Where Should I Open My Restaurant?

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Abstract

We answer a question in real estate that is a variation of a problem that arises in many combinatorics and discrete mathematics courses.

There are three things that matter in property: *location, location, location.*

Unknown. Possibly Lord Samuel of Britain [8]

There is a lot of truth in this opening quote, especially applied to commercial real estate. If you want your restaurant to succeed, customers have to walk through the door.

1 The Problem

A walk on the $\mathbb{Z}^2$ lattice using only the steps $(0, 1)$, which we call 'North', and $(1, 0)$, which we call 'East', is a *lattice path.*

Question 1. *1. How many lattice paths begin at* $(0, 0)$ *and end at* (m, n)*?*

2. What proportion of these paths pass through the point (a, b)*?*

Versions of these elementary exercises appear in many introductory combinatorics textbooks, as lattice paths provide a nice setting to explore identities involving binomial coefficients. See for example [3, Chapter 4], [2, Section 3.5], [4, Section 8.5], and [5, Section 2.6].

Solution. 1. A path from $(0, 0)$ to (m, n) consists of m steps East, and n steps North, and can be uniquely identified with a word of length $m + n$ consisting of m E's and n N's. Therefore, there are $\binom{m+n}{m}$ total lattice paths.

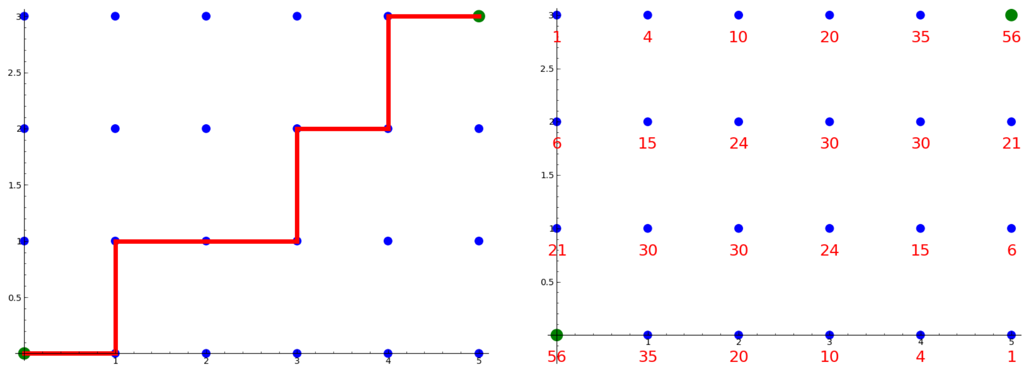

Figure 1: (Left): A path of minimal length from $(0,0)$ to $(5,3)$
(Right): The number of such paths passing through each point.

2. Every lattice path passing through (a,b) consists of two parts: a lattice path from $(0,0)$ to (a,b), followed by a lattice path from (a,b) to (m,n). The number of paths of the first type is $\binom{a+b}{a}$ and paths of the second type are in bijection with lattice paths from $(0,0)$ to $(m-a,n-b)$. Therefore, the proportion of paths passing through (a,b) is

$$F_{m,n}(a,b) := \frac{\binom{a+b}{a}\binom{m-a+n-b}{m-a}}{\binom{m+n}{m}}. \tag{1}$$

This function is our main object of study. We think of the integers m and n being fixed and therefore focus on the numerator, which we denote $f_{m,n}(a,b)$.

Question 2. *In your town, everyone lives in one giant apartment building at $(0,0)$ and works in the local factory at (m,n). For some strange reason, locals choose their walks to work randomly among all minimal lattice paths.*

Nathan has plans to open a restaurant at some point (a,b). If you happen to pass by on your way to work, you will buy a coffee and muffin, arriving at work caffeinated, satisfied, and productive. If you do not pass Nathan's, you arrive at work miserable and tired.

Zoning regulations forbid building on the occupied vertices $(0,0)$ and (m,n), but otherwise Nathan is free to choose where to locate his establishment. Where should he open his restaurant in order to maximize the chance that people will visit?

We rephrase this question in terms of the function defined above.

Question 3. *Given $m,n>0$ what is the maximum value of $f_{m,n}(a,b)$ subject to $0 \le a \le m$, $0 \le b \le n$, and $0 < a+b < m+n$?*

Basic symmetries of binomial coefficients imply that $f_{m,n}(a,b) = f_{n,m}(b,a)$ and $f_{m,n}(a,b) = f_{m,n}(m-a,n-b)$. Therefore, answering this question for

$m \geq n$ and subject to the additional constraint $0 < a + b \leq \frac{m+n}{2}$, is enough to solve it in all instances.

