No-Three-in-Line, Intransitive Dice, and Other Amusements in Mathematics

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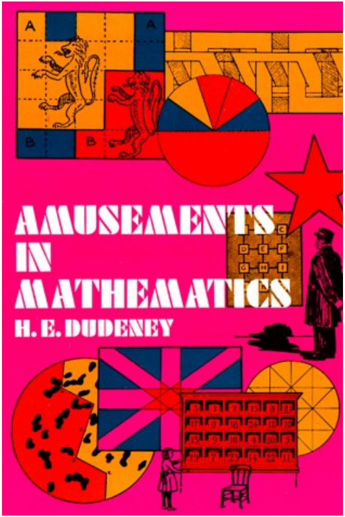
Amusements in Mathematics

Henry Ernest Dudeney, 1917



Amusements in Mathematics: Preface

The history of [Mathematical Puzzles] entails nothing short of the actual story of the beginnings of exact thinking in man. The historian must start from the time when man first succeeded in counting his ten fingers and dividing an apple into two approximately equal parts. Every puzzle that is worthy of consideration can be referred to mathematics and logic. Every man, woman, and child who tried to "reason out" the answer to the simplest puzzle is working, though not necessarily consciously, on mathematical lines...

Amusements in Mathematics: Preface

When a man says, "I have never solved a puzzle in my life", it is difficult to know exactly what he means, for every intelligent individual is doing it every day. The unfortunate inmates of our lunatic asylums are sent there expressly because they cannot solve puzzles - they have lost their power to reason. If there were no puzzles to solve, there would be no questions to ask; and if there were no questions to be asked, what a world it would be! We should all be equally omniscient, and conversation would be useless and idle.

Amusements in Mathematics: Preface

It is possible that some few exceedingly sober-minded mathematicians, who are impatient of any terminology in their favourite science but the academic, and who object to the elusive x and y appearing under any other names, will have wished that various problems had been presented in a less popular dress and introduced with a less flippant phraseology. I can only refer them to the first word of my title and remind them that we are primarily out to be amused - not, it is true, without some hope of picking up morsels of knowledge by the way. If the manner is light, I can only say, in the words of Touchstone, that it is "an ill-favoured thing, sir, but my own; a poor humour of mine, sir."

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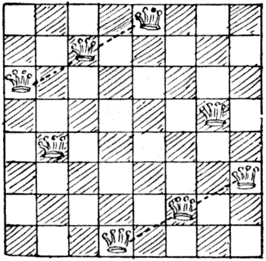
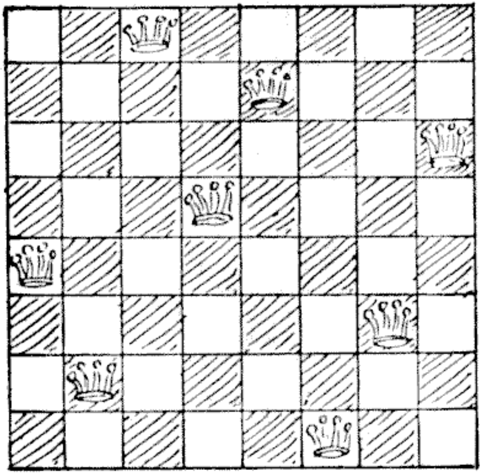
Eight Queens

The queen is by far the strongest piece on the chessboard. If you place her on one of the four squares in the centre of the board, she attacks no fewer than twenty-seven other squares; and if you try to hide her in a corner, she still attacks twenty-one squares. Eight queens may be placed on the board so that no queen attacks another... I show one way in the diagram, and there are in all twelve of these fundamentally different ways.



Can the reader place the eight queens on the board so that no queen shall attack another and so that no three queens shall be in a straight line in any oblique direction?

Eight Queens: Solution



The solution to this puzzle is shown in the diagram. It will be found that no queen attacks another, and also that no three queens are in a straight line in any oblique direction.

No-Three-in-Line

Place two pawns in the middle of the chessboard, one at D4 and the other at E5. Now, place the remaining fourteen pawns (sixteen in all) so that no three shall be in a straight line in any possible direction.

