EXERCISES 4.7.2012 (1)

(1) Let $\bar{M}, M$ be premice and assume M is normally iterable. Let $\sigma : \bar{M} \to M$ be $\Sigma_0^{(n)}$-elementary map. Prove that $\bar{M}$ is normally iterable above $\rho^{n+1}_{\bar{M}}$, that is, any normal iteration with critical points $\geq \rho^{n+1}_{\bar{M}}$ has well-founded last model and has only finitely many truncations.

Hint: Show that ~~iterations~~ normal iterations of $\bar{M}$ above $\rho^{n+1}_{\bar{M}}$ can be copied onto normal iterations of M via the map σ

(2) Let M be iterable premouse and $M \rightsquigarrow M'$ be an iteration with critical points $\geq \rho^{n+1}_{M}$. Let $\sigma : M \to M'$ be a $\Sigma_0^{(n)}$-map. Prove that the iteration does not involve any truncation.

Hint: Imitate the proof of Dodd-Jensen lemma; you need to check that the ~~map~~ copy map have sufficient preservation degree to carry out the copy construction.

(3) Let M be iterable premouse and $\sigma : \bar{M} \to M$ be a $\Sigma_0^{(n)}$-map such that $\rho^{n+1}_{\bar{M}} \leq \mathrm{cr}(\sigma)$. Show that $\bar{M}$ is coiterable with M, and if $M', \bar{M}'$ are the last models in the coiteration then

(a) The iteration $\bar{M} \rightsquigarrow \bar{M}'$ does not involve any truncation

(b) $\bar{M}'$ is an initial segment of M', i.e. $\bar{M}' \trianglelefteq M'$. ($\bar{M}' = M'$ is possible.)

(4) Let $\sigma : \bar{M} \to M$ where M is an acceptable structure. Let U be an ω-complete measure over M on κ. Let $\bar{U}$ be a measure over $\bar{M}$ over $\bar{\kappa}$, and assume $\bar{M}$ is countable. Assume $x \in \bar{U} \Rightarrow \sigma(x) \in U$ all $x \in \mathcal{P}(\bar{\kappa}) \cap \bar{M}$

EXERCISE 4.7-2012 (2)

Assume σ is $\Sigma_0^{(n)}$-preserving where n is such that $\rho_{\bar{M}}^{n+1} \leq \bar{\nu} < \rho_{\bar{M}}^{n}$. Let

$$\pi : \bar{M} \xrightarrow[\nu]{*} \bar{M}'$$

Prove that there is a $\Sigma_0^{(n)}$-elementary $\sigma' : \bar{M}' \to M$ such that the diagram commutes:

$$\begin{tikzcd} M & \\ \bar{M} \arrow[u, "\sigma"] \arrow[r, "\pi"'] & \bar{M}' \arrow[ul, "\sigma'"'] \end{tikzcd}$$

(5) Let M be a premouse such that

(a) If E_ν^M is an M-ultrafilter, i.e. E_ν^M measures all sets in $\mathcal{P}(\mathrm{cr}(E_\nu^M)) \cap M$ then E_ν^M is ω-complete

(b) Every proper initial segment of M is iterable (normally iterable)

Use Exercise (4) to prove that M is normally iterable.

(6) Let M be normally iterable premouse.

(a) Let E_ν^M be a total measure on M (i.e. E_ν^M measures all sets in $\mathcal{P}(\mathrm{cr}(E_\nu^M)) \cap M$.) Let $\langle M_i : i \leq \omega_1 \rangle$ be an iteration of M obtained by applying the image of ν at each step, i.e.

$$\nu_i = \pi_{0i}(\nu)$$

where $\pi_{0i} : M \to M_i$ is the iteration map.

Prove that $E_{\nu_{\omega_1}}^{M_{\omega_1}}$ is ω-complete where $\nu_{\omega_1} = \pi_{0,\omega_1}(\nu)$

(b) More generally, define an iteration $\langle M_i : i \leq \theta \rangle$ where $\theta > |M|$ is regular as follows:

- Let ν^0 = the least ν s.t. E_ν^M ~~such that~~ is a total measure ~~[illegible]~~ over E_ν^M

Construct the iteration $\langle M_i^0 : i \leq \omega_1 \rangle$ as in (a) using E_ν^M. Denote the image of ν^0 by $\nu_{\omega_1}^0$

EXERCISES 4.7.2012 (3)

- Assume $M^{j}_{\omega_1}$ has been defined. Define

r_0^{j+1} = the least $r > r^{j}_{\omega_1}$ such that

$E_r^{M^j_{\omega_1}}$ is a total ~~extend~~ measure over $M^j_{\omega_1}$, and construct the iteration $\langle M_i^{j+1} \mid i \le \omega_1 \rangle$ as in (a). Let $r^{j+1}_{\omega_1}$ be the image of r_0^{j+1} under the map $\pi^{j+1}_{0,\omega_1} : M^j_{\omega_1} \to M^{j+1}_{\omega_1}$

- Form direct limits if j is a limit

The construction stops in $< \Theta$ steps, resulting in a premouse M'. Prove that every total measure $E_r^{M'}$ is ω-complete.

