\tau}}|$).
- $A_\tau = A^{n_\tau, p_\tau \restriction n_\tau}_{N_\tau}$.
- $\tilde{h}_\tau = \tilde{h}^{n_\tau+1}_{N_\tau}$ and $h^\sharp_\tau = h^{n_\tau,p_\tau}_{N_\tau}$.
- If N_τ is pluripotent, $h^*_\tau = h^*_{N_\tau}$.
- For each $k \in \omega$, let $\tilde{\psi}_k(z,y,x,w)$ be a fixed $\Sigma^{(k)}_0$-formula in the language for premice such that

  $$(\exists z^k)\tilde{\psi}_k(z^k, y^0, x^{k+1}, w^0)$$

  constitutes a functionally absolute $\Sigma^{(k)}_1$-definition for the Skolem function $y = \tilde{h}^{k+1}_Q(x,w)$ in any acceptable J-structure Q for this language. Then

$$\tilde{H}_\tau = \{(z,y,x,w) \in H^{n_\tau}_{N_\tau} \times |N_\tau| \times H^{n_\tau+1}_{N_\tau} \times |N_\tau|;\ N_\tau \models \tilde{\psi}_{n_\tau}(z,y,x,w)\}.$$

By our restrictions on τ, the object N_τ is always defined, being the longest initial segment of $\mathbf{L}[E]$ in which τ looks like a successor cardinal.

$$\begin{aligned} \mathcal{S}^0 &= \mathcal{S} - \mathcal{S}^1 \\ \mathcal{S}^1 &= \{\tau \in \mathcal{S};\ (\mu(N_\tau), q(N_\tau)) \text{ is defined}\}; \end{aligned}$$

in other words, $\mathcal{S}^1$ consists of all ordinals τ such that N_τ either admits a strong divisor or is a pluripotent $\mathbf{L}[E]$-level.

Before going further, recall the notation introduced at the end of the subsection on premice and protomice in the previous section. Recall also that if N_τ is pluripotent and does not admit strong divisors then $N(\mathrm{cr}(E^{N_\tau}_{\mathrm{top}}), d_N) = N_\tau$. For $\tau \in \mathcal{S}^1$ we set:

- $q_\tau = q(N_\tau)$, $m_\tau = |q_\tau|$, and $\mu_\tau = \mu(N_\tau)$.
- $d_\tau = d_{N_\tau}$ whenever N_τ is pluripotent.
- $r_\tau = p_\tau - q_\tau$ if N_τ is not pluripotent, and $r_\tau = d_\tau - q_\tau$ if N_τ is pluripotent.
- $M_\tau = N_\tau(\mu_\tau, q_\tau)$.
- $\lambda_\tau = \lambda_{N_\tau}(\mu_\tau, q_\tau)$, $\nu_\tau = \nu_{N_\tau}(\mu_\tau, q_\tau)$ and $\vartheta_\tau = \vartheta_{N_\tau}(\mu_\tau, q_\tau)$.
- $h_\tau = h_{M_\tau}$ and if M_τ is a pluripotent $\mathbf{L}[E]$-level, then $h_\tau = h^*_{M_\tau}$.
- Let $\psi(z, y, x, w)$ be fixed Σ_0-formula in the language for coherent structures such that $(\exists z)\psi(z, y, x, w)$ constitutes functionally absolute Σ_1-definition for the function $y = h_Q(x, w)$ where Q is any acceptable J-structure for this language. Then

$$H_\tau = \{(z, y, x, w) \in |M_\tau| \times |M_\tau| \times H^1_{M_\tau} \times |M_\tau|;\ M_\tau \models \psi(z, y, x, w)\}.$$

We are now ready to give the first approximation to the sequence $\mathcal{C}$. As we have already mentioned, we define $\mathcal{C}$ in such a way that $\mathcal{C} \restriction \mathcal{S}^i$ will be a $\square'_\kappa$-sequence *on* $\mathcal{S}^i$, that is, $C_\tau \subset \mathcal{S}^i$ for any $\tau \in \mathcal{S}^i$ where $i \in \{0, 1\}$. Since $\mathcal{S}^0$ and $\mathcal{S}^1$ partition $\mathcal{S}$, we immediately have that $\mathcal{C}$ is a $\square'_\kappa$-sequence on $\mathcal{S}$. We shall treat the two cases $\tau \in \mathcal{S}^0$ and $\tau \in \mathcal{S}^1$ simultaneously. Let us now define the above mentioned approximations

$$\mathcal{B} = \langle B_\tau;\ \tau \in \mathcal{S}\rangle \quad \text{and} \quad \mathcal{B}^* = \langle B^*_\tau;\ \tau \in \mathcal{S}\rangle.$$

Definition. *Given $\tau \in \mathcal{S}^0$, B_τ is the set of all $\bar\tau \in \mathcal{S}^0 \cap \tau$ satifying:*

- *$N_{\bar\tau}$ is a premouse of the same type as N_τ.*
- *$n_{\bar\tau} = n_\tau$.*
- *There is a map $\sigma_{\bar\tau\tau} : N_{\bar\tau} \to N_\tau$ that is $\Sigma_0^{(n_\tau)}$-preserving with respect to the language for premice and such that*
  - a) *$\bar\tau = \mathrm{cr}(\sigma_{\bar\tau\tau})$ and $\sigma_{\bar\tau\tau}(\bar\tau) = \tau$;*
  - b) *$\sigma_{\bar\tau\tau}(p_{\bar\tau}) = p_\tau$;*
  - c) *to each $\alpha \in p_\tau$ there is a generalized witness $Q_\tau(\alpha)$ for α with respect to N_τ and p_τ such that $Q_\tau(\alpha) \in \mathrm{rng}(\sigma_{\bar\tau\tau})$.*

Solidity witnesses in clause c) are, of course, computed with respect to the language for premice. The map $\sigma_{\bar\tau\tau}$ always has a critical point since, by the restrictions imposed on the elements of $\mathcal{S}$, each N_τ is strictly longer than J^E_τ. Notice that $\sigma_{\bar\tau\tau}$ is uniquely determined; this follows from the soundness of $N_{\bar\tau}$. Given any $x \in N_{\bar\tau}$, there is a $\xi < \kappa$ such that $x = \tilde h_{\bar\tau}(\xi, p_{\bar\tau})$; as $\Sigma_1^{(n_\tau)}$-statements are upward preserved under $\Sigma_0^{(n_\tau)}$-embeddings and Skolem functions $\tilde h_{\bar\tau}$ and $\tilde h_\tau$ have functionally absolute $\Sigma_1^{(n_\tau)}$-definitions, $\sigma(x) = \tilde h_\tau(\xi, p_\tau)$ for any map σ satisfying the requirements of the above definition. Notice also that clause c) in the above definition does not have any influence on the uniqueness of $\sigma_{\bar\tau\tau}$, which will be important in the coming lemmata. One further crucial fact about $\sigma_{\bar\tau\tau}$ is that $\sigma_{\bar\tau\tau}$ is *not* $\Sigma_1^{(n_\tau)}$-preserving, hence

$$\sigma_{\bar\tau\tau} \text{ is non-cofinal at the } n_\tau\text{-th level.} \tag{17}$$

Otherwise $\sigma_{\bar\tau\tau}$ would be $\Sigma_1^{(n_\tau)}$-preserving, so $\tilde h_{\bar\tau}(\xi, p_{\bar\tau})$ would be defined iff $\tilde h_\tau(\xi, p_\tau)$ would. By the soundness of N_τ, we would have $\text{rng}(\sigma_{\bar\tau\tau}) = N_\tau$, and from this the obviously false conslusion $N_{\bar\tau} = N_\tau$. Finally we note that clause c) in the above definition will be used only once in the entire construction, namely in the proof that C_τ is closed, to make sure that certain direct limits are sound. The condition in c) *cannot* be strengthened in the sense that we would require *standard* solidity witnesses to be members of the corresponding ranges; this follows from the fact that we are forced to deal with embeddings with very weak preservation properties which typically do not map standard witnesses to standard witnesses (and similarly in the case of preimages). Thus, the requirement that standard witnesses are in the range of $\sigma_{\bar\tau\tau}$ would damage the coherency of our putative square sequences.

Definition. *Given $\tau \in \mathcal{S}^1$, B_τ is the set of all $\bar\tau \in \mathcal{S}^1 \cap \tau$ satisfying:*

- $(\mu_{\bar\tau}, m_{\bar\tau}) = (\mu_\tau, m_\tau)$.
- *There is a map $\sigma_{\bar\tau\tau} : M_{\bar\tau} \to M_\tau$ that is Σ_0-preserving with respect to the language for coherent structures and such that*
    - a) *$\bar\tau = \text{cr}(\sigma_{\bar\tau\tau})$ and $\sigma_{\bar\tau\tau}(\bar\tau) = \tau$;*
    - b) *$\sigma_{\bar\tau\tau}(q_{\bar\tau}) = q_\tau$;*
    - c) *to each $\alpha \in q_\tau$ there is a generalized witness $Q_\tau(\alpha)$ for α with respect to M_τ and q_τ such that $Q_\tau(\alpha) \in \text{rng}(\sigma_{\bar\tau\tau})$.*

Clause c) is, of course, formulated with respect to the language for coherent structures; in particular, solidity witnesses are computed relative to this language. As before we observe that $\sigma_{\bar\tau\tau}$ is uniquely determined and clause c) in the above definition does not have any influence on its uniqueness. Again, $\sigma_{\bar\tau\tau}$ is *not* Σ_1-preserving. In particular,

$$\sigma_{\bar\tau\tau} \text{ is non-cofinal.} \tag{18}$$

That M_τ is strictly longer than J^E_τ follows from the fact that $\lambda_\tau > \tau$. So again we have $\bar\tau = \mathrm{cr}(\sigma_{\bar\tau\tau})$.

From now on we shall work with a fixed τ. Set

- $n = n_\tau$.

Lemma 3.2. *Let $\tau \in \mathcal{S}^i$ for $i \in \{0,1\}$ and let $\tau^* < \bar\tau$ satisfy all requirements of the definition of B_τ except possibly clause c). Then $\mathrm{rng}(\sigma_{\tau^*\tau}) \subset \mathrm{rng}(\sigma_{\bar\tau\tau})$.*

Proof. Assume w.l.o.g. that $\tau \in \mathcal{S}^0$; the proof for $\tau \in \mathcal{S}^1$ is similar (and, in fact, much simpler). The first step is to show

$$\sup((\sigma_{\tau^*\tau})''\omega\varrho_{\tau^*}) < \sup((\sigma_{\bar\tau\tau})''\omega\varrho_{\bar\tau}). \tag{19}$$

Suppose this is false. Pick a $\xi < \kappa$ such that $\tilde h_{\bar\tau}(\xi, p_{\bar\tau}) = \tau^*$; this is witnessed by some $z \in \mathcal{H}_{\bar\tau}$ in the sense that $\tilde H_{\bar\tau}(z, \tau^*, \xi, p_{\bar\tau})$ holds. By applying $\sigma_{\bar\tau\tau}$ we obtain $\tilde H_\tau(\sigma_{\bar\tau\tau}(z), \tau^*, \xi, p_\tau)$. Choose a $\bar\zeta < \omega\varrho_{\bar\tau}$ such that $z \in S^E_{\bar\zeta}$ where $\zeta = \sigma_{\bar\tau\tau}(\bar\zeta)$. By the failure of (19), there is a $\zeta^* < \omega\varrho_{\tau^*}$ such that $\zeta \le \zeta' \stackrel{\text{def}}{=} \sigma_{\tau^*,\tau}(\zeta^*)$. Then the statement $(\exists u^n \in S^E_{\zeta'})(\exists\delta^n < \tau)\tilde H_\tau(u^n, \delta^n, \xi, p_\tau)$ holds in N_τ and, being a $\Sigma^{(n)}_0$-statement, can be pulled back by $\sigma_{\tau^*\tau}$. It follows that $\tilde\tau = \tilde h_{\tau^*}(\xi, p_{\tau^*})$ is defined and $\sigma_{\tau^*\tau}(\tilde\tau) = \tilde h_{\bar\tau}(\xi, p_{\bar\tau}) = \tau^*$, which contradicts the fact that $\tau^* = \mathrm{cr}(\sigma_{\tau^*\tau})$. This proves (19).

With (19) at hand, the same argument as above, but with the roles of $\bar\tau$ and τ^* swapped, yields the conclusion

$$\tilde h_{\tau^*}(\xi, p_{\tau^*}) \text{ is defined} \longrightarrow \tilde h_{\bar\tau}(\xi, p_{\bar\tau}) \text{ is defined} \tag{20}$$

whenever $\xi < \kappa$. It follows that $\sigma_{\tau^*\tau}(\tilde h_{\tau^*}(\xi, p_{\tau^*})) = \tilde h_\tau(\xi, p_\tau) = \sigma_{\bar\tau\tau}(\tilde h_{\bar\tau})(\xi, p_{\bar\tau})$ whenever $\tilde h_{\tau^*}(\xi, p_{\tau^*})$ is defined. Since N_{τ^*} is sound, this completes the proof of the lemma.

