COUNTING NUMERICAL SEMIGROUPS BY GENUS AND SOME CASES OF A QUESTION OF WILF

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Abstract. The genus of a numerical semigroup is the size of its complement. In this paper we will prove some results about counting numerical semigroups by genus. In 2008, Bras-Amorós conjectured that the ratio between the number of semigroups of genus $g + 1$ and the number of semigroups of genus g approaches ϕ, the golden ratio, as g gets large. Though several recent papers have provided bounds for counting semigroups, this conjecture is still unsolved. In this paper we will show that a certain class of semigroups, those for which twice the genus is less than three times the smallest nonzero element, grows like the Fibonacci numbers, suggesting a possible reason for this conjecture to hold. We will also verify that a 1978 question of Wilf holds for these semigroups and in certain other cases. We will also show that in several situations we can count numerical semigroups of certain genus and multiplicity by counting only semigroups of maximal embedding dimension, and that we can always interpret the number of semigroups of genus g in terms of the number of integer points in a single rational polytope. We also discuss connections with recent work of Blanco, García-Sánchez and Puerto, and mention several further open problems.

1. Introduction

We first recall some important definitions related to numerical semigroups. We will take them from the recently published book of Rosales and García-Sánchez, [10]. This is an excellent reference that discusses the results of many of the other papers in our bibliography.

A numerical semigroup is an additive submonoid S of $\mathbb{N}_0$ such that $\mathbb{N}_0\backslash S$ is finite. Let S be a numerical semigroup and A a subset of S. We say that $A = \{a_1, \ldots, a_n\}$ is a system of generators of S if $S = \{k_1a_1 + \cdots + k_na_n, \mid k_1, \ldots, k_n \in \mathbb{N}\}$. The set A is a minimal system of generators if no proper subset of A is a system of generators. When A is a system of generators of S we will often write $S = \langle A\rangle = \langle a_1, \ldots, a_n\rangle$. It is a standard fact that every numerical semigroup has a unique minimal generating set. It is also straightforward to show that given a submonoid $A \subset \mathbb{N}_0$, $\langle A\rangle$ is a numerical semigroup if and only if $\gcd(A) = 1$. If we divide each element of A by this common factor, then we see that every nontrivial submonoid of $\mathbb{N}_0$ is isomorphic to a unique numerical semigroup.

The smallest nonzero element of S is called the multiplicity of S which we will often denote by $m(S)$. The set $\mathbb{N} \setminus S$ is often denoted by $H(S)$ and is known as the gaps of S. The largest element of $H(S)$ is called the Frobenius number of S which

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we will denote $F(S)$, and $|H(S)|$ is called the genus of S which we will denote $g(S)$. If $S = \mathbb{N}$ then by convention we write $F(S) = -1$. An element $n \in S$ is called a minimal generator if it cannot be written as a sum of smaller elements of S. The embedding dimension of a numerical semigroup, denoted by $e(S)$ is the size of its minimal system of generators. When there is no confusion we will write m, F, g and e, for $m(S), F(S), g(S)$ and $e(S)$.

This paper will focus on several related questions. How many numerical semigroups have genus g? We call this quantity $N(g)$. How many numerical semigroups have genus g and multiplicity m? We call this $N(m, g)$. In [2], Bras-Amorós used an extensive computer analysis to enumerate all of the numerical semigroups with genus at most 50 and noticed a striking pattern in the growth of $N(g)$.

Conjecture 1 ([2]). *We have*

$$\lim_{g\to\infty} \frac{N(g-1)+N(g-2)}{N(g)} = 1, \text{ and } \lim_{g\to\infty} \frac{N(g)}{N(g-1)} = \frac{1+\sqrt{5}}{2}.$$

Although there is computational evidence supporting this conjecture, theoretically little is known. These conjectures give a good understanding of the asymptotic behavior of the sequence $N(g)$. Bras-Amorós also conjectures the following.

Conjecture 2 ([2]). *For each* $g \geq 1$, $N(g-2) + N(g-1) \leq N(g)$.

In fact, a far weaker version of this conjecture is still open.

Conjecture 3. *For each* $g \geq 1$, $N(g-1) \leq N(g)$.

Several recent papers studied gaps in semigroups and used different strategies to give upper and lower bounds for $N(g)$, [1, 2, 3, 4, 5, 9, 17, 25]. Only [1] appears to have considered $N(m, g)$. We will most closely follow the approach of [1, 6] and [13], which exploits a bijection between the set of numerical semigroups of genus g and the set of integer points in a certain rational polytope. We will use this bijection to show the following.

Theorem 1. *Suppose* $2g < 3m$. *Then* $N(m-1, g-1)+N(m-1, g-2) = N(m, g)$.

This result gives us a way to compute $N(m, m+k)$ for any $m > 2k$ when k is fixed and not too large.

If $m(S) = e(S)$ we say that S has maximal embedding dimension. Let $MED(g)$ be the number of maximal embedding dimension semigroups of genus g, and $MED(m, g)$ denote the number which also have multiplicity m. We will relate this smaller classes of semigroups to the collection of all semigroups and show that $N(k + 1) = MED(m, m + k)$ for all $m \geq 2k + 2$.

The other major subject of this paper is the verification of Wilf's question [23], for semigroups with $2g < 3m$ or $F < 2m$. Although Wilf did not phrase this question as a conjecture, we will state it as a conjecture here, since we believe that it is true.

Conjecture 4 ([23]). *Let* S *be a numerical semigroup with embedding dimension* $e(S)$*, Frobenius number* $F(S)$ *and genus* $g(S)$*. Then*

$$e(S) \geq \frac{F(S)+1}{F(S)+1-g(S)}.$$

Special cases of this conjecture have been verified, [8], and Bras-Amorós has checked it for all semigroups of genus at most 50, [2], but a complete solution seems far off. We will discuss the difficulties of extending our arguments to other cases.

In the final section of this paper we will relate our work to some recent results of Blanco, García-Sánchez and Puerto [1], and Zhao [25], and discuss some conjectures which come from looking at tables of data, and further open problems.

2. Apéry Sets and Integer Points of Rational Polytopes

In this section we will explain our approach to counting numerical semigroups which is closely related to the methods of [1, 6] and [13]. This approach relates a numerical semigroup to its Apéry set, a specific generating set, and Apéry sets to the points of a certain rational polytope.

We note that the set of gaps of S uniquely determines S. We now define a concept that will be very important throughout this paper. Consider $n \in S$. For $1 \leq i \leq n-1$ let $w(i)$ be the smallest integer in S which is congruent to i modulo n. The set $\{0, w(1), \ldots, w(n-1)\}$ is called the Apéry set of S with respect to n. Since the Apéry set with respect to n uniquely determines the gaps of S it also uniquely determines S. We will always consider the Apéry set with respect to the multiplicity m of S since this is the Apéry set of smallest size. We will usually just refer to this set of the Apéry set of S, and will usually omit 0. We will often write this set as $\{k_1 m + 1, \ldots, k_{m-1} m + (m-1)\}$ where $k_i \geq 1$. A numerical semigroup with such an Apéry set of size $m-1$ has multiplicity m. We see that every numerical semigroup has a minimal generating set of size at most m. If this inequality is actually an equality, we say that S has maximal embedding dimension. These semigroups will be the subject of Section 4.

It is easy to see that certain sets cannot occur as the Apéry set of a numerical semigroup. For example consider the set $\{11, 25\}$. If this were the Apéry set of a numerical semigroup S then $m = 3$ and $11 \in S$ but $22 \notin S$ which is impossible. We note the following observation which is due to Selmer, [20].

Proposition 2. *Let S be a numerical semigroup with Apéry set $\{k_1 m + 1, \ldots, k_{m-1} m + (m-1)\}$ where $k_i \geq 1$. Then the genus of S is $\sum_{i=1}^{m-1} k_i$.*

Proof. We see that the number of gaps of S which are congruent to i modulo m is exactly k_i. □

The key to our approach is to count semigroups by counting Apéry sets. Recall that a composition of a positive integer n is a way of writing n as a sum of positive integers where the order of these integers does matter.

Proposition 3. *We have that $N(m, g)$ is equal to the number of compositions of g into exactly $m-1$ parts, $\{k_1, \ldots, k_{m-1}\}$ such that $\{k_1 m + 1, \ldots, k_{m-1} m + (m-1)\}$ is the Apéry set of a numerical semigroup.*

Proof. We note that every numerical semigroup of genus g and multiplicity m has a unique Apéry set of the form $\{k_1 m + 1, \ldots, k_{m-1} m + (m-1)\}$ where $k_i \geq 1$ and $g = \sum_{i=1}^{m-1} k_i$. It is clear that this gives a composition of g into exactly $m-1$ parts. So $N(m, g)$ is less than or equal to the number of compositions of this form.

Suppose we have a composition of g into exactly $m-1$ parts $\{k_1, \ldots, k_{m-1}\}$ and that $\{k_1 m + 1, \ldots, k_{m-1} m + (m-1)\}$ is the Apéry set of a numerical semigroup.

Since two different compositions give two different Apéry sets, we see that the number of such compositions is at most $N(m,g)$. $\square$

We have to determine when a composition of g into $m-1$ parts leads to the Apéry set of a numerical semigroup. The following proposition follows directly from the definition of the Apéry set.

Proposition 4. *Consider the set* $\{k_1m+1,\ldots,k_{m-1}m+m-1\}$. *This is the Apéry set of the numerical semigroup* $S=\langle m,k_1m+1,\ldots,k_{m-1}m+m-1\rangle$ *if and only if for all* $1\le i\le m-1,\ (k_i-1)m+i\notin S$.

We can immediately see that certain conditions need to hold in order for $\{m,k_1m+1,\ldots,k_{m-1}m+m-1\}$ to be a valid Apéry set. For example, for each pair (i,j) such that $l=i+j<m$ we see that $(k_l-1)m+l\notin S$ implies that $k_i+k_j\ge k_l$. Similarly for each pair (i,j) such that $l=i+j>m$ we must have $k_i+k_j\ge k_l-1$. The useful result of Kunz [13], and Rosales et. al., [6], is that these conditions completely determine which compositions lead to valid Apéry sets.

