AN ALGORITHM TO COMPUTE ω-PRIMALITY IN A NUMERICAL MONOID

DAVID F. ANDERSON, SCOTT T. CHAPMAN,
NATHAN KAPLAN, AND DESMOND TORKORNOO

Abstract. Let M be a commutative, cancellative, atomic monoid and x a nonunit in M. We define $\omega(x) = n$ if n is the smallest positive integer with the property that whenever $x \mid a_1 \cdots a_t$, where each a_i is an atom, there is a $T \subseteq \{1, 2, \ldots, t\}$ with $|T| \leq n$ such that $x \mid \prod_{k \in T} a_k$. The ω-function measures how far x is from being prime in M. In this paper, we give an algorithm for computing $\omega(x)$ in any numerical monoid. Simple formulas for $\omega(x)$ are given for numerical monoids of the form $\langle n, n+1, \ldots, 2n-1 \rangle$, where $n \geq 3$, and $\langle n, n+1, \ldots, 2n-2 \rangle$, where $n \geq 4$. The paper then focuses on the special case of 2-generator numerical monoids. We give a formula for computing $\omega(x)$ in this case and also necessary and sufficient conditions for determining when x is an atom. Finally, we analyze the asymptotic behavior of $\omega(x)$ by computing $\lim_{x \to \infty} \frac{\omega(x)}{x}$.

1. Introduction

Problems involving non-unique factorizations in monoids and integral domains have gathered much recent attention in the mathematical literature (see the monograph [7] and the references therein). Much of this work was fueled by earlier study of the *elasticity* of a ring of algebraic integers R. The elasticity of R in some sense measures how far R is from being a half-factorial domain (i.e., an integral domain where every irreducible factorization of an element has the same length). A recent paper of Geroldinger and Hassler [8] has introduced a new invariant which essentially measures how far an element of an integral domain or a monoid is from being prime. Their definition is as follows.

Definition 1.1. Let M be a commutative, cancellative, atomic monoid with set of units $M^\times$ and set of irreducibles (or atoms) $\mathcal{A}(M)$. For $x \in M \setminus M^\times$, we define $\omega(x) = n$ if n is the smallest positive integer with the property that whenever $x \mid a_1 \cdots a_t$, where each $a_i \in \mathcal{A}(M)$, there is a $T \subseteq \{1, 2, \ldots, t\}$ with $|T| \leq n$ such that $x \mid \prod_{k \in T} a_k$. If no such n exists, then $\omega(x) = \infty$. For $x \in M^\times$, we define $\omega(x) = 0$. Finally, $\omega(M) = \{\omega(x) : x \in M \setminus M^\times\}$.

It follows easily from the definition that an element $x \in M \setminus M^\times$ is prime if and only if $\omega(x) = 1$. Some basic properties of this function can be found not only in the paper mentioned above [8], but also in [7]. This function is also studied in the context of integral domains in [1]. As with many other constants in the theory of non-unique factorizations, the computation of specific values of $\omega(x)$ is often highly non-trivial, and even in the context of the references already mentioned, there are very few existing calculations. This led us to consider the ω-function in one of the most basic classes of monoids, namely numerical monoids (i.e., additive submonoids of the nonnegative integers $\mathbb{N}_0$). To our surprise, calculations even in this relatively simple additive structure were difficult. We produce in Section 3 an algorithm to compute $\omega(x)$ for any nonzero x in a given numerical monoid S. If S is the monoid $\langle n, n+1, \ldots, 2n-1 \rangle$ (for $n \geq 3$) or $\langle n, n+1, \ldots, 2n-2 \rangle$ (for $n \geq 4$), then we are able

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from the algorithm to give an exact formula for $\omega(x)$ (see Propositions 3.1 and 3.2) which in turn yields the structure of $\omega(S)$. In Section 4, we focus on the case where S requires two generators (i.e., $S = \langle n_1, n_2\rangle$). In this case, Theorem 4.4 yields a formula which in turn gives an exact calculation of $\omega(S)$ (Theorem 4.7). Using these two theorems, we obtain an interesting characterization of the generators n_1 and n_2 (Theorem 4.5). While the results of Sections 3 and 4 yield sets $\omega(S)$ of the form $[k, \infty) \cap \mathbb{N}$ for $k \geq 1$, we show in Example 3.2 that this is not always the case. We close by considering the asymptotic behavior of $\omega(x)$ (Theorem 4.9).