Note that I purposely do not say queens, because by the words "any possible direction" I go beyond attacks on diagonals. The pawns must be regarded as mere points in space – at the centres of the squares.

No-Three-in-Line

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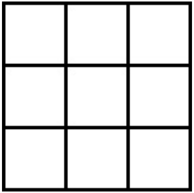
Can you place 16 points in an 8×8 grid such that no three lie on a line?

No-Three-in-Line: 3×3

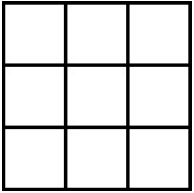
What about a 3×3 grid?



What is the maximum number of points, no three on a line?

No-Three-in-Line: 3×3

What about a 3×3 grid?



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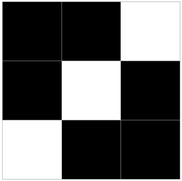
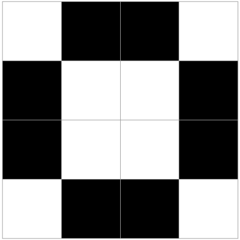


No-Three-in-Line: Larger Grids



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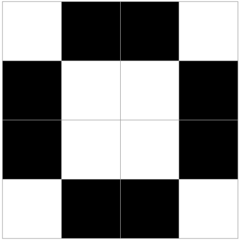

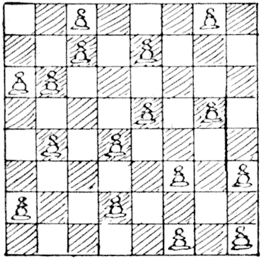

No-Three-in-Line: Upper Bound

- There are n vertical lines that cover all of the points of our $n \times n$ grid.
- A collection of points with no three on any line has at most 2 points on any vertical line.
- So, we can have **at most** $2n$ **points** in an $n \times n$ grid with no three on a line.

Can we always find $2n$ such points?

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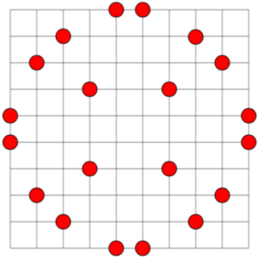
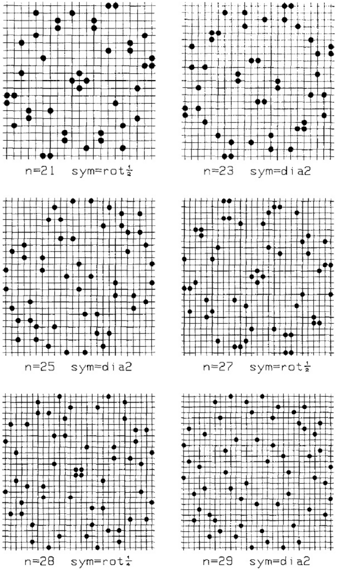


No-Three-in-Line: Larger Grids

Flammenkamp [1992,1998]



No-Three-in-Line: What is known?

For all $n \leq 46$ and $n = 48, 50, 52$, we **can** find $2n$ points in an $n \times n$ grid, no three in a line.

What about 47?

No-Three-in-Line: What is known?

For all $n \leq 46$ and $n = 48, 50, 52$, we **can** find $2n$ points in an $n \times n$ grid, no three in a line.

What about 47?

There are $\binom{47^2}{94}$ ways to choose 94 points in a 47×47 grid.
Huge number: 168 digits.

How many ways can we pick n points, one in each row and one in each column? $n!$
There are at most $(47!)^2$ collections of 94 points, at most two on each vertical and on each horizontal line.
This has **only** 119 digits.

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We do not know of any n for which you cannot get $2n$ points in an $n \times n$ grid, no three in a line.

No-Three-in-Line: Lower Bounds

If n is prime, can get n points, no three in a line.

- Label the points (x, y) with $1 \leq x, y \leq n$.
- Take the set of points such that n divides $x^2 + y^2 - 1$. This is a 'smooth conic curve'.
- There are either $n - 1$, n, or $n + 1$ such points. No three lie on a line.

