DELTA SETS OF NUMERICAL MONOIDS USING NON-MINIMAL SETS OF GENERATORS

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Dedicated to Professor William W. Smith on the occasion of his retirement from the faculty at the University of North Carolina at Chapel Hill

Abstract. Several recent papers have studied the structure of the delta set of a numerical monoid. We continue this work with the assumption that the generating set S chosen for the numerical monoid M is not necessarily minimal. We show that for certain choices of S, the resulting delta set can be made (in terms of cardinality) arbitrarily large or small. We close with a close analysis of the case where $M = \langle n_1, n_2, in_1 + jn_2\rangle$ for nonnegative i and j.

1. Introduction

The study of delta sets of various commutative cancellative atomic monoids has been a frequent topic in the recent literature (see the bibliography for numerous references). Several papers have addressed this subject with respect to a numerical monoid M ([1], [3], [5] and [6]). In a recent paper [2] which computes delta sets of certain arithmetical congruence monoids, several of the proofs relied on factoring elements in numerical monoids using generating sets which may not be minimal. Hence, the question arises as to how the delta set of a numerical monoid changes as its set of generators changes. In this paper, we address this issue and show (among other things) that no matter the structure of $\Delta(M)$, generating sets S_1 and S_2 of M can be found so that $|\Delta^{S_1}(M)|$ is arbitrarily large, and $|\Delta^{S_2}(M)| = 1$.

Before reviewing our results, we will cover some basic definitions and notation as outlined in [9]. Let M be a commutative cancellative atomic monoid with set $\mathcal{A}(M)$ of irreducible elements and set $M^\times$ of units. For $m \in M\backslash M^\times$, set $\mathcal{L}(m) = \{\, t \in \mathbb{N} \mid \exists\, x_1, \ldots, x_t \in \mathcal{A}(M) \text{ with } m = x_1 \cdots x_t \,\}$. The set $\mathcal{L}(m)$ is called the *set of lengths* of m. For any $m \in M\backslash M^\times$, we define $L(m) = \sup \mathcal{L}(m)$ and $\ell(m) = \inf \mathcal{L}(m)$. Moreover, if $m \in M\backslash M^\times$ and $\mathcal{L}(m) = \{x_1, \ldots, x_n\}$ with $x_1 < x_2 < \cdots < x_n$, then the *delta set* of m is $\Delta(m) = \{x_i - x_{i-1} | 2 \leq i \leq n\}$, and the *delta set* of M is $\Delta(M) = \bigcup_{m\in M\backslash M^\times} \Delta(m)$. By a fundamental result of Geroldinger [8, Lemma 3], if $d = \gcd \Delta(M)$ and $|\Delta(M)| < \infty$, then $\{d\} \subseteq \Delta(M) \subseteq \{d, 2d, \ldots, kd\}$ for some $k \in \mathbb{N}$. A general review of known results involving delta sets can be found in [9, Section 6.7].

A numerical monoid M is any submonoid of the nonnegative integers (denoted $\mathbb{N}_0$) under addition. We will say that the integers $n_1 < n_2 < \cdots < n_k$ generate M if $M = \{a_1n_1 + \cdots + a_kn_k \mid a_i \in \mathbb{N}_0 \text{ for all } i\}$ and denote this by $M = \langle n_1, n_2, \ldots, n_k\rangle$. Each numerical monoid M has a unique minimal (in terms of cardinality and set inclusion) set of generators. If $n_1, \ldots, n_k$ are the minimal

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generators of M, then $\mathcal{A}(M) = \{n_1, \ldots, n_k\}$. If, for M as written above, we have $\gcd(n_1, \ldots, n_k) = 1$, then M is called *primitive*. We note that every numerical monoid is clearly isomorphic to a primitive numerical monoid. If M is a primitive numerical monoid, then there exists an integer $F(M) \notin M$ such that $m > F(M)$ implies that $m \in M$. The integer $F(M)$ is known as the *Frobenius number* of M.

Much is known about the structure of $\Delta(M)$ for a numerical monoid M. Of particular interest in our current work are the following results.

Proposition 1.1. *Let $M = \langle n_1, \ldots, n_k \rangle$ be a primitive numerical monoid with $n_1, \ldots, n_k$ a minimal system of generators.*

(1) $|\Delta(M)| < \infty$ *and* $\min \Delta(M) = \gcd\{n_i - n_{i-1} \,|\, 2 \le i \le k\}$ [3, Proposition 2.9].
(2) *If $M = \langle n, n+k, n+2k, \ldots, n+bk \rangle$, then $\Delta(M) = \{k\}$* [3, Theorem 3.9].
(3) *For any k and v in $\mathbb{N}$ there exists a three generated numerical monoid M with $\Delta(M) = \{k, 2k, \ldots, vk\}$* [3, Corollary 4.8].
(4) *The sequence $\{\Delta(x)\}_{x \in M}$ is eventually periodic* [5].

Our approach will be slightly different than that of [3] and [5] and we extend the definitions of the previous page as follows. For any numerical monoid M, let $S = \{n_1, n_2, \ldots, n_k\}$ be an arbitrary generating set for M. For $x \in M \setminus M^\times$, set $\mathcal{F}^S(x) = \{(x_1, \ldots, x_k) \subset \mathbb{N}_0^k \mid x = x_1 n_1 + \ldots + x_k n_k\}$. We will refer to $\mathcal{F}^S(x)$ as the set of factorizations of x in S. For $x \in M \setminus M^\times$, the set

$$\mathcal{L}^S(x) = \{l \in \mathbb{N} \mid \exists\ (x_1, \ldots, x_t) \in \mathcal{F}^S(x) \text{ with } l = x_1 + \cdots + x_l\}$$

will be referred to as the set of lengths of x with respect to S. Set $L^S(x) = \max \mathcal{L}^S(x)$ and $\ell^S(x) = \min \mathcal{L}^S(x)$. Next, if we have $\mathcal{L}^S(x) = \{l_1, \ldots, l_n\}$, with $l_1 < l_2 < \ldots < l_n$, then

$$\Delta^S(x) = \{l_i - l_{i-1} \mid 2 \le i \le n\}$$

is known as the delta set of x with respect to S and $\Delta^S(M) = \bigcup_{x \in M \setminus M^\times} \Delta^S(x)$ the delta set of M with respect to S.