Throughout, we will use the standard notation from the theory of non-unique factorizations as outlined in [7]. Let M be a commutative, cancellative monoid. The nonunit $x \in M$ is an *atom* if whenever $x = yz$, then either $y \in M^\times$ or $z \in M^\times$. As mentioned above, $\mathcal{A}(M)$ will denote the set of atoms of M. If every $x \in M \setminus M^\times$ can be written as a product of elements from $\mathcal{A}(M)$, we say that M is *atomic*. An element $x \in M\setminus M^\times$ is *prime* if $x \mid yz$ implies $x \mid y$ or $x \mid z$. Let $x \in M \setminus M^\times$. We say $a_1 \cdots a_n$ is a *factorization* of x if each $a_i \in \mathcal{A}(M)$ and $x = a_1 \cdots a_n$, and the length of this factorization is n. Finally, for $x \in M\setminus M^\times$, we define the *length set* of x, denoted by $\mathcal{L}(x)$, as $\mathcal{L}(x) = \{n \mid \; x = a_1 \cdots a_n, \text{ where } a_i \in \mathcal{A}(M)\}$. We let $L(x) = \sup \mathcal{L}(x)$ and $l(x) = \inf \mathcal{L}(x)$. If $L(x) < \infty$ for every $x \in M \setminus M^\times$, then M is called a *bounded factorization monoid* (or *BFM*).

A submonoid S of $\mathbb{N}_0$ under addition is called a *numerical monoid.* Every numerical monoid S has a unique minimal generating set, where the generators are precisely the atoms of the monoid. A numerical monoid $S = \langle n_1, \ldots, n_t\rangle$ is *primitive* if $\gcd(n_1, \ldots, n_t) = 1$. Every primitive numerical monoid S has the property that $\mathbb{N}_0 \setminus S$ is finite. Thus, there is a greatest element of $\mathbb{N}_0$ not in S, which we call the *Frobenius number* of S and denote by $F(S)$. Since every numerical monoid is isomorphic to a primitive numerical monoid, we will only concern ourselves with primitive numerical monoids. A good general reference on numerical monoids is [6]. Properties of the length sets of a numerical monoid have been studied extensively in the recent literature (see for example [2], [3], [4], and [5]).

We note here some recent work of Omidali [9] on the catenary and tame degree of certain numerical monoids. A numerical monoid S is said to be generated by a generalized arithmetic sequence if $S = \langle a, ha + d, \ldots, ha + xd\rangle$ where a, d, h and x are positive integers and $\gcd(a, d) = 1$. In [9, Theorem 3.10], the author shows for such an S that $\omega_{\min}(S) = \max\{\omega(a), \omega(ha+d), \ldots, \omega(ha+xd)\}$ coincides with the tame degree (denoted $t(S)$) of S (see [7, Chapter 3] for the definition of $t(S)$ and a summary of known facts concerning this constant). In general, it is clear that there are elements $x \in S$ with $\omega(x) \neq t(x)$. By our Proposition 3.1, the set $\omega(S)$ is unbounded, but by [9, Theorem 3.10], $t(S) = \{t(x) \mid x \in S\}$ is finite. We leave the question of whether or not there is a strong relationship between $\omega(x)$ and $t(x)$ in a general numerical monoid to future work.

2. Background and Basic Results

We begin with a basic, but important, observation.