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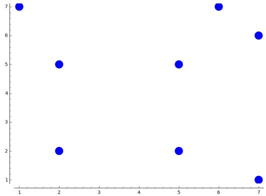

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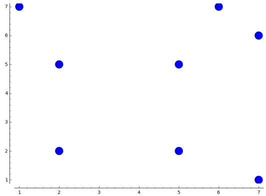

Hall, Jackson, Sudberry, Wild ('75):
If $n = 2p$ where p is prime, divide grid into 4 $p \times p$ grids and put a 'large' subset of a conic in each. Get $3p$ points, no three on a line.

No-Three-in-Line: Guy/Kelly Conjecture

Let $f(n)$ be the maximum number of points in an $n \times n$ grid, no three on a line. For large n we know

$$(3/2 - \epsilon)n \leq f(n) \leq 2n.$$

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Conjecture (Guy and Kelly, '68)

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Idea behind the conjecture:

Theorem (Guy and Kelly)

The number of triples of collinear points in an $n \times n$ grid is

$$\frac{3}{\pi^2} n^4 \log(n) + O(n^4).$$

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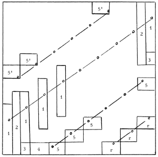

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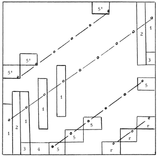

Three points (x_1, y_1), (x_2, y_2), (x_3, y_3) lie on a line if and only if the slope of the line connecting the first two equals the slope of the line connecting the last two. That is

$$\det \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} = x_1y_2 - x_1y_3 - x_2y_1 + x_2y_3 + x_3y_1 - x_3y_2 = 0.$$

How many points satisfy this equation subject to
$1 \leq x_1, x_2, x_3, y_1, y_2, y_3 \leq n$?

No-Three-in-Line: How to get to the Guy/Kelly Conjecture?

Suppose we have a bunch of points, $p_1, \ldots, p_k$, no three on a line. Pick a point q. **What is the probability that it is not on a line connecting any two points from our set**?

No-Three-in-Line: How to get to the Guy/Kelly Conjecture?

Suppose we have a bunch of points, $p_1, \ldots, p_k$, no three on a line. Pick a point q. **What is the probability that it is not on a line connecting any two points from our set**?

The probability that q is not on the line connecting p_1 and p_2 is the number of non-collinear triples divided by the total number of triples.

Assume everything is independent. Compute the expected number of collections of k points, no three on a line.

Intransitive Dice: Six Sided-Dice (Effron/Gardner)

Consider the following four six-sided dice:

$$\begin{aligned} A &= (0,0,4,4,4,4), & B &= (3,3,3,3,3,3), \\ C &= (2,2,2,2,6,6), & D &= (1,1,1,5,5,5). \end{aligned}$$

If we roll A and B there are 36 equally likely outcomes. In 24 of these outcomes the number showing on A is larger than the number showing on B. We say that A beats B with probability $2/3$, or just that 'A beats B'.

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- A beats B with probability $2/3$.
- B beats C with probability $2/3$.
- C beats D with probability $2/3$.
- D beats A with probability $2/3$.

Let A have sides $(a_1, \ldots, a_n)$ and B have sides $(b_1, \ldots, b_n)$.
We see that A beats B if and only if

$$\sum_{i,j} \operatorname{sign}(a_i - b_j) > 0.$$

Intransitive Dice: What is a die?

Definition

An n-sided die has n faces chosen from $\{1, 2, \ldots, n\}$ such that the sum of the faces is $n(n+1)/2$. We do not care about the arrangement of the faces, so $(2, 3, 1)$ and $(1, 2, 3)$ define the same die.

Example:

- For $n = 3$ there are two dice, $(2, 2, 2)$ and $(3, 2, 1)$.
- For $n = 4$ there are five dice,
  $(4, 4, 1, 1)$, $(4, 3, 2, 1)$, $(4, 2, 2, 2)$, $(3, 3, 3, 1)$, $(3, 3, 2, 2)$.

Definition

A triple of dice (A, B, C) is intransitive if A beats B, B beats C, and C beats A.

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Definition

A triple of dice (A, B, C) is intransitive if A beats B, B beats C, and C beats A.