Regarding the proof of (20), notice that this time we have to consider also those points in $\mathrm{rng}(\sigma_{\tau^*\tau})$ which are *not* elements of the n-th reduct. To make sure that "pulling back" in the above argument can be applied here, we need to know that the relation $(\exists x^0)(x^0 = \tilde{h}_\tau(\xi, p_\tau))$ is uniformly $\Sigma_1^{(n)}$, i.e. it can be expressed in the form $(\exists u^n) H'_\tau(u^n, \xi, p_\tau)$ where H'_τ is uniformly $\Sigma_0^{(n)}$. This is a consequence of the fact that $\tilde{h}_\tau$ is a good $\Sigma_1^{(n)}$-function, so the substitution of $\tilde{h}_\tau$ for v^0 in the relation defined by $\Sigma_1^{(0)}$-formula $(\exists x^0)(x^0 = v^0)$ yields the required result. To each $\xi < \kappa$, if z witnesses the left side of (20), i.e. if $H'_{\tau^*}(z, \xi, p_{\tau^*})$, then $H'_\tau(\sigma_{\tau^*,\tau}(z), \xi, p_\tau)$ and we can bound $\sigma_{\tau^*,\tau}(z)$ by some $\sigma_{\bar\tau\tau}(S^E_{\bar\zeta})$ where $\bar\zeta < \omega\varrho_{\bar\tau}$. Pulling back by $\sigma_{\bar\tau\tau}$ then yields $(\exists u^n \in S^E_{\bar\zeta}) H'_{\bar\tau}(u^n, \xi, p_{\bar\tau})$, and thus the right side of (20). □(Lemma 3.2)

Lemma 3.3. *Let $\tau \in \mathcal{S}^i$ for $i \in \{0,1\}$ and $\bar\tau \in B_\tau$. Then $B_\tau \cap \bar\tau = B_{\bar\tau} - \min(B_\tau)$.*

Proof. Suppose w.l.o.g. that $\tau \in \mathcal{S}^0$. Pick a $\tau^* \in B_\tau \cap \bar\tau$. We first show that $\tau^* \in B_{\bar\tau}$. By the previous lemma, $\mathrm{rng}(\sigma_{\tau^*\tau}) \subset \mathrm{rng}(\sigma_{\bar\tau\tau})$, so we can define a map $\sigma : N_{\tau^*} \to N_{\bar\tau}$ by $\sigma = (\sigma_{\bar\tau\tau})^{-1} \circ \sigma_{\tau^*\tau}$. It is a routine to verify that σ satisfies all requirements imposed on $\sigma_{\tau^*\bar\tau}$ except possibly clause c), which we verify now. Given $\bar\alpha \in p_{\bar\tau}$, let $\alpha = \sigma_{\bar\tau\tau}(\bar\alpha)$ and $\sigma_{\tau^*\bar\tau}(\alpha^*) = \bar\alpha$. The definition of B_τ guarantees that we have a generalized witness $Q_\tau(\alpha)$ for α with respect to N_τ and p_τ in the range of $\sigma_{\tau^*\tau}$, and we know that the $\sigma_{\bar\tau,\tau}$-preimage $\bar{Q}(\bar\alpha)$ of $Q_\tau(\alpha)$ is in the range of σ. But "Q is a generalited witness for α with respect to N and p" is a $\Pi_1^{(n)}$-statement, so it is downward preserved under $\Sigma_0^{(n)}$-maps. It follows that $\bar{Q}(\bar\alpha)$ is a generalized witness for $\bar\alpha$ with respect to $N_{\bar\tau}$ and $p_{\bar\tau}$. This proves that $\tau^* \in B_{\bar\tau}$ and thus the inclusion $\subset$.

Let $\tau' = \min(B_\tau)$. Pick a $\tau^* \in B_{\bar\tau} - \tau'$ that is larger than τ'. Define an embedding $\sigma : N_{\tau^*} \to N_\tau$ by $\sigma = \sigma_{\bar\tau\tau} \circ \sigma_{\tau^*\bar\tau}$. Again, σ meets all requirements in the above definition except possibly clause c), which suffices to conclude that $\sigma = \sigma_{\tau^*\tau}$. Regarding c), if $Q(\alpha) \in \mathrm{rng}(\sigma_{\tau'\tau})$ is a generalized witness for $\alpha \in p_\tau$ with respect to N_τ and p_τ, then $Q(\alpha)$ is in the range of $\sigma_{\tau^*\tau}$ by the previous lemma, so $\tau^* \in B_\tau$. This proves the inclusion $\supset$. □(Lemma 3.3)

By the previous lemma, the sequence $\mathcal{B}$ is almost coherent; the only deficiency of $\mathcal{B}$ is that the initial segments of B_τ might grow as τ decreases. This can be fixed by adding all potential initial segments to each B_τ. For $\tau \in \mathcal{S}$ we set

- $\tau(0) = \tau$;
- $\tau(i+1) \simeq \min(B_{\tau(i)})$;
- $\ell_\tau =$ the least i such that $B_{\tau(i)} = \varnothing$.

The number ℓ_τ is defined for every $\tau \in \mathcal{S}$, otherwise $\langle \tau_i \rangle_i$ would constitute an infinite decreasing sequence of ordinals. We are now ready to define a fully coherent sequence $\mathcal{B}^*$. Given any $\tau \in \mathcal{S}$,

- $B^*_\tau = B_{\tau(0)} \cup \cdots \cup B_{\tau(\ell_\tau - 1)}$
- $\sigma^*_{\bar\tau\tau} = \sigma_{\tau(1)\tau(0)} \circ \cdots \circ \sigma_{\tau(j)\tau(j-1)} \circ \sigma_{\bar\tau\tau(j)}$ whenever $\bar\tau \in B^*_\tau$ and j is such that $\bar\tau \in B_{\tau(j)}$.

Lemma 3.4. *$\mathcal{B}^*$ is a coherent sequence.*

Proof. Pick a $\tau \in \mathcal{S}$ and a $\bar\tau \in B^*_\tau$. Assume w.l.o.g. that $\bar\tau > \tau(\ell_\tau - 1) = \min(B^*_\tau)$. We first observe

$$\min(B_{\bar\tau}) \in B^*_\tau \quad \text{and} \quad B_{\bar\tau} = B^*_\tau \cap [\min(B_{\bar\tau}), \bar\tau) \tag{21}$$

This follows from Lemma 3.3. Let j be such that $\bar\tau \in B_{\tau(j)}$. Then $\tau(j+1) \in B_{\bar\tau}$, and either $\tau(j+1) = \min(B_{\bar\tau})$ or else $B_{\bar\tau} \cap \tau(j+1)$ is an end-tail of $B_{\tau(j+1)} \subset B^*_\tau$. This proves the first part of (21). To see the second part, we observe that $B_{\bar\tau}$ agrees with $B_{\tau(j)}$ on $[\tau(j+1), \bar\tau)$ and $B_{\bar\tau}$ agrees with $B_{\tau(j+1)}$ on $[\min(B_{\bar\tau}), \tau(j+1))$.

Define $\bar\tau(i)$ from $\bar\tau$ the same way $\tau(i)$ was defined from τ. Let $\bar\ell = \ell_{\bar\tau}$. Using (21) we inductively show that $B_{\bar\tau(i)}$ is a segment of B^*_τ for all $i < \bar\ell$. It follows that $B^*_{\bar\tau}$ is a (not necessarily initial) segment of B^*_τ. But $B^*_{\bar\tau}$ must be in fact an initial segment of B^*_τ; otherwise $\bar\tau(\bar\ell - 1) > \tau(\ell_\tau - 1)$ which would mean that $B_{\bar\tau(\bar\ell-1)}$ is nonempty. Contradiction. $\square$(Lemma 3.4)

Since each $\sigma^*_{\bar\tau\tau}$ is the unique $\Sigma^{(n)}_0$-preserving map from $N_{\bar\tau}$ to N_τ with critical point $\bar\tau$ sending $p_{\bar\tau}$ to p_τ, Lemma 3.2 guarantees that $\mathrm{rng}(\sigma^*_{\tau^*\tau}) \subset \mathrm{rng}(\sigma^*_{\bar\tau\tau})$ whenever $\tau^* < \bar\tau$ are in $\mathcal{S}^0$. On $\mathcal{S}^1$, the situation is analogous. Thus, we can define $\sigma^*_{\tau^*\bar\tau}$

by $\sigma^*_{\tau^*\bar\tau} = (\sigma^*_{\bar\tau\tau})^{-1} \circ \sigma^*_{\tau^*\tau}$ for *any* $\tau^* \leq \bar\tau$ from $B^*_\tau \cup \{\tau\}$. It follows immediately that

(22)
$\sigma^*_{\tau^*\bar\tau} : N_{\tau^*} \to N_{\bar\tau}$ is the unique map that is $\Sigma^{(n)}_0$-preserving with respect to the language for premice, has critical point τ^*, and sends τ^* to $\bar\tau$ and p_{τ^*} to $p_{\bar\tau}$ whenever $\tau \in \mathcal{S}^0$ and $\tau^* \leq \bar\tau$ are in $B^*_\tau \cup \{\tau\}$
and
$\sigma^*_{\tau^*\bar\tau} : M_{\tau^*} \to M_{\bar\tau}$ is the unique map that is Σ_0-preserving with respect to the language for coherent structures, has critical point τ^*, and sends τ^* to $\bar\tau$ and q_{τ^*} to $q_{\bar\tau}$ whenever $\tau \in \mathcal{S}^1$ and $\tau^* \leq \bar\tau$ are in $B^*_\tau \cup \{\tau\}$.

To complete the definition of the square sequence, we will make use of the following crucial lemma, whose proof will constitute the rest of this section.

Lemma 3.5. *For every $\tau \in \mathcal{S}$ of uncountable cofinality, B_τ is a c.u.b. subset of τ on a tail-end. In other words, there is a $\bar\tau < \tau$ such that $B_\tau - \bar\tau$ is closed and unbounded in τ.*

To each $\tau \in \mathcal{S}$, let β_τ be the least $\beta \in B^*_\tau \cup \{\tau\}$ such that $B^*_\tau - \beta$ is closed in τ. The ordinal β_τ is always defined and if $\mathrm{cf}(\tau) > \omega$, then $\beta_\tau < \tau$, as follows from the previous lemma and the fact that B_τ is a tail-end of B^*_τ. Set

$$C^*_\tau = B^*_\tau - \beta_\tau \qquad \text{and} \qquad \mathcal{C}^* = \langle C^*_\tau;\, \tau \in \mathcal{S}\rangle$$

Given $\bar\tau \in C^*_\tau$, we know that $\bar\tau \in B^*_\tau$, $\bar\tau \geq \beta_\tau$ and that $B^*_\tau, B^*_{\bar\tau}$ cohere. It follows that $\beta_{\bar\tau} = \beta_\tau$. Consequently, $C^*_{\bar\tau} = B^*_{\bar\tau} - \beta_{\bar\tau} = B^*_\tau \cap \bar\tau - \beta_\tau = C^*_\tau \cap \bar\tau$. The sequence $\mathcal{C}^*$ thus satisfies all requirements of a $\square'_\kappa$-sequence with the only exception that the sets C^*_τ might have large order type.

We now observe that for $\tau \in \mathcal{S}^1$, the order type of C^*_τ is always small. To each $\tau^* < \bar\tau$ from C^*_τ we have the map $\sigma^*_{\tau^*\bar\tau} : M_{\tau^*} \to M_{\bar\tau}$ satisfying (22), so $\mathrm{dom}(F_{\tau^*}) \subset \mathrm{dom}(F_{\bar\tau})$ and this inclusion is strict, as $\sigma^*_{\tau^*\bar\tau}$ is non-cofinal. Hence $\vartheta_{\tau^*} < \vartheta_{\bar\tau}$. It follows that $\vartheta_{\bar\tau} \mapsto \bar\tau$ is a strictly monotone enumeration of C^*_τ with domain contained in ϑ_τ. But $\vartheta_\tau \leq \mu^+ \leq \kappa$ since $\mu_\tau < \kappa$, so $\mathrm{otp}(C^*_\tau) \leq \mathrm{otp}(\vartheta_\tau) \leq \kappa$.

The above discussion shows that for $\tau \in \mathcal{S}^1$, our construction already yields a $\square'_\kappa$-sequence. For $\tau \in \mathcal{S}^0$ this is not obvious and to arrange that the order types are small, we will replace the sets C^*_τ by suitably chosen subsets.

Let $X_\tau(\xi)$ be the $\Sigma^{(n)}_1$-hull of $\{\xi, p_\tau\}$ in N_τ. We define sequences $\langle \tau_\iota\rangle, \langle \xi^\tau_\iota\rangle$ in the following way:

- $\tau_0 = \min(C^*_\tau \cup \{\tau\})$
- $\xi^\tau_\iota \simeq$ the least $\xi < \kappa$ such that $X_\tau(\xi)$ is not contained in $\mathrm{rng}(\sigma^*_{\tau_\iota \tau})$
- $\tau_{\iota+1} =$ the least $\bar\tau \in C^*_\tau \cup \{\tau\}$ such that $X_\tau(\xi^\tau_\iota)$ is contained in $\mathrm{rng}(\sigma^*_{\bar\tau\tau})$
- $\tau_\gamma = \sup\{\tau_\iota;\, \iota < \gamma\}$ for limit γ
- $\iota_\tau =$ the least ι such that $\tau_\iota = \tau$

If $\tau_\iota < \tau$, then ξ^τ_ι is always defined; just choose a $\xi < \kappa$ such that $\tau_\iota = \tilde h_\tau(\xi, p_\tau)$ and observe that $X_\tau(\xi)$ is not contained in $\mathrm{rng}(\sigma^*_{\tau_\iota\tau})$, as $\tau_\iota = \mathrm{cr}(\sigma^*_{\tau_\iota\tau})$. Set

$$C'_\tau = \{\tau_\iota;\, \iota < \iota_\tau\}$$

Lemma 3.6. *$\langle C'_\tau;\, \tau \in \mathcal{S}^0\rangle$ is a $\square'_\kappa$-sequence on $\mathcal{S}^0$.*

Proof. Given $\iota < \iota_\tau$, the above definition immediately yields that $\tau_{\iota+1} > \tau_\iota$. It follows that $\langle \tau_\iota;\, \iota < \iota_\tau\rangle$ is a normal sequence, hence C'_τ is closed. Furthermore, if τ is not ω-cofinal, C'_τ is unbounded since each $X_\tau(\xi^\tau_\iota)$ is countable and therefore contained in $\mathrm{rng}(\sigma^*_{\bar\tau\tau})$ for sufficiently large $\bar\tau \in C^*_\tau$ (recall that C^*_τ is c.u.b. in τ in this case). We next observe that if $\bar\iota < \iota$, then $X_\tau(\xi^\tau_\iota)$ is not contained in $\mathrm{rng}(\sigma^*_{\tau_{\bar\iota}\tau})$, as $\mathrm{rng}(\sigma^*_{\tau_{\bar\iota}\tau}) \subset \mathrm{rng}(\sigma^*_{\tau_\iota\tau})$. So $\xi^\tau_\iota < \xi^\tau_{\bar\iota}$ would contradict the minimality of $\xi^\tau_{\bar\iota}$. Furthermore, since $X_\tau(\xi^\tau_{\bar\iota}) \subset \mathrm{rng}(\sigma^*_{\tau_{\bar\iota+1}\tau}) \subset \mathrm{rng}(\sigma^*_{\tau_\iota\tau})$, the ordinals ξ^τ_ι and $\xi^\tau_{\bar\iota}$ must be distinct. This proves that $\langle \xi^\tau_\iota;\, \iota < \iota_\tau\rangle$ is a *strictly* increasing sequence of ordinals smaller than κ. As an immediate consequence we have $\iota_\tau \le \kappa$ and that $\iota \mapsto \tau_\iota$ is a strictly monotone enumeration of C'_τ. So $\mathrm{otp}(C'_\tau) \le \kappa$. It only remains to prove that the sequence $\langle C'_\tau;\, \tau \in \mathcal{S}^0\rangle$ is coherent.