Proposition 2.1. *Let M be a commutative, cancellative, atomic BFM.*

(a) $\omega(xy) \leq \omega(x) + \omega(y)$ *for all x and $y \in M \setminus M^\times$.*
(b) *The set $\omega(M)$ is unbounded.*

Proof. The proof of (a) can be found in [Lemma 3.3 [8]]. For (b), let $x \in M \setminus M^\times$ and $x = a_1 \cdots a_t$ be a longest factorization of x. Since there can be no subset $T \subseteq \{1, 2, \ldots, t\}$ with $|T| < t$ such that $x \mid \prod_{i\in T} a_i$, we have $\omega(x) \geq t = L(x)$. Thus, $\omega(M)$ is unbounded since $\{L(x) : \; x \in M \setminus M^\times\}$ is unbounded. □

Let $S = \langle n_1, \ldots, n_t \rangle$ be a minimally generated, primitive numerical monoid with $1 < n_1 < \cdots < n_t$. Let $\bar{n} = (n_1, \ldots, n_t) \in \mathbb{N}_0^t$ be the vector representation of the generating set of S. Then for each $x \in S$, there exists a vector $\bar{x} = (x_1, \ldots, x_t) \in \mathbb{N}_0^t$ such that $x = \bar{x} \cdot \bar{n} = x_1 n_1 + \cdots + x_t n_t$. Thus, we can represent elements of S as vectors in $\mathbb{N}_0^t$.

Definition 2.1. (1) Let $\bar{x} = (x_1, \ldots, x_t) \in \mathbb{N}_0^t$ and $\bar{y} = (y_1, \ldots, y_t) \in \mathbb{N}_0^t$. We say that $\bar{x}$ *subsumes* $\bar{y}$ (or $\bar{y}$ *is subsumed by* $\bar{x}$) if $\bar{x} \neq \bar{y}$ and $x_i \geq y_i$ for all $1 \leq i \leq t$.

(2) For each $1 \leq i \leq t$, define $m_i : S \to \mathbb{N}_0$ by $m_i(x) = \min \{d : \ dn_i - x \in S, \ d \in \mathbb{N}_0\}$.

From the definition of $m_i(x)$, we see that the function m_i computes the smallest multiplier such that x divides that multiple of the i^{th} generator in the monoid S.

Definition 2.2. Let $x \in S \setminus S^\times$; we define

(1) $\mathcal{D}(x) = \{\bar{v} \in \mathbb{N}_0^t : \ x \mid \bar{v} \cdot \bar{n}\}$,
(2) $\mathcal{F}(x) = \{\bar{v} \in \mathcal{D}(x) : \ x = \bar{v} \cdot \bar{n}\}$, and
(3) $\mathcal{Y}(x) = \{\bar{y} \in \mathcal{D}(x) : \ \forall \ \bar{c} \in \mathcal{D}(x), \ \bar{y} \text{ does not subsume } \bar{c}\}$.

The set $\mathcal{D}(x)$ is the set of factorizations of elements of S divisible by x. The set $\mathcal{F}(x)$ is the factorization set of x. Let $(0, \ldots, 0, m_i(x), 0, \ldots, 0) \in \mathbb{N}_0^t$ such that $m_i(x)$ is in the i^{th} component for all $1 \leq i \leq t$. It follows from the definition of $\mathcal{Y}(x)$ that $(0, \ldots, 0, m_i(x), 0, \ldots, 0) \in \mathcal{Y}(x)$, and so $\mathcal{Y}(x)$ is nonempty. Using the above notation, we obtain an initial representation for $\omega(x)$.

Proposition 2.2. $\omega(x) = \max \{\sum_1^t y_i : \ \bar{y} \in \mathcal{Y}(x)\}$.