(7) Let $\sigma : \bar{M} \to M$ be the inverse of the collapsing map coming from collapsing $h^n_M(\rho^n \cup \{p_M\})$. Fill in the details in the proof that

(a) if $M', \bar{M}'$ are the last models in the coiteration of M with $\bar{M}$ then
  (i) $M' = \bar{M}'$ and there is no truncation on either side
  (ii) $\mathcal{P}(\rho^n_M) \cap \bar{M} = \mathcal{P}(\rho^n_M) \cap M$

Here of course we need to assume that M is iterable.

(8) In the situation from (7), prove that $\rho^k_{\bar{M}} = \rho^k_M$ for all $k \geq n$.

(9) Let M be iterable premouse, $\bar{M}$ be a premouse, and $\sigma : \bar{M} \xrightarrow[\Sigma_0^{(n)}]{} M$ be such that $\sigma \upharpoonright \rho^{n+1}_{\bar{M}} = id$.

Assume also that $\bar{M}$ is $(n+1)$-sound and M is sound. Prove that $\bar{M}$ is an initial segment of M.

<u>Hint</u> this is a coiteration argument.

(10) Let M be iterable, $\sigma : \bar{M} \to M$ be Σ_0-elementary, and $\sigma \upharpoonright \kappa = id$. Let $\tau = \kappa^{+\bar{M}}$. Prove that $E^{\bar{M}} \upharpoonright \tau = E^{M} \upharpoonright \tau$.

<u>Hint</u> ~~[illegible]~~ Let $\beta < \tau$ be such that $\bar{M} \| \beta$ projects to κ. Apply (9) to $\sigma \upharpoonright (\bar{M} \| \beta) : \bar{M} \| \beta \to M \| \sigma(\beta)$

(11) Let $\mathcal{T}$ be an iteration tree with models M_i. Recall that if i is the immediate $\mathcal{T}$-predecessor of $j+1$ then $M^{\mathcal{T}}_{j+1} = \mathrm{Ult}(M_i, E^{\mathcal{T}}_j)$ where $E^{\mathcal{T}}_j$ is an extender in $M^{\mathcal{T}}_i$ such that

$$\nu_{\rho_j}^{M^{\mathcal{T}}_{j+1}} = \nu_{\rho_j}^{M^{\mathcal{T}}_j}$$

where $\rho_j = \mathrm{length}(E^{\mathcal{T}}_j)$. Prove:

$$i < j \implies \nu_{\rho_i}^{M^{\mathcal{T}}_i} = \nu_{\rho_i}^{M^{\mathcal{T}}_j}$$

(i.e. complete the proof that was sketched in the lecture.)

(12) ~~Exer~~ Let $\mathcal{T}$ be an iteration tree on M of limit length and let S be a cofinal subset of $\delta(\mathcal{T})$. Then there is at most one branch - cofinal branch through $\mathcal{T}$ such that $S \subseteq \mathrm{rng}(\pi^{\mathcal{T}}_b)$

(13) Let $\mathcal{T}$ be an iteration tree with two cofinal branches. Then $\mathrm{cf}(\mathrm{lh}(\mathcal{T})) = \omega$. (Of course $\mathcal{T}$ is of limit length.)

EXERCISES 4.4.2012 (5)

(14) Assume $\mathcal{T}$ is an iteration tree of limit length θ. Let $\alpha < \theta$ be a limit ordinal, and let

$$b_\alpha = \text{the branch } [0, \alpha)_{\mathcal{T}}$$

that is $\xi \in b_\alpha$ iff $0 \leq_{\mathcal{T}} \xi <_{\mathcal{T}} \alpha$. Prove that $\langle b_\alpha \mid \alpha \in \lim \cap \theta \rangle$ is a coherent sequence of clubs

(15) Prove that if $\mathcal{T}$ is a normal iteration tree of length θ. If θ is measurable then $\mathcal{T}$ has a unique cofinal well-founded branch. The same is true if θ is weakly compact.

(16) Suppose $\square(\theta)$ fails and T is a normal iteration tree of length θ. Then $\mathcal{T}$ has a cofinal well-founded branch.

(17) Given two iterable premice M, N, write

$M \equiv_{DJ} N$ iff in the coiteration of M with N, there is no truncation on either side and the last models agree

$M <_{DJ} N$ iff in the coiteration of M with N, there is a truncation on the N-side or the last model on the M-side is a proper initial segment of the last model on the N-side

Prove: (a) $\equiv_{DJ}$ is an equivalence relation on all iterable ~~pre~~ premice

(b) $\leq_{DJ}$ is a linear preordering on all iterable premice when "$\leq_{DJ}$"

has the obvious meaning. i.e. $\leq_{DJ}$ is reflexive, transitive, and total, i.e. for any pair M, N of iterable premice we have $M \leq_{DJ} N$ or $N \leq_{DJ} N$

ⓒ $<_{DJ}$ is well-founded,

So $\leq_{DJ}$ is a prewellordering of all iterable premice. This prewellordering is called "Dodd-Jensen" prewellordering.