Consider the following set of inequalities:

$$\begin{aligned} x_i &\ge 1 && \text{for all } i\in\{1,\ldots,m-1\}\\ x_i+x_j &\ge x_{i+j} && \text{for all } 1\le i\le j\le m-1,\ i+j\le m-1\\ x_i+x_j+1 &\ge x_{i+j-m} && \text{for all } 1\le i\le j\le m-1,\ i+j>m\\ x_i &\in \mathbb{Z} && \text{for all } i\in\{1,\ldots,m-1\}. \end{aligned}$$

Proposition 5 ([6, 13])**.** *There is a one to one correspondence between solutions* $\{k_1,\ldots,k_{m-1}\}$ *to the above inequalities and the Apéry sets of numerical semigroups with multiplicity* m. *If we add the condition that* $\sum_{i=1}^{m-1}k_i=g$, *then there is a one to one correspondence between solutions* $\{k_1,\ldots,k_{m-1}\}$ *to the above inequalities and the Apéry sets of numerical semigroups with multiplicity* m *and genus* g.

Each of the above inequalities defines a half space in $\mathbb{R}^{m-1}$, so their intersection defines a rational polyhedral cone. If we fix the sum of the k_i, then each $k_i\le g$, and we see that this polyhedron is bounded, and therefore is a rational polytope. Therefore, we can use the theory of integer points in rational polytopes to try to count semigroups. This is discussed extensively in [1]. We see that adding the extra condition $\sum_{i=1}^{m-1}k_i=g$ amounts to taking the intersection of our $m-1$ dimensional polyhedral cone with a hyperplane defined by g.

We point out that there is a very similar result for maximal embedding dimension semigroups, also due to Rosales et. al., [6]. We note that the condition that an Apéry set element gives a minimal generator requires stricter bounds on sums of the other Apéry set elements.

Proposition 6 ([6])**.** *We have that* $MED(m,g)$ *is equal to exactly the number of* $m-1$ *tuples* $\{k_1,\ldots,k_{m-1}\}$ *satisfying the following set of inequalities:*

$$\begin{aligned} x_i &\ge 1 && \textit{for all } i\in\{1,\ldots,m-1\}\\ x_i+x_j &\ge x_{i+j}+1 && \textit{for all } 1\le i\le j\le m-1,\ i+j\le m-1\\ x_i+x_j &\ge x_{i+j-m} && \textit{for all } 1\le i\le j\le m-1,\ i+j>m\\ x_i &\in \mathbb{Z} && \textit{for all } i\in\{1,\ldots,m-1\},\\ \sum_{i=1}^{m-1} x_i &= g. \end{aligned}$$

We aim to prove a corollary which outlines the general structure of $N(m, g)$. Suppose $m \geq 3$, since we can easily determine $N(2, g)$ and $MED(2, g)$. We want to define a function which is eventually a quasipolynomial following the language of [7], that is, functions $f(n)$ such that there is a period s and a collection of polynomials $f_i(n)$ for each $1 \leq i \leq s$ such that for $n \gg 0$, $f(n) = f_i(n)$ for all $i \equiv n \pmod{s}$. The degree of the quasipolynomial is the largest degree of the f_i. We denote the set of such functions by $QP_{\gg 0}$. We will prove the following.

Proposition 7. *For fixed m and $g \gg 0$, $N(m, g)$ is eventually a quasipolynomial of degree $m - 2$ with period depending on m. That is, there exists some period s such that for each $1 \leq i \leq s$ there is a polynomial $f_i(g)$ such that $N(m, g) = f_i(g)$ whenever $g \equiv i \pmod{s}$ and $g \gg 0$.*

For fixed m, and $g \gg 0$, $MED(m, g)$, is also eventually a quasipolynomial of degree $m - 2$.

We will give this result in a few steps. First we will note that $N(m, g)$ is given by the number of integer points inside a certain polytope of dimension exactly $m - 2$. A straightforward application of a theorem of Chen, Li and Sam [7], allows us to conclude that $N(m, g)$ is eventually a quasipolynomial in g of degree at most $m-2$. We will show that $N(m, g)$ is related to the number of integer points of a related polytope of dimension $m - 2$ for which we can apply a classical theorem of Ehrhart to count integer points. Finally, we give a lower bound for $N(m, g)$ in terms of the number of integer points of this polytope and conclude that $N(m, g)$ is eventually a quasipolynomial of degree exactly $m - 2$.

We will focus on the inequalities giving $N(m, g)$. The argument for $MED(m, g)$ is extremely similar. Recall that $N(m, g)$ is given by the integer points satisfying the following inequalities:

$$\begin{aligned} x_i &\geq 1 && \text{for all } i \in \{1, \ldots, m-1\} \\ x_i + x_j - x_{i+j} &\geq 0 && \text{for all } 1 \leq i \leq j \leq m-1,\ i + j \leq m - 1 \\ x_i + x_j - x_{i+j-m} &\geq -1 && \text{for all } 1 \leq i \leq j \leq m-1,\ i + j > m, \end{aligned}$$

and also satisfying $\sum_{i=1}^{m-1} x_i = g$.

The last condition corresponds to taking a hyperplane through the polyhedral cone defined by the above inequalities, giving a polytope $P(g)$. We set $x_{m-1} = g - \sum_{i=1}^{m-2} x_i$ and can give our polytope in terms of inequalities which contain only the variables $x_1, \ldots, x_{m-2}$ and the parameter g. For each fixed value of g we see that this polytope has dimension at most $m - 2$. We can see that it has dimension equal to $m - 2$ by noting that it contains the points $x_1 = \cdots = x_{m-1} = \frac{g}{m-1}$ and for any very small ε relative to g, and for any $1 \leq i \leq m - 1$ it also contains the point $x_i = \frac{g}{m-1} + \varepsilon$ and $x_j = \frac{g}{m-1} - \frac{\varepsilon}{m-2}$ for all $j \neq i$. It is easy to see that any convex set containing these points is at least $m-2$ dimensional, and therefore that for any $g > 0$, $P(g)$ has dimension exactly $m - 2$.

We now give a result of [7] which implies that the number of integer points inside of $P(g)$ grows as a quasipolynomial in g of degree at most $m - 2$.

Theorem 8 (Theorem 2.1 in [7])**.** *For $n \gg 0$, define a rational polytope $P(n) = \{x \in \mathbb{R}^d \mid V(n)x \geq c(n)\}$, where $V(x)$ is an $r \times d$ matrix, and $c(x)$ is an $r \times 1$ column vector, both with entries in $\mathbb{Z}[x]$. Then $\#(P(n) \cap \mathbb{Z}^d) \in QP_{\gg 0}$.*

After substituting for x_{m-1}, we can represent each of the inequalities defining the sets which count $N(m, g)$ and $MED(m, g)$ in terms of a row of an $r \times (m - 2)$

matrix and $r \times 1$ column matrix, for some r. Each x_i satisfies $1 \leq x_i \leq g$, and the value of x_{m-1} is determined by the values of the other x_i, so clearly there are at most g^{m-2} integer points inside of $P(g)$. Therefore we see that the degree of this quasipolynomial is at most $m-2$. We need to do a little work to show that the degree of this quasipolynomial is exactly $m-2$ and not something smaller.

We need only give a lower bound for the number of points inside of $P(g)$ which is a quasipolynomial of degree $m-2$ minus something which is eventually a quasipolynomial of degree at most $m-3$. Since we know that the number of points inside of $P(g)$ is eventually a quasipolynomial, we will know that its degree is at least $m-2$. We already know that its degree is at most $m-2$, and conclude that it must be exactly $m-2$.

We next consider the polytope $P'(g)$ defined by the following related inequalities:

$$\begin{aligned} x_i &\geq 0 && \text{for all } i \in \{1, \ldots, m-1\} \\ x_i + x_j - x_{i+j} &\geq 0 && \text{for all } 1 \leq i \leq j \leq m-1,\ i+j \leq m-1 \\ x_i + x_j - x_{i+j-m} &\geq 0 && \text{for all } 1 \leq i \leq j \leq m-1,\ i+j > m, \end{aligned}$$

and $\sum_{i=1}^{m-1} x_i = g$. As above, we can see that for each value of g this polytope is $m-2$ dimensional. In fact, there is a very clear relationship between the polytopes resulting from two different values of g. Let $0 < g_1 < g_2$. We claim that a point $p \in P'(g_1)$ if and only if $\frac{g_2}{g_1}p \in P'(g_2)$. Suppose $p \in P'(g_1)$. Then clearly $\frac{g_2}{g_1}(x_i + x_j - x_k) \geq 0$ for all $i + j \equiv k \pmod{m}$ and $\frac{g_2}{g_1}x_i \geq 0$. Similarly if $\frac{g_2}{g_1}p \in P'(g_2)$, then we can divide each coordinate by $\frac{g_2}{g_1}$ and see that $p \in P'(g_1)$. Therefore, these inequalities define a family of dilations of a single rational polytope of dimension $m-2$. We recall the following theorem of Ehrhart which is discussed in [7].

Theorem 9 (Ehrhart)**.** *Let $P \subset \mathbb{R}^d$ be a polytope with rational vertices. Then the function $L_P(n)$ which is the number of integer points inside of nP is a quasipolynomial of degree* $\dim P$.

We apply this theorem and see that the number of integer points in $P'(g)$ grows as a quasipolynomial in g of dimension exactly $m-2$. We will now give a lower bound for the number of points of $P(g)$, completing the proof.

Proof. A lower bound on the number of points in $P(g)$ is the number of integer points satisfying the inequalities defining $P'(g)$ except that each $x_i \geq 0$ is replaced with $x_i \geq 1$. This is equal to the number of integer points in $P'(g)$ minus the number of integer points for which at least one $x_i \geq 0$ is actually an equality. A lower bound for this is the number of points in $P'(g)$ minus the number of points in $P'(g)$ for which $x_1 = 0$, minus the number of points for which $x_2 = 0$, and so on, for each $1 \leq i \leq m-1$ subtracting the number of integer points of $P'(g)$ for which $x_i = 0$.