We break the remainder of our work into two sections. In Section 2, we examine how the set $\Delta^S(M)$ can vary from $\Delta(M)$. In Theorem 2.1 we show for all positive integers n, there is a generating set S_1 of M so that $|\Delta^{S_1}(M)| > n$. Conversely, in Theorem 2.2 we show that there is always a generating set S_2 such that $\Delta^{S_2}(M) = \{1\}$. In Section 3, for primitive $M = \langle n_1, n_2 \rangle$, we examine closely the behavior of the set $\Delta^S(M)$ for $S = \{n_1, n_2, in_1 + jn_2\}$ for $i, j \ge 0$. After some basic structure results, we are able to show the following for such an S.

(1) If $S = \{n_1, n_2, n_1 + n_2\}$, then $\Delta^S(M) = \{1, 2, \ldots, n_2 - n_1\}$ (Proposition 3.8).
(2) $\Delta^S(M) = \Delta(M)$ if and only if $i + j - 1 = n_2 - n_1$ (Theorem 3.5).
(3) We give exact conditions for when $|\Delta^S(M)| = 1$ (Theorem 3.6).
(4) If $i + j = 2$, then $\Delta^S(M) = \{1, 2, ..., k\}$ for some k (Proposition 3.7).
(5) If $\Delta^S(M) = \{1, k\}$, then $k = 2$ (Theorem 3.12).

We note that for numerical monoids, the proofs of Proposition 3.5 in [3] and Proposition 1.4.4 in [9] are still valid when we lose the minimality of our generators. Hence, we will freely use the following throughout the remainder of our work.

Proposition 1.2. $\gcd(\Delta^S(M)) = \min(\Delta^S(M)) = \gcd\{n_{i+1} - n_i, 1 \le i < t\}$.

2. Variations in $\Delta^S(M)$ For Different Generating Sets

We start by proving that the delta set of a numerical monoid can be made arbitrarily large with non-minimal generating sets.

Theorem 2.1. *For any primitive numerical monoid M and all $n \in \mathbb{N}$, there is a finite generating set S such that $|\Delta^S(M)| > n$.*

Proof. We will proceed by showing that for all finite generating sets $S \subset M$, there exists a finite generating set $S' \subset M$ such that $S \subset S'$ and $|\Delta^S(M)| < |\Delta^{S'}(M)|$. Then we can find a series of finite generating sets $S_0, S_1, \ldots, S_n$, where S_0 is the minimal generating set, and $|\Delta^{S_i}(M)| \geq i$.

Let $F(M)$ be the Frobenius number of M. Let d be the largest element of $\Delta^S(M)$. Let s be the largest element of S. Let k be the smallest integer such that $\Delta^S(M) \subset \bigcup_{i=1}^k \Delta^S(i)$. Such a value will always exist, since for each $l \in \Delta^S(M)$, there is a least element $y \in M$ such that $l \in \Delta^S(y)$.

Now choose $m \in \mathbb{N}$ such that $m > \max\{F(M), k, s(d+1)\}$, and let $S' = S \cup \{m\}$. We see since $m > k$, that $\Delta^S(M) \subset \Delta^{S'}(M)$, because given $x \in M$ with $x < m$ no factorization of x includes m, and the factorizations of x in S' are precisely those in S. Since $m > F(M)$ there is a factorization of m in S and $l^S(m) \geq \frac{m}{s} > d+1$. We see that the set of factorizations of m in the generating set S' is precisely the set of factorizations of m in S, as well as the factorization $m = 1 \cdot m$, of length 1. Thus, we see that $l^S(m) - 1 \in \Delta^{S'}(m)$. Since $l^S(m) - 1 > d + 1 - 1 > d$ and $l^S(m) - 1 \in \Delta^{S'}(M)$ the proof is complete. □

If we choose our generating set S to include many small elements of M then we can show that $\Delta^S(M)$ is small.

Theorem 2.2. *Let M be a primitive numerical monoid, with minimal generating set $\{n_1, n_2, \ldots, n_k\}$. For all $N \geq 2n_k$, if we let $S = \{m \in M$ such that $m \leq N\}$, then $\Delta^S(M) = \{1\}$.*

Proof. We can write $S = \{n_1, \ldots, n_k, n_{k+1}, \ldots, n_r\}$ where the first k generators are the minimal generators of M and $n_{k+1} < \cdots < n_r$ are the generators we have added to S.

Suppose $\Delta^S(M) \neq \{1\}$. Then there exists some $x \in M$ such that we have two factorizations of x in S,

$$x = \sum_{i=1}^{r} b_i n_i = \sum_{i=1}^{r} c_j n_i,$$

where each $b_i, c_i \geq 0$, $\sum_{i=1}^r b_i = B$, $\sum_{i=1}^r c_i = C$, $C - B \geq 2$ and x does not have any factorizations of length between B and C. In particular, we may assume that x has no factorizations of length $B+1$ or $C-1$.

Suppose $b_i \geq 0$ for some $i \geq k+1$. Since n_i is not a minimal generator of M we can write it as a sum of two elements of S and clearly we have a factorization of x of length $B+1$. So we have $b_i = 0$ for all $i \geq k+1$ and therefore $x \leq Bn_k$.

Suppose there exist i, j such that $c_i, c_j > 0$ and $n_i + n_j \leq N$. Then $n_i + n_j \in S$ and we clearly have a factorization of length $C-1$. So the factorization of length C contains at most one generator less than $\frac{N}{2}$. Therefore $x \geq n_1 + (C-1)\frac{N}{2}$.

We have $n_1 + (C-1)\frac{N}{2} \leq x \leq Bn_k$. Since $C - 1 > B$ we have $\frac{N}{2} < n_k$, which contradicts our choice of N. Therefore there does not exist $x \in M$ such that $\Delta^S(x)$ contains an element greater than 1 and $\Delta^S(M) = \{1\}$. □

3. The Structure of $\Delta^S(M)$ when $S = \{n_1, n_2, in_1 + jn_2\}$

3.1. **Basic Structure Results.** We now will find restrictions on what sort of delta sets are obtainable, presenting both general principles and specific examples. Throughout the rest of this paper we will assume that M is primitive. We begin with a useful lemma. Let $M = \langle n_1, n_2 \rangle$ be a numerical monoid, and let $s = in_1 + jn_2$ with $0 \leq j,\ 0 \leq i < n_2$. This implies that $in_1 + jn_2$ is the shortest possible factorization of s in M.