Theorem 1. *Let m and n be positive integers with $m \geq n$. The maximum value of $f_{m,n}(a,b)$ subject to $0 \leq a \leq m,\ 0 \leq b \leq n$, and $0 < a + b \leq \frac{m+n}{2}$ is given by $f_{m,n}(1,0) = \binom{m+n-1}{m-1}$ if $m > n$ and $f_{m,n}(1,1) = 2\binom{m+n-2}{m-1}$ if $m = n$.*

We say that a point (a', b') *gives a maximum* if $f_{m,n}(a', b')$ solves the optimization problem in Question 3

The solution we give is elementary, but not obvious. Instead of solving a two variable optimization problem we restrict to certain single variable refinements, finding maximum values by analyzing ratios of consecutive terms. We first focus on the 'square case' $m = n$, and use it in the proof of the general case. In the final section we discuss connections to the Gamma function, the hypergeometric distribution, and higher-dimensional lattice paths.

2 The Proof

Before giving a formal proof of Theorem 1 we give some intuition for why it is true. Since all of our paths start at $(0,0)$ and end at (m,n) it seems reasonable that points (a,b) that are close to $(0,0)$ or to (m,n) will have many paths passing through them. Since the most direct path from $(0,0)$ to (m,n) in $\mathbb{R}^2$ is given by the straight line $y = \frac{n}{m}x$, we expect that most lattice paths do not stray too far form this line.

In order to test this intuition we consider $f_{m,n}(a,b)$ for some points close to the origin, those with $a + b \in \{1,2\}$. We compare these points by taking the ratio of the corresponding function values. Suppose that $n \geq 2$. Since $m \geq n$, we have

$$\frac{f_{m,n}(1,0)}{f_{m,n}(0,1)} = \frac{\binom{m+n-1}{m-1}}{\binom{m+n-1}{n-1}} = \frac{m}{n} \geq 1.$$

Similarly, $f_{m,n}(2,0) \geq f_{m,n}(0,2)$. We also have

$$\frac{f_{m,n}(2,0)}{f_{m,n}(1,1)} = \frac{m-1}{2n}.$$

Which of $(2,0)$ and $(1,1)$ has more paths passing through it depends on the ratio $\frac{m-1}{n}$, which is closely related to the slope of the line $y = \frac{n}{m}x$.

It is easy to check that $f_{m,n}(1,0) \geq f_{m,n}(2,0)$, so the only other comparison to make is

$$\frac{f_{m,n}(1,0)}{f_{m,n}(1,1)} = \frac{m+n-1}{2n},$$

which is at least 1 except in the square case, $m = n$. This square case is the only situation in which the point $(1,1)$ actually lies on the line $y = \frac{n}{m}x$, making up for the fact that it is two steps from the origin instead of one.

Our goal is to make this reasoning more precise. We first prove Theorem 1 in the square case, which involves optimizing $f_{n,n}(a,b)$ restricted to two different kinds of diagonal lines. We then use this result to prove the theorem for general rectangular grids, inducting on the size of $m - n$.

2.1 The Square Case

Proof of Theorem 1 for $m = n$. The idea of the proof is to turn this two variable optimization problem into a single variable one. For each $k \in [1, 2n-1]$, we find the maximum value of $f_{n,n}(a, b)$ restricted to the diagonal line $a + b = k$. By symmetry, we need only consider $k \in [1, n]$.

Diagonals: $a + b = k$

Consider the set of all $(a, k-a)$ where $a \in [0, k]$. We claim that

$$\frac{f_{n,n}(a-1, k-a+1)}{f_{n,n}(a, k-a)} \geq 1, \text{ if and only if } a \geq \frac{k+1}{2}.$$

As we move along the diagonal consisting of points $(k, 0), (k-1, 1), \ldots, (0, k)$ the value of this function increases as we take steps 'Northwest' until we cross the vertical line $a = \frac{k+1}{2}$. When k is odd the two points closest to the central diagonal $(\frac{k+1}{2}, \frac{k-1}{2})$ and $(\frac{k-1}{2}, \frac{k+1}{2})$ give the same maximum value. When k is even the maximum value occurs at $(\frac{k}{2}, \frac{k}{2})$.