We first note that if we consider the inequalities defining the $m-2$ dimensional polytope $P'(g)$ and impose an extra equality condition that $x_i = 0$ for some i, we get a new set linear inequalities which do not involve x_i. This gives a polytope of dimension at most $m-3$. Applying Theorem 8, we know that the number of integer points inside such a polytope is eventually a quasipolynomial of degree at most $m-3$.

For each i, the number of points of $P'(g)$ satisfying $x_i = 0$ is eventually a quasipolynomial of degree at most $m-3$. Since this holds for each i, the sum of the number of these points for different values of i is also eventually a quasipolynomial in g of dimension at most $m-3$. Therefore we have a lower bound for the number of points inside of $P(g)$ which is a quasipolynomial of degree $m-2$ minus something which is eventually a quasipolynomial of degree at most $m-3$. This completes the proof that for fixed m, $N(m,g)$ is eventually a quasipolynomial in g of degree exactly $m-2$. □

In further work we would like to investigate the properties of these polytopes and their related quasipolynomials, including the coefficients of their leading terms and their periods, and whether they are in fact actual quasipolynomials and not just eventually quasipolynomials.

3. The Fibonacci-Like Behavior of Certain Semigroups

We will begin by providing a table of values of $N(m,g)$ so that the reader can get a sense of the type of patterns that emerge in this data. These values were computed using a program written in SAGE based on Proposition 5, [21]. Our values agree with the values contained in [1], and with the values of $N(g)$ given in [2].

In this section we will prove Theorem 1, giving some indication of why Bras-Amorós' conjecture may hold. We will use two lemmas to set up a bijection between the set of semigroups with multiplicity m and genus g where $2g < 3m$, and the set of semigroups of multiplicity $m-1$ and genus either $g-1$ or $g-2$.

Lemma 10. *Suppose* $2g < 3m$ *and that* $S = \langle m, k_1 m + 1, \ldots, k_{m-1} m + m - 1\rangle$ *is a semigroup of genus* $\sum_{i=1}^{m-1} k_i = g$ *with each* $k_i \geq 1$. *Then* $k_{m-1} \leq 2$.

Lemma 11. *Suppose* $2g < 3m+2$ *and that* $S = \langle m, k_1 m + 1, \ldots, k_{m-1} m + m - 1\rangle$ *is a semigroup of genus* $\sum_{i=1}^{m-1} k_i = g$ *with each* $k_i \geq 1$. *Then for each* $1 \leq i \leq m-1$, $k_i \leq 3$.

We will first prove the theorem and then give the proof of the two lemmas.

Proof of Theorem 1. Suppose $S = \langle m, k_1 m+1, \ldots, k_{m-1}m+m-1\rangle$ is a semigroup of genus $\sum_{i=1}^{m-1} k_i = g$ with each $k_i \geq 1$. By the previous two lemmas, $k_{m-1} = 1$ or 2 and $1 \leq k_i \leq 3$ for all i.

We claim that $S' = \langle m-1, k_1(m-1)+1, \ldots, k_{m-2}(m-1)+m-2\rangle$ is a numerical semigroup of genus $\sum_{i=1}^{m-2} k_i = g - k_{m-1}$ and multiplicity $m-1$. We need only check that for all pairs (i,j) with $l = i+j < m-1$, we have $k_i + k_j \geq k_l$ and that for all pairs (i,j) with $l = i+j > m-1$ we have $k_i + k_j + 1 \geq k_{l-(m-1)}$. The first condition holds because $\{k_1 m + 1, \ldots, k_{m-1} m + m - 1\}$ is the Apéry set of a numerical semigroup by assumption, and the second condition holds because each $k_i \geq 1$, so $k_i + k_j + 1 \geq 3$ and each $k_{l-m} \leq 3$ by our second lemma.

We now consider starting with $S' = \langle m-1, k_1(m-1)+1, \ldots, k_{m-2}(m-1)+m-2\rangle$, a numerical semigroup of multiplicity $m-1$ and genus either $g-1$ or $g-2$. We note that $2g < 3m$ implies that $2(g-1) = 2g-2 < 3m-2 = 3(m-1)+1$ and therefore by Lemma 11 we see that each $k_i \leq 3$. Now consider $S = \langle m, k_1 m+1, \ldots, k_{m-2}m + m-2, lm+m-1\rangle$, where l is either 1 or 2 depending on whether the genus of S' is $g-1$ or $g-2$. We will show that $\{k_1 m+1, \ldots, k_{m-2}m+m-2, lm+m-1\}$ satisfies the inequalities necessary for it to be the Apéry set of a numerical semigroup of multiplicity m and genus g. For each pair (i,j) with $l = i+j < m-1$ we have

$g\backslash m$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	N(g)
0	1																										1
1		1																									1
2		1	1																								2
3		1	2	1																							4
4		1	2	3	1																						7
5		1	2	4	4	1																					12
6		1	3	6	7	5	1																				23
7		1	3	7	10	11	6	1																			39
8		1	3	9	13	17	16	7	1																		67
9		1	4	11	16	27	28	22	8	1																	118
10		1	4	13	22	37	44	44	29	9	1																204
11		1	4	15	24	49	64	72	66	37	10	1															343
12		1	5	18	32	66	85	116	116	95	46	11	1														592
13		1	5	20	35	85	112	172	188	182	132	56	12	1													1001
14		1	5	23	43	106	148	239	288	304	277	178	67	13	1												1693
15		1	6	26	51	133	191	325	409	492	486	409	234	79	14	1											2857
16		1	6	29	61	163	237	441	559	754	796	763	587	301	92	15	1										4806
17		1	6	32	68	196	301	573	750	1094	1246	1282	1172	821	380	106	16	1									8045
18		1	7	36	80	236	369	737	1015	1534	1841	2074	2045	1759	1122	472	121	17	1								13467
19		1	7	39	89	282	444	945	1334	2106	2601	3227	3356	3217	2580	1502	578	137	18	1							22464
20		1	7	43	104	330	541	1193	1737	2840	3561	4812	5301	5401	4976	3702	1974	699	154	19	1						37396
21		1	8	47	115	390	658	1490	2231	3793	4822	6939	8020	8721	8618	7556	5204	2552	836	172	20	1					62194
22		1	8	51	133	456	784	1847	2851	4967	6490	9752	11657	13643	14122	13594	11258	7178	3251	990	191	21	1				103246
23		1	8	55	143	525	940	2262	3582	6426	8638	13439	16375	20640	22364	22740	21150	16462	9730	4087	1162	211	22	1			170963
24		1	9	60	164	608	1114	2766	4475	8252	11355	18309	22518	30282	34252	36614	36334	32408	23640	12981	5077	1353	232	23	1		282828
25		1	9	64	181	697	1307	3353	5512	10519	14756	24663	30588	43234	50772	57340	59354	57484	48870	33370	17068	6239	1564	254	24	1	467224

TABLE 1. Number of Numerical Semigroups of Multiplicity m and Genus g

$k_i + k_j \geq k_l$ by our assumption on S'. For any pair (i, j) with $m - 1 = i + j$ we have $k_i + k_j \geq l$ because $l \leq 2$. For any $l = i + j > m - 1$ we have $k_i + k_j + 1 \geq 3 \geq k_{l-(m-1)}$, completing the proof. $\square$

We will now complete this argument by proving our two lemmas.

Proof of Lemma 10. We argue by contradiction. Suppose that $\{k_1 m + 1, \ldots, k_{m-1} m + m - 1\}$ is the Apéry set of a numerical semigroup with genus $g = \sum_{i=1}^{m-1} k_i$, $2g < 3m$, and that $k_{m-1} \geq 3$. By the inequalities implying that we have a valid Apéry set, for each $1 \leq i \leq m - 2$, we must have $k_i + k_{(m-1)-i} \geq 3$. We sum over the k_i twice and get

$$2g = 2k_{m-1} + \sum_{i=1}^{m-2} (k_i + k_{(m-1)-i}) \geq 3(m - 2) + 6 = 3m$$

which is a contradiction. $\square$

Proof of Lemma 11. We argue by contradition. Suppose that $\{k_1 m + 1, \ldots, k_{m-1} m + m - 1\}$ is the Apéry set of a numerical semigroup with genus $g = \sum_{i=1}^{m-1} k_i$, $2g < 3m + 2$, and that there exists some i with $k_i \geq 4$. By the inequalities implying that we have a valid Apéry set, for each $1 \leq j \leq i - 1$, $k_j + k_{i-j} \geq 4$ and for $i + 1 \leq j \leq m - 1$ we have $k_j + k_{m+i-j} \geq 3$. We sum over the k_i twice and get

$$2g = 2k_i + \sum_{j=1}^{i-1} (k_j + k_{i-j}) + \sum_{j=i+1}^{m-1} (k_j + k_{m+i-j}) \geq 8 + 4(i - 1) + 3(m - i - 1) \geq 3m + 2,$$

since $i \geq 1$, giving a contradiction. $\square$

We can use the recursion of Theorem 1 to give exact formulas for $N(m, g)$ for several families of pairs of (m, g). We begin with a trivial base case, which is certainly already known.

Proposition 12. *For any $m \geq 1$, $N(m, m - 1) = 1$.*

Proof. It is clear that any semigroup with smallest element m contains the $m - 1$ gaps $1, 2, \ldots, m - 1$. If there are no more gaps in the semigroup then it contains the elements $m, m + 1, \ldots, 2m - 1$. Therefore there is exactly one semigroup with multiplicity m and genus $m - 1$, and minimal generating set $\{m, m + 1, \ldots, 2m - 1\}$. $\square$

Proposition 13. *For all $k \geq 0$ there exists a monic polynomial of degree $k + 1$, $f_k(x)$ such that for all $m > 2k$, $N(m, m + k) = \frac{1}{(k+1)!} f_k(m)$.*

Proof. We will prove this by induction on k. First consider $k = 0$. We will show that for any $m \geq 1$ we have $N(m, m) = m - 1$. For $m = 1$ we see that $N(1, 1) = 0$. Suppose the formula holds for all $1 \leq i \leq m - 1$. We consider $N(m, m)$. Since $3m > 2m$ we can apply Theorem 1 and see that $N(m, m) = N(m - 1, m - 1) + N(m - 1, m - 2) = m - 2 + 1 = m - 1$ by the induction hypothesis and the previous proposition. This completes the $k = 0$ case.