Lemma 3.1. *Let $M = \langle n_1, n_2 \rangle$ be a primitive numerical monoid and $S = \{n_1, n_2, in_1 + jn_2\}$ with $i < n_2$. Then $i + j - 1 \in \Delta^S(M)$.*

Proof. The shortest factorization of $in_1 + jn_2$ in S has length 1. Every other factorization is of the form $in_1 + jn_2 = (i + kn_2) \cdot n_1 + (j - kn_1) \cdot n_2$ for some $-\lfloor i/n_2 \rfloor \leq k \leq \lfloor j/n_1 \rfloor$. But $i < n_2$, so $k \geq 0$ and $\mathcal{L}^S(in_1 + jn_2) = \{1\} \cup \{i + j + k(n_2 - n_1) \mid 0 \leq k \leq \lfloor j/n_1 \rfloor\}$ and thus $i + j - 1 \in \Delta^S(M)$. □

It will often be useful to consider the cases $j \neq 0$ and $j = 0$ separately. If $j = 0$ we can write $n_2 = ki + c$ for unique integers $k \geq 0$ and $0 \leq c < i$.

Proposition 3.2. *Let $M = \langle n_1, n_2 \rangle$, $S = \{n_1, n_2, in_1\}$ with $2 \leq i < n_2$ and $n_2 = ki + c$ for unique integers $k \geq 0$ and $0 \leq c < i$. Then if $in_1 < n_2$, we have $\{i - 1,\ k + c - n_1\} \subseteq \Delta^S(M)$.*

Proof. Note that $n_2 - in_1 = i(k - n_1) + c > 0$ implies $k \geq n_1$. We will show that $\{i-1, k+c-n_1\} = \Delta^S(n_1n_2)$. Any factorization of n_1n_2 is of the form $n_1n_2 = x_1n_1 + x_2n_2 + x_3(in_1)$ with $x_1, x_2, x_3 \geq 0$. If $x_2 > 0$ then since $n_1n_2 - x_2n_2 \equiv 0 \pmod{n_1}$ we must have $x_2 = n_1$. Every factorization with $x_2 = 0$ is of the form $(c + li)n_1 + in_1(k - l)$ for $0 \leq l \leq k$, so

$$\mathcal{L}^S(n_1n_2) = \{n_1\} \cup \{c + k + l(i - 1) \mid 0 \leq l \leq k\}.$$

We can order the elements of $\mathcal{L}^S(n_1n_2)$ from least to greatest as $\{n_1, k+c, k+c+(i-1), \ldots, ki+c\}$. We see that $\Delta^S(n_1n_2) = \{k + c - n_1, i - 1\}$. □

We now state a theorem proved in [6, Theorem 3.1], and derive a useful corollary. We note that the proof of this theorem does not require the generating set of M to be minimal. Let $k_1 = \min\{k \mid k > 0,\ kn_1 \in \langle n_2, n_3 \rangle\}$ and $k_3 = \min\{k \mid k > 0,\ kn_3 \in \langle n_1, n_2 \rangle\}$.

Theorem 3.3. *Let $M = \langle n_1, n_2, n_3 \rangle$ be a numerical monoid with $n_1 < n_2 < n_3$. Then $\max(\Delta(M)) = \max(\Delta(k_1n_1) \cup \Delta(k_3n_3))$.*

We will use this theorem to explicitly calculate $\max(\Delta^S(M))$ in certain cases. We will now compute $\max(\Delta^S(M))$ for the main type of numerical monoid that we will study in this paper.

Corollary 3.4. *Let $M = \langle n_1, n_2 \rangle$ be a numerical monoid, and let $s = in_1 + jn_2$ with $0 \leq j,\ 0 \leq i < n_2$ and $i + j \geq 2$. Let $S = \{n_1, n_2, s\}$ and if $j = 0$, then let k and c be as in Proposition 3.2. It follows that*

(1) *If $j \neq 0$ $\max(\Delta^S(M)) = \max\{n_2 - n_1, i + j - 1\}$.*
(2) *If $j = 0$ and $n_2 < s$, $\max(\Delta^S(M)) = i - 1$.*
(3) *If $j = 0$ and $s < n_2$, $\max(\Delta^S(M)) = \max\{i - 1, k + c - n_1\}$.*

Proof. (1) Every $x \neq 0$ in $\langle n_2, s \rangle$ satisfies $x, x - n_2 \in M$. The smallest multiple of n_1 satisfying this condition is n_1n_2, so $k_1 = n_2$. If $i \neq 0$ then $n_1n_2 - s \notin M$ and the only factorizations of n_1n_2 are $(n_2, 0, 0)$ and $(0, n_1, 0)$. Therefore $\Delta^S(n_1n_2) = \{n_2 - n_1\}$. If $i = 0$ then $n_2n_1 = n_1n_2 = (n_1 - kj)n_2 + ks$ for all $0 \leq k \leq \lfloor n_1/j \rfloor$. In this case,

$$\mathcal{L}^S(n_1n_2) = \{n_2\} \cup \{n_1 - k(j - 1) \mid 0 \leq k \leq \lfloor n_1/j \rfloor\}.$$

Thus $\{n_2 - n_1\} \subseteq \Delta^S(n_1n_2) \subseteq \{n_2 - n_1, i + j - 1\}$.

Since $s \in M$, $k_3 = 1$. The factorizations of s in $\langle n_1, n_2 \rangle$ are of the form $s = (i+kn_2)n_1+(j-kn_1)n_2$ for any $0 \leq k \leq \lfloor j/n_1 \rfloor$. Note that $(i, j, 0)$ is the shortest factorization of s by hypothesis. Thus $\mathcal{L}^S(s) = \{1\} \cup \{i + j + k(n_2 - n_1) \mid 0 \leq k \leq \lfloor j/n_1 \rfloor\}$, and $\{i + j - 1\} \subseteq \Delta^S(s) \subseteq \{n_2 - n_1, i + j - 1\}$.

Therefore $\Delta^S(k_1n_1)\cup\Delta^S(k_3s) = \{n_2-n_1, i+j-1\}$ and by Theorem 3.3 we have $\max(\Delta^S(M)) = \max\{n_2-n_1, i+j-1\}$.

(2) Suppose $x \in \langle n_2, s\rangle$ with $x \neq 0$. Then either $x - n_2 \in M$ or $x - s \in M$. The least kn_1 with $kn_1 - n_2 \in M$ is n_1n_2 and $in_1 = s < n_1n_2$. Thus s is the least multiple of n_1 in $\langle n_2, s\rangle$ and $k_1 = i$. We see that $\mathcal{L}^S(s) = \{1, i\}$ and so $\Delta^S(s) = \{i-1\}$.

Since $s \in M,\ k_3 = 1$. Therefore $\Delta^S(k_1n_1)\cup\Delta^S(k_3s) = \Delta^S(s) = \{i-1\}$ and by Theorem 3.3 we have $\max(\Delta^S(M)) = i-1$.

