DELTA SETS OF NUMERICAL MONOIDS ARE EVENTUALLY PERIODIC

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ABSTRACT. Let M be a numerical monoid (i.e., an additive submonoid of $\mathbb{N}_0$) with minimal generating set $\langle n_1, \ldots, n_t \rangle$. For $m \in M$, if $m = \sum_{i=1}^t x_i n_i$, then $\sum_{i=1}^t x_i$ is called a *factorization length* of m. We denote by $\mathcal{L}(m) = \{m_1, \ldots, m_k\}$ (where $m_i < m_{i+1}$ for each $1 \leq i < k$) the set of all possible factorization lengths of m. The Delta set of m is defined by $\Delta(m) = \{m_{i+1} - m_i \mid 1 \leq i < k\}$ and the Delta set of M by $\Delta(M) = \cup_{0 \neq m \in M} \Delta(m)$. In this paper, we expand on the study of $\Delta(M)$ begun in [2] and [3] by showing that the delta sets of a numerical monoid are eventually periodic. More specifically, we show for all $x \geq 2kn_2n_k^2$ in M that $\Delta(x) = \Delta(x + n_1 n_k)$.

Let M be a commutative cancellative monoid with set $M^\bullet$ of nonunits and $\mathcal{A}(M)$ of irreducible elements. We assume that M is atomic (i.e., every nonunit can be written as a product of irreducible elements). Problems involving the factorization properties of elements in M into irreducible elements have been a frequent topic in the mathematical literature over the past 20 years (see [6] and the references cited therein). Most of this work entails a study of the length set of an element $x \in M$ which is defined as

$$\mathcal{L}(x) = \{l \mid \exists\, a_1, \ldots, a_l \in \mathcal{A}(M) \text{ such that } x = a_1 \cdots a_l\}.$$

If $\mathcal{L}(x) = \{l_1, \ldots, l_j\}$ with $l_1 < l_2 < \ldots < l_j$, then define the delta set of x as the set of consecutive differences of lengths,

$$\Delta(x) = \{l_{i+1} - l_i \mid 1 \leq i < j\}.$$

The delta set of M is defined as

$$\Delta(M) = \bigcup_{x \in M^\bullet} \Delta(x).$$

The set $\Delta(M)$ has been widely studied. In particular, [10] studies the delta set of a Krull monoid with finite divisor class group, [6, Section 6.7] examines the specific case of block monoids, while [2] and [3] focus on the case where M is a numerical monoid. However, the exact structure of $\Delta(M)$ is known for very few monoids.

Our work in this paper answers a question raised by Paul Baginski in 2004 which was motivated by the work on numerical monoids in [2]. Using computer data generated by programs similar to those in [1], Baginski conjectured for numerical monoids that the sequence $\{\Delta(x)\}_{x \in M}$ is eventually periodic. We affirm this in Theorem 1 and find an upper bound for the fundamental period. As a a result, in Corollary 3 we show that the problem of computing the delta set of a numerical monoid M can be done in finite time with a bound derived from the minimal generating set of M.

A numerical monoid M is any submonoid of the nonnegative integers (denoted $\mathbb{N}_0$) under addition. We will say that the integers $n_1 < n_2 < \ldots < n_k$ generate M if $M = \{a_1 n_1 + \cdots + a_k n_k \mid a_i \in \mathbb{N}_0 \text{ for all } i\}$ and denote this by

$$M = \langle n_1, n_2, \ldots, n_k \rangle.$$

Each numerical monoid M has a unique minimal (in terms of cardinality) set of generators. Henceforth, we shall assume that a given generating set for a numerical monoid M is minimal. If for M

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as written above, we have $\gcd(n_1, \ldots, n_k) = 1$, then M is called *primitive*. We note that every numerical monoid is clearly isomorphic to a primitive numerical monoid. If M is a primitive numerical monoid, then there exists an integer $F(M) \notin M$ such that $m > F(M)$ implies that $m \in M$. $F(M)$ is known as the *Frobenius number* of M and its computation has been a central focus of research for over 100 years (see [11]). As with much work concerning numerical monoids, the Frobenius number will play a central role in our argument. The monograph [5, Chapter 10] is a good general reference on numerical monoids and [9] contains an in depth study of the Frobenius number. Our main result now follows.