Now suppose that this proposition holds for each nonnegative integer less than k. That is, for each $0 \leq i < k$ we have a monic polynomial $f_i(x)$ of degree $i + 1$ such that for all $m > 2i$ we have $N(m, m + i) = \frac{1}{(i+1)!} f_i(m)$.

Consider some $m > 2k$. Then by Theorem 1 we have

$$N(m, m+k) - N(m-1, m+k-1) = N(m-1, m+k-2) = \frac{1}{k!} f_{k-1}(m-1),$$

by induction since $m - 1 > 2(k-1)$.

Suppose we find a monic polynomial $f_k(x)$ of degree $k+1$ such that

$$\frac{1}{(k+1)!}(f_k(m) - f_k(m-1)) = \frac{1}{k!} f_{k-1}(m-1)$$

for all $m > 2k$, and $\frac{1}{(k+1)!} f_k(2k+1) = N(2k+1, 3k+1)$. We claim that $\frac{1}{(k+1)!} f_k(m) = N(m, m+k)$ for all $m > 2k$.

We argue by contradiction. Consider the minimal $m > 2k$ such that $\frac{1}{(k+1)!} f_k(m) \neq N(m, m+k)$. Since we have assumed $\frac{1}{(k+1)!} f_k(2k+1) = N(2k+1, 3k+1)$, we see that $m \geq 2k+2$. We have $N(m, m+k) - N(m-1, m+k-1) = \frac{1}{k!} f_{k-1}(m-1)$ by Theorem 1 and the induction hypothesis, and $N(m-1, m+k-1) = \frac{1}{(k+1)!} f_k(m-1)$ by the minimality of m. We see that $\frac{1}{(k+1)!} f_k(m-1) + \frac{1}{k!} f_{k-1}(m-1) = \frac{1}{k+1!} f_k(m)$, also by our assumptions about $f_k(x)$. This contradicts the claim that $N(m, m+k) \neq \frac{1}{k+1!} f_k(m)$.

Therefore, we need only show that there exists a monic polynomial $f_k(x)$ of degree $k+1$ such that $\frac{1}{(k+1)!}(f_k(m) - f_k(m-1)) = \frac{1}{k!} f_{k-1}(m-1)$ for all $m > 2k$ and $\frac{1}{(k+1)!} f_k(2k+1) = N(2k+1, 3k+1)$. We substitute $m+1$ for m. We will show that for each constant C, there is a monic polynomial $f_k(x)$ such that

$$\frac{1}{(k+1)!}(f_k(m+1) - f_k(m)) = \frac{1}{k!} f_{k-1}(m)$$

for all $m \geq 2k$ with constant term C. We can then choose C such that $\frac{1}{(k+1)!} f_k(2k+1) = N(2k+1, 3k+1)$, giving our desired polynomial and completing the proof.

We let $f_k(x) = \sum_{i=0}^{k+1} a_i x^i$ with $a_{k+1} = 1$ and $f_{k-1}(x) = \sum_{j=0}^{k} b_j x^j$ with $b_k = 1$. We will show that there exist choices $\{a_0, \ldots, a_k\}$ satisfying our desired properties. We have

$$f_k(x+1) - f_k(x) = \sum_{i=0}^{k+1} a_i((x+1)^i - x^i) = \sum_{i=0}^{k+1} a_i \sum_{0 \leq j < i} \binom{i}{j} x^j = \sum_{j=0}^{k+1} \left(\sum_{i=j+1}^{k+1} a_i \binom{i}{j} \right) x^j.$$

For all $0 \leq j \leq k$ we want

$$\sum_{i=j+1}^{k+1} a_i \binom{i}{j} = (k+1) b_j.$$

We label these equations by the value of j from 0 to k. Equation k is automatically verified since $\binom{k+1}{k} a_{k+1} = (k+1) b_k = k+1$.

Now suppose we have chosen $\{a_k, a_{k-1}, \ldots, a_{l+1}\}$ such that equations $k, k-1, \ldots, l$ are verified. We will show that there is a unique choice of a_l so that equation $l-1$ holds. We want

$$\sum_{i=l}^{k+1} a_i \binom{i}{l-1} = (k+1) b_{l-1},$$

which implies

$$a_l = \frac{(k+1)b_{l-1} - \sum_{i=l+1}^{k+1} a_i \binom{i}{l-1}}{\binom{l}{l-1}} = \frac{(k+1)b_{l-1} - \sum_{i=l+1}^{k+1} a_i \binom{i}{l-1}}{l}.$$

We see that the value of a_l is determined by the values $\{a_{k+1}, a_k, \ldots, a_{l+1}, b_k, b_{k-1}, \ldots, b_{l-1}\}$. In this way, the set $\{a_{k+1}, a_k, \ldots, a_1\}$ is determined by the set $\{b_k, b_{k-1}, \ldots, b_0\}$. We see that we have constructed $f_k(x)$ so that for all $m \geq 2k$,

$$\frac{1}{(k+1)!}(f_k(m+1) - f_k(m)) = \frac{1}{k!} f_{k-1}(m).$$

We note that we are still free to choose the value of a_0. We choose it such that $\frac{1}{(k+1)!} f_k(2k+1) = N(2k+1, 3k+1)$. This completes the proof.

□

We can compute some of these $f_k(m)$ explicitly using the method described in this proof.

Corollary 14. *For* $m \geq 1$, $N(m, m-1) = 1$.
For $m \geq 1$, $N(m, m) = m - 1$.
For $m \geq 2$, $N(m, m+1) = \frac{m^2-3m+4}{2}$.
For $m \geq 4$, $N(m, m+2) = \frac{m^3-6m^2+17m}{6}$.
For $m \geq 6$, $N(m, m+3) = \frac{m^4-10m^3+47m^2-38m+48}{24}$.
For $m \geq 8$, $N(m, m+4) = \frac{m^5-15m^4+105m^3-225m^2+374m+240}{120}$.
For $m \geq 10$, $N(m, m+5) = \frac{m^6-21m^5+205m^4-795m^3+1954m^2+96m+2880}{720}$.
For $m \geq 12$, $N(m, m+6) = \frac{m^7-28m^6+364m^5-2170m^4+7819m^3-7882m^2+22056m+10080}{5040}$.
For $m \geq 14$,
$N(m, m+7) = \frac{m^8-36m^7+602m^6-5040m^5+25529m^4-58044m^3+135148m^2-17520m+887040}{40320}$.

We point out that the method of proof described above does not explain why our formula holds for $N(2k, 3k)$ for each listed value of k. We include these cases because the explicit computation agrees.

We next point out how Theorem 1 can fail in cases where $2g \geq 3m$. First consider the example of $N(5, 10) = 22$ compared to $N(4, 8) + N(4, 9) = 9 + 11 = 20$. We see that the list of $\{k_1, k_2, k_3, k_4\}$ leading to valid Apéry sets of semigroups $\{5k_1 + 1, \ldots, 5k_4 + 4\}$ with multiplicity 5 and genus 10 is:

$$\{1,2,3,4\}, \{3,2,3,2\}, \{3,2,1,4\}, \{3,3,1,3\}, \{3,4,1,2\}, \{4,1,3,2\}, \{4,2,3,1\}, \{4,3,2,1\},$$
$$\{2,4,2,2\}, \{3,4,2,1\}, \{3,2,2,3\}, \{2,2,3,4\}, \{2,2,3,3\}, \{2,2,4,2\}, \{3,1,4,2\}, \{2,3,2,3\},$$
$$\{2,3,3,2\}, \{2,4,1,3\}, \{4,2,2,2\}, \{2,4,3,1\}, \{3,3,2,2\}, \{3,3,3,1\}.$$

We also consider the set Apéry sets $\{k_1, k_2, k_3\}$ leading a semigroup of multiplicity 4 and genus either 8 or 9:

$$\{2,3,4\}, \{2,4,3\}, \{3,2,4\}, \{3,3,3\}, \{3,4,2\}, \{4,1,4\}, \{4,2,3\}, \{4,3,2\}, \{5,1,3\}, \{5,2,2\},$$
$$\{5,3,1\}, \{2,2,4\}, \{2,3,3\}, \{2,4,2\}, \{3,1,4\}, \{3,2,3\}, \{3,3,2\}, \{4,1,3\}, \{4,2,2\}, \{4,3,1\}.$$

We notice that not only does Theorem 1 not hold, but neither Lemma 10 nor Lemma 11 holds, as there are semigroups in these sets with k_{m-1} equal to 3 and 4, and with other k_i equal to $3, 4$ and 5.

We also note that it is not always the case that $N(m-1, g-1) + N(m-1, g-2) \leq N(m, g)$. For example, $85 = N(7, 12) < N(6, 11) + N(6, 10) = 86$.

It seems that the growth behavior of $N(m, g)$ for $2g \geq 3m$ is much more complicated than in the $2g < 3m$ case.

4. Maximal Embedding Dimension Semigroups

In this section we will discuss some connections between general numerical semigroups and maximal embedding dimension semigroups. Let $MED(m, g)$ be the number of maximal embedding dimension numerical semigroups of multiplicity m and genus g. We begin this section with a table analogous to the table appearing in the previous section. These values were computed using a very similar SAGE program to the one used to compute $N(m, g)$, [21].

We will now work towards showing that $N(k+1) = MED(m, m+k)$ for any $m \geq 2k+2$. This allows us to represent $N(g)$ as the number of integer points inside a single polytope, instead of a union of several polytopes. The key to our argument will be Proposition 6.