Consider the ratio of consecutive terms

$$\frac{f_{n,n}(a-1, k-a+1)}{f_{n,n}(a, k-a)} = \frac{a}{k-a+1} \cdot \frac{n-k+a}{n-a+1}. \tag{2}$$

This is at least 1 if and only if

$$a(n-k+a) - (k-a+1)(n-a+1) = (n+1)(2a-(k+1)) \geq 0. \tag{3}$$

Since $n+1$ is positive, this holds for $a \in \left[\frac{k+1}{2}, k\right]$.

Diagonals: $a = b$ **and** $a = b + 1$

Since $f_{n,n}(a, b)$ restricted to each line $a + b = k$ reaches a maximum either at $f_{n,n}\left(\frac{k}{2}, \frac{k}{2}\right)$ or at $f_{n,n}(\frac{k+1}{2}, \frac{k-1}{2})$, the symmetry of $f_{n,n}(a, b)$ implies that we need only show that $f_{n,n}(1, 1)$ is a maximum among all $f_{n,n}(a, a)$ for $a \in \left[1, \frac{n}{2}\right]$ and that $f_{n,n}(1, 0)$ is a maximum among all $f_{n,n}(a+1, a)$ for $a \in \left[0, \frac{n-1}{2}\right]$.

We consider the ratio

$$\frac{f_{n,n}(a+k, a)}{f_{n,n}(a+k+1, a+1)} = \frac{(a+1)(a+k+1)}{(2a+k+2)(2a+k+1)} \cdot \frac{(2n-2a-k)(2n-2a-k-1)}{(n-a)(n-a-k)}$$

in the special cases $k = 0$ and $k = 1$. When $k = 0$, this simplifies to

$$\frac{(a+1)(2n-2a-1)}{(2a+1)(n-a)},$$

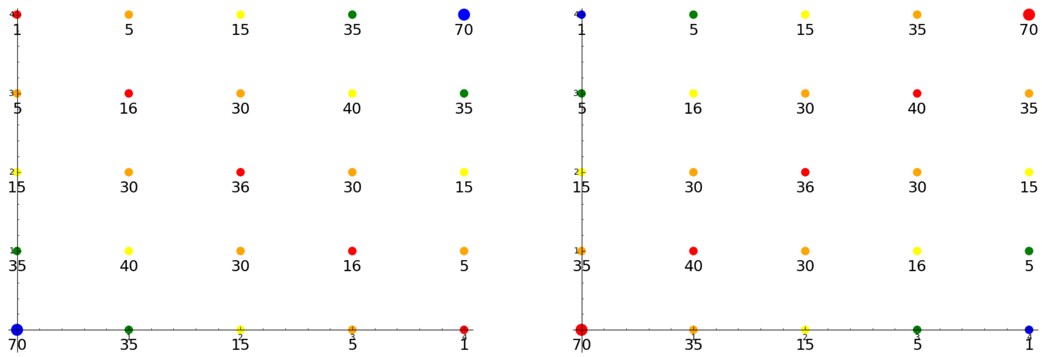

Figure 2: Values of $f_{4,4}(a,b)$. In the first plot vertices are grouped along diagonal lines $a + b = k$. In the second plot vertices are group along diagonal lines $b = a + k$.

which is greater than 1 if and only if

$$(a+1)(2n-2a-1) - (2a+1)(n-a) = n - 1 - 2a \geq 0.$$

This holds for $a \in \left[0, \frac{n-1}{2}\right]$. When $k = 1$, our ratio simplifies to

$$\frac{(a+2)(2n-2a-1)}{(2a+3)(n-a)},$$

which is greater than 1 if and only if

$$(a+2)(2n-2a-1) - (2a+3)(n-a) = n - 2 - 2a \geq 0.$$

This holds for $a \in \left[0, \frac{n-2}{2}\right]$.

We see that the maximum value of $f_{n,n}(a,a)$ subject to $a \in \left[1, \frac{n}{2}\right]$ is given by $f_{n,n}(1,1)$ and that the maximum value of $f_{n,n}(a+1,a)$ subject to $a \in \left[0, \frac{n-1}{2}\right]$ is given by $f_{n,n}(1,0)$. Noting that $f_{n,n}(1,1) > f_{n,n}(1,0)$, completes the the proof of Theorem 1 in the square case. □

In our argument for the general rectangular case we require a slight refinement of the square case.