Pick a $\bar\tau \in C'_\tau$. Assume that $\bar\tau > \min(C'_\tau)$; otherwise there is nothing to prove. Since $\bar\tau \in C^*_\tau$, we know that $C^*_{\bar\tau} = C^*_\tau \cap \bar\tau$. By induction on ι we show that $\bar\tau_\iota \simeq \tau_\iota$ whenever $\iota < \iota_{\bar\tau}$. For $\iota = 0$ this follows immediately and the same applies to limit ι. It remains to prove that $\bar\tau_{\iota+1} \simeq \tau_{\iota+1}$, granted that this equality holds with ι in place of $\iota + 1$. Here we utilize the following fact.

$$\text{(23)} \qquad \text{If } X_\tau(\xi) \subset \mathrm{rng}(\sigma^*_{\bar\tau\tau}), \text{ then } X_\tau(\xi) = (\sigma^*_{\bar\tau\tau})''X_{\bar\tau}(\xi).$$

The inclusion $\supset$ in the conclusion of (23) follows immediately, as $\Sigma^{(n)}_1$-statements are upward preserved under $\sigma^*_{\bar\tau,\tau}$. Now suppose $y \in X_\tau(\xi)$. This means that there is an $i \in \omega$ and $\zeta < \omega\varrho_\tau$ such that $(\exists u^n \in S^E_\zeta)\tilde H_\tau(u^n, y, \langle i, \xi\rangle, p_\tau)$; take ζ to be the least such. Then ζ is uniquely characterized by the following $\Sigma_0^{(n)}$-statement

(24)
$$(\exists u^n \in S^E_\zeta)\tilde{H}_\tau(u^n, y, \langle i,\xi\rangle, p_\tau) \;\&\; (\forall v^n \in S^E_\zeta)(\forall u^n \in v^n)\neg\tilde{H}_\tau(u^n, y, \langle i,\xi\rangle, p_\tau),$$

Since $y = \tilde{h}_\tau(\langle i,\xi\rangle, p_\tau)$ and $\tilde{h}$ is a good $\Sigma_1^{(n)}$-function, substituting $\tilde{h}_\tau(\langle i,\xi\rangle, p_\tau)$ for y in (24) yields that ζ is $\Sigma_1^{(n)}$-definable over N_τ from ξ and p_τ, so $\zeta \in X_\tau(\xi)$. By our assumption that $X_\tau(\xi) \subset \mathrm{rng}(\sigma^*_{\bar\tau,\tau})$, there are $\bar y, \bar\zeta \in N_{\bar\tau}$ such that $y = \sigma^*_{\bar\tau\tau}(\bar y)$ and $\zeta = \sigma^*_{\bar\tau\tau}(\bar\zeta)$. Then (24) holds in $N_{\bar\tau}$ with $\bar\tau, \bar\zeta, \bar y$ and $p_{\bar\tau}$ in place of τ, ζ, y and p_τ; this witnesses that $(\exists u^n)\tilde{H}_{\bar\tau}(u^n, \bar y, \langle i,\xi\rangle, p_{\bar\tau})$, i.e. that $\bar y = \tilde{h}_{\bar\tau}(\langle i,\xi\rangle, p_{\bar\tau}) \in X_{\bar\tau}(\xi)$. This proves (23).

Now suppose $\tau_\iota = \bar\tau_\iota < \bar\tau$. Recall that $\sigma^*_{\tau'\tau} = \sigma^*_{\bar\tau\tau} \circ \sigma^*_{\tau'\bar\tau}$ for any $\tau' < \bar\tau$. By the easy part of (23), for any $\xi < \kappa$ we have

$$X_\tau(\xi) \subset \mathrm{rng}(\sigma^*_{\tau_\iota\tau}) \longrightarrow X_{\bar\tau}(\xi) \subset \mathrm{rng}(\sigma^*_{\tau_\iota\bar\tau}).$$

It follows that $\xi^{\bar\tau}_\iota \geq \xi^\tau_\iota$. Since we are assuming that $\bar\tau \in C'_\tau$ and $\tau_\iota < \bar\tau$, we have $\tau_{\iota+1} \leq \bar\tau$, so $X_\tau(\xi^\tau_\iota) \subset \mathrm{rng}(\sigma^*_{\bar\tau\tau})$. Using the full strength of (23) we obtain

$$X_\tau(\xi^\tau_\iota) = (\sigma^*_{\bar\tau\tau})'' X_{\bar\tau}(\xi^\tau_\iota), \tag{25}$$

so $X_{\bar\tau}(\xi^\tau_\iota) \not\subset \mathrm{rng}(\sigma^*_{\bar\tau_\iota,\bar\tau})$. It follows that $\xi^{\bar\tau}_\iota = \xi^\tau_\iota$. Letting ξ_ι be this common value, (25) guarantees that for every $\tau' \in [\bar\tau_\iota, \bar\tau] \cap C^*_\tau$,

$$X_\tau(\xi_\iota) \subset \mathrm{rng}(\sigma^*_{\tau'\tau}) \longleftrightarrow X_{\bar\tau}(\xi_\iota) \subset \mathrm{rng}(\sigma^*_{\tau'\bar\tau}),$$

which in turn implies that $\bar\tau_{\iota+1} = \tau_{\iota+1}$. □(Lemma 3.6)

It is now obvious that if we define $\mathcal{C} = \langle C_\tau ; \tau \in \mathcal{S}\rangle$ by

$$C_\tau = \begin{cases} C'_\tau & \text{if } \tau \in \mathcal{S}^0 \\ C^*_\tau & \text{if } \tau \in \mathcal{S}^1, \end{cases}$$

then $\mathcal{C}$ is a $\square'_\kappa$-sequence on $\mathcal{S}$. To complete the construction, we have to give the proof of Lemma 3.5. The rest of this section is devoted to this task.

When $\tau \in \mathcal{S}^0$. Let $\tau \in \mathcal{S}^0$ be a limit point of $\mathcal{S}$ with uncountable cofinality. We first define an approximation D to B_τ. The set D is the set of all $\bar\tau \in \mathcal{S} \cap \tau$ satifying:

- $N_{\bar\tau}$ is a premouse of the same type as N_τ.
- $n_{\bar\tau} = n_\tau$.

- There is a map $\sigma_{\bar\tau\tau}: N_{\bar\tau} \to N_\tau$ which is $\Sigma_0^{(n_\tau)}$-preserving with respect to the language for premice and such that
  a) $\bar\tau = \mathrm{cr}(\sigma_{\bar\tau\tau})$ and $\sigma_{\bar\tau\tau}(\bar\tau) = \tau$;
  b) $\sigma_{\bar\tau\tau}(p_{\bar\tau}) = p_\tau$;
  c) to each $\alpha \in p_\tau$ there is a generalized witness $Q_\tau(\alpha)$ for α with respect to N_τ and p_τ such that $Q_\tau(\alpha) \in \mathrm{rng}(\sigma_{\bar\tau\tau})$.

Generalized witnesses in clause c) are, of course, computed in the language for premice. The only difference between D and B_τ is that we also allow ordinals from $\mathcal{S}^1$ to be elements of D. Later we prove that there are only boundedly many such ordinals in τ. Obviously, $B_\tau \subset D$.

Lemma 3.7. *D is unbounded in τ.*

Proof. Given a $\tau' < \tau$, we shall find a $\tilde\tau \geq \tau'$ in D. Form the elementary hull of $\{N_\tau, \tau', \mathcal{S}\}$ in $J^E_{\kappa^{++}}$. Let H be the transitive collapse of this hull and let $\sigma_0 : H \to J^E_{\kappa^{++}}$ be the associated uncollapsing map. Notice that both κ and κ^+ are in $\mathrm{rng}(\sigma_0)$. We set:

- $\bar N, \bar\kappa, \bar\tau, \bar{\mathcal{S}} = \sigma_0^{-1}(N_\tau, \kappa, \tau, \mathcal{S})$;
- $\sigma = \sigma_0 \restriction \bar N : \bar N \to N_\tau$;
- $\tilde\tau = \sup(\sigma''\bar\tau)$.

As $\mathcal{S}$ is a c.u.b. subset of τ, its σ_0-preimage $\bar{\mathcal{S}}$ is c.u.b. in $\bar\tau$. It follows that $\sigma''\bar{\mathcal{S}}$ is cofinal in $\tilde\tau$, so $\tilde\tau$ is a limit point of $\mathcal{S}$. Thus, $\tilde\tau \in \mathcal{S}$. Notice also that $\tilde\tau < \tau$, as $\mathrm{rng}(\sigma)$ is countable, and that $\tau' < \tilde\tau$, as $\tau' \in \mathrm{rng}(\sigma)$. Our aim is to show that $\tilde\tau \in D$.

Since N_τ is in the range of σ_0, all first order properties of N_τ are downward preserved under σ_0. In particular, $\bar N$ is premouse of the same type as N_τ. Recall that N_τ is not pluripotent, as $\tau \in \mathcal{S}^0$ and $(\mu(N_\tau), q(N_\tau))$ is always defined for pluripotent N_τ. Thus, either $\mathrm{cr}(E^{\bar N}_{\mathrm{top}}) \geq \bar\kappa$ or else $\omega\varrho^1_{\bar N} > \bar\kappa$. Working in the language for premice, the following is standard to verify:

- $\bar\tau = \bar\kappa^{+\bar N}$;
- $\omega\varrho^\omega_{\bar N} = \omega\varrho^{n+1}_{\bar N} = \bar\kappa < \omega\varrho^n_{\bar N}$ where $n = n_\tau$;
- $\sigma(p_{\bar N}) = p_\tau$;
- $\bar N$ is sound and solid;
- $\sigma(W^{\bar\alpha, p_{\bar N}}_{\bar N}) = W^{\alpha, p_\tau}_{N_\tau}$ whenever $\bar\alpha \in p_{\bar N}$ and $\alpha = \sigma(\bar\alpha)$.

By the interpolation lemma (Lemma 1.2), the canonical extension $\tilde{\sigma} : \bar{N} \to \tilde{N}$ of $\sigma \restriction J^{E^{\bar{N}}}_{\bar{\tau}} : J^{E^{\bar{N}}}_{\bar{\tau}} \to J^{E}_{\tilde{\tau}}$ exists, and is $\Sigma_0^{(n)}$-preserving and cofinal. Also, there is a $\Sigma_0^{(n)}$-preserving embedding $\sigma' : \tilde{N} \to N_\tau$ satisfying $\sigma' \restriction \tilde{\tau} = \mathrm{id}$, $\sigma'(\tilde{\tau}) = \tau$ and $\sigma' \circ \tilde{\sigma} = \sigma$. Notice also that $\tilde{\tau} = \kappa^{+\tilde{N}} = \tilde{\sigma}(\bar{\tau})$. Our aim is to show that $\tilde{N} = N_{\tilde{\tau}}$, $n_{\tilde{\tau}} = n$ and $\sigma' = \sigma_{\tilde{\tau}\tau}$, which will prove the lemma. At this point we are only able to state these preservation properties with respect to the language for coherent structures, since it is not clear that $\tilde{N}$ is a premouse of the same type as N_τ and $\bar{N}$, and that the constant sympbol $\dot{\gamma}$ is interpreted in $\tilde{N}$ in the correct way.

We would like to apply the condensation lemma to the embedding $\sigma' : \tilde{N} \to N_\tau$, so let us verify that this is possible. We first show that $\tilde{N}$ is a premouse of the same type as $\bar{N}$ and N_τ. This will also take care of the first clause in the definition of D. Assume that N_τ is active, as otherwise there is nothing to prove. To see that $\tilde{N}$ is a potential premouse, we have to check the weak amenability of the top extender, as the rest is automatically preserved. Notice that either $n > 0$, in which case $\tilde{\sigma}$ is Σ_2-preserving, or else $n = 0$, in which case $\tilde{\sigma}$ is Σ_0-preserving and cofinal and, being a coarse pseudoultrapower embedding, maps successor cardinals of $\bar{N}$ cofinally into their images. In the former case, the potential premousehood of $\bar{N}$ is automatically upward preserved under σ. In the latter case we have $\bar{\mu} \stackrel{\text{def}}{=} \mathrm{cr}(E^{\bar{N}}_{\text{top}}) \geq \bar{\kappa}$ (recall that this follows from our assumption that N_τ is not pluripotent), so $\bar{\vartheta} \stackrel{\text{def}}{=} \bar{\mu}^{+\bar{N}}$ is mapped cofinally into its image $(\tilde{\sigma}(\bar{\mu}))^{+\tilde{N}}$. It follows that $E^{\tilde{N}}_{\text{top}}$ measures all subset of $\mathrm{cr}(E^{\tilde{N}}_{\text{top}})$ in $\tilde{N}$. Now if $\bar{N}$ is of type A or B then $\tilde{N}$ is of the same type by the preservation properties of $\tilde{\sigma}$ and Lemma 1.5. If $\bar{N}$ is of type C then $n > 0$, again by our assumption that N_τ is not pluripotent. Then either $n > 1$, in which case $\tilde{\sigma}$ is $\Sigma_2^{(1)}$-preserving, or else $n = 1$, in which case $\tilde{\sigma}$ is $\Sigma_0^{(1)}$-preserving and cofinal, and so $Q^{(1)}$-preserving. In either case, $\tilde{N}$ is a type C premouse by b) of the same lemma. It follows that the preservation properties of the maps $\tilde{\sigma}$ and σ' hold with respect to the language for premice.