(3) Using exactly the same argument as above, we see that s is the least multiple of n_1 in $\langle n_2, s\rangle$ and $k_1 = i$. We have $\mathcal{L}^S(s) = \{1, i\}$ and $\Delta^S(s) = \{i-1\}$.

We compute $k_3 = \min\{k \mid k > 0,\ kn_2 \in \langle n_1, s\rangle\}$. Every $x \in \langle n_1, s\rangle$ satisfies $x - n_1 \in M$. The smallest multiple of n_2 with $kn_2 - n_1 \in M$ is n_1n_2, so $k_3 = n_1$. In the proof of Proposition 3.2 we showed that $\Delta^S(n_1n_2) = \{i-1, k+c-n_1\}$. Therefore $\Delta(s)\cup\Delta(n_1n_2) = \{i-1, k+c-n_1\}$, and by Theorem 3.3 we have $\max(\Delta^S(M)) = \max\{i-1, k+c-n_1\}$. □

3.2. **Sets S for which $|\Delta^S(M)| = 1$.** Let $M = \langle n_1, n_2\rangle$ and $S = \{n_1, n_2, s\}$ for $s = in_1 + jn_2$ with $s = in_1 + jn_2$ with $0 \le j,\ 0 \le i < n_2$ and $i + j \ge 2$ be a nonminimal generating set for M. In this section we investigate when $|\Delta(M)| = |\Delta^S(M)| = 1$. We first determine precisely when the addition of a nonminimal generator produces an identical delta set.

Theorem 3.5. *$\Delta(M) = \Delta^S(M)$ if and only if $i + j - 1 = n_2 - n_1$.*

Proof. Recall that $\Delta(M) = \{n_2 - n_1\}$. First suppose $\Delta(M) = \Delta^S(M)$. By Lemma 3.1, we know $i + j - 1 \in \Delta^S(M)$. Therefore $i + j - 1 = n_2 - n_1$.

Now suppose $i+j-1 = n_2-n_1$. We first note that $s = in_1+jn_2 \ge (i+j)n_1 = (n_2-n_1)n_1+n_1 > n_2$. By Proposition 1.2, $\min(\Delta^S(M)) = \gcd\{n_2 - n_1, s - n_2\}$. We have

$$s - n_2 = (i + j - 1)n_1 + (j - 1)(n_2 - n_1) = (n_2 - n_1)(n_1 + j - 1).$$

This implies $\min(\Delta^S(M)) = \gcd\{n_2 - n_1, (n_2 - n_1)(n_1 + j - 1)\} = n_2 - n_1$.

We will now apply Corollary 3.4 to find $\max(\Delta^S(M))$. If $j = 0$ then $\max(\Delta^S(M)) = i - 1 = n_2 - n_1$. If $j \neq 0$ then $\max(\Delta^S(M)) = \max\{n_2 - n_1, i + j - 1\} = n_2 - n_1$.

We have $\max(\Delta^S(M)) = n_2 - n_1 = \min(\Delta^S(M))$. Therefore $\Delta^S(M) = \{n_2 - n_1\} = \Delta(M)$. □

We now determine necessary and sufficient conditions for the addition of a non-minimal generator to yield a singleton delta set.

Theorem 3.6. *$|\Delta^S(M)| = 1$ if and only if one of the following two conditions holds:*

(1) *$i + j - 1 = n_2 - n_1$.*
(2) *$j = 0$ and there exists a positive integer $l \le \lfloor \frac{n_2}{i} \rfloor + 1$ such that $l(i - 1) = n_2 - n_1$.*

Proof. Suppose $|\Delta^S(M)| = 1$. Lemma 3.1 implies that $i + j - 1 \in \Delta^S(M)$. If $i + j - 1 = n_2 - n_1$ then condition (1) holds. Suppose $i + j - 1 \neq n_2 - n_1$. If $n_1n_2 - s \notin M$ then the only factorizations of n_1n_2 in S are $(n_2, 0, 0)$ and $(0, n_1, 0)$ and $\Delta^S(n_1n_2) = \{n_2 - n_1\}$. This implies $|\Delta^S(M)| \ge 2$, which is a contradiction. Therefore $n_1n_2 - in_1 - jn_2 \in M$. Hence either $i = 0$ or $j = 0$. Suppose $i = 0$. Then the two longest factorizations of n_1n_2 have length n_2 and n_1, so $n_2 - n_1 \in \Delta^S(n_1n_2)$ and $|\Delta^S(M)| \ge 2$. So $i \neq 0$ and thus $j = 0$.

Corollary 3.4 and the fact that $|\Delta^S(M)| = 1$ yields $i - 1 \in \Delta^S(M)$. Combining this with Proposition 1.2 (in either the case $in_1 < n_2$ or $n_2 < in_1$) yields an l with $l(i - 1) = n_2 - n_1$. Now suppose $l > \lfloor n_2/i \rfloor + 1$. Then n_1n_2 has the factorizations $(0, n_1, 0)$ and $(n_2 - ki, 0, k)$ for

$0 \leq k \leq \lfloor n_2/i \rfloor$. Thus $\mathcal{L}^S(n_1n_2) = \{n_1, n_2 - k(i-1) \mid 0 \leq k \leq \lfloor n_2/i \rfloor\}$. But $n_1 = n_2 - l(i-1)$, so we have $\mathcal{L}^S(n_1n_2) = \{n_2 - k(i-1) \mid 0 \leq k \leq \lfloor n_2/i \rfloor \text{ or } k = l\}$. Thus the consecutive difference between each pair of terms is $(i-1)$, except for the last pair which has a difference of $(i-1)(l - \lfloor n_2/i \rfloor)$. Since $l > \lfloor n_2/i \rfloor + 1$, we have $(l - \lfloor n_2/i \rfloor) > 1$ and $(i-1)(l - \lfloor n_2/i \rfloor) \neq i - 1$. Thus $|\Delta^S(n_1n_2)| = 2$ and $|\Delta^S(M)| \geq 2$.

Now suppose that $i + j - 1 = n_2 - n_1$. By Theorem 3.5 $|\Delta^S(M)| = 1$. So suppose $j = 0$ and there exists a positive integer $l \leq \lceil n_2/i \rceil$ such that $l(i-1) = n_2 - n_1$. In either the case $in_1 < n_2$ or $n_2 < in_1$, Proposition 1.2 and a simple calculation yields

$$\min(\Delta^S(M)) = (i-1)\gcd\{l, n_1\}.$$

Since $l(i-1) = n_2 - n_1$ we have $n_1 + l(i-1) = n_2$. So $\gcd\{l, n_1\} \mid n_2$ and $\gcd\{n_1, n_2\} = 1$ implies $\gcd\{l, n_1\} = 1$ and $\min(\Delta^S(M)) = i - 1$.