Lemma 1. *For $n \geq 5$ the largest value of $f_{n,n}(a,b)$ where*

$$(a,b) \notin \{(0,0),(n,n),(1,1),(n-1,n-1)\}$$

is given by $f_{n,n}(1,0)$.

The $n \geq 5$ assumption is necessary. For $n = 4$ the largest value of $f_{4,4}(a,b)$ where $(a,b) \notin \{(0,0),(4,4),(1,1),(3,3)\}$ is given by $f_{4,4}(2,2) = 36 > f_{4,4}(1,0) = 35$.

Proof. A point (a', b') with $0 < a' + b' = k \leq n$ maximizing $f_{n,n}(a, b)$ subject to the conditions of Lemma 1 either gives the maximum value of of $f_{n,n}(a, b)$ among all points on the diagonal line $a+b = k$, or has $a'+b' = 2$, in which case we note that $f_{n,n}(1, 0) \geq f_{n,n}(2, 0)$. Therefore, we need only consider points on the diagonal lines $a = b$ and $a = b + 1$. By the analysis in the proof above, we need only consider the ratio

$$\frac{f_{n,n}(1,0)}{f_{n,n}(2,2)} = \frac{\binom{2n-1}{n-1}}{6\binom{2n-4}{n-2}} = \frac{(2n-1)(2n-2)(2n-3)}{6n(n-1)^2}.$$

This is at least 1 for $n \geq 5$, completing the proof of Lemma 1. □

2.2 The Rectangular Case

Proof of Theorem 1: The Rectangular Case. The idea of the proof is to divide the set of lattice paths from $(0, 0)$ to (m, n) into two disjoint sets: paths passing through $(m - 1, n)$, and paths passing through $(m, n - 1)$. If $(1, 0)$ gives a maximum for $f_{m-1,n}(a, b)$ and for $f_{m,n-1}(a, b)$, then we conclude that $(1, 0)$ gives a maximum for $f_{m,n}(a, b)$.

We proceed by a kind of double induction. For a given value of n we induct on m. The case $n = 0$ is trivial. We give the argument for $n = 1$ in detail, and then adapt it to the general situation. We compute that $f_{2,1}(1, 0) = f_{2,1}(1, 1)$. Since $(1, 0)$ gives a maximum for $f_{2,1}(a, b)$ and for $f_{3,0}(a, b)$, we conclude that $(1, 0)$ also gives a maximum for $f_{3,1}(a, b)$. Suppose that $m \geq 3$ and that $(1, 0)$ gives a maximum for $f_{m,1}(a, b)$. Obviously, $(1, 0)$ gives a maximum for $f_{m+1,0}(a, b)$, so we conclude that $(1, 0)$ gives a maximum for $f_{m+1,1}(a, b)$.

We now argue by induction on n. Suppose that for $k \in [1, n - 1]$ and any $m > k$ we know that $(1, 0)$ gives a maximum for $f_{m,k}(a, b)$. We show that $(1, 0)$ also gives a maximum for $f_{m,n}(a, b)$.

For a fixed n, the base case of our induction is to show that $(1, 0)$ gives a maximum for $f_{n+1,n}(a, b)$. We verify this explicitly for $n = 3$ and $n = 4$. For $n \geq 5$ we use the result of Lemma 1, that $f_{n,n}(1, 0) \geq f_{n,n}(a, b)$ for all pairs (a, b) satisfying $0 < a+b \leq n$ except $(a, b) = (1, 1)$. By the induction hypothesis, $(1, 0)$ gives a maximum among all $f_{n+1,n-1}(a, b)$. Combining these facts shows that $f_{n+1,n}(1, 0) \geq f_{n+1,n}(a, b)$ for all pairs (a, b) satisfying $0 < a+b \leq n$ except $(a, b) = (1, 1)$. The explicit computation that $f_{n+1,n}(1, 0) = f_{n+1,n}(1, 1)$ shows that $(1, 0)$ gives a maximum for $f_{n+1,n}(a, b)$.

Now we suppose that $(1, 0)$ gives a maximum for $f_{m,n}(a, b)$. By induction $(1, 0)$ gives a maximum for $f_{m+1,n-1}(a, b)$. We conclude that $(1, 0)$ gives a maximum for $f_{m+1,n}(a, b)$, completing the proof.