Form now on work in the language for premice. As a next step, we have to verify that the fine structural requirements are met. More precisely,

$$\omega\varrho^{\omega}_{\tilde{N}} = \omega\varrho^{n+1}_{\tilde{N}} \leq \kappa \text{ and } \tilde{N} \text{ is sound.}$$

Let $\tilde{p} = \tilde{\sigma}(p_{\bar{N}})$. By the properties of pseudoultrapowers, every $a \in \tilde{N}$ is of the form $\tilde{\sigma}(f)(\xi)$ where $\xi < \kappa$ and f is a good $\Sigma_1^{(n-1)}(\bar{N})$-function in some $\bar{q} \in \bar{N}$. Fix a functionally absolute $\Sigma_1^{(n-1)}$-definition $\varphi(u,v,z)$ for f, i.e. φ is such that $y = f(x)$ iff $\bar{N} \models \varphi(x, \bar{q}, y)$. Setting $\tilde{q} = \tilde{\sigma}(\bar{q})$, $\varphi(x, \tilde{q}, y)$ defines $\tilde{\sigma}(f)$ over $\tilde{N}$. Pick an $\eta < \bar{\kappa}$ such that $\bar{q} = \tilde{h}_{\bar{N}}^{n+1}(\eta, p_{\bar{N}})$; this is possible, as $\bar{N}$ is sound and $\omega\varrho_{\bar{N}}^{n+1} = \bar{\kappa}$. From the fact that this identity is upward preserved under $\tilde{\sigma}$ and that substituing $\tilde{h}_{\tilde{N}}^{n+1}$ into φ results in a $\Sigma_1^{(n)}$-statement (recall that $\tilde{h}_{\bar{N}}^{n+1}$ is a *good* $\Sigma_1^{(n)}(\bar{N})$-function) we infer that a is the unique object z satisfying

$$\tilde{N} \models \varphi(\xi, \tilde{h}_{\tilde{N}}^{n+1}(\tilde{\sigma}(\eta), \tilde{p}, z),$$

so a is $\Sigma_1^{(n)}(\tilde{N})$-definable in the parameter $\tilde{p}$ and ordinals $\xi, \tilde{\sigma}(\eta) < \kappa$. This proves that $\tilde{h}_{\tilde{N}}^{n+1}(\kappa \cup \{\tilde{p}\}) = \tilde{N}$. As an immediate consequence we have $\omega\varrho_{\tilde{N}}^{n+1} \leq \kappa$. In fact, $\omega\varrho_{\tilde{N}}^{n+1} = \kappa$ since κ is a cardinal in $\mathbf{L}[E]$ and our construction takes place there. Thus, $\tilde{p} \in R_{\tilde{N}}^{n+1}$. To see that $\tilde{N}$ is sound, it suffices to prove that $\tilde{p} = p_{\tilde{N}}$. By our construction, $\sigma'(\tilde{p}) = p_\tau$ and $W_{N_\tau}^{\alpha, p_\tau} \in \operatorname{rng}(\sigma) \subset \operatorname{rng}(\sigma')$ whenever $\alpha \in p_\tau$, so for every $\tilde{\alpha} \in \tilde{p}$, $W_{N_\tau}^{\alpha, p_\tau} \in \operatorname{rng}(\sigma')$ is a generalized witness for $\alpha = \sigma'(\tilde{\alpha})$ with respect to N_τ and p_τ. Recall that the property of being such a witness is $\Pi_1^{(n)}$ whenever α is not smaller than the $(n+1)$-st projectum, and therefore is downward preserved under σ'. So $\sigma'^{-1}(W_{N_\tau}^{\alpha, p_\tau})$ is a generalized witness for $\tilde{\alpha}$ with respect to $\tilde{N}$ and $\tilde{p}$ in $\tilde{N}$ whenever $\tilde{\alpha} \in \tilde{p}$. As $\tilde{p} \in P_{\tilde{N}}^* = P_{\tilde{N}}^{n+1}$, we have $\tilde{p} = p_{\tilde{N}}$. This completes the proof of the fact that $\tilde{N}$ is sound. Moreover, we have shown that $\sigma'(\tilde{p}) = p_\tau$.

Now apply the condensation lemma to $\sigma' : \tilde{N} \to N_\tau$. Clause a) is impossible, as $\kappa^{+\tilde{N}} = \tilde{\tau} < \tau = \kappa^{+N_\tau}$. Clause c) is false, as it would imply that $\tilde{N}$ is not sound. Finally, d) is not the case, as $E_{\tilde{\tau}} = \varnothing$. It follows that $\tilde{N}$ is a proper initial segment of N_τ, and thus of $\mathbf{L}[E]$. Since $\tilde{\tau} = \kappa^{+\tilde{N}}$ and $\tilde{N}$ projects to κ, $\tilde{N}$ is the collapsing level for $\tilde{\tau}$ in $\mathbf{L}[E]$, i.e. $\tilde{N} = N_{\tilde{\tau}}$. That $n_{\tilde{\tau}} = n$ is clear from the discussion in the previous paragraph. Also, $\sigma' = \sigma_{\tilde{\tau}\tau}$, as $\sigma' \restriction \kappa = \mathrm{id}$ and $\sigma'(p_{\tilde{\tau}}) = p_\tau$ and σ' is fully determined by these two properties. To see that $\tilde{\tau}$ is in D, it suffices to show that σ' satisfies clause c) in the definition of D. But this also follows immediately from the previous paragraph. $\square$(Lemma 3.7)

Lemma 3.8. *D is closed in τ.*

Proof. Let $\tilde{\tau}$ be a limit point of D. So $\tilde{\tau} \in \mathcal{S}$, as $\mathcal{S}$ is closed. To see that $\tilde{\tau}$ has the required properties, we shall use the properties of direct limits developed in the fine structure subsection in the preliminaries. Form the direct limit $\langle \tilde{N}, \sigma_{\bar{\tau}\tilde{\tau}};\ \bar{\tau} \in D \cap \tilde{\tau} \rangle$

of the diagram $\langle N_{\bar\tau}, \sigma_{\tau^*\bar\tau};\ \tau^* \le \bar\tau\ \&\ \tau^*, \bar\tau \in D \cap \tilde\tau\rangle$. This direct limit is well-founded, as there is a Σ_0-preserving embedding $\sigma : \tilde N \to N_\tau$ defined by $\sigma : \sigma_{\bar\tau\tilde\tau}(x) \mapsto \sigma_{\bar\tau\tau}(x)$. From now on consider $\tilde N$ to be transitive. Notice that $\sigma \circ \sigma_{\bar\tau\tilde\tau} = \sigma_{\bar\tau\tau}$. For $\xi < \tilde\tau$ we have $\sigma_{\tilde\tau\tau}(\xi) = \sigma_{\bar\tau\tau}(\xi) = \xi$ where $\bar\tau$ is such that $\xi < \bar\tau < \tilde\tau$, so $\sigma_{\tilde\tau\tau} \restriction \tilde\tau = \mathrm{id}$. Also, the thread $\langle\bar\tau;\ \bar\tau \in D \cap \tilde\tau\rangle$ clearly represents $\tilde\tau$ in $\tilde N$, so $\tilde\tau = \sigma_{\bar\tau\tilde\tau}(\bar\tau)$ and $\sigma_{\tilde\tau\tau}(\tilde\tau) = \tau$. It follows that $\tilde\tau = \kappa^{+\tilde N}$ and $\tilde\tau = \mathrm{cr}(\sigma_{\tilde\tau\tau})$. By the discussion of direct limits at the end of the fine structure subsection in the preliminaries, the maps $\sigma_{\bar\tau\tilde\tau}$ are in fact $\Sigma_0^{(n)}$-preserving where $n = n_\tau$. As in the previous lemma, these preservation properties hold with respect to the language for coherent structures, as it is not clear that $\tilde N$ is a premouse of the same type as N_τ and the constant $\dot\gamma$ is correctly interpreted in $\tilde N$. Our aim is to show that $\tilde N = N_{\tilde\tau}$, and again we intend to use the condensation lemma. Thus, we have to show that the asuumptions are met.

The first step towards this is the verification that $\tilde N$ is a premouse of the same type as N_τ. To see that $\tilde N$ is a potential premouse, recall that Π_2-properties which hold on a tail-end are upward preserved under direct limit maps. We know that each N_{τ_ι} is of the same type as N_τ. If they are of type A, then so is $\tilde N$, as the statement "$(\forall\bar\lambda < \lambda_N)(\bar\lambda$ is not a cutpoint of $E^N_{\mathrm{top}})$" is $\Pi_2(N)$ for any N. If N_τ is of type B, then $\sigma_{\tau^*\bar\tau}(\gamma_{N_{\tau^*}}) = \gamma_{N_{\bar\tau}}$, so $\sigma_{\tau^*\bar\tau}(\lambda^*_{N_{\tau^*}}) = \lambda^*_{N_{\bar\tau}}$, as λ^*_N is the largest cardinal in $J^{E^N}_{\gamma_N}$ for any N. Set $(\tilde\gamma, \tilde\lambda^*) = \sigma_{\bar\tau\tilde\tau}(\gamma_{N_{\bar\tau}}, \lambda^*_{N_{\bar\tau}})$; the preservation properties of the direct limit maps then guarantee that $\tilde\lambda^*$ is a cutpoint of $E^{\tilde N}_{\mathrm{top}}$ and $E^{\tilde N}_{\mathrm{top}} \restriction \tilde\lambda^* = E^{\tilde N}_{\tilde\gamma}$. Now, exactly as in the case of type A premice, we can show that there are no cutpoints of $E^{\tilde N}_{\mathrm{top}}$ larger than $\tilde\lambda^*$. Hence $\tilde N$ is of type B as well, $\tilde\lambda^* = \lambda^*_{\tilde N}$ and $\tilde\gamma = \gamma_{\tilde N}$. It remains to discuss the case where N_τ is of type C. Then $\omega\varrho^1_{N_{\bar\tau}} = \lambda_{N_{\bar\tau}} > \kappa$, as both N_τ and all $N_{\bar\tau}$ are in $\mathcal{S}$. It follows that $n > 0$, so $\Pi_2^{(1)}$-statements which hold on a tail-end of $D \cap \tilde\tau$ are upwards preserved under the direct limit maps. Notice also that $\omega\varrho^1_{\tilde N} = \bigcup\{\sigma''_{\bar\tau\tilde\tau}\omega\varrho^1_{N_{\bar\tau}};\ \bar\tau \in D \cap \tilde\tau\} = \bigcup\{\sigma''_{\bar\tau\tilde\tau}\lambda_{N_{\bar\tau}};\ \bar\tau \in D \cap \tilde\tau\} = \lambda_{\tilde N}$. That $\tilde N$ is of type C can be expressed by the $Q^{(1)}$-statement

$$(\forall\zeta^1)(\exists\bar\lambda^1 \ge \zeta^1)(\bar\lambda^1 \text{ is a cutpoint of } E^{\tilde N}_{\mathrm{top}}),$$

so it is true in $\tilde N$ by the preservation properties of the direct limit maps. Now given any $\bar\lambda < \lambda_{\tilde N}$,

$$E^{\tilde N}_{\mathrm{top}} \,|\, \bar\lambda = \{\langle x, y\rangle;\ (\exists y')(\langle x, y'\rangle \in E^{\tilde N}_{\mathrm{top}}\ \&\ y = y' \cap \bar\lambda\},$$

so $E^{\tilde N}_{\rm top}\,|\,\bar\lambda$ is $\Sigma_1(N)$-definable in $\bar\lambda$. But we have seen that $\bar\lambda < \omega\varrho^1_{\tilde N} = \lambda_{\tilde N}$, so $E^{\tilde N}_{\rm top}\,|\,\bar\lambda$ being a bounded subset of $\lambda_{\tilde N}$, must be in $\tilde N$. This proves that $\tilde N$ is a type C premouse. From now on we know that the preservation properties of all maps $\sigma\bar\tau\tilde\tau$ and σ hold with respect to the language for premice.

Let $\tilde p = \sigma_{\bar\tau\tilde\tau}(p_{\bar\tau})$ for $\bar\tau \in D\cap\tilde\tau$. Given an $x \in \tilde N$, there is a $\bar\tau \in D\cap\tilde\tau$ and an $\bar x \in N_{\bar\tau}$ such that $x = \sigma_{\bar\tau\tilde\tau}(\bar x)$. By the soundness of $N_{\bar\tau}$, there is an $\xi < \kappa$ satisfying $\bar x = \tilde h_{\bar\tau}(\xi, p_{\bar\tau})$. This statement, being $\Sigma^{(n)}_1$, is upward preserved by $\sigma_{\bar\tau\tilde\tau}$, so $x = \tilde h^{n+1}_{\tilde N}(\xi,\tilde p)$. Since x was arbitrary, $\tilde N = \tilde h^{n+1}_{\tilde N}(\kappa\cup\{\tilde p\})$. As in the proof of the previous lemma we conclude that $\omega\varrho^\omega_{\tilde N} = \omega\varrho^{n+1}_{\tilde N} = \kappa$ and $\tilde p \in R^{n+1}_{\tilde N}$. Now notice that $\sigma_{\tilde\tau\tau}(\tilde p) = p_\tau$ and that for any $\alpha \in p_{N_\tau}$, there is a generalized witness for α with respect to N_τ and p_τ in $\mathrm{rng}(\sigma)$. The latter follows from the fact that $\mathrm{rng}(\sigma) \supset \mathrm{rng}(\sigma_{\bar\tau\tau})$ and that $\mathrm{rng}(\sigma_{\bar\tau\tau})$ contains such witnesses, as is ensured by c) in the defintion of D. From now on we can literally repeat the proof of the previous lemma and infer that $\tilde N$ is sound and $\tilde p = p_{\tilde N}$, and, consequently, that $\tilde N = N_{\tilde\tau}$, $n_{\tilde\tau} = n$ and $\sigma = \sigma_{\tilde\tau\tau}$. Thus, $\tilde\tau \in D$. $\square$(Lemma 3.8)

Lemma 3.9. *D is a subset of $\mathcal{S}^0$ on a tail-end, i.e. there is a $\bar\tau < \tau$ such that $D - \bar\tau \subset \mathcal{S}^0$.*

Proof. Suppose the contrary, i.e. let $\langle\tau_\iota;\,\iota<\delta\rangle$ be an increasing sequence cofinal in τ such that each τ_ι is in $\mathcal{S}^1$. Due to the pigeonhole principle, we can assume w.l.o.g. that $|q_{\tau_\iota}| = m$ for some fixed $m \in \omega$. Furthermore, we can assume that the sequence $\langle\mu_{\tau_\iota};\,\iota<\delta\rangle$ is monotone (not necessarily strictly), as we can always replace it by a monotone subsequence $\langle\mu_{\tau_{\iota(\xi)}};\,\xi<\delta'\rangle$ where $\iota(\xi)$ is inductively defined by

$$\begin{aligned}\iota^*(\xi) &= \sup\{\iota(\bar\xi)+1;\,\bar\xi<\xi\}\\ \iota(\xi) &\simeq \text{the least } \iota \text{ such that } \iota^*(\xi)\le\iota<\delta \text{ and } \mu_{\tau_\iota} = \min\{\mu_{\tau_\eta};\,\iota^*(\xi)\le\eta<\delta\}\end{aligned}$$

where $\sup(\varnothing) = 0$. Let $\mu = \sup\{\mu_{\tau_\iota};\,\iota<\delta\}$, let q be the bottom segment of p_τ with exactly m elements and let $r = p_\tau - q$. Notice that $\mu \le \kappa$ and that $q = \sigma_{\tau_\iota\tau}(q_{\tau_\iota})$ and $r = \sigma_{\tau_\iota\tau}(r_{\tau_\iota})$ for all $\iota < \delta$.