We now compute $\max(\Delta^S(M))$. If $in_1 > n_2$ then Corollary 3.4 implies $\max(\Delta^S(M)) = i - 1$. If $in_1 < n_2$ then we can write $n_2 = ki + c$ for some unique integers $k \geq 1$ and $0 \leq c < i$. Then Corollary 3.4 implies $\max(\Delta^S(M)) = \max\{i-1, k+c-n_1\}$. We have $l(i-1) = n_2 - n_1 = ki + c - n_1$, so $l = \frac{ki+c-n_1}{i-1} = k + \frac{k+c-n_1}{i-1}$. Since $l \leq \lfloor n_2/i \rfloor + 1$ we have $l \leq k + 1$. This implies $\frac{k+c-n_1}{i-1} \leq 1$ and therefore $i - 1 \geq k + c - n_1$. Then $\max(\Delta^S(M)) = i - 1$. Therefore $\max(\Delta^S(M)) = i - 1 = \min(\Delta^S(M))$, and $\Delta^S(M) = \{i - 1\}$. □

3.3. **Intervals as Delta Sets.** It is difficult to give explicit formulas for $\Delta^S(M)$. In this section we will consider $M = \langle n_1, n_2 \rangle$ with certain nonminimal generating sets and prove the following result. In the following, for an integer $k > 1$ we set $[1, k] = \{1, 2, 3, \ldots, k\}$.

Theorem 3.7. *Let $M = \langle n_1, n_2 \rangle$ and $S = \{n_1, n_2, s\}$ with $s = in_1 + jn_2,\ i, j \geq 0$ and $i + j = 2$. Then $\Delta^S(M) = [1, k]$ for some k.*

There are three cases depending on the value of i. We will consider each separately. We begin with $i = j = 1$.

Proposition 3.8. *Let $M = \langle n_1, n_2 \rangle$ and $S = \{n_1, n_2, n_1 + n_2\}$. Then $\Delta^S(M) = \{1, 2, \ldots, n_2 - n_1\}$.*

Proof. Since $\gcd\{n_1 + n_2 - n_2, n_2 - n_1\} = \gcd\{n_1, n_2\} = 1$, Proposition 1.2 implies $\min(\Delta^S(M)) = 1$. Corollary 3.4 implies that $\max(\Delta^S(M)) = n_2 - n_1$. Let $k = 2 + \lceil \frac{n_2}{n_1} \rceil$. Let $y_0, y_1, \ldots, y_{n_1-1}$ be defined so that $y_m = (kn_1 + m)n_2$. We will show that $\bigcup_{m=0}^{n_1-1} \Delta(y_m) = [1, n_2 - n_1]$.

Every solution of the equation $(kn_1 + m)n_2 = x_1n_1 + x_2n_2$ with $x_1, x_2 \geq 0$ satisfies $x_1 = (kn_2 - ln_2)$ and $x_2 = m + ln_1$ for some integer $l \geq 0$. In particular, every factorization of $(kn_1 + m)n_2$ with respect to S has the form $(kn_1 + m)n_2 = (x_1 + x_3)n_1 + (x_2 + x_3)n_2$. So every factorization of $(kn_1 + m)n_2$ satisfies $x_1 + x_3 = (k - l)n_2$ and $x_2 + x_3 = ln_1 + m$. Since $x_1, x_2, x_3 \geq 0$, we must have $0 \leq l \leq k$ and $x_3 \leq \min\{ln_1 + m, (k - l)n_2\}$.

We will see that the set of factorization lengths of $(kn_1 + m)n_2$ is a union of intervals and we will compute the gaps between them. A factorization (x_1, x_2, x_3) of x has length $x_1 + x_2 + x_3 = (k - l)n_2 + ln_1 + m - x_3$.

For a fixed m, let

$$I_l = [(k - l)n_2 + ln_1 + m - \min\{ln_1 + m, (k - l)n_2\}, (k - l)n_2 + ln_1 + m].$$

We see that for $0 \leq l < k,\ \max(I_l) = n_2 - n_1 + \max(I_{l+1})$. So we have $\max(I_k) < \max(I_{k-1}) < \cdots < \max(I_0)$, and whenever $\min(I_l) - \max(I_{l+1}) > 0$ we have $\min(I_l) - \max(I_{l+1}) \in \Delta(y_m)$.

Let $t = \frac{kn_2 - m}{n_1+n_2}$. For $l \le \lfloor t \rfloor$ we have $ln_1 + m \le (k-l)n_2$. This implies that $\min(I_l) = (k-l)n_2$. We have

$$\min(I_l) - \max(I_{l+1}) = (k-l)n_2 - [(ln_1 - m) + (k-l)n_2 - (n_2 - n_1)] = (n_2 - n_1) - (ln_1 + m).$$

We see that $\lfloor \frac{n_2-n_1}{n_1} \rfloor \le \frac{n_2-n_1}{n_1}$, and since $k = 2 + \lceil \frac{n_2}{n_1} \rceil \ge 2 + \frac{n_2}{n_1}$ and $m \le n_1$, we have

$$\lfloor t \rfloor \ge \frac{kn_2 - m}{n_1 + n_2} - 1 \ge \frac{2n_2 + \frac{n_2^2}{n_1} - n_1 - (n_1 + n_1)}{n_1 + n_2} \ge \frac{\frac{n_2^2}{n_1} - n_1}{n_1 + n_2} = \frac{n_2^2 - n_1^2}{n_1(n_1 + n_2)} = \frac{n_2 - n_1}{n_1}.$$

Therefore we have that $\lfloor \frac{n_2-n_1}{n_1} \rfloor \le \lfloor t \rfloor$. We conclude that $\{(n_2 - n_1) - m - ln_1\} \in \Delta(y_m)$ for all $0 \le l \le \lfloor \frac{n_2-n_1-m}{n_1} \rfloor$. We see that $\bigcup_{m=0}^{n_1-1} \Delta(y_m) = [1, n_2 - n_1]$, completing the proof. □

We will now consider $i = 0,\ j = 2$ and $i = 2,\ j = 0$.

Proposition 3.9. *Let $M = \langle n_1, n_2 \rangle$ and $S = \{n_1, n_2, 2n_2\}$. Then $\Delta^S(M) = [1, n_2 - n_1]$.*

Proof. Since $\gcd\{n_2, n_2 - n_1\} = \gcd\{n_1, n_2\} = 1$, Proposition 1.2 implies $\min(\Delta^S(M)) = 1$. By Corollary 3.4, we know that $\max(\Delta^S(M)) = n_2 - n_1$. Let $y_0, y_1, \ldots, y_{n_2-n_1}$ be defined so that $y_l = n_1n_2 + 2ln_2$. We will show that $(n_2 - n_1) - l \in \Delta(y_l)$, and therefore $\bigcup_{l=0}^{n_2-n_1} \Delta(y_l) = [1, n_2 - n_1]$.