Proof. Let $\bar{k} = (k_1, \ldots, k_t) \in \mathcal{Y}(x)$ such that $\sum_1^t k_i = \max \{\sum_1^t y_i : \ \bar{y} \in \mathcal{Y}(x)\}$. Since, for every $\bar{c} \in \mathcal{D}(x)$ we have that $\bar{c}$ is not subsumed by $\bar{k}$, we conclude that x does not divide any subsum of the sum $\bar{k} \cdot \bar{n} = k_1 n_1 + \cdots + k_t n_t$. Thus, we have $\omega(x) \geq \sum_1^t k_i$ because $x \mid \bar{k} \cdot \bar{n}$. But, since $\sum_1^t k_i = \max \{\sum_1^t y_i : \ (y_1, \ldots, y_t) \in \mathcal{Y}(x)\}$, we also have that $\omega(x) \leq \sum_1^t k_i$. Therefore, $\omega(x) = \sum_1^t k_i$. □

3. ω-Measure in General Numerical Monoids

Our main result of this section is an algorithm which computes $\omega(x)$ for any nonunit x in a given primitive numerical monoid.

The Omega Algorithm $(x, \{n_1, \ldots, n_t\})$**.**

Input: $\{n_1, \ldots, n_t\}$, the primitive set of generators for S, and $0 \neq x \in S$.

Output: $\omega(x)$.

(1) Compute $m_i(x)$, for $1 \leq i \leq t$. Let $M = \max \{m_i(x) : \ 1 \leq i \leq t\}$.
(2) Solve $U_0(x) = \left\{(d_1, \ldots, d_t) \in \mathbb{N}_0^t : \ \sum_1^t d_i = M, \ d_i < m_i(x)\right\}$, and
$V_0(x) = \left\{(d_1, \ldots, d_t) \in \mathbb{N}_0^t : \ \sum_1^t d_i > M, \ d_i < m_i(x)\right\}$.
(3) Set $U_1(x) = \{\bar{v} \in U_0(x) : \ x \mid \bar{v} \cdot \bar{n}\}$.
Set $V_1(x) = \{\bar{v} \in V_0(x) : \ \exists \ \bar{c} \in U_1(x) \text{ such that } \bar{v} \text{ subsumes } \bar{c}\}$.
Set $V_2(x) = V_0(x) \setminus V_1(x)$.
(4) Set $V_3(x) = \{\bar{v} \in V_2(x) : \ x \mid \bar{v} \cdot \bar{n}\}$.
Set $V_4(x) = \{\bar{v} \in V_3(x) : \ \exists \ \bar{c} \in V_3(x) \text{ such that } \bar{v} \text{ subsumes } \bar{c}\}$.
(5) Set $\mathcal{W}(x) = V_3(x) \setminus V_4(x)$.

(6) If $\mathcal{W}(x)$ is not empty, then $\omega(x) = \max\ \{\sum_1^t d_i :\ (d_1, \ldots, d_t) \in \mathcal{W}(x)\}$.
Otherwise, $\omega(x) = M$.

Proof. Let $x \in S$, $M = \max\ \{m_i(x) :\ 1 \leq i \leq t\}$, and $\mathcal{W}(x)$ be as constructed above. We will show that $\mathcal{W}(x) = \{\bar{v} \in \mathcal{Y}(x) :\ \sum_1^t v_i > M\}$. Then, if $\mathcal{W}(x)$ is not empty, it is clear that $\omega(x) = \max\ \{\sum_1^t d_i :\ (d_1, \ldots, d_t) \in \mathcal{W}(x)\} = \max\ \{\sum_1^t y_i :\ \bar{y} \in \mathcal{Y}(x)\}$.