We show that (μ, q) is a divisor of N_τ by verifying all clauses in the definition of a divisor. From the proof of previous lemma we know that $\langle N_\tau, \sigma_{\tau_\iota\tau};\,\iota<\delta\rangle$ is the direct limit of the diagram $\langle N_{\tau_\iota}, \sigma_{\tau_{\bar\iota}\tau_\iota};\,\bar\iota\le\iota<\delta\rangle$. We shall use properties of fine structural direct limits discussed in the fine structure subsection in the preliminaries without further reference. Notice first that

$$\omega\varrho_\tau = \bigcup_{\iota<\delta} \sigma''_{\tau_\iota\tau}\omega\varrho_{\tau_\iota}. \tag{26}$$

We further observe that

$$\tilde{h}_\tau(\mu \cup \{r\}) = \bigcup_{\iota<\delta} \sigma''_{\tau_\iota\tau}\tilde{h}_{\tau_\iota}(\mu_{\tau_\iota} \cup \{r_{\tau_\iota}\}). \tag{27}$$

The inclusion $\supset$ follows from the fact that $\Sigma_1^{(n)}$-statements are upward preserved under $\sigma_{\tau_\iota\tau}$, so if $y = \tilde{h}_{\tau_\iota}(\xi, r_{\tau_\iota})$ for some $\xi < \mu_{\tau_\iota}$ then $\sigma_{\tau_\iota\tau}(y) = \tilde{h}_\tau(\xi, r)$. The converse follows from the properties of direct limits that $\Sigma_1^{(n)}$ statements are downward preserved on a tail-end. So if $y = \tilde{h}_\tau(\xi, r)$, then for a sufficiently large $\iota < \delta$, the value $y_\iota = \tilde{h}_{\tau_\iota}(\xi, r_{\tau_\iota})$ is defined and $\sigma_{\tau_\iota\tau}(y_\iota) = y$. This proves (27). Now as $(\mu_{\tau_\iota}, q_{\tau_\iota})$ is a divisor for N_{τ_ι} for any $\iota < \delta$, each hull $\tilde{h}_{\tau_\iota}(\mu_{\tau_\iota} \cup \{r_{\tau_\iota}\})$ is cofinal in $\omega\varrho_{\tau_\iota}$. This together with (26) and (27) yields that $\tilde{h}_\tau(\mu \cup \{r\})$ is cofinal in $\omega\varrho_\tau$, which proves clause c) in the definition of divisor.

To verify d), we show

$$\tilde{h}_\tau(\mu \cup \{r\}) \cap (\max(q) + 1) \subset \mu. \tag{28}$$

Recall that we have set $\max(\varnothing) = \kappa$ here. Fix a $\zeta \leq \max(q)$ such that $\zeta = \tilde{h}_\tau(\xi, r)$ for some $\xi < \mu$. Pick an $\iota < \delta$ such that $\bar{\zeta} = \tilde{h}_{\tau_\iota}(\xi, r_{\tau_\iota})$ is defined and so $\zeta = \sigma_{\tau_\iota\tau}(\bar{\zeta})$. Hence $\bar{\zeta} \leq \max(q_{\tau_\iota})$. As $(\mu_{\tau_\iota}, q_{\tau_\iota})$ is a divisor, $\bar{\zeta} < \mu_{\tau_\iota}$. So $\zeta = \sigma_{\tau_\iota\tau}(\bar{\zeta}) = \bar{\zeta} < \mu_{\tau_\iota} \leq \mu$, which proves (28). Thus, setting λ to be the least ordinal in $\tilde{h}_\tau(\mu, r) - \mu$, we have $\lambda > \max(q_\tau)$. On the other hand, $\lambda \leq \sigma_{\tau_\iota\tau}(\lambda_{\tau_\iota}) < \omega\varrho_\tau$; the former inequality is a consequence of the fact that $\sigma_{\tau_\iota\tau}(\lambda_{\tau_\iota})$ is obviously in $\tilde{h}_\tau(\mu \cup \{r\})$. This verifies a), b) and d) and completes the proof of the claim.

Recall that N_τ as well as all N_{τ_ι} *fail* to be pluripotent, as $\tau \in \mathcal{S}^0$ and all N_{τ_ι} are of the same type as N_τ. This means that divisors of these premice are determined by the standard parameters, and not by the Dodd parameters. Moreover, (μ, q) is not strong, as no $\tau \in \mathcal{S}^0$ admits a strong divisor. The remaining part of the argument makes a substantial use of Lemma 2.22 as well as of its proof. By that lemma, $p_{N'_\tau(\mu)}$ is a proper lengthening of $r' = \pi'^{-1}(r)$ where $\pi' : N'_\tau(\mu) \to N_\tau$ is the uncollapsing embedding associated with the $\Sigma_1^{(n)}$-hull $\tilde{h}_\tau(\mu_\tau \cup \{p_\tau\})$. Let β' be the largest element of $p_{N'_\tau(\mu)} - r'$. Then $W^{\beta',r'}_{N'_\tau(\mu)} \in N'_\tau(\mu)$, as the preservation properties of π' combined with the condensation lemma guarantee that $N'_\tau(\mu)$ is solid. Letting $Q = \pi'(W^{\beta',r'}_{N'_\tau(\mu)})$ and $t = \pi'(t')$ where t' is the preimage of r' under the associated canonical witness map, $\langle Q, t\rangle$ is a generalized witness, in the weak sense introduced immediately above Lemma 2.16, for $\beta = \pi'(\beta')$ with respect to N_τ and r. We have to consider this weaker notion of a solidity witness, as it might happen that $\beta < \kappa = \omega\varrho^{n+1}_{N_\tau}$ while $\langle Q, t\rangle$ contains information merely about $\Sigma^{(n)}_1(N_\tau)$-definable subsets of β. Let $\xi, \eta < \mu$ be such that $\langle Q, t\rangle = \tilde{h}_\tau(\xi, p_\tau)$ and $\beta = \tilde{h}_\tau(\eta, p_\tau)$; such ordinals exist, as $\mathrm{rng}(\pi') = \tilde{h}_\tau(\mu_\tau \cup \{p_\tau\})$. Let $\iota < \delta$ be sufficiently large so that $\eta, \xi < \mu_{\tau_\iota}$ and both $\bar\beta = \tilde{h}_{\tau_\iota}(\eta, p_{\tau_\iota})$ and $\bar T = \tilde{h}_{\tau_\iota}(\xi, p_{\tau_\iota})$ are defined. Then $\sigma_{\tau_\iota\tau}(\bar\beta, \bar T) = (\beta, \langle Q, t\rangle)$, so $\bar T$ is of the form $\langle \bar Q, \bar t\rangle$, and is a generalized witness, in the weak sense, for $\bar\beta$ with respect to N_{τ_ι} and r_{τ_ι}; to see this, recall that being a generalized witness in the weak sense is a $\Pi^{(n)}_1$-property, and therefore is downward preserved under the $\Sigma^{(n)}_0$-preserving map $\sigma_{\tau_\iota\tau}$. Moreover, both $\bar\beta$ and $\langle \bar Q, \bar t\rangle$ are in the range of $\pi'_{\tau_\iota} = \tilde{h}_{\tau_\iota}(\mu_\iota \cup \{p_\iota\})$ by our choice of η, ξ and ι. Finally $\bar\beta \geq \mu_{\tau_\iota}$, as $\beta \geq \mu$ (this follows easily from the definition of β) and $\mu_\iota \leq \mu$. Thus, if $(\bar\beta, \langle \bar Q, \bar t\rangle) = \pi'_{\tau_\iota}(\bar\beta', \langle \bar Q', \bar t'\rangle)$, then $\bar\beta' \geq \mu_{\tau_\iota}$ and $\langle \bar Q', \bar t'\rangle$ is a generalized witness for $\bar\beta'$ with respect to $N'_{\tau_\iota}(\mu_{\tau_\iota})$ and r'_{τ_ι} that is an element of $N'_{\tau_\iota}(\mu_{\tau_\iota})$. Notice that $\langle \bar Q', \bar t'\rangle$ turns out to be a generalized witness in the strong sense, as the $(n+1)$-st projectum of $N'_{\tau_\iota}(\mu_{\tau_\iota})$ is precisely μ_{τ_ι}. By the first part of the proof of Lemma 2.18, the standard witness $W' \stackrel{\text{def}}{=} W^{\bar\beta', r'_{\tau_\iota}}_{N'_{\tau_\iota}(\mu_{\tau_\iota})}$ is an element of $N'_{\tau_\iota}(\mu_{\tau_\iota})$ as well. Let t'_{τ_ι} be the preimage of r'_{τ_ι} under the associated canonical witness map. Since this map is $\Sigma^{(n)}_1$-preserving, for any set A which is $\Sigma^{(n)}_1(N'_{\tau_\iota}(\mu_{\tau_\iota}))$ in the parameter r'_{τ_ι} we can find some A' which is $\Sigma^{(n)}_1(W')$-definable in the parameter t'_{τ_ι} such that $A \cap \bar\beta' = A' \cap \bar\beta' \in N'_{\tau_\iota}(\mu_{\tau_\iota})$. Then $r'_{\tau_\iota} \notin P^{n+1}_{\tau_\iota}$, as $\bar\beta' \geq \mu_{\tau_\iota}$. In other words, $p_{N'_{\tau_\iota}(\mu_{\tau_\iota})}$ is a proper lengthening of r'_{τ_ι}. By Lemma 2.22, the divisor $(\mu_{\tau_\iota}, q_{\tau_\iota})$ is not strong, a contradiction. $\square$(Lemma 3.9)

Let $\bar\tau$ be minimal with the property that $D - \bar\tau \subset \mathcal{S}^0$. By the previous lemmata, $\bar\tau < \tau$ and $B_\tau - \bar\tau = D - \bar\tau$ is closed and unbounded in τ. This completes the proof of Lemma 3.5 for $\tau \in \mathcal{S}^0$.

When $\tau \in \mathcal{S}^1$. Here we adopt the same strategy as for $\tau \in \mathcal{S}^0$. We again assume that τ is a limit point of $\mathcal{S}$, of uncountable cofinality, but now, of course, $\tau \in \mathcal{S}^1$.

Let D be the set of all $\bar\tau \in \mathcal{S}^1 \cap \tau$ satisfying:

- (μ_τ, m_τ) determines a strong divisor of $N_{\bar\tau}$, i.e. setting $q^*_{\bar\tau}$ to be the bottom segment of $p_{\bar\tau}$ of length m_τ, the pair $(\mu_\tau, q^*_{\bar\tau})$ is a strong divisor of $N_{\bar\tau}$.
- Setting $M^*_{\bar\tau} = N_{\bar\tau}(\mu_\tau, q^*_{\bar\tau})$, there is a map $\sigma_{\bar\tau\tau} : M^*_{\bar\tau} \to M_\tau$ which is Σ_0-preserving with respect to the language for coherent structures and such that
  a) $\bar\tau = \mathrm{cr}(\sigma_{\bar\tau\tau})$ and $\sigma_{\bar\tau\tau}(\bar\tau) = \tau$;
  b) $\sigma_{\bar\tau\tau}(q^*_{\bar\tau}) = q_\tau$;
  c) to each $\alpha \in q_\tau$ there is a generalized witness $Q_\tau(\alpha)$ for α with respect to M_τ and q_τ such that $Q_\tau(\alpha) \in \mathrm{rng}(\sigma_{\bar\tau\tau})$.

Generalized witnesses in clause c) are, of course, computed in the language for coherent structures. Notice that $M^*_{\bar\tau}$ is *never* a pluripotent $\mathbf{L}[E]$-level, even if M_τ is, as its top extender cannot measure all subsets of μ_τ. This follows from the fact that $\sigma_{\bar\tau\tau}$ is not cofinal. We first prove that D is closed unbounded in τ, and then that $(\mu_\tau, q^*_{\bar\tau}) = (\mu_{\bar\tau}, q_{\bar\tau})$ on a tail-end of D. The proofs of the following lemmata are, as expected, variants of the proofs of the corresponding lemmata from the previous case adapted to protomice.

Lemma 3.10. *D is unbounded in τ.*

Proof. Let $\tau' < \tau$; we have to find a $\tilde\tau \geq \tau'$ in D. Let $M = N_\tau(\mu_\tau, q_\tau)$. It might happen (and it often will!) that $M = N_\tau$ is a pluripotent premouse. Recall that M can only be of type A or B, as $\tau \in \mathcal{S}$. The following argument will be carried out in the language for coherent structures even in the case where M is a type B premouse.

As before construct $\bar M, \bar\kappa, \bar\tau \in H$ and an embedding $\sigma : H \to J^E_{\kappa^{++}}$ such that $\sigma(\bar M, \bar\kappa, \bar\tau, \bar\mu, \bar\vartheta) = M, \kappa, \tau, \mu_\tau, \vartheta$ where $\vartheta = \vartheta(M)$. Also, setting $\tilde\tau = \sup(\sigma''\bar\tau)$, it follows as before that $\tau' \leq \tilde\tau < \tau$ and that $\tilde\tau$ is a limit point of $\mathcal{S}$, so $\tilde\tau$ is an element of $\mathcal{S}$. From the preservation properties of σ we infer the following facts about $\bar M$:

- $\bar M$ is a coherent structure that is a premouse just in case that M is. If M is a premouse, then M is pluripotent and $\bar M$ is a premouse of the some type as M. If M is a protomouse, then the top extender of $\bar M$ is not weakly amenable with respect to $\bar M$.
- $\bar\tau = \bar\kappa^{+\bar M}$.
- $\omega\varrho^\omega_{\bar M} = \omega\varrho^1_{\bar M} = \bar\kappa$.

- If $\bar{q}$ is the standard parameter of $\bar{M}$, then $\sigma(\bar{q}) = q_\tau$.
- $\bar{\vartheta}$ is closed in $\bar{M}$ relative to $\bar{q}$, see Lemma 2.24 and the preceding definition.
- $\bar{M}$ is sound and solid;
- $\sigma(W^{\bar{\alpha},\bar{q}}_{\bar{M}}) = W^{\alpha,q_\tau}_{M}$ whenever $\bar{\alpha} \in \bar{q}$ and $\alpha = \sigma(\bar{\alpha})$.