Since $n_1n_2 + l2n_2$ is a multiple of n_2, given a factorization $(n_1 + 2l)n_2 = x_1n_1 + (x_2 + 2x_3)n_2$ we know that $x_1 = mn_2$ for $0 \le m \le 1 + \left\lfloor \frac{2l}{n_1} \right\rfloor$.

Let A_m denote the set of lengths of factorizations of y_l with $x_1 = mn_2$. We see that for such a factorization $0 \le x_3 \le l + \left\lfloor \frac{n_1(1-m)}{2} \right\rfloor$. We see that $\max(A_m) = mn_2 + 2l + n_1(1-m)$ and therefore $\min(A_m) = mn_2 + l + n_1(1-m) - \left\lfloor \frac{n_1(1-m)}{2} \right\rfloor$. We see that $\min(A_0) < \min(A_1) < \cdots < \min(A_{1+\left\lfloor \frac{2l}{n_1} \right\rfloor})$ and that $\min(A_1) - \max(A_0) = (n_2 + l) - (2l + n_1) = (n_2 - n_1) - l \in \Delta(y_l)$, completing the proof. □

Proposition 3.10. *Let $M = \langle n_1, n_2 \rangle$ and $S = \{n_1, n_2, 2n_1\}$. If $2n_1 > n_2$, then $\Delta^S(M) = \{1\}$. If $2n_1 < n_2$ then $\Delta^S(M) = [1, \lceil n_2/2 \rceil - n_1]$.*

Proof. If $2n_1 > n_2$ then $\max(\Delta^S(M)) = i - 1 = 1$ by Corollary 3.4, so $\Delta^S(M) = \{1\}$ and we are done. So suppose $2n_1 < n_2$. We can write $n_2 = 2k + c$ for unique integers $k \ge 0$ and $0 \le c \le 1$. By that same corollary, we know $\max(\Delta^S(M)) = k + c - n_1 = \lceil n_2/2 \rceil - n_1$. Let $y_0, y_1, \ldots, y_{\lceil \frac{n_2}{2} \rceil - n_1}$ be defined so that $y_l = (n_2 + 2l)n_1$. We will prove that $\left\lceil \frac{n_2}{2} \right\rceil - n_1 - l \in \Delta(y_l)$, and taking the union of the sets $\Delta(y_l)$ will complete the proof.

Since y_l is a multiple of n_1, whenever we have $y_l = (x_1 + 2x_3)n_1 + x_2n_2$ we know that $x_2 = mn_1$ for some $0 \le m \le 1 + \left\lfloor \frac{2l}{n_2} \right\rfloor$. Since $l \le \lceil \frac{n_2}{2} \rceil - n_1 \le \frac{n_2+1}{2} - n_1$, we have

$$0 < \frac{2l}{n_2} \le \frac{n_2 + 1 - 2n_1}{n_2} \le 1 + \frac{1 - 2n_1}{n_2} < 1,$$

and therefore $0 \le m \le 1$.

Let A_0 denote the set of factorizations with $m = 0$ and let A_1 denote the set of factorizations with $m = 1$. We see that $\max(A_1) = n_1 + 2l$ and $\min(A_0) = l + \left\lfloor \frac{n_2}{2} \right\rfloor + n_2 - 2\left\lfloor \frac{n_2}{2} \right\rfloor = l + n_2 - \left\lfloor \frac{n_2}{2} \right\rfloor$.

Therefore $\min(A_0) - \max(A_1) = \left\lceil \frac{n_2}{2} \right\rceil - n_1 - l \in \Delta(y_l)$, completing the proof. □

Combining the last three propositions proves Theorem 3.7. We will now consider one more case in which we can explicitly describe $\Delta^S(M)$.

Theorem 3.11. *Let $M = \langle n_1, n_2\rangle$ and $S = \{n_1, n_2, in_1 + jn_2\}$ such that $i + j - 1 = k(n_2 - n_1)$ for $k > 0$, $j \geq 0$, and $0 \leq i < n_2$. Then $\Delta^S(M) = \{n_2 - n_1, 2(n_2 - n_1), \ldots, k(n_2 - n_1)\}$.*

Note that the case $k = 1$ is implied by Theorem 3.5.

Proof. We first note that $in_1 + jn_2 \geq (i + j)n_1 = k(n_2 - n_1)n_1 + n_1$. Therefore $in_1 + jn_2 - n_2 \geq (kn_1 - 1)(n_2 - n_1) > 0$, so $in_1 + jn_2 > n_2$. We will now compute $\max(\Delta^S(M))$ and $\min(\Delta^S(M))$. Since $n_2 - n_1 \leq i + j - 1$, by Corollary 3.4 we see that $\max(\Delta^S(M)) = i + j - 1 = k(n_2 - n_1)$. Since $in_1 + jn_2 = (k(n_2 - n_1) + 1)n_1 + j(n_2 - n_1)$, we see that

$$\gcd\{n_2 - n_1, in_1 + jn_2 - n_1\} = \gcd\{n_2 - n_1, (k + j)(n_2 - n_1)\} = n_2 - n_1.$$

By Proposition 1.2 we have $\min(\Delta^S(M)) = n_2 - n_1$.

Let $y_0, y_1, \ldots, y_{k-1}$ be defined so that $y_l = in_1 + jn_2 + ln_1n_2$. We will prove that $(k - l)(n_2 - n_1) \in \Delta(y_l)$, and taking the union of the sets $\Delta(y_l)$ will complete the proof.

It is clear that the shortest factorization of y_l which does not contain any factors of $in_1 + jn_2$ has length $ln_1 + j + i = ln_1 + k(n_2 - n_1) + 1$. The longest factorization of x which contains exactly one factor of $in_1 + jn_2$ has length $1 + ln_2$. We have $ln_1 + k(n_2 - n_1) + 1 - (ln_2 - 1) = (k - l)(n_2 - n_1) \geq 0$.