Theorem 15. *Fix $k \geq 0$ and suppose $m \geq 2k+2$. Then $MED(2k+2, 3k+2) = MED(m, m+k)$.*

Proof. We will prove this result by showing that $MED(m, m+k)$ is equal to the number of $2k+1$ tuples $\{k_1, \ldots, k_{2k+1}\}$ satisfying the following set of inequalities:

$$\begin{aligned} x_i &\in \mathbb{Z} && \text{for all } i \in \{1, \ldots, 2k+1\} \\ 1 \leq x_i &\leq 2 && \text{for all } i \in \{1, \ldots, 2k+1\} \\ x_i + x_j &\geq x_{i+j} + 1 && \text{for all } 1 \leq i \leq j \leq 2k+1,\ i+j \leq 2k+1 \\ \sum_{i=1}^{2k+1} x_i &= 3k+2. \end{aligned}$$

We have omitted the third set of inequalities from Proposition 6, because when $1 \leq x_i \leq 2$, for all $1 \leq i \leq 2k+1$ then it is automatically the case that $x_i + x_j \geq x_k$ for all (i, j, k).

We argue by contradiction. Suppose there exists some $m \geq 2k+3$ for which $MED(m, m+k) \neq MED(2k+2, 3k+2)$.

First, suppose there exists a maximal embedding dimension semigroup given by $\{k_1, \ldots, k_{m-1}\}$ where some $k_r \geq 3$. For each pair $i \neq r-i$, we must have $k_i + k_{r-i} \geq 3$ by Proposition 6. If $2i \equiv r$ modulo m then $k_i \geq 2$. We sum over the k_i twice, noting that $k_r = 3$ and get

$$2g = 2m + 2k \geq 3(m-2) + 6 = 2m + m > 2m + 2k.$$

This is a contradiction, so we assume that $1 \leq k_i \leq 2$ for $1 \leq i \leq 2k+1$.

We next show that there cannot exist an $r \geq 2k+2$ with $k_r = 2$. We argue by contradiction and suppose there exists such an r. Then for each pair $i, r-i$ with $i < r$, we must have $k_i + k_{r-i} \geq 3$. If $2i \equiv r$ modulo m then $k_i \geq 2$. We sum over the k_i twice, noting that $k_r = 2$ and get

$$2g \geq 2(m-2) + (r-1) + 4 > 2m + 2k,$$

which is a contradiction.

We have seen that $k_r = 1$ for all $r \geq 2k+2$ and that $k_r = 1$ or 2 for all r. We conclude that $MED(m, m+k)$ is given by the size of the solution set of the inequalities given above.

$g\backslash m$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	N(g)
0	1																										1
1		1																									1
2		1	1																								2
3		1	1	1																							3
4		1	1	1	1																						4
5		1	2	2	1	1																					7
6		1	2	3	2	1	1																				10
7		1	2	4	2	2	1	1																			13
8		1	3	5	4	4	2	1	1																		21
9		1	3	7	5	6	4	2	1	1																	30
10		1	3	8	8	9	4	4	2	1	1																41
11		1	4	10	10	14	7	7	4	2	1	1															61
12		1	4	12	13	19	12	10	7	4	2	1	1														86
13		1	4	14	16	25	18	17	9	7	4	2	1	1													119
14		1	5	16	22	35	25	26	16	12	7	4	2	1	1												173
15		1	5	19	24	45	37	39	24	20	12	7	4	2	1	1											241
16		1	5	21	32	56	50	58	37	30	17	12	7	4	2	1	1										334
17		1	6	24	35	74	68	82	59	48	27	23	12	7	4	2	1	1									474
18		1	6	27	43	93	89	111	90	75	41	37	23	12	7	4	2	1	1								663
19		1	6	30	51	114	114	156	127	117	67	58	31	23	12	7	4	2	1	1							922
20		1	7	33	61	142	149	212	184	171	107	93	48	39	23	12	7	4	2	1	1						1297
21		1	7	37	68	172	192	281	252	254	167	148	78	61	39	23	12	7	4	2	1	1					1807
22		1	7	40	80	205	237	368	349	357	256	226	125	99	60	39	23	12	7	4	2	1	1				2499
23		1	8	44	89	246	301	485	468	500	384	355	194	157	96	67	39	23	12	7	4	2	1	1			3484
24		1	8	48	104	292	369	619	619	699	557	540	315	247	150	111	67	39	23	12	7	4	2	1	1		4835
25		1	8	52	115	340	444	786	805	967	797	798	495	395	235	179	103	67	39	23	12	7	4	2	1	1	6676

TABLE 2. Number of Maximal Embedding Dimension Numerical Semigroups of Multiplicity m and Genus g

It is easy to check that this number is equal to $MED(2k+2, 3k+2)$. We need only note that no numerical semigroup with maximal embedding dimension and this multiplicity and genus can have any $k_r \geq 3$ since this would imply that $g \geq m - 1 + 2 + \lfloor \frac{m-1}{2} \rfloor > m + k$. $\square$

We note that this is the optimal value of m for the statement to hold. For example $MED(m, m+2) = 4$ if and only if $m \geq 6 = 2k+2$.

We next state a simple observation as a lemma.

Lemma 16. *Suppose $S \subset \mathbb{N}_0$ is a numerical semigroup of genus k. Then $x \geq 2k$ implies $x \in S$.*

Proof. We first recall that a set $S \subset \mathbb{N}$ with finite complement is a numerical semigroup if and only if it satisfies $0 \in S$ and $i, j \in S$, implies $i + j \in S$.

Suppose $x \geq 2k$ and $x \notin S$. Consider the pairs $\{(1, x-1), (2, x-2), \ldots, (\lfloor \frac{x}{2} \rfloor, \lceil \frac{x}{2} \rceil)\}$. For each pair $(i, x-i)$ at least one of i or $x-i$ is not in S. Since there are at least k such pairs, we get a contradiction. $\square$

We can now relate $N(g)$ and $MED(m, g)$.

Theorem 17. *For each $k \geq 0$ we have $N(k+1) = MED(2k+2, 3k+2)$.*

Proof. We recall from Proposition 6 and the proof of Theorem 15 that there is a bijective correspondence between maximal embedding dimension numerical semigroups of multiplicity $2k+2$ and genus $3k+2$ and solutions $\{k_1, \ldots, k_{2k+1}\}$ to the inequalities

$$\begin{aligned} x_i &\in \mathbb{Z} && \text{for all } i \in \{1, \ldots, 2k+1\} \\ 1 \leq x_i &\leq 2 && \text{for all } i \in \{1, \ldots, 2k+1\} \\ x_i + x_j &\geq x_{i+j} + 1 && \text{for all } 1 \leq i \leq j \leq 2k+1,\ i+j \leq 2k+1 \\ \sum_{i=1}^{2k+1} x_i &= 3k+2. \end{aligned}$$

We note that whenever $1 \leq x_i \leq 2$ for all $1 \leq i \leq 2k+1$, it automatically holds that for any triple $1 \leq i_1, i_2, i_3 \leq 2k+1$ we have $x_{i_1} + x_{i_2} \geq x_{i_3}$.

We claim that there is a bijection between this set and the set of numerical semigroups $S \subset \mathbb{N}$ of genus $k+1$. We associate each set $\{k_1, \ldots, k_{2k+1}\}$ with $k_{w_1} = \cdots = k_{w_{k+1}} = 2$ satisfying these inequalities, to the set $S = \mathbb{N} \setminus \{w_1, \ldots, w_{k+1}\}$. We will show that this is a bijection in two steps. First we will show that each set $\{k_1, \ldots, k_{2k+1}\}$ leads to a unique semigroup of genus $k+1$, and then we will show that each semigroup of genus $k+1$ gives a unique solution $\{k_1, \ldots, k_{2k+1}\}$ to the above inequalities.

First consider some set $\{k_1, \ldots, k_{2k+1}\}$ with $k_{w_1} = \cdots = k_{w_{k+1}} = 2$ giving a maximal embedding dimension semigroup of multiplicity $2k+2$ and genus $3k+2$. Let $S = \mathbb{N} \setminus \{w_1, \ldots, w_{k+1}\}$. We need only show that for any $i, j \in S$, we have $i + j \in S$. We will show the contrapositive.

Suppose $z \notin S$. We consider every way of writing z as a sum of two positive integers, $\{(1, z-1), (2, z-2), \ldots, (\lfloor \frac{z}{2} \rfloor, \lceil \frac{z}{2} \rceil)\}$. Choose some pair (i, j). We must show that either i or j is not in S. We see that $k_z = 2$ and that for each $i + j = z$ we have $k_i + k_j > k_z = 2$. Therefore, either $k_i = 2$ or $k_j = 2$, and we conclude that either i or j is not in S. Therefore S is a numerical semigroup of genus $k+1$.

Since a semigroup is determined by its complement, is clear that different solutions $\{k_1, \ldots, k_{2k+1}\}$ lead to different semigroups S.

Next suppose S is a numerical semigroup of genus $k+1$ with gap set $\{w_1, \ldots, w_{k+1}\}$. By Lemma 16 we see that each $w_i \leq 2k+1$. Consider the set $\{k_1, \ldots, k_{2k+1}\}$ where k_i is equal to 2 if i is equal to some w_j, and equals 1 otherwise. In order to check that this $2k+1$-tuple gives a solution to the above inequalities, we need only check that for each k_z equal to 2, we cannot write $z = i + j$ with k_i and k_j equal to 1. Suppose that there were some triple (z, i, j) for which this was possible. Then we would have $i, j \in S$, but $z \in \{w_1, \ldots, w_{k+1}\}$ implying $z \notin S$, and giving a contradiction. This completes the proof.

□

Corollary 18. *For each $g \geq 0,\ 0 \leq k \leq g$ we have*

$$N(g-k) = MED(2g+k, 3g-1).$$

Proof. This is a direct application of the previous Theorem, noting that $N(g) = MED(2g, 3g-1)$ and using Theorem 15. □

We now see that we can read off values of $N(g)$, by examining values of $MED(m, g)$, where we either consider one relatively large value of m and vary g, or one large value of g and vary m.

It is not so surprising that a strong connection exists between numerical semigroups of genus g and a particular set of numerical semigroups of maximal embedding dimension. The following result of Rosales [18], indicates that the semigroups of maximal embedding dimension can model the entire set of numerical semigroups.