Form the canonical extension $\tilde{\sigma} : \bar{M} \to \tilde{M}$ of $\sigma \upharpoonright J^{E^{\bar{M}}}_{\bar{\tau}} : J^{E^{\bar{M}}}_{\bar{\tau}} \to J^{E}_{\tilde{\tau}}$. Let $\sigma' : \tilde{M} \to M$ be the embedding given by the interpolation lemma. Notice that, as $\omega\varrho^1_{\bar{M}} = \bar{\kappa}$, this extension is a coarse extension. Exactly as in the proof of Lemma 3.7 we infer:

- $\tilde{M}$ is a coherent structure, and $\tilde{\sigma} : \bar{M} \to \tilde{M}$ is Σ_0-preserving and cofinal;
- $\tilde{\sigma}$ is a coarse pseudoultrapower map and $\tilde{\sigma}(\bar{\kappa}, \bar{\tau}) = (\kappa, \tilde{\tau})$;
- $\mathrm{cr}(\sigma') = \tilde{\tau}$ and $\sigma'(\tilde{\tau}) = \tau$;
- $h_{\tilde{M}}(\kappa \cup \{\tilde{q}\}) = \tilde{M}$ where $\tilde{q} = \tilde{\sigma}(\bar{q})$;
- $\omega\varrho^\omega_{\tilde{M}} = \omega\varrho^1_{\tilde{M}} = \kappa$ and $\tilde{q} \in R_{\tilde{M}}$;
- letting $\bar{t}$ be the preimage of $\bar{q}$ under the canonical witness map associated with $W^{\bar{\alpha},\bar{q}}_{\bar{M}}$,

  $$\langle Q(\tilde{\alpha}), \tilde{t}\rangle = \tilde{\sigma}(\langle W^{\bar{\alpha},\bar{q}}_{\bar{M}}), \bar{t}\rangle)$$

  is a generalized witness for $\tilde{\alpha} = \tilde{\sigma}(\bar{\alpha})$ with respect to $\tilde{M}$ and $\tilde{q}$ that is an element of $\tilde{M}$;
- $\tilde{q} = p_{\tilde{M}}$;
- $\tilde{M}$ is solid and sound;
- $\sigma'(p_{\tilde{M}}) = q_\tau$ and $\sigma'(Q(\tilde{\alpha})) = W^{\alpha,q_\tau}_{M}$ whenever $\tilde{\alpha} \in \tilde{q}$ and $\alpha = \sigma'(\tilde{\alpha})$, as $\sigma' \circ \tilde{\sigma} = \sigma$.

We note that the equality $\tilde{q} = p_{\tilde{M}}$ follows from its two preceding clauses. Since $\tilde{q} \in P_{\tilde{M}}$, we do not need to know beforehands that $\tilde{M}$ is solid. See the subsection on fine structure, the part on solidity, in the preliminaries.

Applying Lemma 2.19 (resp. Lemma 2.20), we conclude that $\tilde{M} = N_{\tilde{\tau}}(\mu_\tau, \tilde{q})$. Thus, $\tilde{M}$ is always a protomouse, even if M is a premouse. This is caused by the fact that σ' is non-cofinal. Let $\bar{F}, \tilde{F}$ and F be the top extender of $\bar{M}, \tilde{M}$ and M, respectively. So $\mathrm{cr}(\tilde{F}) = \mu_\tau$. Recall that $\bar{F}$ and $\tilde{F}$ determine cofinal maps from $\bar{\vartheta}$ and $\tilde{\vartheta} = \vartheta(\tilde{M})$ to $\mathrm{ht}(\bar{M})$ and $\mathrm{ht}(\tilde{M})$, respectively. Also, these maps have the same Σ_1-definition over the respective structures (in $\bar{M}$, for instance, the map is defined by $\mathrm{otp}(a) \mapsto \mathrm{otp}(\bar{F}(a))$). This together with the fact that $\tilde{\sigma}$ is cofinal implies that

$\tilde{\sigma}$ maps $\bar{\vartheta}$ cofinally into $\tilde{\vartheta}$. To see that $\tilde{\tau} \in D$, it suffices to prove that $(\mu_\tau, \tilde{q})$ is a strong divisor of $N_{\tilde{\tau}}$. By Lemma 2.24, this is equivalent to showing that $\tilde{\vartheta}$ is closed in $\tilde{M}$ relative to $\tilde{q}$.

Assume we have established:

(29) There are unboundedly many $\vartheta^* < \vartheta$ that are closed in M relative to q_τ.

It follows that there are unboundedly many $\bar{\vartheta}^* < \bar{\vartheta}$ which are closed in $\bar{M}$ relative to $\bar{q}$. That $\bar{\vartheta}^*$ is closed in $\bar{M}$ relative to $\bar{q}$ is equivalent to

$$(\forall w)(\forall f \in J^{E^{\bar{M}}}_{\bar{\vartheta}^*})(\forall \xi < \mu_\tau)[w = \bar{F}(f)(\bar{q}, \xi) \ \longrightarrow\ w \cap \bar{\mu} \in J^{E^{\bar{M}}}_{\bar{\vartheta}^*}].$$

This is a Π_1-statement, so the same holds of $\tilde{\sigma}(\bar{\vartheta}^*), \tilde{q}$ and $\tilde{F}_{\bar{\vartheta}^*}$ in $\tilde{M}$, which proves that $\tilde{\sigma}(\bar{\vartheta}^*)$ is closed in $\tilde{M}$ relative to $\tilde{q}$. So $\tilde{\vartheta}$ is closed in $\tilde{M}$ relative to $\tilde{q}$, as $\tilde{\sigma}$ maps $\bar{\vartheta}$ cofinally into $\tilde{\vartheta}$.

To complete the proof, we have to verify (29). If M is a premouse, (29) follows by a simple closure argument. We now give a more general argument which also goes through for protomice. Notice that $\mathrm{cf}(\vartheta) = \mathrm{cf}(\tau) > \omega$, as

$$\vartheta' \mapsto \sup(\{F(f)(q_\tau, \xi);\ f : \mu_\tau \to \mu_\tau \ \&\ f \in J^E_{\vartheta'} \ \&\ \xi \in \kappa\} \cap \tau)$$

is a monotone map mapping ϑ cofinally into τ. This follows from the regularity of τ in M and the fact that for each ϑ^*, the values on the right side are determined by F_{ϑ^*}, which is an element of M. Suppose (29) fails. Let $\delta = \mathrm{cf}(\vartheta)$ and let $\langle \vartheta'_\xi; \xi < \delta\rangle$ be a normal sequence unbounded in ϑ such that ϑ'_0 is above all ordinals that are closed in M relative to q_τ. To each $\xi < \delta$ fix a map $f_\xi : \mu_\tau \to \mathcal{P}(\mu_\tau)$ and an ordinal $\eta_\xi < \mu_\tau$ such that $F(f_\xi)(q_\tau, \eta_\xi) \cap \mu_\tau \notin J^E_{\vartheta'_\xi}$. For limit ξ, let $g(\xi)$ be the least $\bar{\xi}$ such that $f_\xi \in J^E_{\vartheta'_{\bar{\xi}}}$. Then g is a regressive function defined on all limit ordinals below δ, so it has a constant value, say ξ^*, on some stationary $S \subset \delta$. Thus, for $\xi \in S$ we have $f_\xi \in J^E_{\vartheta^*}$ where $\vartheta^* = \vartheta'_{\xi^*}$. As μ_τ is the largest cardinal in J^E_ϑ, there is an enumeration $f^* \in J^E_\vartheta$ of all functions $f : \mu_\tau \to \mathcal{P}(\mu_\tau)$ which are in $J^E_{\vartheta^*}$. Define $f' : \mu_\tau \to \mathcal{P}(\mu_\tau)$ by

$$f'(\zeta) = \{\prec\xi, \eta, \eta'\succ;\ \eta' \in f^*(\xi)(\zeta, \eta) \ \&\ \eta, \xi < \mu_\tau\}.$$

Clearly, $f' \in J^E_\vartheta$ and $F(f')(q_\tau)$ is a subset of λ_τ. As ϑ is closed, $F(f')(q_\tau) \cap \mu_\tau \in J^E_{\vartheta'_{\xi'}}$ for some $\xi' \in S$. Hence

$$\{\eta' < \mu_\tau;\ \prec\xi, \eta, \eta'\succ \in F(f')(q_\tau)\} \in J^E_{\vartheta_{\xi'}}$$

whenever $\eta, \xi < \mu_\tau$. In particular, this must be true of $\tilde{\xi}$ and $\eta_{\xi'}$ where $\tilde{\xi} < \mu_\tau$ is such that $f^*(\tilde{\xi}) = f_{\xi'}$. Then

$$\{\eta' < \mu_\tau;\ \prec\tilde{\xi}, \eta_{\xi'}, \eta'\succ \in F(f')(q_\tau)\} = F(f_{\xi'})(q_\tau, \eta_{\xi'}) \cap \mu_\tau,$$

and $F(f_{\xi'})(q_\tau, \eta_{\xi'}) \cap \mu_\tau \notin J^E_{\vartheta_{\xi'}}$ by our choice of $f_{\xi'}$ and $\eta_{\xi'}$. Contradiction.

$\square$(Lemma 3.10)

Lemma 3.11. *D is closed in τ.*

Proof. We shall closely follow the proof of Lemma 3.8. Let $\tilde{\tau} < \tau$ be a limit point of D. Then $\tilde{\tau} \in \mathcal{S}$. Form the direct limit $\langle \tilde{M}, \sigma_{\bar{\tau}\tilde{\tau}};\ \bar{\tau} \in D \cap \tilde{\tau}\rangle$ of the diagram $\langle M^*_{\bar{\tau}}, \sigma_{\tau^*\bar{\tau}};\ \tau^* \leq \bar{\tau}\ \&\ \tau^*, \bar{\tau} \in D \cap \tilde{\tau}\rangle$. As before we have the Σ_0-preserving map $\sigma : \tilde{M} \to M$ defined by $\sigma_{\bar{\tau}\tilde{\tau}}(x) \mapsto \sigma_{\bar{\tau}\tau}(x)$, so $\tilde{M}$ is well-founded and we can consider it to be transitive. The arguments from the proof of Lemma 3.8 can be modified in a straightforward way to obtain the following properties of $\tilde{M}$ and σ. In the clauses below, $\bar{\tau}$ is an arbitrary element of $D \cap \tilde{\tau}$.

- $\tilde{M}$ is a coherent structure.
- $\sigma_{\bar{\tau}\tilde{\tau}}(\bar{\tau}) = \tilde{\tau}$ and $\sigma \circ \sigma_{\bar{\tau}\tilde{\tau}} = \sigma_{\bar{\tau}\tau}$.
- $\mathrm{cr}(\sigma) = \tilde{\tau}$ and $\sigma(\tilde{\tau}) = \tau$.
- $h_{\tilde{M}}(\kappa \cup \{\tilde{q}\}) = \tilde{M}$ where $\tilde{q} = \sigma_{\bar{\tau}\tilde{\tau}}(q^*_{\bar{\tau}})$, so $\omega\varrho^\omega_{\tilde{M}} = \omega\varrho^1_{\tilde{M}} = \kappa$ and $\tilde{q} \in R_{\tilde{M}}$.
- $\sigma(\tilde{q}) = q_\tau$.
- Let $\tilde{\alpha} \in \tilde{q}$ and $\sigma(\tilde{\alpha}) = \alpha$. So $\alpha \in q_\tau$. If $\langle Q(\alpha), t(\alpha)\rangle$ is a generalized witness for α with respect to M_τ and q_τ and $\langle Q(\alpha), t(\alpha)\rangle \in \mathrm{rng}(\sigma_{\bar{\tau}\tau})$ for some $\bar{\tau} \in D \cap \tau$ then $\langle Q(\tilde{\alpha}), t(\tilde{\alpha})\rangle \stackrel{\text{def}}{=} \sigma^{-1}(\langle Q(\alpha), t(\alpha)\rangle)$ is a generalized witness for $\tilde{\alpha}$ with respect to $\tilde{M}$ and $\tilde{q}$.
- $\tilde{q} = p_{\tilde{M}}$.
- $\tilde{M}$ is sound and solid.

The very first clause follows from the fact that the direct limit $\tilde{M}$ satisfies Π_2-statements which hold on a tail-end of $D \cap \tilde{\tau}$; the rest is clear. By the condensation Lemma 2.19 (resp. Lemma 2.20), $\tilde{M} = N_{\tilde{\tau}}(\mu_\tau, \tilde{q})$. Thus, $\tilde{M}$ is always a protomouse.