Suppose we have a factorization of x with at least two factors of $in_1 + jn_2$. Then it has length at most $(ln_1n_2 - in_1 - jn_2)/n_1 + 2 = ln_2 - i - jn_2/n_1 + 2 \leq ln_2 + 1$. Therefore $ln_1 + j + i$ and $1 + ln_2$ are consecutive elements in $\mathcal{L}^S(x)$ and we have $(k - l)(n_2 - n_1) \in \Delta^S(y_l)$. □

3.4. **Delta Set $\{1, t\}$.** In this section we will continue to work with $M = \langle n_1, n_2\rangle$ and $S = \{n_1, n_2, in_1 + jn_2\}$. We will show for $t > 2$ that the set $T = \{1, t\}$ is not equal to $\Delta^S(M)$ for any values i, j. In particular, we will prove the following:

Theorem 3.12. *Let $M = \langle n_1, n_2\rangle$ and let $s = in_1 + jn_2$ with $0 \leq j$, $0 \leq i < n_2$. Let $S = \{n_1, n_2, s\}$. Then if $\Delta^S(M) = \{1, t\}$, $t = 2$.*

This proof divides into three major cases: (1) $i + j = 2$, (2) $i + j > 2, j \neq 0$, and (3) $i + j > 2, j = 0$. The first case follows from Proposition 3.7. We know that $\Delta^S(M)$ must be an interval, and thus $t = 2$. The rest of the proof will follow from two propositions.

Proposition 3.13. *Let $M = \langle n_1, n_2\rangle$ and let $s = in_1$ with $3 \leq i < n_2$. Let $S = \{n_1, n_2, s\}$. Then if $\Delta^S(M) = \{1, t\}$, $t = 2$.*

Proof. Lemma 3.1 implies that $i - 1 \in \{1, t\}$. Since $i \geq 3$ we have $i - 1 = t$. We can write $n_2 = ki + c$ for unique integers $k \geq 0$ and $0 \leq c < i$. We consider n_1n_2, as we did in the proof of Proposition 3.2. Any factorization of n_1n_2 is of the form $n_1n_2 = x_1n_1 + x_2n_2 + x_3(in_1)$ with $x_1, x_2, x_3 \geq 0$. If $x_2 > 0$ then since $n_1n_2 - x_2n_2 \equiv 0 \pmod{n_1}$ we must have $x_2 = n_1$. Every factorization with $x_2 = 0$ is of the form $(c + li)n_1 + in_1(k - l)$ for $0 \leq l \leq k$, so

$$\mathcal{L}^S(n_1n_2) = \{n_1\} \cup \{c + k + l(i - 1) \mid 0 \leq l \leq k\}.$$

We first suppose that $n_1 \geq k + c$. Since $n_1 < n_2$, we must have $n_1 \in [c + k + (l - 1)(i - 1), c + k + l(i - 1))$ for some $1 \leq l \leq k$. If $n_1 \neq c + k + (l - 1)(i - 1)$ then we have $\{n_1 - (c + k + (l - 1)(i - 1)), c + k + l(i - 1) - n_1\} \subseteq \Delta^S(M)$. Clearly both of these elements are at most $i - 2$ and their sum is $i - 1$. Therefore they must both be equal to 1, and so $t = 2$. If $n_1 = c + k + l(i - 1)$ for some $0 \leq l < k$ then Proposition 1.2 implies

$$\min(\Delta^S(M)) = \gcd\{(i - 1)n_1, ki + c - n_1\} = \gcd\{(i - 1)n_1, (k - l)(i - 1)\} = i - 1 = t \neq 1,$$

which is a contradiction.

So we can suppose $n_1 < k+c$. Then we have $k+c-n_1 \in \Delta^S(n_1n_2)$ so we must have $k+c-n_1 = i-1$ or $k+c-n_1 = 1$. In the first case we have $k = i-1+n_1-c$. Then $ki+c-in_1 = (i-c)(i-1)$. Proposition 1.2 implies $\min(\Delta^S(M)) = \gcd\{(i-1)n_1, (i-c)(i-1)\} \geq i-1 > 1$ which is a contradiction.

So we have $k+c-n_1 = 1$. Consider $2n_1n_2 = 2n_1(ki+c)$. Any factorization of $2n_1n_2$ is of the form $2n_1n_2 = x_1n_1 + x_2n_2 + x_3(in_1)$ with $x_1, x_2, x_3 \geq 0$. Since $2n_1n_2 - x_2n_2 \equiv 0 \pmod{n_1}$, we must have $x_2 \in \{0, n_1, 2n_1\}$. There a unique factorization with $x_2 = 2n_1$ and it has length $2n_1$.

Let A_1 be the set of factorizations of $2n_1n_2$ with $x_2 = n_1$. We have factorizations $2n_1n_2 = n_1n_2 + (c+li)n_1 + (k-l)in_1$, for any $0 \leq l \leq k$. Such a factorization has length $n_1 + c + k + l(i-1)$ for $0 \leq l \leq k$.

Let A_0 be the set of factorizations of $2n_1n_2$ with $x_2 = 0$. We have $2n_1n_2 = (2c+li)n_1 + (2k-l)in_1$, for any $0 \leq l \leq 2k$ if $2c < i$ and any $-1 \leq l \leq 2k$ if $2c \geq i$, since $2cn_1 = in_1 + (2c-i)n_1$. Such a factorization has length $2c + 2k + l(i-1)$ for l in the allowed range.

The shortest two factorizations in A_1 have lengths $n_1 + c + k = 2n_1 + 1$ and $2n_1 + 1 + (i-1)$ respectively. There exists a factorization in A_0 of length $2c + 2k = 2n_1 + 2$. The next longest factorization in A_0 has length $2c + 2k + (i-1)$. Therefore $2n_1 + 2$ and $2n_1 + 1 + (i-1)$ are consecutive factorization lengths of $2n_1n_2$ and $i - 2 \in \Delta^S(2n_1n_2)$. This implies $i-2 = 1$ and $t = 2$. □

Proposition 3.14. *Let $M = \langle n_1, n_2 \rangle$, $S = \{n_1, n_2, s\}$ with $s = in_1 + jn_2$ with $i+j > 2$, $0 \leq i < n_2$, and $j > 0$. Suppose $\Delta^S(M) = \{1, t\}$ for some $t \geq 2$. Then $t = 2$.*

Proof. Lemma 3.1 implies $i+j-1 \in \Delta^S(M)$. Since $i+j-1 > 1$ we must have $i+j = t+1$. We now consider n_1n_2. Since $j > 0$ we have $s > n_2$. The longest factorization of n_1n_2 in S has length n_2 and the second longest has length n_1. Therefore $n_2 - n_1 \in \Delta^S(n_1n_2)$. If $n_2 - n_1 = i+j-1$ then by Theorem 3.5 we have $\Delta^S(M) = \{n_2 - n_1\}$ which is a contradiction. So $n_2 - n_1 \neq t$ implies that $n_2 - n_1 = 1$.