□

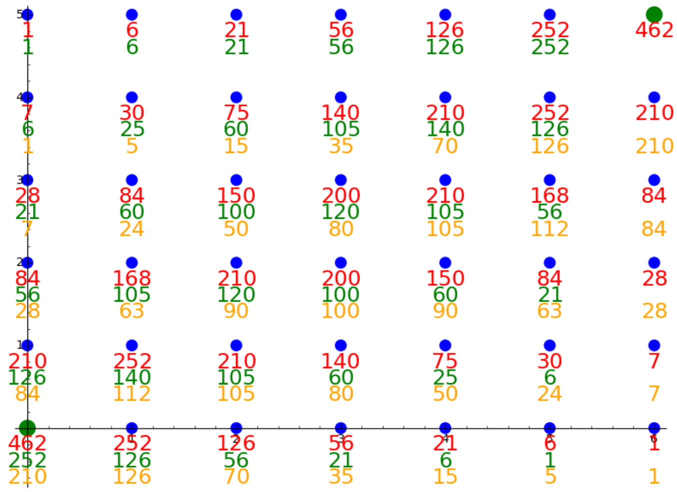

Figure 3: Values of $f_{6,5}(a,b)$ (Red), $f_{5,5}(a,b)$ (Green), and $f_{6,4}(a,b)$ (Orange). Comparing $f_{6,5}(1,0)$ and $f_{6,5}(1,1)$, noting that $(1,0)$ gives the second largest value of $f_{5,5}(a,b)$ and gives a maximum for $f_{6,4}(a,b)$, implies that $(1,0)$ gives a maximum for $f_{6,5}(a,b)$.

3 The Hypergeometric Distribution, the Gamma Function, and *The Jetsons*

The Hypergeometric Distribution

One of the most appealing aspects of Question 3 is that it takes an elementary subject not obviously related to statistics, counting lattice paths, and leads to a fundamental discrete probability distribution.

Question 4. *Suppose there are $m+n$ students in a class, with $a+b$ girls and the rest boys. If we randomly choose m students, what is the probability that exactly a of them are girls?*

By basic counting, first choosing the girls and then choosing the boys, we see that the answer is exactly the expression $F_{m,n}(a,b)$.

This is the basis for the *hypergeometric distribution*, which is essential to understanding sampling without replacement from a finite population. Usually it is introduced in the following form. In a set of n elements, n_1 are red and the rest are black. If we choose exactly r elements at random without replacement, then the probability that exactly k of our choices are red is given by

$$\frac{\binom{n_1}{k}\cdot\binom{n-n_1}{r-k}}{\binom{n}{r}}.$$

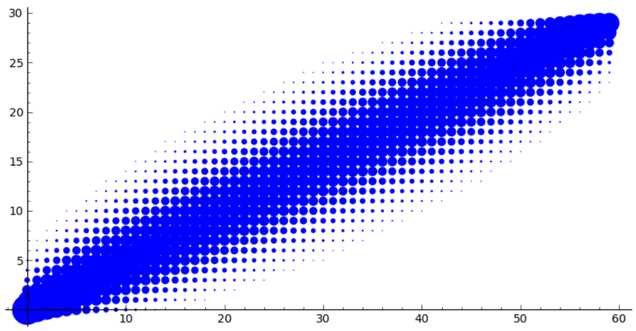

Figure 4: A plot showing relative sizes of $F_{60,30}(a,b)$, where larger points represent larger values.

For more of the basics of the hypergeometric distribution and some applications, see [6, Section II.6]. Considering more sophisticated types of lattice paths and their generating functions leads to certain hypergeometric series that have number theoretic applications in the theory of partitions [1].

The Gamma Function

We have emphasized finding the maximum value of $F_{m,n}(a,b)$ among all points except $(0,0)$ and (m,n), but it is also interesting to consider a birds-eye view of this function over its entire domain. In Figure 4 we give an example for $m=60,\ n=30$, where the circle at (a,b) is large if the corresponding value of $F_{60,30}(a,b)$ is large.

Points that are not close to the main diagonal $y=\frac{1}{2}x$ do not have many paths passing through them. Given a lattice path from $(0,0)$ to (m,n) we can compute its maximum distance from this line. How large do we expect this maximum to be? We leave this and more refined statistical questions about these lattice paths to the interested reader.