Theorem 1. *Given a primitive numerical monoid* $M = \langle n_1, \ldots, n_k \rangle$, *we have for all* $x \geq 2kn_2n_k^2$ *that* $\Delta(x) = \Delta(x + n_1n_k)$.

Before proceeding, we note that work which is in some sense related to Baginski's conjecture has already appeared in the literature. Papers by Hassler [7] and Foroutan and Hassler [4] examine the factorization properties of the powers, x^n, of some fixed nonunit x of a commutative cancellative atomic monoid M. In fact, that M in Theorem 1 is periodic follows from [4, Proposition 3.1 (2)] (or [6, Proposition 4.9.6]) in the following manner [8]. Pick $a \in M$ and $b_1, \ldots, b_n \in M$ such that $M = \{b_i + ja | 1 \leq i \leq n, j \geq 0\}$. Then [4, Theorem 3.1 (1)] gives a bound N^* and, for each $i \in \{1, \ldots, n\}$, bounds B_i such that $\Delta(b_i + ja) = \Delta(b_i + la)$ for all $j, l \geq B_i$ with $j \equiv l \mod N^*$. Now, if we put $N := aN^*$ and $B := a \max\{B_1, \ldots, B_n\} + \max\{b_1, \ldots, b_n\}$, then a simple calculation shows that $\Delta(x) = \Delta(y)$ for all $x, y \in M$ for which $x, y \geq B$ and $x \equiv y \mod N$. We note that this argument does not yield the bounds or the period given in Theorem 1 in terms of the minimal generating set of M.

Before continuing to our proof, we will require some additional notation. Let

$$\mathsf{Z}(x) = \{(a_1, \ldots, a_k) \mid x = \sum_{i=1}^{k} a_i n_i \text{ with } a_i \in \mathbb{N}_0 \text{ for all } i\}$$

denote the set of factorizations of x in M. For all $(a_1, a_2, \ldots, a_k) \in \mathsf{Z}(x)$, it follows that $(a_1 + n_k, a_2, \ldots, a_k)$ and $(a_1, \ldots, a_{k-1}, a_k + n_1)$ are both in $\mathsf{Z}(x + n_1n_k)$. Define $\varrho : \mathsf{Z}(x) \to \mathsf{Z}(x + n_1n_k)$ by

$$\varrho(a_1, a_2, \ldots, a_k) = (a_1 + n_k, a_2, \ldots, a_k).$$

We also define $\phi : \mathsf{Z}(x) \to \mathsf{Z}(x + n_1n_k)$ by

$$\phi(a_1, \ldots, a_{k-1}, a_k) = (a_1, \ldots, a_{k-1}, a_k + n_1).$$

We see that if f is a factorization of x of length l, then $\varrho(x)$ has length $l + n_k$ and $\phi(x)$ has length $l + n_1$. Thus, ϱ and ϕ preserve ordering by length of factorization. We in fact can say more. The proof of the following assertion is left to the reader.

Lemma 2. *Let* $M = \langle n_1, \ldots, n_k \rangle$ *be a primitive numerical monoid.*

(1) *If* l_1, l_2 *are elements in* $\mathcal{L}(x)$, *then* l_1, l_2 *are consecutive elements in* $\mathcal{L}(x)$ *if and only if* $l_1 + n_k, l_2 + n_k$ *are consecutive elements in the length set of factorizations in the image of* ϱ.
(2) *Similarly,* l_1, l_2 *are consecutive elements in* $\mathcal{L}(x)$ *if and only if* $l_1 + n_1, l_2 + n_1$ *are consecutive elements in the length set of factorizations in the image of* ϕ.

Proof of Theorem 1. If $k = 1$, then $\Delta(M) = \emptyset$ for all nonunits $x \in M$ and the result follows. If $k = 2$, then by [2, Proposition 3.1] $\Delta(M) = \{n_2 - n_1\}$ and the result again easily follows. So, assume $k > 2$ and note throughout our argument that by hypothesis $x \geq 2kn_2n_k^2$. If $(a_1, \ldots, a_k) \in \mathsf{Z}(x + n_1n_k)$ with length l and $a_1 \geq n_k$, then we see that $(a_1 - n_k, \ldots, a_k) \in \mathsf{Z}(x)$, $\varrho(a_1 - n_k, \ldots, a_k) = (a_1, \ldots, a_k)$, and thus $(a_1, \ldots, a_k)$ is in the image of ϱ. Hence, for all $l \in \mathcal{L}(x + n_1n_k)$ not in the length set of the image ϱ, we see that all factorizations $(a_1, \ldots, a_k)$ of length l must have $a_1 < n_k$. Among factorizations of $x + n_1n_k$ of length l not in the image of ϱ, choose one with a maximal coefficient of n_1. If we denote this factorization as $(a_1, \ldots, a_k)$, we will demonstrate that $a_i < n_k - n_1$ for all $i = 2, \ldots, k-1$.