To complete the proof, we have to show that $(\mu_\tau, \tilde{q})$ is a strong divisor of $\tilde{M}$. We again show that $\tilde{\vartheta} = \vartheta(\tilde{M})$ is closed in $\tilde{M}$ relative to $\tilde{q}$. So pick an $f \in J^E_{\tilde{\vartheta}}$ and $\xi < \mu_\tau$, and fix some $\bar{\tau} \in D \cap \tilde{\tau}$ such that $\tilde{F}(f) \in \mathrm{rng}(\sigma_{\bar{\tau}\tilde{\tau}})$ where $\tilde{F}$ is the top extender of $\tilde{M}$. Letting $\bar{F}$ be the top extender of $M^*_{\bar{\tau}}$, we have

$$\tilde{F}(f)(\tilde{q}, \xi) \cap \mu_\tau = \sigma_{\bar{\tau}\tilde{\tau}}(\bar{F}(f)(q^*_{\bar{\tau}}, \xi) \cap \mu_\tau) = \bar{F}(f)(q^*_{\bar{\tau}}, \xi) \cap \mu_\tau \in J^E_{\vartheta_{\bar{\tau}}} \subset J^E_{\tilde{\vartheta}}.$$

The second equality is a consequence of the fact that $\sigma_{\bar{\tau}\tilde{\tau}}(\mu_\tau) = \mu_\tau$, and the membership relation follows from assumption that $(\mu_\tau, q^*_{\bar{\tau}})$ is a strong divisor of $N_{\bar{\tau}}$. (Notice that this in fact proves that $\vartheta_{\bar{\tau}}$ is closed in $\tilde{M}$ relative to $\tilde{q}$ for each $\bar{\tau} \in D \cap \tilde{\tau}$. Also, the set of all $\vartheta_{\bar{\tau}}$ is c.u.b. in $\tilde{\vartheta}$.) □(Lemma 3.11)

Lemma 3.12. *There is a $\bar{\tau} < \tau$ such that for every $\tau' \in D - \bar{\tau}$ we have $(\mu_\tau, q^*_{\tau'}) = (\mu_{\bar{\tau}}, q_{\bar{\tau}})$. Consequently, $D - \bar{\tau} = B_\tau - \bar{\tau}$.*

Proof. We will again follow the proof of the corresponding Lemma 3.9. Suppose for a contradiction that $\langle \tau_\iota;\, \iota < \delta \rangle$ is an increasing sequence cofinal in τ such that $(\mu_{\tau_\iota}, q_{\tau_\iota}) \neq (\mu_\tau, q^*_{\tau_\iota})$. By the definition of $(\mu_{\tau_\iota}, q_{\tau_\iota})$, each q_{τ_ι} is a bottom part of $q^*_{\tau_\iota}$, say $q^*_{\tau_\iota} = q_{\tau_\iota} \cup s_{\tau_\iota}$. Moreover, $\mu_{\tau_\iota} > \mu_\tau$ for every $\iota < \delta$. This follows again from the definition of $(\mu_{\tau_\iota}, q_{\tau_\iota})$ if $q_{\tau_\iota} = q^*_{\tau_\iota}$, and from Lemma 2.26 otherwise. Arguing as in the proof of Lemma 3.9, we can assume w.l.o.g. that $\langle \mu_{\tau_\iota};\, \iota < \delta \rangle$ is a monotone sequence and all q_{τ_ι} have a fixed size, say m. Then $\sigma_{\tau_{\bar{\iota}}\tau_\iota}(q_{\tau_{\bar{\iota}}}, s_{\tau_{\bar{\iota}}}) = (q_{\tau_\iota}, s_{\tau_\iota})$ whenever $\bar{\iota} \leq \iota < \delta$, and we can set (recall the notation from the beginning of this section, in particular that introduced in the subsection "Defining the square sequence")

$$\begin{aligned} q &= \sigma_{\tau_\iota \tau}(q_{\tau_\iota}); \\ s &= \sigma_{\tau_\iota \tau}(s_{\tau_\iota}); \\ r &= r_\tau; \\ \mu &= \sup_{\iota < \delta} \mu_{\tau_\iota}. \end{aligned}$$

The former two values clearly do not depend on ι. Also, s might be empty, in which case $q = q_\tau$. We first observe that (μ, q) is a divisor of N_τ. As q is a bottom part of q_τ and $\mu > \mu_\tau$, this divisor cannot be strong by the definition of (q_τ, μ_τ). Using a reflection argument we then derive a contradiction to the fact that all $(\mu_{\tau_\iota}, q_{\tau_\iota})$ are strong.

Recall that $M = N_\tau(\mu_\tau, q_\tau)$. To see that (μ, q) is a divisor of N_τ, we first verify clause c) from the definition of a divisor. Assuming that M is a protomouse, $\tilde{h}_\tau(\mu \cup \{r \cup s\}) \supset \tilde{h}_\tau(\mu_\tau \cup \{r\})$ whenever N_τ is not pluripotent, and $h_\tau(\mu \cup \{r \cup s\}) \supset h_\tau(\mu \cup \{r\})$ whenever N_τ is pluripotent. In either case, the hull on the left is cofinal in $\omega\varrho_\tau$, as the hull on the right is. If $M = N_\tau$ then $h_\tau(\mu \cup \{s\})$ is cofinal in N_τ by the elementary properties of coherent structures ($r = \varnothing$ in this case). Now it suffices to show that $\tilde{h}_\tau(\mu \cup \{r \cup s\}) \cap (\max(q)+1) = \mu$ in the former case where M is a protomouse and N_τ is not pluripotent, and that $h_\tau(\mu \cup \{r \cup s\}) \cap (\max(q)+1) = \mu$ in the latter two cases where either M is a protomouse and N_τ is pluripotent or else $M = N_\tau$. This will guarantee that (μ, q) has the rest of the properties of divisors. Suppose first that M is a protomouse, and pick a ζ from the above intersection. Let F_τ and F_{τ_ι} be the top extender of M and $M^*_{\tau_\iota}$, respectively. By Lemma 2.13, ζ is of the form $F_\tau(f)(\xi, s)$ for some $f : \mu_\tau \to \mu_\tau$ and $\xi < \mu$. For ι is sufficiently large such that $f \in \mathrm{dom}(F_{\tau_\iota})$ and $\xi < \mu_{\tau_\iota}$ set $\zeta_\iota = F_{\tau_\iota}(f)(\xi, s_{\tau_\iota})$, then clearly $\sigma_{\tau_\iota\tau}(\zeta_\iota) = \zeta$. Since $\zeta \le \max(q)$, we have $\zeta_\iota \le \max(q_{\tau_\iota})$. Again by Lemma 2.13, $\zeta_\iota \in \tilde{h}_{\tau_\iota}(\mu_{\tau_\iota} \cup \{r_{\tau_\iota}\})$ if N_{τ_ι} is not pluripotent, and $\zeta_\iota \in h_{\tau_\iota}(\mu_{\tau_\iota} \cup \{r_{\tau_\iota}\})$ if N_{τ_ι} is pluripotent; this follows from the fact that $(\mu_{\tau_\iota}, q_{\tau_\iota})$ is a divisor of N_{τ_ι} (recall that M_{τ_ι} is a protomouse). Hence $\zeta_\iota < \mu_{\tau_\iota}$ and, consequently, $\zeta = \sigma_{\tau_\iota\tau}(\zeta_\iota) < \mu_{\tau_\iota} \le \mu$. This completes the proof that (μ, q) is a divisor if M is a protomouse. If $M = N_\tau$, the argument is the same, but we have to use Lemma 2.14 instead of Lemma 2.13, i.e. this time ζ will be of the form $F_\tau(f)(\xi, s, \mu_\tau)$. Clearly, the presence of μ_τ as a parameter here will not do any harm, as $\mu_\tau < \mu$.

Recall the notation introduced in Section 2, the subsection on strong divisors. $\pi' : N'_\tau(\mu) \to N_\tau$ is the uncollapsing map associated with the hull $\tilde{h}_\tau(\mu \cup \{p_\tau\})$ if N_τ is not pluripotent, and with $h_\tau(\mu \cup \{d_\tau\})$ otherwise. Let $\pi'(r', s') = (r, s)$. As we have already mentioned, the divisor (μ, q) cannot be strong. This means that for some β' such that $\mu \le \beta' < \min(s')$ we have

$$W^{\beta', r' \cup s'}_{N'_\tau(\mu)} \in N'_\tau(\mu). \tag{30}$$

To see this, we have to discuss two cases. The notion of solidity witness used here is the weaker one introduced immediately above Lemma 2.16. That is, forming solidity witnesses is sensitive to definability degree $\Sigma_1^{(n_\tau)}$ in the language corresponding to N_τ. If N_τ is not pluripotent or else if $\lambda_{N^*_\tau}(\mu_\tau, r \cup s) > \mu_\tau$, then (30) follows from

Lemma 2.22. We know that $p_{N'_\tau(\mu)}$ is a proper lengthening of $r' \cup s'$, so it suffices to let β' be the maximum of $p_{N'_\tau(\mu)} - (r' \cup s')$. If $\lambda_{N^*_\tau}(\mu_\tau, r \cup s) = \mu_\tau$ (so N_τ is pluripotent and $r \cup s = \varnothing$), we use Lemma 2.23. By the remark after this lemma, $W^{\mu,\varnothing}_{N'(\mu)} = N^*(\mu, d_\tau) \in N'(\mu)$, so we let $\beta' = \mu$ in this case. Let t' be the preimage of $p_{N'_\tau(\mu)} - (\beta'+1)$ under the canonical witness map. We now apply π' to $W^{\beta', r' \cup s'}_{N'(\mu)}$ and infer that:

- $\langle Q, t\rangle = \langle \pi'(W^{\beta', r'\cup s'}_{N'(\mu)}), \pi'(t')\rangle$ is a generalized witness for $\beta = \pi'(\beta')$ with respect to N_τ and $r \cup s$.
- $\langle Q, t\rangle \in h_\tau(\mu \cup \{d_\tau\})$ or $\langle Q, t\rangle \in \tilde{h}_\tau(\mu \cup \{p_\tau\})$, depending on whether N_τ is pluripotent or not.
- $\langle W^{\beta, r\cup s}_{N_\tau}, t^*, \beta\rangle \in h_\tau(\mu \cup \{d_\tau, \vartheta_\tau\})$ or $\langle W^{\beta, r\cup s}_{N_\tau}, t^*, \beta\rangle \in \tilde{h}_\tau(\mu \cup \{p_\tau, \vartheta_\tau\})$, depending on whether N_τ is pluripotent or not. Here t^* is the preimage of $(r \cup s) - (\beta+1)$ under the canonical witness map.
- $\langle W^{\beta,s}_M, w, \beta\rangle \in h_\tau(\mu \cup \{d_\tau, \vartheta_\tau\})$ or $\langle W^{\beta,s}_M, w, \beta\rangle \in \tilde{h}_\tau(\mu \cup \{p_\tau, \vartheta_\tau\})$, depending on whether N_τ is pluripotent or not. Here w is the preimage of $s-(\beta+1)$ under the canonincal withess map.
- $\langle W^{\beta,s}_M, w, \beta\rangle = F(f)(\xi, \vartheta_\tau, q_\tau)$ for some $f \in \mathrm{dom}(F)$ and $\xi < \mu$. Here, F is the top extender of M.

The second clause follows from the first one by the fact that generalized witnesses for the n-th level are preserved under $\Sigma^{(n)}_1$-preserving embeddings. The third clause follows from the second by the first part of Lemma 2.18: Just notice that $W^{\beta, r\cup s}_{N_\tau}$ is Σ_0-definable from Q, β and ϑ_τ and both Q and β are in $\tilde{h}_\tau(\mu \cup \{p_\tau, \vartheta_\tau\})$ (we assume w.l.o.g. that either N_τ fails to be pluripotent or else $\lambda_{N^*_\tau}(\mu_\tau, r_\tau \cup s) > \mu_\tau$; the proof in the other case is similar). The step from the third clause to the fourth one is a straightforward application of d) from Lemma 2.16 if M is a protomouse; if $M = N_\tau$, then there is nothing to prove, as $W^{\beta,s}_M = W^{\beta, r\cup s}_{N_\tau}$. The last clause is obtained from the fourth one by Lemmata 2.13 and 2.14, depending on whether $M \neq N_\tau$ or $M = N_\tau$. Notice that in the latter case, Lemma 2.14 requires adding μ_τ as an additional parameter. This is, however superfluous in the above application, as μ_τ is Σ_1-definable from ϑ_τ, so we can "absorb" μ_τ in f and ϑ_τ.

Pick an ι sufficiently large such that $\xi < \mu_{\tau_\iota}$ and $f \in \mathrm{dom}(F^*_{\tau_\iota})$, where $F^*_{\tau_\iota}$ is the top extender of $M^*_{\tau_\iota}$. Then $F^*_{\tau_\iota}(f)(\xi, \vartheta_\tau, q^*_{\tau_\iota})$ is of the form $\langle Q_\iota, w_\iota, \beta_\iota\rangle$ where

$\langle Q_\iota, w_\iota\rangle$ is a generalized witness for $\beta_\iota = \sigma^{-1}_{\tau_\iota\tau}(\beta)$ with respect to $M^*_{\tau_\iota}$ and s_{τ_ι}. Also, $\beta_\iota \geq \sigma^{-1}_{\tau_\iota\tau}(\mu) = \mu \geq \mu_{\tau_\iota}$. Furthermore, we have:

- $\langle Q_\iota, w_\iota\rangle \in h_{\tau_\iota}(\mu_{\tau_\iota} \cup \{d_{\tau_\iota}\})$ or $\langle Q_\iota, w_\iota\rangle \in \tilde{h}_{\tau_\iota}(\mu_{\tau_\iota} \cup \{p_{\tau_\iota}\})$ depending on whether N_{τ_ι} is pluripotent or not.
- $\langle W^{\beta_\iota, s_{\tau_\iota}}_{M^*_{\tau_\iota}}, w^*_\iota\rangle \in h_{\tau_\iota}(\mu_{\tau_\iota} \cup \{d_{\tau_\iota}\})$ or $\langle W^{\beta_\iota, s_{\tau_\iota}}_{M^*_{\tau_\iota}}, w^*_\iota\rangle \in \tilde{h}_{\tau_\iota}(\mu_{\tau_\iota} \cup \{p_{\tau_\iota}\})$ depending on whether N_{τ_ι} is pluripotent or not. Here w^*_ι is the preimage of $s_{\tau_\iota} - (\beta_\iota + 1)$ under the canonical witness map.
- $\langle W^{\beta_\iota, r_{\tau_\iota}\cup s_{\tau_\iota}}_{N_{\tau_\iota}}, t_\iota\rangle \in h_{\tau_\iota}(\mu_{\tau_\iota} \cup \{d_{\tau_\iota}\})$ or $\langle W^{\beta_\iota, r_{\tau_\iota}\cup s_{\tau_\iota}}_{N_{\tau_\iota}}, t_\iota\rangle \in \tilde{h}_{\tau_\iota}(\mu_{\tau_\iota} \cup \{p_{\tau_\iota}\})$ depending on whether N_{τ_ι} is pluripotent or not. Here t_ι is the preimage of $r_{\tau_\iota} \cup s_{\tau_\iota} - (\beta_\iota + 1)$ under the canonical witness map.

The first clause follows from Lemma 2.13, as $M^*_{\tau_\iota}$ is a protomouse. The second clause is a consequence of the first one and the second part of Lemma 2.18, which guarantees that $\langle W^{\beta_\iota, s_{\tau_\iota}}_{M^*_{\tau_\iota}}, w^*_\iota\rangle$ is Σ_0-definable from $\langle Q_\iota, w_\iota\rangle, \beta_\iota$ and $\vartheta^*_{\tau_\iota} \stackrel{\text{def}}{=} \vartheta(M^*_{\tau_\iota})$. Now it suffices to notice that $\vartheta^*_{\tau_\iota} < \mu_{\tau_\iota}$ and both $\langle Q_\iota, w_\iota\rangle$ and β_ι are in the hull in question, so the same also holds of $\langle W^{\beta_\iota, s_{\tau_\iota}}_{M^*_{\tau_\iota}}, w^*_\iota\rangle$. The third clause then follows from d) of Lemma 2.16. Recall that $r_{\tau_\iota} = r^*_{\tau_\iota} \cup s_{\tau_\iota}$ where $r^*_{\tau_\iota} = p_{\tau_\iota} - q^*_{\tau_\iota}$ (resp. $r^*_{\tau_\iota} = d_{\tau_\iota} - q^*_{\tau_\iota}$, if N_{τ_ι} is pluripotent).