Proposition 19 ([18])**.** *There is a one to one correspondence between the set of numerical semigroups with multiplicity m and Frobenius number F, and the set of numerical semigroups with maximal embedding dimension, Frobenius number $F+m$, multiplicity m, and each other minimal generator greater than $2m$.*

We note that the bijection comes from adding m to each element of the Apéry set of the original numerical semigroup except the multiplicity, leading to a semigroup of the same multiplicity but with maximal embedding dimension. If we start with a maximal embedding dimension semigroup of multiplicity m where each nonzero Apéry set element is greater than $2m$, then we get a numerical semigroup of the same multiplicity if we subtract m from each of them. Either by applying this proposition or using the same argument that leads to it, we can prove the following.

Proposition 20. *There is a one to one correspondence between the set of numerical semigroups with multiplicity m and genus g, and the set of numerical semigroups with maximal embedding dimension, genus $g+m-1$, multiplicity m, and every other minimal generator greater than $2m$.*

The following bound follows directly from this proposition.

Corollary 21. *We have $N(m, g) \leq MED(m, g+m-1)$.*

This result leads us naturally to consider how many semigroups of maximal embedding dimension, genus g and multiplicity m also have some other minimal generator less than $2m$. We first point out that if this is not the case, then each Apéry set element must be of the form $k_i m + i$ with $k_i \geq 2$ and therefore $g \geq 2m-2$. For the first time we are forced to consider the prime factorization of the multiplicity.

Proposition 22. *Suppose $g \geq 0$ and $m \geq 2$. Then $N(m,g) = MED(m, g+m-1)$ if and only if m is prime and $g > \frac{(m-2)(m-1)}{2}$.*

Proof. Our main tool is Proposition 20. We need to show that if m is not prime, or m is prime and $g \leq \frac{(m-2)(m-1)}{2}$, we can find a semigroup of maximal embedding dimension, multiplicity m, genus g and with some generator other than the multiplicity less than $2m$. In the case that m is prime and $g > \frac{(m-2)(m-1)}{2}$ we must show that no such semigroups exist. We will consider the cases with m prime and m composite separately.

First suppose that m is composite. Then we can write it as $m = pm'$ where p is the smallest prime dividing m. Given any $g \geq m-1$, we can write $g = m' - 1 + (m-m')k + r$ where $0 \leq r < m - m'$ and $k \geq 1$ in a unique way. We will consider our Apéry set elements in two groups. Let $i_1 < i_2 < \cdots < i_{m-m'}$ be an enumeration of the elements of 1 to $m-1$ which are not divisible by p. Consider the Apéry set with elements

$$\{m+p, \ldots, m+(m'-1)p,$$
$$(k+1)m + i_1, (k+1)m+i_2, \ldots, (k+1)m + i_r, km + i_{r+1}, \ldots, km + i_{m-m'}\}.$$

Once we show that this is a numerical semigroup of maximal embedding dimension, we see easily that the genus is $m' - 1 + (k+1)r + k(m-m'-r) = m' - 1 + (m-m')k + r = g$.

Suppose that this set does not give a numerical semigroup of maximal embedding dimension. We will use Proposition 6. There must be three elements $k_a m + a$, $k_b m + b$, $k_c m + c$ such that either $k_a + k_b \leq k_c$, $a < b < c$, and $a + b = c$, or $k_a + k_b \leq k_c - 1$ and $a + b = m + c$. Clearly this implies that $k_c > k_a, k_b$, and so $k_c > 1$. This shows that c is not divisible by p, and that it is not possible for both a and b to be divisible by p. So at least one of k_a and k_b is equal to k or $k+1$, and so $k_a + k_b \geq k+1$. Since $k_c \leq k+1$, we see that $k_a + k_b > k_c - 1$, and therefore we must have $a + b = c$ and $k_a + k_b = k_c = k+1$.

Since $a, b < c$ and $\max\{k_a, k_b\} \geq k$, then $\max\{k_a, k_b\} \geq k_c$. So, we cannot have $k_a + k_b = k_c = k+1$. This is a contradiction, showing that this set gives a valid maximal embedding dimension numerical semigroup, and that it has genus g, multiplicity m and and another Apéry set element of size less than $2m$.

From now on we suppose that m is prime. We will first show that if $g \leq \frac{(m-2)(m-1)}{2}$ then there is a maximal embedding dimension semigroup of multiplicity m, genus $g+m-1$, and with some minimal generator other than the multiplicity less than $2m$. We write the elements of our Apéry set as $\{k_1 m + 1, \ldots, k_{m-1} m + m - 1\}$. Consider the Apéry set given by the following k_i values:

$$\{k+1, \ldots, k+1, k, \ldots, k, k-1, k-2, \ldots, 2, 1\}.$$

We increase terms with $k_i = k$ to $k_i = k+1$ from left to right until we have $\{k+1, \ldots, k+1, k, k-1, \ldots, 1\}$, at which point we increase terms $k_i = k+1$ to $k_i = k+2$ from left to right. This process can begin with the semigroup with each $k_i = 1$ and continues until we reach the semigroup with $\{k_1, \ldots, k_{m-1}\}$ given by $\{m-1, m-2, \ldots, 1\}$, which has genus $\frac{m(m-1)}{2} = \frac{(m-2)(m-1)}{2} + m - 1$. Now we need only show that at each step our set leads to a maximal embedding dimension numerical semigroup.

Suppose that it does not. We again use Proposition 6. There must be three elements $k_a m + a$, $k_b m + b$, $k_c m + c$ such that either $k_a + k_b \leq k_c$, $a < b < c$ and

$a + b = c$, or $k_a + k_b \leq k_c - 1$ and $a + b = m + c$. In the first case we see that $a, b < c$ implies that $k_a > k_c$, so this is not possible. Therefore, we can suppose that the second case holds. We have $k_c > k_a, k_b$ and therefore, $c < a, b$. We write $c = (a + b) - m$. We want to consider all pairs a, b such that $a + b$ is constant. We see that one of the pairs which always has the smallest value of $k_a + k_b$ is given by $a = m - 1,\ b = c + 1$. However, since $k_c - k_{c+1} \leq 1$, it is not possible for $k_a + k_b \leq k_c - 1$. Therefore, this Apéry set gives a numerical semigroup of maximal embedding dimension.

Now we consider the other direction. Suppose that m is prime and $g > \frac{(m-2)(m-1)}{2}$. We must show that there are no maximal embedding dimension semigroups of multiplicity m, genus $g + m - 1$ and with a generator other than the multiplicity of size less than $2m$. We recall a result of Sylvester from 1884, that if $S = \langle a, b \rangle$ is a numerical semigroup generated by two elements it has genus $\frac{ab-a-b+1}{2}$, [10]. Since m is prime, each $m + i$ with $1 \leq i \leq m - 1$ is relatively prime to m. Therefore $\langle m, m + i \rangle$ is a semigroup with genus $\frac{m(m+i)-2m-i+1}{2}$. Any semigroup of multiplicity m containing $m + i$ must contain this semigroup. We want to see how adding the condition of maximal embedding dimension changes this bound on the genus.

We use Proposition 6 again. If we write our Apéry set elements as $\{k_1 m + 1, \ldots, k_{m-1} m + (m - 1)\}$, then we see that $k_i = 1$. Maximal embedding dimension implies that $k_i + k_i \geq k_{2i}$, where we take $2i$ modulo m. So $k_{2i} \leq 2k_i$ and similarly we see that $k_i + 2k_i \geq k_i + k_{2i} \geq k_{3i}$. Continuing in this way, we see that $k_{li} \leq lk_i$. Since m is prime, i is a generator of $\mathbb{Z}/m\mathbb{Z}$ of order m. Since $k_i = 1$, summing over l we see that the genus of such a semigroup is at most $\sum_{l=1}^{m-1} l = \frac{m(m-1)}{2} = \frac{(m-2)(m-1)}{2} + m - 1$. This shows that for any maximal embedding dimension semigroup with multiplicity m and any other generator less than $2m$, we have $g \leq \frac{(m-2)(m-1)}{2} + m - 1$, completing the proof.

□

We have shown that in most circumstances the inequality $N(m, g) \leq MED(m, g+m-1)$ is not an equality, however we should be able to show that these quantities are closely related. That is, for large values of g relative to some fixed composite m, the number of maximal embedding dimension semigroups of multiplicity m and genus g and which contain another generator other than the multiplicity less than $2m$, should be small relative to $MED(m, g)$. More formally, we conjecture the following.

Conjecture 5. *Fix* $m \geq 2$. *Then* $\liminf_{g \to \infty} \frac{N(m, g-(m-1))}{MED(m,g)} = 1$.

This should not be difficult to verify in simple cases, for example $m = 4$. It will become more complicated as m has more prime factors, and therefore there are more nontrivial subgroups of $\mathbb{Z}/m\mathbb{Z}$.

5. Some Cases of Wilf's Question

In this section we will consider the relationship between the Apéry set of a semigroup, its genus, and its number of minimal generators. First recall that $F(S)$, the Frobenius number of S is the largest natural number not in S and that $e(S)$, the embedding dimension of S, is the number of minimal generators of S. Where it will not cause confusion we will often omit the S. Wilf's question, Conjecture 4, asks whether $e(S) \geq \frac{F(S)+1}{F(S)+1-g(S)}$. We recall some results of Dobbs and Matthews [8], establishing certain special cases.

Proposition 23 (Dobbs, Matthews)**.** *If S is a semigroup with $e \leq 3$ or equal to $m-1$, $F+1-g \leq 4$, or $g \leq \frac{3(F+1)}{4}$, then Conjecture 4 holds.*

We note that for any m there is a unique semigroup with multiplicity m and $F < m$, which has minimal generating set $\{m, m+1, \ldots, 2m-1\}$, and that equality holds for Conjecture 4 in this case.

Our main result is the following.

Theorem 24. *Let S be a semigroup such that $2g < 3m$ or such that $F < 2m$. Then Conjecture 4 holds.*

We first note that given the Apéry set of a numerical semigroup $\{k_1 m+1, \ldots, k_{m-1}m+m-1\}$, it is trivial to find F, the Frobenius number. Let $K = \max_{1 \leq i \leq m-1} k_i$, and let j be the largest i such that $k_i = K$. Then we have $(K-1)m + j = F$.