So suppose that $a_i \geq n_k - n_1$. Let $b_1 = a_1 + n_k - n_i$, $b_i = a_i + n_1 - n_k, b_k = a_k + n_i - n_1$, and $b_j = a_j$ for all other j. From $a_i \geq n_k - n_1$ we see that $b_j \geq 0$ for all j. We see that

$$\begin{aligned} b_1n_1 + b_in_i + b_kn_k &= (a_1 + n_k - n_i)n_1 + (a_i + n_1 - n_k)n_i + (a_k + n_i - n_1)n_k \\ &= a_1n_1 + a_in_i + a_kn_k. \end{aligned}$$

If $2 < i < k-1$, then

$$\begin{aligned} \sum_{j=1}^{k} b_jn_j &= b_1n_1 + b_in_i + b_kn_k + \sum_{j=2}^{j=i-1} b_jn_j + \sum_{j=i+1}^{j=k-1} b_jn_j \\ &= a_1n_1 + a_in_i + a_kn_k + \sum_{j=2}^{j=i-1} a_jn_j + \sum_{j=i+1}^{j=k-1} a_jn_j \qquad (1) \\ &= \sum_{j=1}^{k} a_jn_j = x + n_1n_k. \end{aligned}$$

We also have

$$\begin{aligned} \sum_{j=1}^{k} b_j &= b_1 + b_i + b_k + \sum_{j=2}^{j=i-1} b_j + \sum_{j=i+1}^{j=k-1} b_j \\ &= (a_1 + n_k - n_i) + (a_i + n_1 - n_k) + (a_k + n_i - n_1) \qquad (2) \\ &\quad + \sum_{j=2}^{j=i-1} a_j + \sum_{j=i+1}^{j=k-1} a_j \\ &= \sum_{j=1}^{k} a_j = l. \end{aligned}$$

This implies that $(b_1, \ldots, b_k)$ is also a factorization of $x+n_1n_k$ of length l, but $b_1 = a_1+n_k-n_i > a_1$, which contradicts our choice of a_1 as the maximal coefficient of n_1 in such a factorization. With minor adjustments to the inequalities in (1) and (2), the argument also works for the cases $i = 2$ and $i = k-1$ and hence $a_i < n_k - n_1$ for all $i = 2, \ldots, k-1$.

Now that we have bounded a_i for $i < k$, we will note that since $a_kn_k \leq x$, we have $a_k \leq x/n_k$. Thus,

$$l = a_1 + \sum_{i=2}^{k-1} a_i + a_k < n_k + (k-2)(n_k - n_1) + \frac{x}{n_k} < \frac{x}{n_k} + kn_k.$$

This inequality holds for all $l \in \mathcal{L}(x + n_1n_k)$ not in the image of ϱ , and hence we have verified the following claim.

Claim A: If $l \in \mathcal{L}(x + n_1n_k)$ such that $l \geq x/n_k + kn_k$, then l is in the length set of the image of ϱ.

If $(a_1, \ldots, a_k) \in \mathrm{Z}(x + n_1n_k)$ with length l, $a_k \geq n_1$, we see that $(a_1, \ldots, a_k - n_1) \in \mathrm{Z}(x)$, and $\phi(a_1, \ldots, a_k - n_1) = (a_1, \ldots, a_k)$, so that $(a_1, \ldots, a_k)$ is in the image of ϕ. Thus, for all $l \in \mathcal{L}(x+n_1n_k)$ not in the length set of the image ϕ, we see that all factorizations $(a_1, \ldots, a_k)$ of length l must have $a_k < n_1$.

Among such factorizations of $x + n_1n_k$ of length l, choose one with a maximal coefficient of n_k. If we denote this factorization as $(a_1, \ldots, a_k)$, then again we have $a_i < n_k - n_1$ for all $i = 2, \ldots, k-1$. Otherwise, as before we can define $b_1 = a_1 + n_k - n_i, b_i = a_i + n_1 - n_k, b_k = a_k + n_i - n_1$, and $b_j = a_j$

for all other j. Then again we see that $(b_1, \ldots, b_k)$ is another factorization of $x + n_1 n_k$ of length l. But, $b_k = a_k + n_i - n_1 > a_k$, contradicting the fact that a_k is the maximal coefficient of n_k in such a factorization.