Thus

$$S = \{n_1, n_1+1, (i+j)n_1 + j\}.$$

Let l be the least positive integer such that either $li \geq n_2$ or $lj \geq n_1$. We see that $ls = lin_1 + ljn_1 + lj = lin_1 + lj(n_1+1)$.

Suppose $li \geq n_1+1$. Since $i < n_1+1$ we have $l \geq 2$. We can write $ls = (l-m)s + min_1 + mj(n_1+1)$ for any $0 \leq m \leq l$. By the definition of l, $m < l$ implies that $min_1 + mjn_2$ has only one factorization in M. When $m = l$ we also have the extra factorization,

$$ls = (li - (n_1+1))n_1 + (lj + n_1)(n_1+1).$$

The definition of l implies that $(l-1)j < n_1$. Since $l \geq 2$ we have $lj < 2n_1$. We may also have one more factorization of the form $ls = (li + (n_1+1))n_1 + (lj - n_1)(n_1+1)$ if $lj \geq n_1$. We see that $\mathcal{L}^S(ls) = \{l + m(i+j-1) \mid 0 \leq m \leq l\} \cup \{l(i+j) - 1\}$, with one extra factorization of length $l(i+j)+1$ if $lj \geq n_1$. We see that there are three consecutive factorization lengths,

$$l + (l-1)(i+j-1) = (l-1)(i+j) + 1 < l(i+j) - 1 < l(i+j).$$

Therefore $(l(i+j)-1) - ((l-1)(j-1)+1) = i+j-2 \in \Delta^S(ls)$. Since $i+j-2 \neq t = i+j-1$, we have $i+j-2 = 1$ which implies $t = 2$.

Now suppose $lj \geq n_1$ and $li < n_1+1$. We will end up considering two cases based on the value of $\lfloor lj/n_1 \rfloor$. We see that $i \leq j$. Consider $(l+1)s$. We can write $(l+1)s = (l+1-m)s + min_1 + mj(n_1+1)$ for $m \leq l+1$. When $m < l$, $min_1 + mjn_2$ has only one factorization in M. We will group factorizations by the value of m. For $m < l$ there is one factorization for each value of m and it has length $l+1+m(i+j-1)$. The longest of these has length $l+1+(l-1)(i+j-1) = l(i+j)-(i+j-2) < l(i+j)$.

Let A_l denote the set of all factorizations of $(l+1)s$ with $m=l$. We want to compute the length of the longest factorization in this set. We have the factorization

$$(l+1)s = s + (li + (n_1+1))n_1 + (lj - n_1)(n_1+1)$$

which has length $2+l(i+j)$. If $\lfloor lj/n_1 \rfloor = 1$ then this is the maximum factorization length in A_l. First suppose $\left\lfloor \frac{lj}{n_1} \right\rfloor = 1$.

Let A_{l+1} denote the set of all factorizations of $(l+1)s$ with $m=l+1$. We want to compute the length of the shortest factorization in A_{l+1}. We have a factorization of length $(l+1)(i+j)$. If $(l+1)i < n_1+1$ then this is the length of the shortest factorization of A_{l+1}. When $(l+1)i \geq n_1+1$ we also have a factorization

$$(l+1)s = ((l+1)i - (n_1+1))n_1 + ((l+1)j + n_1)(n_1+1),$$

which has length $(l+1)(i+j)-1$. Since $li < n_1+1$ we have $(l+1)i < 2(n_1+1)$ and $(l+1)(i+j)-1$ is the length of the shortest factorization in A_{l+1}.

First suppose $(l+1)i < n_1+1$. Then we have consecutive factorization lengths $2+l(i+j) < (l+1)(i+j)$. We have $(l+1)(i+j)-(2+l(i+j)) = i+j-2 \in \Delta^S((l+1)s)$ and therefore $i+j-2=1$ which implies $t=2$.

Now suppose $(l+1)i \geq n_1+1$. Then we have consecutive factorization lengths $2+l(i+j) < (l+1)(i+j)-1$. We have $(l+1)(i+j)-1-(2+l(i+j)) = i+j-3 \in \Delta^S((l+1)s)$ and therefore $i+j=4$ or $t=2$. Suppose $i+j=4$. Since $i<j$ we have only two possibilities $(i,j) \in \{(0,4),(1,3)\}$. Since $(l+1)i \geq n_1+1$ we cannot have $i=0$. So $i=1$ and $l \geq n_1$. Since $n_1 \geq 2$ we have $(l-1)j \geq 3(n_1-1) \geq n_1$, contradicting $(l-1)j < n_1$. Therefore $t=2$.

Now suppose $\lfloor lj/n_1 \rfloor = k > 1$. Since $(l-1)j < n_1$, we must have $l=1$. Now $\left\lfloor \frac{j}{n_1} \right\rfloor = k > 1$ and $n_1 \geq 2$ implies $j \geq 4$. We have $j = kn_1 + c$ for some $0 \leq c < n_1$. So $k = \frac{j-c}{n_1} \leq \frac{j}{n_1} \leq j-2$.

Consider $2s = (2-m)s + min_1 + mj(n_1+1)$ for $0 \leq m \leq 2$. Let A_1 be the set of factorizations of $2s$ with $m=1$. The longest factorization in this set is $2s = s + (i + k(n_1+1))n_1 + c(n_1+1)$, which has length $1+i+j+k$. Let A_2 be the set of factorizations of $2s$ with $m=2$. If $2i < n_1+1$ then the shortest factorization in this set has length $2(i+j)$. If $2i \geq n_1+1$ then since $i < n_1+1$ we have $2i < 2(n_1+1)$ and the shortest factorization in this set has length $2(i+j)-1$.

Suppose $2i < n_1+1$. We have $2(i+j)-(1+i+j+k) = i+j-1-k = t-k$. If $k < i+j-2$, then $\Delta^S(2s)$ contains an element other than 1 or t, which is a contradiction. If not, then we must have $k = j-2$ so $i=0,\ j=4,\ n_1=k=2$ and $c=0$. This gives $M = \langle 2,3 \rangle,\ j=4$ and $S=\{2,3,12\}$. In this case $2 \in \Delta^S(21)$, so $\Delta^S(M) \neq \{1,t\}$.

Suppose $2i \geq n_1+1$. Since $n_1 \geq 2$ we have $i \geq 2$. We have $2(i+j)-1-(1+i+j+k) = i+j-2-k = t-1-k$. Since $i+j-3 \geq j-1 > k$, $\Delta^S(2s)$ contains an element other than 1 or t, which is a contradiction. □

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