Let $\pi'_\iota : N'_{\tau_\iota}(\mu_{\tau_\iota}) \to N_{\tau_\iota}$ be the associated uncollapsing map and let

$$(\beta'_\iota, r'_{\tau_\iota}, Q'_\iota, t'_\iota) = \pi'^{-1}_\iota(\beta_\iota, r_{\tau_\iota}, W^{\beta_\iota, r_{\tau_\iota}\cup s_{\tau_\iota}}_{N_{\tau_\iota}}, t_\iota).$$

Then $\langle Q'_\iota, t'_\iota\rangle$ is a generalized witness for β'_ι with respect to $N'_{\tau_\iota}(\mu_{\tau_\iota})$ and r'_{τ_ι}. Notice also that $\beta'_\iota \geq \mu_{\tau_\iota}$, as $\beta_\iota \geq \mu_{\tau_\iota}$ and $\pi'_\iota \restriction \mu_{\tau_\iota} = \text{id}$. Now we can proceed exactly as in the proof of Lemma 3.9. First of all, once $\langle Q'_\iota, t'_\iota\rangle$ is in $N'_{\tau_\iota}(\mu_{\tau_\iota})$, we know that also the standard witness W'_ι for β'_ι with respect to $N'_{\tau_\iota}(\mu_{\tau_\iota})$ and r'_{τ_ι} is in $N'_{\tau_\iota}(\mu_{\tau_\iota})$. Consider two cases. Granting that either N_{τ_ι} is not pluripotent or else $\lambda_{N^*_{\tau_\iota}}(\mu_{\tau_\iota}, r_{\tau_\iota}) > \mu_{\tau_\iota}$, we conclude that every subset of $\beta'_\iota \geq \mu_{\tau_\iota}$ which is $\Sigma^{(n_{\tau_\iota})}_1(N'_{\tau_\iota}(\mu_{\tau_\iota}))$ in r'_{τ_ι} is $\mathbf{\Sigma}^{(n_{\tau_\iota})}_1(W'_\iota)$, and therefore is an element of $N'_{\tau_\iota}(\mu_{\tau_\iota})$. Thus, $p_{N'_{\tau_\iota}(\mu_{\tau_\iota})}$ must be a *proper* lengthening of r'_{τ_ι}. By Lemma 2.22, this means that $(\mu_{\tau_\iota}, q_{\tau_\iota})$ cannot be strong, a contradiction. In the remaining case where N_{τ_ι} is pluripotent and $\lambda_{N^*_{\tau_\iota}}(\mu_{\tau_\iota}, r_{\tau_\iota}) = \mu_{\tau_\iota}$, the same argument yields that $E^{N'_{\tau_\iota}(\mu_{\tau_\iota})}_{\text{top}} | \mu_{\tau_\iota}$, being a cutback of the top extender of W'_ι, is an element of $N'_{\tau_\iota}(\mu_{\tau_\iota})$. By Lemma 2.23, the divisor $(\mu_{\tau_\iota}, d_{\tau_\iota})$ is not strong, which again contradicts the choice of $(\mu_{\tau_\iota}, d_{\tau_\iota})$. $\Box$(Lemma 3.12)

Let $\bar{\tau}$ be the least as in the previous lemma. Then $B_\tau - \bar{\tau} = D - \bar{\tau}$ is closed and unbounded in τ. This completes the proof of Lemma 3.5 for $\tau \in \mathcal{S}^1$.

Proof of Main Theorem. The construction from the previous subsections proves Theorem 3.1 and thus the implication d) $\longrightarrow$ a) of Main Theorem. Notice also that a) $\longrightarrow$ b) is trivial, so it suffices to focus on the proofs of b) $\longrightarrow$ c) and c) $\longrightarrow$ d)

Proof of b) $\longrightarrow$ c)**.** Given a subcompact cardinal κ, we show that $\Box_{\kappa,<\kappa}$ fails. Assume for a contradiction that $\mathcal{C} = \langle \mathcal{C}_\alpha;\ \kappa < \alpha < \kappa^+\ \&\ \lim(\alpha)\rangle$ is such a sequence. Using a simple coding device, $\mathcal{C}$ can be treated as a subset of κ^+. Fix a cardinal μ, a sequence $\bar{\mathcal{C}} = \langle \bar{\mathcal{C}}_\alpha;\ \mu < \alpha < \mu^+\ \&\ \lim(\alpha)\rangle$ and an embedding

$$\sigma : \langle H_{\mu^+}, \bar{\mathcal{C}}\rangle \longrightarrow \langle H_{\kappa^+}, \mathcal{C}\rangle$$

witnessing the subcompactness of κ relative to $\mathcal{C}$. We derive a contradiction by showing that $\mathrm{cf}(\mu^+) \le \mu$. Letting $\nu = \sup(\sigma''\mu^+)$, we know that $\mathrm{cf}(\nu) = \mu^+$. Fix a $C \in \mathcal{C}_\nu$; then C is a closed unbounded subset of ν. We show that each proper initial segment of C is of order type strictly less than ν.

Notice that there are cofinally many $\bar{\gamma}$ in μ^+ such that $\mathrm{cf}(\bar{\gamma}) = \omega$ and $\sigma(\bar{\gamma}) = \sup(\sigma''\bar{\gamma}) \in C$. This follows from the standard closure argument and the fact that σ is continuous at limits of length strictly less than μ. (Given a $\bar{\gamma}_0$, inductively pick a $\bar{\gamma}_{n+1}$ such that $\sigma(\bar{\gamma}_n) < \gamma' < \sigma(\bar{\gamma}_{n+1})$ for some $\gamma' \in C$; then $\bar{\gamma} = \sup_n \bar{\gamma}_n$ meets the above requirements.) Now pick any such $\bar{\gamma}$. Since $\gamma = \sigma(\bar{\gamma})$ is ω-cofinal, $\mathrm{otp}(C) < \kappa$ for all $C \in \mathcal{C}_\gamma$. This follows from the regularity of κ (see the discussion below the definition of subcompactness in the introduction). Applying the regularity of κ to the fact that $|\mathcal{C}_\gamma| < \kappa$, we find an $\varepsilon < \kappa$ such that $\mathrm{otp}(C) < \varepsilon$ whenever $C \in \mathcal{C}_\gamma$. The least such ε, being definable from $\mathcal{C}_\gamma \in \mathrm{rng}(\sigma)$, must be in $\mathrm{rng}(\sigma)$ as well, so $\varepsilon < \mu$. Thus, $C \cap \gamma$, being an element of $\mathcal{C}_\gamma$, must be of ordertype strictly less than μ. This completes the proof of the implication b) $\longrightarrow$ c).

Proof of c) $\longrightarrow$ d)**.** We argue in $\mathbf{L}[E]$. Starting from the hypothesis that there are stationarily many $\nu < \kappa^+$ such that $E_\nu \neq \varnothing$, we show that κ is subcompact. Assume for a contradiction that this is not the case; let A be the $<_{\mathbf{L}[E]}$-least subset of $\tau = \kappa^+$ witnessing this. Notice that $A \in J^E_{\tau^+}$, the support $|J^E_\tau|$ of J^E_τ is precisely H_τ and that any elementary embedding $\sigma : \langle H_{\mu^+}, \bar{A}\rangle \to \langle H_\tau, A\rangle$ of our interest, being a bounded subset of H_τ, is an element of H_τ. Consequently, the fact that A witnesses the failure of subcompactness for κ can be expressed as $M \models \varphi(A,\tau)$ where $M = \mathbf{L}[E] \,||\, \tau^+$ and $\varphi(A,\tau)$ is the Σ_1-statement

$$(\exists H)[H = |J^E_\tau| \;\&\; \neg(\exists \kappa,\mu,\vartheta,\bar{H},\bar{A},\sigma \in H)\tilde{\varphi}(\mu,\kappa,\vartheta,\sigma,\bar{H},H,\bar{A},A)];$$

here, letting Sat_N be the satisfaction predicate for the structure N, (recall that Sat_N is uniformly Σ_0-definable in the parameter N), $\tilde{\varphi}(\kappa,\mu,\vartheta,\sigma,\bar{H},H,\bar{A},A)$ is the conjunction of the following formulae:

- κ is the largest cardinal in H;
- $\mu < \kappa \;\&\; \vartheta = \mu^{+H} \;\&\; \bar{H} = (H_\vartheta)^H \;\&\; \bar{A} \subset \vartheta$;
- $\sigma : \bar{H} \to H \;\&\; \mathrm{cr}(\sigma) = \mu$;
- $(\forall x \in \bar{H})(\forall \psi(v) \in \mathrm{Fml})[\mathrm{Sat}_{\langle \bar{H},\bar{A}\rangle}(\psi(v),x) \;\longleftrightarrow\; \mathrm{Sat}_{\langle H,A\rangle}(\psi(v),\sigma(x))]$.

A standard closure argument shows that the set of all ordinals $\nu < \tau$ satisfying $\nu \cap h_M(\nu \cup \{\tau\}) = \nu$ is closed unbounded in τ. Since we assume that $E_\nu \neq \varnothing$ for stationarily many ν, we can pick a ν satisfying both

$$\nu \cap h_M(\nu \cup \{\tau\}) = \nu \quad \text{and} \quad E_\nu \neq \varnothing. \tag{31}$$

Let M' be the transitive collapse of $h_M(\nu \cup \{\tau\})$ and let σ' be the associated uncollapsing map. Then M' is a passive premouse, $\omega\varrho^1_{M'} \leq \nu = \mathrm{cr}(\sigma')$ and $\sigma'(\nu) = \tau$; this last equality follows from the first part of (31). Thus, $\nu = \kappa^{+M'}$. By the condensation lemma, M' is solid and its standard parameter is universal, so $\tilde{M} = \mathrm{core}_\nu(M')$ exists. Let $\tilde{\sigma} : \tilde{M} \to M'$ be the associated core map. Notice that $\nu = \kappa^{+\tilde{M}}$, as any surjection $g : \alpha \to \nu$ such that $g \in \tilde{M}$ and $\alpha < \nu$ would give rise to the surjection $\tilde{\sigma}(g) \restriction \alpha : \alpha \to \nu$ that is an element of M'. Consequently, the critical point of $\tilde{\sigma}$, being at least ν, must be *strictly* larger than ν, as every ordinal between κ and $\tilde{\sigma}(\nu)$ has M'-cardinality equal to κ. Letting $\sigma = \sigma' \circ \tilde{\sigma} : \tilde{M} \to M$, the above discussion can be summarized as follows:

- $\omega\varrho^1_{\tilde{M}} \leq \nu$ and $\tilde{M}$ is sound above ν;
- $\mathrm{cr}(\sigma) = \nu$ and $\sigma(\nu) = \tau$;
- σ is Σ_1-preserving.

These conclusions enable us to apply the condensation lemma to $M, \tilde{M}$ and σ; we show that clause d) of the condensation lemma must be true. Indeed, both a) and b) are impossible, as $E_\nu^{\tilde{M}} = \varnothing \neq E_\nu$; the equality on the left follows from the fact that ν is a cardinal in $\tilde{M}$. Clause c) fails, as it requires that $\tilde{M} = \mathrm{Ult}^*(M\,||\,\zeta, E_\alpha)$ where $M\,||\,\zeta$ is the longest initial segment of M which does not collapse ν and E_α is an extender with critical point κ for some $\alpha \geq \nu$. In our case, however, $\zeta = \nu$, so the only candidate for α would be ν. This yields an immediate contradiction, as $\mathrm{cr}(E_\nu)$ is certainly smaller than κ. It follows that d) of the condensation lemma holds. Set $M' = \mathrm{Ult}(M\,||\,\nu, E_\nu)$. Notice that $\tilde{M}$ is in fact a proper initial segment of $\bar{M} = M'\,||\,\nu^{+M'} = \mathrm{Ult}(M\,||\,\vartheta^{+M}, E_\nu)$ where $\vartheta = \mu^{+M}$ and $\mu = \mathrm{cr}(E_\nu)$, as $\tilde{M}$ projects to ν.

We are now ready to derive the final contradiction. First we observe that A is in the range of σ. Indeed, as σ is Σ_1-preserving and $\sigma(\nu) = \tau$, there is a set $B \in \tilde{M}$ such that $\tilde{M} \models \varphi(B, \nu)$. The $<_{\tilde{M}}$-least such set, call it $\tilde{A}$, must be the preimage of A under σ, as otherwise $A <_M \sigma(\tilde{A})$ and the preservation properties of σ would force the existence of some $\tilde{A}^* <_{\tilde{M}} \tilde{A}$ for which $\varphi(\tilde{A}^*, \nu)$ holds in $\tilde{M}$, a contradiction. Notice also that $\tilde{A}$ is the $<_{\tilde{M}}$-least set B such that $\varphi(B, \nu)$ holds in $\tilde{M}$, as for any $\tilde{A}' <_{\tilde{M}} \tilde{A}$ satisfying $\varphi(\tilde{A}', \nu)$, the set $\sigma(\tilde{A}') <_M A$ would witness the failure of subcompactness of κ, which would contradict our choice of A. Finally, as $\tilde{M}$ is a proper initial segment of $\bar{M}$, the set $\tilde{A}$ satisfies $\varphi(\tilde{A}, \nu)$ in $\bar{M}$ and $\tilde{A}$ is $<_{\bar{M}}$-minimal with this property. This means that $\tilde{A}$ is definable in $\bar{M}$ from the single parameter ν; it follows that $\tilde{A}$ is in the range of the associated ultrapower map $\bar{\pi} : M\,||\,\vartheta^{+M} \to \bar{M}$. Set $\bar{A} = \bar{\pi}^{-1}(\tilde{A})$ and $\pi = \sigma \circ \bar{\pi}$. Notice that σ defined merely on a proper initial segment of $\bar{M}$; however, this will not do any harm, since the objects of our interest H_τ and $\tilde{A}$ are in the domain of σ. Then $\pi(H_\vartheta, \bar{A}) = (H_\tau, A)$, so

$$\pi \restriction \bar{H} : \langle H_\vartheta, \bar{A} \rangle \longrightarrow \langle H_\tau, A \rangle$$

guarantees that A cannot be a witness to the failure of subcompactness of κ. Contradiction. This completes the proof of c) $\longrightarrow$ d) and thus that of Main Theorem.

□(Main Theorem)

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