We then get

$$l \geq a_1 = \frac{x - \sum_{i=2}^{k-1} a_i n_i - a_k n_k}{n_1} > \frac{x - (k-2)(n_k - n_1)n_k - n_1 n_k}{n_1} > \frac{x - kn_k^2}{n_1}.$$

This inequality holds for all $l \in \mathcal{L}(x + n_1 n_k)$ not in the image of ϕ , and hence we have verified a second claim.

Claim B: If $l \in \mathcal{L}(x + n_1 n_k)$ such that $l \leq (x - kn_k^2)/n_1$, then l is in the length set of the image of ϕ.

Now we observe by Proposition 2.9.4 in [6] that

$$F(M) \leq (n_1 - 1)(n_2 + \ldots + n_k) - n_1 < (k-1)n_1 n_k - (n_2 + \ldots + n_k) < (k-1)n_1 n_k - n_2.$$

Choose the integer i such that $x + n_1 n_k \equiv kn_1 n_k + i \pmod{n_2}$ with $1 \leq i \leq n_2$. We then choose the integer s such that $x + n_1 n_k = sn_2 + (kn_1 n_k - n_2 + i)$. From $x \geq 2kn_2 n_k^2$ we can see easily that $s \geq 0$.

If we let $r = kn_1 n_k - n_2 + i$, we see that

$$r - n_1 n_k > (k-1)n_1 n_k - n_2 > F(M)$$

implies that $r - n_1 n_k \in M$. Let $(c_1, c_2, \ldots, c_k)$ be an arbitrary element of $\mathrm{Z}(r - n_1 n_k)$. We then see that $(c_1 + n_k, c_2, \ldots, c_k) \in \mathrm{Z}(r)$ and this also gives us that

$$(c_1 + n_k, c_2 + s, c_3, \ldots, c_k) \in \mathrm{Z}(x + n_1 n_k).$$

We wish to bound the length of this factorization.

First, we see that as $s = (x + n_1 n_k - r)/n_2$, we have $s < x/n_2$ from $r > n_1 n_k$. We also see that $s > (x - kn_1 n_k)/n_2$ from $r < (k+1)n_1 n_k$. Setting $c = n_k + c_1 + \ldots + c_k$, it follows from the previous paragraph that the length we wish to bound is $s + c$. We can bound $c \leq r/n_1$ since $(c_1 + n_k, c_2, \ldots, c_k)$ is a factorization of r. We then see

$$c \leq \frac{r}{n_1} \leq \frac{kn_1 n_k}{n_1} = kn_k.$$

Thus we get $s + c < x/n_2 + kn_k$, so overall we have

$$\frac{x - kn_1 n_k}{n_2} < s < s + c < \frac{x}{n_2} + kn_k.$$

Now let $y \in \Delta(x + n_1 n_k)$, and l_1, l_2 be consecutive elements in $\mathcal{L}(x + n_1 n_k)$ such that $l_2 - l_1 = y$. Since $s + c \in \mathcal{L}(x + n_1 n_k)$, we have either $s + c \leq l_1$ or $l_2 \leq s + c$. We now consider the first case. Since $2kn_2 n_k^2 \leq x$ we have

$$kn_k^2(n_2 + n_1) < (n_k - n_2)x.$$

Hence,

$$kn_k\left(1 + \frac{n_1}{n_2}\right) < \left(\frac{1}{n_2} - \frac{1}{n_k}\right)x$$

and thus

$$kn_k + \frac{kn_1 n_k}{n_2} < \frac{x}{n_2} - \frac{x}{n_k}.$$

Finally,

$$\frac{x}{n_k} + kn_k < \frac{x - kn_1 n_k}{n_2} < s + c.$$

and thus, $l_2 > l_1 \geq s+c > x/n_k+kn_k$. By Claim A, l_1, l_2 are both in the length set of the image of ϱ. Moreover, they must be consecutive elements in this length set. Thus, by Lemma 2, $l_1 - n_k, l_2 - n_k$ are consecutive elements of $\mathcal{L}(x)$. Thus, we see $y = (l_2 - n_k) - (l_1 - n_k) \in \Delta(x)$.