Another appealing aspect of Question 3 is that it is an elementary example of an optimization problem where the correct first step is not to take a derivative and set it to zero. Attempting to go down this path does lead to interesting mathematics. Binomial coefficients are defined in terms of factorials, which are initially defined only for non-negative integers. However, there is a natural continuous setting in which to consider this problem that involves the gamma function $\Gamma(t)$. This function plays an important role in complex analysis and analytic number theory. Figure 4 suggests that the discrete values of $F_{m,n}(a,b)$ can be continuously interpolated in a nice way.

The *Gamma function* is defined by

$$\Gamma(t)=\int_0^\infty x^{t-1}e^{-x}dx,$$

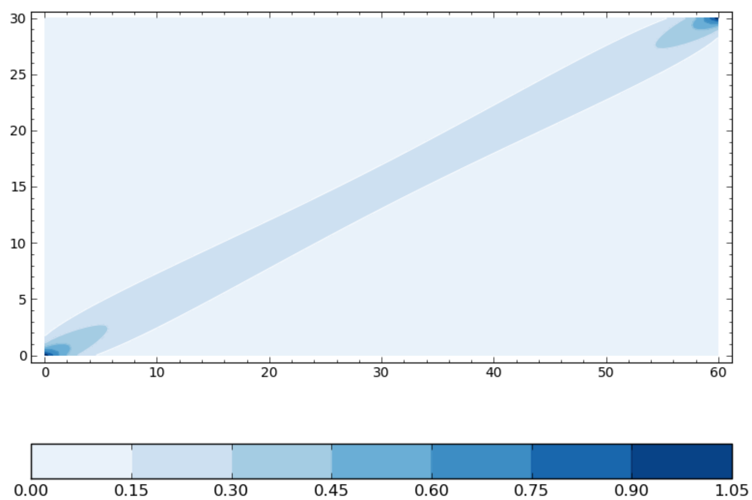

Figure 5: A plot showing the relative sizes of the continuous version of $F_{60,30}(a,b)$ defined by the gamma function.

for all complex number t except negative integers and zero. It is a standard exercise to show that for n a positive integer, $\Gamma(n) = (n-1)!$. A continuous version of the binomial coefficient is then given by

$$\binom{x}{y} = \frac{\Gamma(x+1)}{\Gamma(y+1)\cdot\Gamma(x-y+1)}.$$

For much more on the gamma function and its role in number theory, see [7, Chapter 3].

In Figure 5 we give a contour plot of the continuous version of the function $F_{60,30}(a,b)$. We see the same phenomenon we saw in Figure 4. This function is small away from the line $y = \frac{1}{2}x$ and takes its largest values very close to the origin and to the point $(60, 30)$.

Giving a continuous function that agrees with $F_{m,n}(a,b)$ suggests a way to try to find the maximum value of this function subject to the constraints of Question 3 by taking derivatives. The derivative of the gamma function is given in terms of the polygamma function. This approach leads to some more advanced complex analysis, but it is not clear that this helps us determine where to open our restaurant.

Restaurants in $\mathbb{Z}^n$

We end this paper with a generalization of Question 3 to higher dimensions. We considered all potential locations for our restaurant on a rectangular grid. Suppose we finally reach the situation promised by *The Jetsons* decades ago. You live at $(0,0,0)$ and take your aerocar to your office at point (a_1, a_2, a_3). You always follow the rules of space-traffic, only driving along edges of the integer lattice $\mathbb{Z}^3$. Your minimum length drive takes $a_1 + a_2 + a_3$ steps. The number of such drives is given by the trinomial coefficient $\binom{a_1+a_2+a_3}{a_1,a_2,a_3} = \frac{(a_1+a_2+a_3)!}{a_1!a_2!a_3!}$.

Question 5. *Which point in $\mathbb{Z}^3$ other than $(0,0,0)$ and (a_1,a_2,a_3) is visited on the maximum number of drives? That is, where is the best place to open your 3-dimensional space cantina? More generally, we can ask the same question in $\mathbb{Z}^n$. If people live at the origin and work at $(a_1,\ldots,a_n)$, where should I open my n-dimensional restaurant in order to maximize visits?*

The main ideas of the discussion at the start of Section 2 carry over to this higher dimensional setting. The ideal location should be both 'close' to the origin and not 'too far' from the line connecting the origin to the endpoint $(a_1,\ldots,a_n)$. A similar approach of maximizing first over hyperplanes $\sum_{i=1}^n x_i = k$ and then over lines orthogonal to them seems promising. We leave this as an exercise for the interested reader.

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