We first show that $\bar{k} \in \mathcal{W}(x)$ implies $\bar{k} \in \mathcal{Y}(x)$. Since $\bar{k} \in \mathcal{W}(x)$ implies $\bar{k} \in V_3(x)$, we have $\bar{k} \in \mathcal{D}(x)$. Now we will show that $\bar{k}$ does not subsume any $\bar{c} \in \mathcal{D}(x)$. Suppose that there is $\bar{b} \in \mathcal{D}(x)$ such that $\bar{k}$ subsumes $\bar{b}$ and $\sum_1^t b_i < M$. Then there exists $\bar{c} \in \mathcal{D}(x)$ with $\sum_1^t c_i = M$ and $b_i \leq c_i \leq k_i$ for all $1 \leq i \leq t$. Since $\bar{k} \in \mathcal{W}(x)$ implies $\bar{k} \notin V_1(x)$, $\bar{k}$ does not subsume any $\bar{c} \in \mathcal{D}(x)$ with $\sum_1^t c_i \leq M$. Since $\bar{k} \in \mathcal{W}(x)$ implies $\bar{k} \notin V_4(x)$, $\bar{k}$ does not subsume any $\bar{c} \in \mathcal{D}(x)$ with $\sum_1^t c_i > M$. Therefore, $\mathcal{W}(x) \subset \mathcal{Y}(x)$. It is clear that $\bar{y} \in \mathcal{Y}(x)$ is in $\mathcal{W}(x)$ if and only if $\sum_1^t y_i > M$.

If $\mathcal{W}(x)$ is empty, then $\sum_1^t y_i \leq M$ for all $\bar{y} \in \mathcal{Y}(x)$. Since there exists an integer $j \in \{1, 2, \ldots, t\}$ such that $(0, \ldots, 0, m_j(x), 0, \ldots, 0) \in \mathcal{Y}(x)$ and $m_j(x) = M$, we have $\max\ \{\sum_1^t y_i :\ \bar{y} \in \mathcal{Y}(x)\} = M = \omega(x)$. □

Example 3.1. The Omega Algorithm can be readily programmed using any standard computer algebra package. To demonstrate this, we compute the omega values of the generators for some three-generated numerical monoids.

$\langle n_1, n_2, n_3\rangle$	Ordering of Omega Values
$\langle 5, 7, 17\rangle$	$\omega(5) = 5 < \omega(7) = 7 < \omega(17) = 9$
$\langle 5, 7, 11\rangle$	$\omega(5) = 3 < \omega(7) = \omega(11) = 5$
$\langle 4, 5, 6\rangle$	$\omega(4) = 2 < \omega(5) = 4 > \omega(6) = 3$
$\langle 6, 9, 11\rangle$	$\omega(6) = \omega(9) = 3 < \omega(11) = 7$
$\langle 7, 11, 17\rangle$	$\omega(7) = \omega(11) = \omega(17) = 5$
$\langle 6, 7, 11\rangle$	$\omega(6) = 4 > \omega(7) = 3 < \omega(11) = 5$
$\langle 7, 8, 12\rangle$	$\omega(7) = 5 > \omega(8) = \omega(12) = 4$

It is unclear whether or not the remaining 6 orderings (such as $\omega(n_1) = \omega(n_2) > \omega(n_3)$) are possible.

In Propositions 3.1 and 3.2, we apply the algorithm to two specific classes of numerical monoids and obtain exact formulas for $\omega(x)$ and $\omega(S)$.

Proposition 3.1. *Let $S = \langle n, n+1, \ldots, 2n-1\rangle$, for an integer $n \geq 3$. If $0 \neq x \in S$, then*

$$\omega(x) = \left\lceil \frac{x}{n} \right\rceil + 1$$

and thus $\omega(S) = \{2, 3, 4, 5, \ldots\}$.

Proof. Let $n \geq 3$ be an integer and $S = \langle n, n+1, \ldots, 2n-1\rangle$. Then $S = \{0, n, n+1, n+2, \ldots\}$. Let $x \in S$. Then there exist unique positive integers q and r such that $x = qn + r$, where $q = \lfloor \frac{x}{n} \rfloor$ and $0 \leq r \leq n-1$. We consider two cases, where $r = 0$ or $r > 0$.

If $r = 0$, we first show that $\omega(x) \leq q + 1$. The sum of any $q + 1$ atoms is at least $(q + 1)n$. Since $(q + 1)n - x = n \in S$, then x divides the sum of any $q + 1$ atoms and $\omega(x) \leq q + 1$.