Question

Pick three n-sided dice at random.

What is the probability that you get an intransitive triple?

Intransitive Dice: A Conjecture

Intransitive dice seem to be not all that rare...

Conjecture (Conrey, Gabbard, Grant, Liu, Morrison)

As n goes to infinity, the probability that three n-sided dice give an intransitive triple approaches $1/4$.

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Let $A = (a_1, \ldots, a_n)$ be an n-sided die.
Identify A with a point in $(w_1, \ldots, w_n) \in \mathbb{Z}^n$, where w_j is the number faces equal to j.
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- Each w_j is a nonnegative integer.
- $\sum_{j=1}^{n} w_j = n$.
- $\sum_{j=1}^{n} j \cdot w_j = n(n+1)/2$.

The set of n-sided dice are the integer points in a polytope in $\mathbb{R}^n$.

Intransitive Dice: How Many Dice?

Theorem (Takács,CGGLM)

As n goes to infinity, the number of n-sided dice is asymptotic to

$$\frac{\sqrt{3}}{2\pi} \cdot \frac{4^n}{n^2}.$$

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A die is a partition of $n(n+1)/2$ into exactly n parts where each part is of size at most n.

Theorem (Hardy, Ramanujan)

Let $p(n)$ be the number of partitions of n. As n goes to infinity, $p(n)$ is asymptotic to

$$\frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right).$$

Intransitive Dice: What About Ties?

Conjecture (CGGLM)

As n goes to infinity the probability that a random pair of n-sided dice tie goes to 0.

Intransitive Dice: What About Ties?

Conjecture (CGGLM)

As n goes to infinity the probability that a random pair of n-sided dice tie goes to 0.

Let A be identified with the point $(w_1, \ldots, w_n)$ and B be identified with the point $(v_1, \ldots, v_n)$. (That is, A has exactly w_j faces equal to j.)

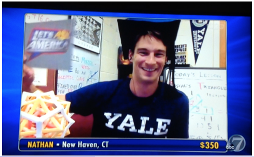
These dice tie if and only if

$$\sum_{i<j} v_i w_j - \sum_{i>j} v_i w_j = 0.$$

Just like counting collinear triples in a grid, this is a question about integer solutions to a single quadratic polynomial.

Machine Learning and Mathematical Finance

Machine Learning and Mathematical Finance



Game Show Math

- Ten contestants ranked from 10 to 1.
- Each gets asked a question in order.
- If you get your question wrong, you only get \$25,000.
- If you get it right and nobody ranked higher that you gets their question right, you get \$1,000,000.
- If you get it right and somebody else ranked than you also gets it right, you get the amount of money you came in with.

Suppose you get asked a question and you're 50% sure you know the right answer, **what is the expected value of a guess**?

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Suppose you get asked a question and you're 50% sure you know the right answer, **what is the expected value of a guess**?

- Suppose every starts with the same amount of money, say \$50,000.
- Suppose that questions are drawn from a 'nice' distribution.
- Suppose that everyone does the same expected value calculation you do and they will guess if they decide that it is expected value positive to do so.

Suppose you're in 8$^{\text{th}}$ place. How does the expected value vary with the distribution? How high could it possibly be?

Tournament of Ten

Tournament of Ten

Here's how the tournament works:
http://tinyurl.com/hhbyqq5

Here's what happened...
http://tinyurl.com/zlmk63d

How Many People Have Ever Been Alive?

Year	Population	Births per 1,000	Births Between Benchmarks
50,000 B.C.	2	-	-
8000 B.C.	5,000,000	80	1,137,789,769
1 A.D.	300,000,000	80	46,025,332,354
1200	450,000,000	60	26,591,343,000
1650	500,000,000	60	12,782,002,453
1750	795,000,000	50	3,171,931,513
1850	1,265,000,000	40	4,046,240,009
1900	1,656,000,000	40	2,900,237,856
1950	2,516,000,000	31-38	3,390,198,215
1995	5,760,000,000	31	5,427,305,000
2011	6,987,000,000	23	2,130,327,622

Source: http://tinyurl.com/jjdpz45