We recall that Lemma 11 implies that if $2g < 3m$ then $K \leq 3$. In fact, Lemma 10 implies that $k_{m-1} < 3$ and we see that $F \leq 3m-2$. We will prove Theorem 24 in two parts. The first part is easy.

Lemma 25. *Suppose S is a semigroup with $2g < 3m$ and $F > 2m$. Then Conjecture 4 holds.*

Proof. Note that $g < \frac{3m}{2} < \frac{3(2m+1)}{4} < \frac{3(F+1)}{4}$, since $F \geq 2m+1$, so this follows directly from Proposition 23. $\square$

We now consider the case where $F < 2m$. Note that it is possible for this to hold but $2g > 3m$. For example, consider $S = \langle m, 2m+1, \ldots, 3m-1 \rangle$, which satisfies $g = 2m-2$ which is at least $\frac{3m}{2}$ whenever $m \geq 4$, but $F = 2m-1$. We will prove Conjecture 4 for these cases as well.

Proposition 26. *Suppose S is a semigroup with $F < 2m$. Then Conjecture 4 holds. Moreover, equality holds if and only if $S = \langle m, m+1, \ldots, 2m-1 \rangle$, or $S = \langle 3, 4 \rangle$.*

Proof. We can suppose that $m < F < 2m$, and correspondingly there is an element of the Apéry set of S of the form $2m+i$ for some $1 \leq i \leq m-1$. Let $2m+c$ be the maximal such generator. Therefore $F(S)+1 = m+c+1$.

The idea of the proof is to first reduce to the particular case where $c = m-1$, and then to use the fact that both $e(S)$ and $F(S)+1-g(S)$ are related to the number of Apéry set elements of the form $m+i$.

The Apéry set of S must be of the form $\{k_1 m+1, k_2 m+2, \ldots, k_{c-1}m+(c-1), 2m+c, m+(c+1), \ldots, m+(m-1)\}$, where each $k_i \in \{1,2\}$ because $2m+i \in S$ for all i and we know that $m+c \notin S$ and $m+i \in S$ for all $c+1 \leq i \leq m-1$. Let S' be the semigroup with multiplicity $c+1$ and Apéry set given by $\{k_1(c+1)+1, \ldots, k_{c-1}(c+1)+c-1, 2(c+1)+c\}$. It is clear that this Apéry set gives a numerical semigroup since each $k_i \leq 2$. We see that $F(S') = 2c+1$.

Let R be the number of $k_i = 1$ with $1 \leq i \leq c-1$. If $c = 1$ we set $R = 0$. Now by counting the number of terms of the Apéry set which are $m+i$ and the number which are $2m+i$, we see that

$$g(S) = (m-c-1) + 2 + 2(c-1-R) + R = m+c-R-1,$$

and that

$$g(S') = R + 2(c - R) = 2c - R.$$

Therefore

$$F(S) + 1 - g(S) = m + c + 1 - (m + c - R - 1) = R + 2,$$

and

$$F(S') + 1 - g(S') = 2(c + 1) - (2c - R) = R + 2.$$

We see that an Apéry set element of the form $k_i m + i$ with $i \leq c$ is a minimal generator of S if and only if $k_i(c+1) + i$ is a minimal generator of S'. Therefore, we see that $e(S') = e(S) - (m - 1 - c)$, since there are no minimal generators corresponding to $m + j$ for $c + 1 \leq j \leq m - 1$, and since every generator of S of the form $m + j$ for some $c + 1 \leq j \leq m - 1$ is minimal.

Now suppose that $e(S) \leq \frac{F(S)+1}{F(S)+1-g(S)}$. We claim that unless $c = m - 1$, $e(S') < \frac{F(S')+1}{F(S')+1-g(S')}$. This follows from the fact that

$$\begin{aligned} \left(\tfrac{F(S')+1}{F(S')+1-g(S')} - e(S')\right) - \left(\tfrac{F(S)+1}{F(S)+1-g(S)} - e(S)\right) &= \\ \tfrac{2(c+1)-(m+(c+1))}{R+2} - (e(S) - (m - (c+1)) + e(S) &= \\ \tfrac{(c+1)-m}{R+2} + (m - (c+1)). \end{aligned}$$

Since $c + 1 \leq m$, we see that this is positive unless $c = m - 1$.

Therefore, if there exists a semigroup with $c \neq m - 1$ in which Conjecture 4 is violated, or equality holds, then there exists a related semigroup with $c = m - 1$ in which Conjecture 4 does not hold. We will show that this does not happen, completing the proof.

Consider $S = \langle m, k_1 m + 1, \ldots, k_{m-2} m + m - 2, 2m + m - 1 \rangle$. We see that m is always a minimal generator and each Apéry set element of the form $m + i$ is minimal, so $e \geq R + 1$. Suppose $e \leq \frac{F+1}{F+1-g}$. Then $(R+1)(R+2) \leq 2m$.

Each nonminimal Apéry set element is of the form $(m + i) + (m + j)$ where $m + i$ and $m + j$ are Apéry set elements with $1 \leq i, j \leq m - 2$. Since there are R elements of the form $m + i$, we see that there are at most $\frac{R^2+R}{2}$ nonminimal Apéry set elements. Therefore $e \geq m - \frac{R^2+R}{2}$. This implies $(m - \frac{R^2+R}{2})(R + 2) \leq 2m$. We have

$$2m - \left(m - \frac{R^2 + R}{2}\right)(R + 2) = \frac{R(R+1)(R+2)}{2} - mR,$$

which is at least 0 if and only if $(R + 1)(R + 2) \geq 2m$. Therefore, Conjecture 4 is never violated for these semigroups, and equality holds if and only if $e = m - \frac{R^2+R}{2}$ and $(R + 1)(R + 2) = 2m$. We see that $m = \frac{R^2+R}{2} + (R + 1)$, and therefore, every Apéry set element not of the form m or $m + i$, must be the sum of two Apéry set elements $m + i, m + j$, and all such sums must lead to distinct nonminimal Apéry set elements.

Suppose equality holds in the Conjecture 4. Clearly $m + 1$ must be an Apéry set element, and therefore $2m + 2$ must be an Apéry set element. If $m = 3$, we have the semigroup with Apéry set $\{3, 4, 8\}$, where $F + 1 = 6$, $g = 3$, $R = 1$ and $e = 2$. Equality holds in Conjecture 4 for this case. If $m > 3$, we can continue as above and see that $m + 3$ is an Apéry set element, as is $2m + 4$. If $m + 5$ is an Apéry set element, then there are two distinct ways to write $2m + 6$ as a sum of

Apéry set elements: $(m+3)+(m+3),\ (m+1)+(m+5)$. However, if $2m+5$ is an Apéry set element, then it is also a minimal generator since there is no way to write 5 as a sum of two elements of $\{0,1,3\}$. Therefore, there are no other cases where $m < F < 2m$ and equality holds.

□

We have tried quite hard to extend these arguments to other cases, in particular where $2m < F < 3m$ but $2g \geq 3m$. A very slight variation of the above argument works for $F = 2m+1$ and $F = 2m+2$. We can write our Apéry set as,

$$\{3m+1, k_2m+2, \ldots, k_{m-1}m+m-1\},\ \{k_1m+1, 3m+2, k_3m+3, \ldots, k_{m-1}m+m-1\},$$

in the first and second cases, respectively. If the number of Apéry set elements with $k_i = 1$ is R, then in the first case, we see again that the number of nonminimal elements is at most $\frac{R^2+R}{2}$. In the second case, k_1 must be 2 or 3 since $2m+2 \notin S$, and so the number of nonminimal elements is at most $\frac{R^2+R}{2}$. In the first case, $F+1-g = R+3$, and in the second $F+1-g$ is $R+3$ if $k_1 = 3$, and $R+4$ if $k_1 = 2$. The argument proceeds as above.

As soon as $F = 2m+3$, the situation changes somewhat. For example, if $m+1$ is an Apéry set element, then both $2m+2$ and $3m+3$ can be nonminimal Apéry set elements. It is no longer true that the only way to have a nonminimal Apéry set element is to add two minimal generators $(m+i),\ (m+j)$. While we can deal with this case with some effort, it seems that when $F = 2m+c$ with c not very small, the situation becomes much more complicated.

One of the issues that is relevant in this analysis is whether we can find large sets of generators of the form $m+i$ such that all of their pairwise sums are distinct. Given a semigroup, we identify the minimal generator $m+i$ with $i \in \mathbb{Z}/m\mathbb{Z}$ and let A be the set of all such i. We recall the following definition. Let G be a group, usually either $\mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$, and $S \subset G$ be a subset. We say that S is a Sidon set if there is no solution to $s_1+s_2 = s_3+s_4$ with $s_1, s_2, s_3, s_4 \in S$ aside from the trivial solutions $\{s_1, s_2\} = \{s_3, s_4\}$. We see that we are asking for relatively large Sidon sets of $\mathbb{Z}/m\mathbb{Z}$, or at least sets where there are not too many solutions to the above equality. It turns out that there are ways to find such sets. We give a construction of Ruzsa from [14].

Suppose p is a prime and θ a generator of $\mathbb{Z}/p\mathbb{Z}^\times$. For $1 \leq i < p$ let $a_{t,i}$ be the element of $\mathbb{Z}/(p^2-p)\mathbb{Z}$ defined by

$$a_{t,i} \equiv t \pmod{p-1}, \text{ and } a_{t,i} \equiv i\theta^t \pmod{p}.$$

Let $\text{Ruzsa}(p,\theta,k) = \{a_{t,k} :\ 1 \leq t < p\} \subset \mathbb{Z}_{p^2-p}$. Ruzsa [19], showed that this gives a Sidon set of p elements of $\mathbb{Z}/(p^2-p)\mathbb{Z}$, leading to $\frac{p^2+p}{2}$ distinct sums. In [14], several other approaches to constructing Sidon sets, and other sets with few overlaps in pairwise sums, are discussed.

