DISCUSSION 6/25/12

1. If U is a normal measure on κ w.r.t./over $M \models ZFC^-$, say $M = J_\tau$ then $\tau \leq$ well-founded part of $ult(M, U)$. We are not assuming $U \in M$.

2. U is weakly amenable w.r.t. M. iff
   a) If $f : \kappa \to \mathcal{P}(\kappa)$ and $f \in M$ then $\{\xi < \kappa \mid f(\xi) \in U\} \in M$.
   b) $\mathcal{P}(\kappa) \cap M = \mathcal{P}(\kappa) \cap Ult(M, U)$.

3. U is weakly amenable w.r.t. M then κ is weakly compact in M, and $\{\alpha < \kappa \mid \alpha$ weakly compact in $M\} \in U$.

4. M, M' transitive, arbitrary. $\sigma : M \to M'$ Σ_0-preserving and cofinal
   a) σ is Σ_1-preserving
   b) Let $A \subseteq M$ s.t. (M, A) is amenable, i.e. $x \cap A \in M$ all $x \in M$.

Let $A' = \bigcup\{\sigma(A \cap x) \mid x \in M\}$. Then (M', A') is amenable and $\sigma : (M, A) \to (M', A')$ is Σ_0-preserving wrt language $\{\in, A\}$.

5. Let (M, A) be amenable, U be an ultrafilter over M, and $Ult((M,A),U)$ well-founded, say $(M', A') = Ult((M,A),U)$. Then A' is as in 4.

6. If $M = \langle J_\tau, U \rangle$ is a premouse and $\langle M_i \mid i < \lambda \rangle$ is an iteration then the ultrapower maps are
   - fully elementary in the language $\{\in\}$.
   - Σ_0-preserving in the language $\{\in, \dot{U}\}$
   - cofinal, hence Σ_1-preserving in the language $\{\in, U\}$.

7. "Shift" Lemma:
(Successor step in the copying construction)
Let $\sigma : \bar{M} \to M$ be Σ_0-preserving, $\sigma(\bar{\kappa}) = \kappa$, $\bar{U}$ a normal measure over $\bar{M}$ on $\bar{\kappa}$, U a normal measure over M on κ.
s.t. $X \in \bar{U} \Rightarrow \sigma(X) \in U$. Let $M' = Ult(M, U)$ and $\bar{M}' = Ult(\bar{M}, \bar{U})$.

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Then there is a unique Σ_0-preserving map $\sigma' : \bar{M}' \to M'$ s.t.

- $\sigma'(\bar{\kappa}) = \kappa$
- The diagram

$$\begin{tikzcd} M \arrow[r, "\pi_U"'] & M' \\ \bar{M} \arrow[u, "\sigma"] \arrow[r, "\pi_{\bar{U}}"'] & \bar{M}' \arrow[u, "\sigma'"'] \end{tikzcd}$$

commutes.

The map is defined by $\sigma'(\pi(f)(\bar{\kappa})) = \pi \circ \sigma(f)(\kappa)$

In fact: $\sigma' \upharpoonright \bar{\kappa}^{+\bar{M}} = \sigma \upharpoonright \bar{\kappa}^{+\bar{M}}$.

8. Use the shift lemma to complete the proof of the copying construction.

Find an example of J_α and a normal measure (in ZFC) U over J_α s.t. U on κ, $J_\alpha \models ZFC^- + \kappa^+$ exists such that J_α can see $\wp(\kappa)$ as an element and s.t. $Ult(J_\alpha, U)$ well-founded, but U not weakly amenable over J_α.

9. If $\tau = (\kappa^+)^{J_\alpha}$ then $J_\tau \models ZFC^-$.

10. If $\tau = (\kappa^+)^{J_\alpha}$ and $f : \kappa \twoheadrightarrow \gamma$ is a surjection then there is $a \subseteq \kappa$ that codes a well-ordering of order-type γ.
Easier: $a \subseteq \kappa \times \kappa$ s.t. a is a well-ordering of order-type γ.

11. If $\tau = (\kappa^+)^{J_\alpha}$, $a \in J_\alpha$ and $a \subseteq \kappa$ then $a \in J_\tau$ (like the proof of GCH).

12. $p \in R_\alpha \Rightarrow p \in P_\alpha$.
(diagonalization arguement.)

13. ρ_α is a $\underset{\sim}{\Sigma}_1$-cardinal over J_α.

14. If κ is a cardinal in J_α then $(H_\kappa)^{J_\alpha} = J_\kappa$.

15. $<^*$ is a well-ordering on $[On]^{<\omega}$.

16. Try to check the details about B_τ.

17. Prove B_τ is closed and
$\bar{\tau} \in B_\tau \Rightarrow B_{\bar{\tau}} = B_\tau \cap \bar{\tau}$.

18. Try to prove the claim:
$\tau^* < \bar{\tau}$ in $B_\tau \Rightarrow \delta_{\tau^*} < \delta_{\bar{\tau}}$.