In the second case our assumption $x \geq 2kn_2n_k^2$ implies

$$(n_2 - n_1)x > kn_2n_k(n_k + n_1)$$

and hence

$$\left(\frac{1}{n_1} - \frac{1}{n_2}\right) x > kn_k \left(\frac{n_k}{n_1} + 1\right).$$

Finally

$$\frac{x}{n_1} - \frac{x}{n_2} > \frac{kn_k^2}{n_1} + kn_k$$

implies that

$$\frac{x - kn_k^2}{n_1} > \frac{x}{n_2} + kn_k > s + c.$$

Thus $l_1 < l_2 \leq s+c < (x-kn_k^2)/n_1$ and by Claim B, l_1, l_2 are both in the length set of the image of ϕ. Thus, again by Lemma 2, $l_1 - n_1, l_2 - n_1$ are consecutive elements of $\mathcal{L}(x)$ and $y \in \Delta(x)$ follows as before. Therefore, $\Delta(x + n_1n_k) \subset \Delta(x)$.

Next, let $z \in \Delta(x)$, and l_1, l_2 be consecutive elements in $\mathcal{L}(x)$ with $l_2 - l_1 = z$. Let f_1, f_2 be factorizations of length l_1, l_2, respectively. We first remind ourselves that $sn_2 + r = x + n_1n_k$, and $(c_1, \dots, c_k)$ is a factorization of $r - n_1n_k$ of length $c - n_k$. Thus, $(c_1, c_2+s, c_3, \dots, c_k)$ is a factorization of x of length $s + c - n_k$, so that $s + c - n_k \in \mathcal{L}(x)$.

We have either $l_1 \geq s + c - n_k$ or $l_2 \leq s + c - n_k$. Consider the first case. We see that the lengths of $\varrho(l_1), \varrho(l_2)$ are $l_1 + n_k, l_2 + n_k$, respectively. Moreover, by Lemma 2, these must be consecutive elements in the length set of the image of ϱ. We thus have

$$l_2 + n_k > l_1 + n_k \geq s + c \geq x/n_k + kn_k.$$

Hence, all values between $l_1 + n_k$ and $l_2 + n_k$ in the set $\mathcal{L}(x + n_1n_k)$ must be in the image set of ϱ. However, the existence of such an element would contradict the fact that $l_1 + n_k, l_2 + n_k$ are consecutive in the length set of the image of ϱ. Thus, $l_1 + n_k$ and $l_2 + n_k$ are in fact consecutive elements in $\mathcal{L}(x + n_1n_k)$, and thus $z \in \Delta(x + n_1n_k)$.

In the second case we see that the lengths of $\phi(l_1), \phi(l_2)$ are $l_1 + n_1, l_2 + n_1$, respectively, and again these must be consecutive elements in the length set of the image of ϕ. From

$$l_1 + n_1 < l_2 + n_1 \leq s + c - n_k + n_1 < s + c < \frac{x - kn_k^2}{n_1}$$

we see that all values between $l_1 + n_1$ and $l_2 + n_1$ in the set $\mathcal{L}(x + n_1n_k)$ must be in the image set of ϕ. The existence of such an element contradicts the fact that $l_1 + n_1, l_2 + n_1$ are consecutive in the image set of ϕ. Hence, $l_1 + n_1, l_2 + n_1$ are consecutive elements of $\mathcal{L}(x + n_1n_k)$, and thus $z \in \Delta(x + n_1n_k)$. Therefore, we know $\Delta(x) \supset \Delta(x + n_1n_k)$ and $\Delta(x) = \Delta(x + n_1n_k)$, completing the proof. □

We close with this immediate corollary.

Corollary 3. *Let $M = \langle n_1, \dots, n_k \rangle$ be a primitive numerical monoid, then if we set $N = 2kn_2n_k^2 + n_1n_k$, we have:*

$$\Delta(M) = \bigcup_{x \in M, x < N} \Delta(x)$$

Proof. Let y be some element in $\Delta(M)$ not in the above union. If we then let z be the least value in M with $y \in \Delta(z)$, we see that $z \geq N$ implies $z - n_1 n_k \geq 2kn_2n_k^2$. But then we have $\Delta(z - n_1 n_k) = \Delta(z)$, so that $y \in \Delta(z - n_1 n_k)$, a contradiction. □

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