We also point out that it is not enough in general to find a small set $A \subset \mathbb{Z}/m\mathbb{Z}$ such that $|A+A|$ is very large or all of $\mathbb{Z}/m\mathbb{Z}$. Two generators $m+i_1$ and $m+i_2$ such that $i_1+i_2 > F(S)-m$ cannot add together to give a nonminimal Apéry set element, since our largest Apéry set element must be $F(S)+m$. We are therefore asking for $|A+A|$ to be very large, and for almost all of the sums to be less than $F(S)$. This is related to the 'postage stamp problem' and the problem of finding small 2-bases of the set $[1,m]$, [11, 24].

Few of the recent papers which have considered approaches to counting numerical semigroups have addressed Wilf's question. We believe that one reason for this is that there is no obvious relationship between the number of minimal generators of a semigroup, $e(S)$, and its Apéry set or genus. Consider the set of inequalities which tell us whether an $(m-1)$-tuple of positive integers gives a valid Apéry set. If there exist some pair (i, j) satisfying $i + j = l < m$ and $k_i + k_j = k_l$, then we see that $k_l m + l$ cannot be a minimal generator. When some of our inequalities turn to equalities, we gain information about which generators cannot be minimal, but it does not seem too easy to combine this information with the tools we have used above.

6. Some Further Questions

In this section we will discuss some open questions and possibilities for further work. We would like to be able to use the nice recurrence satisfied by $N(m, g)$ when $2g < 3m$ to prove Bras-Amorós' conjecture on $N(g)$. This would be possible if we could show that as g grows large, almost all semigroups satisfy $2g < 3m$. It is not clear whether this will turn out to be the case.

Let $R(g)$ be the number of semigroups of genus g for which $2g < 3m$. We provide some data:

Genus	N(g)	R(g)	R(g) / N(g)
15	2857	1715	.6002
16	4806	2555	.5316
17	8045	3778	.4696
18	13467	7611	.5652
19	22464	11389	.5070
20	37396	16926	.4526
21	62194	33680	.5415
22	103246	50606	.4901
23	170963	75565	.4420
24	282828	112049	.3962

It is not at all clear whether $R(g)/N(g)$ approaches a limit, or if it does, what that limit would be. These questions are related to recent work of Zhao [25], in which he gives constructions of large families of semigroups and in particular focuses on those for which the Frobenius number is at most $3m$. Let $T(g)$ denote the number of such semigroups of genus g. The following is Conjecture 4.1 in [25].

Conjecture 6 ([25])**.** *We have*

$$\lim_{g\to\infty} \frac{T(g)}{N(g)} = 1.$$

This conjecture along with a related one on the behavior of $T(g)$ would imply Bras-Amorós' conjecture. We note that every semigroup satisfying $2g < 3m$ has $F < 3m$, but not conversely. It would be interesting to investigate the connection between $R(g)$ and $T(g)$ using some of the tools of [25].

When looking at values of $N(m, g)$ we noticed that the following pattern appears to hold.

Conjecture 7. *For any $m \geq 2$, $N(m,g) \leq N(m,g+1)$.*

If we fix the multiplicity m then we have seen that there is a certain polyhedral cone in $\mathbb{R}^{m-1}$ which determines whether a set $\{k_1, \ldots, k_{m-1}\}$ leads to a valid Apéry set of a numerical semigroup. If we fix the genus, then we are taking a hyperplane through this cone. For fixed m, as we increase g we are looking at how these polytopes resulting from different hyperplanes relate to each other. As we have seen above, $N(m,g)$ is eventually given by a quasipolynomial in g. That is, there exists some period s and some positive integer k such that for each residue class modulo s there exists a polynomial $f_i(x)$ such that for all $g \geq k$ congruent to i modulo s we have $N(m,g) = f_i(g)$. It is trivial to show that $N(2,g) = 1$ for all $g \geq 1$, and a little case analysis leads to $N(3,g) = \lfloor \frac{g}{3} \rfloor + 1$. The quasipolynomials for $m = 4$ and 5 have also been computed, but are quite complicated. See [1] for a detailed discussion.

One worthwhile exercise would be to verify that for $m = 4, 5$, Conjecture 7 holds. It would also be worth trying to use some of the techniques of geometric combinatorics to investigate the period and coefficients of the quasipolynomial associated to $N(m,g)$.

Looking extensively at the data for $N(m,g)$ presents some other interesting patterns. If we fix m and g and consider $N(m+k, g+k)$, we see $2(g+k) < 3(m+k)$ if and only if $k > 2g - 3m$. Once we pass this value of k, it is relatively easy to understand the behavior of $N(m+k, g+k)$ as it is determined by a polynomial discussed in Section 3. Interestingly, the following also appears to be true.

Conjecture 8. *For all $m > 1$ and $g > 1$ we have $N(m,g) \leq N(m+1, g+1)$.*

We recall that a sequence $a_1, \ldots, a_n$ is unimodal if there exists some j such that $a_1 \leq a_2 \leq \cdots \leq a_j \geq a_{j-1} \geq a_n$. Data suggests that the following two sequences are unimodal.

Conjecture 9. *For any fixed $g \geq 1$ the sequence defined by $a_i = N(i,g)$ is unimodal. We note that $a_i = 0$ for all $i \geq g+2$.*

For any fixed $g \geq 1$, the sequence defined by $b_i = N(i, g-i)$ is unimodal.

If these conjectures hold, it would be interesting to find which term of the sequence is maximal. This does not have to be unique. For example, $N(8,12) = N(9,12) = 116$, which is the largest value of $N(m,12)$. In general, for fixed g the largest value of $N(m,g)$ seems to occur when m is approximately $2g/3$.

Though $N(m,g)$ and $MED(m,g)$ are closely related, these last two conjectures do not hold for maximal embedding dimension semigroups. For example

$$2 = MED(3,5) < 3 = MED(4,6) > 2 = MED(5,7) < 4 = MED(6,8).$$

Also,

$$3 = MED(3,9) < 7 = MED(4,9) > 5 = MED(5,9) < 6 = MED(6,9) > 4 = MED(7,9).$$

Finally,

$$80 = MED(5,22) < 172 = MED(6,21) > 149 = MED(7,20) < 156 = MED(8,19).$$

It does not appear that as many of the sequences to be found within a table of $MED(m,g)$ values are unimodal. Some of the nonmonotonic behavior of $MED(m,g)$ appears to be related to the prime factorization of m, relative to the nearby values

$m-1$ and $m+1$. Looking at the chart of values of $MED(m,g)$ it seems that these values are larger when m has many prime factors. The beginning of a possible explanation for this comes from Proposition 22.

It does appear that for fixed m, $MED(m,g)$ is nondecreasing. This would imply that $N(g)$ is also nondecreasing.

Conjecture 10. *For any* $m \geq 2,\ g \geq 0,\ MED(m,g) \leq MED(m,g+1)$.

There are several other types of numerical semigroups which we have not touched upon in this paper. The book [10] is an excellent reference. For example, a symmetric numerical semigroup is one for which $g(S) = \frac{F(S)+1}{2}$ and a pseudo-symmetric is one for which $g(S) = \frac{F(S)+2}{2}$. We recall that a numerical semigroup S is Arf if and only if for any $x, y, z \in S$ with $x \geq y \geq z$, we have $x+y-z \in S$. Proposition 3.12 in [10] implies that an Arf semigroup necessarily has maximal embedding dimension. A saturated numerical semigroup S is one for which if $s, s_1, \ldots, s_r \in S$ with $s_i \leq s$ for all $1 \leq i \leq r$ and $z_1, \ldots, z_r \in \mathbb{Z}$ such that $z_1 s_1 + \cdots + z_r s_r \geq 0$, we have $s + z_1 s_1 + \cdots z_r s_r \in S$. Lemma 3.31 in [10] shows that every saturated numerical semigroup is Arf.

At this point there are several 'density' questions we could ask. Given a numerical semigroup of genus g, what are the chances it has maximal embedding dimension? It is very unlikely that $\lim_{g\to\infty} \frac{MED(g)}{N(g)} = 1$, but for fixed m it is possible that $\lim_{g\to\infty} \frac{MED(m,g)}{N(m,g)} = 1$. Similarly, what are the chances that it is symmetric or pseudo-symmetric? This is almost certainly 0. What about the chances that a maximal embedding dimension of genus at most g is Arf, or saturated, or that an Arf semigroup of genus at most g is saturated?

There is no shortage of such questions to ask. They all build towards a single somewhat vague question. What properties does a 'generic' semigroup of genus g have, for large values of g?

We will end this section with one further direction to investigate. Some of the terminology that is now standard for numerical semigroups comes from algebraic geometry, such as referring to the cardinality of the gap set as the genus. We recall that if we have a projective, irreducible, algebraic curve defined over an algebraically closed field, and a nonsingular point P, we can associate to P a numerical semigroup called the Weierstrass semigroup. In 1980, Buchweitz showed that not every numerical semigroup can arise as a Weierstrass semigroup, and in particular, that the following criterion must hold. For each $n \geq 2$, we must have that the n-fold sum of the set of gaps of the semigroup must be bounded above by $(2n-1)(g-1)$. For example, the semigroup $\langle 13, 14, 15, 16, 17, 18, 20, 22, 23\rangle$ has 16 gaps, $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 19, 21, 24, 25\}$, and the 2-fold sum of these gaps is $[2, 50] \setminus \{39, 41, 47\}$. Therefore since $46 > 3(16-1)$, this semigroup does not satisfy Buchweitz's criterion. See [16] for a list of such semigroups and [12, 22] for more information on this criterion.

Komeda has suggested studying the percentage of numerical semigroups of genus g which satisfy the Buchweitz criterion, [12]. Now that we have a better understanding of the possible gaps of a numerical semigroup, particularly in cases where $2g < 3m$ or $F < 3m$, it would be interesting to try to attack this problem.

Perhaps the largest omission in this discussion of counting semigroups is that we focused on ordering semigroups by genus instead of by Frobenius number. We can ask very similar questions to the ones addressed in this paper by instead considering the number of numerical semigroups of multiplicity m and Frobenius number F. See [1] and [15] for more on this interesting question.

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