To see that $\omega(x) = q + 1$, consider $(q - 1)n + 2(n + 1)$. We know that x divides this sum of $q + 1$ atoms, but since $(q - 1)n + 2(n + 1) - x - n = 2 < n$, we see that x does not divide any subsum of q atoms. So $\omega(x) = q + 1 = \lceil \frac{x}{n} \rceil + 1$.

If $r > 0$, we first show that $\omega(x) \leq q + 2$. The sum of any $q + 2$ atoms is at least $(q + 2)n$. Since $(q + 2)n - x = 2n - r > n \in S$, then x divides the sum of any $q + 2$ atoms and $\omega(x) \leq q + 2$. Since $x \mid (q + 2)n$, but $(q + 1)n - x = n - r \in [1, n - 1]$, we see that x does not divide any subsum of $q + 1$ atoms, and therefore $\omega(x) \geq q + 2$. So $\omega(x) = q + 2 = \lceil \frac{x}{n} \rceil + 1$. The structure of $\omega(S)$ now easily follows. □

Proposition 3.2. *Let $S = \langle n, n + 1, \ldots, 2n - 2 \rangle$, for an integer $n \geq 4$, $x \in S$, and let k be the unique integer such that $kn < x \leq (k + 1)n$. Then,*

$$\omega(x) = \begin{cases} k + 3, & \text{if } k + 2 \text{ divides } x - (kn + 1), \\ k + 2, & \text{otherwise.} \end{cases}$$

Moreover, it follows that

$$\omega(S) = \begin{cases} \{2, 3, 4, 5, \ldots\} & \text{if } n \text{ is even} \\ \{3, 4, 5, 6, \ldots\} & \text{if } n \text{ is odd} \end{cases}.$$

Proof. Since $S = \{0, n, n + 1, \ldots, 2n - 2, 2n, 2n + 1, \ldots\}$, we note that $x \mid y$ if and only if $y - x = 0$, or $y - x \geq n$ and $y - x \neq 2n - 1$. We first see that $\omega(x) \leq k + 3$ by noting that $(k + 3)n - x \geq 2n$, and therefore x divides the sum of any $k + 3$ atoms.

Suppose that there is a sum of $k + 3$ atoms such that x does not divide any subsum of $k + 2$ atoms. Since $(k + 2)n - x \geq n$, this is possible if and only if the difference between any subsum of $k + 2$ atoms and x is equal to $2n - 1$. In this case, all of the $k + 3$ atoms must be the same, say $n + t$ for $0 \leq t < n - 1$ and $(k + 2)(n + t) - x = 2n - 1$. This is possible if and only if $x = kn + (k + 2)t + 1$. In this case, we have $\omega(x) \geq k + 3$, and therefore $\omega(x) = k + 3$. Otherwise, if x divides $k + 3$ atoms, then it divides some subsum of $k + 2$ atoms and $\omega(x) \leq k + 2$.

We suppose that there is no t such that $x = kn + (k + 2)t + 1$, so in particular $x \neq kn + 1$, and show that $\omega(x) \geq k + 2$. If $x = (k + 1)n$, then $x \mid kn + 2(n + 1)$, but x does not divide any subsum of $k + 1$ atoms. If $x \not\equiv 0, 1 \pmod{n}$, then $x \mid (k + 2)n$, but $(k + 1)n - x \in [1, n - 2] \notin S$. Therefore $\omega(x) \geq k + 2$, and so $\omega(x) = k + 2$.

For the second assertion, no matter the parity of n, we have $kn + 2 \in S$ for all $k \geq 2$. By our formula, $\omega(kn + 2) = k + 2$, and hence $\{4, 5, 6, 7, \ldots\} \subseteq \omega(S)$. If n is even, then easy computations yield $\omega(n) = 2$ and $\omega(n + 2) = 3$, which completes the top formula. If n is odd, then $\omega(n) = 3$. Since $k > 0$ for all other elements of S, the second formula follows. □

We close this section by showing that $\omega(S)$ does not always consist of an *interval* of integers. We will first require a proposition.

Proposition 3.3. *Let $S = \langle n_1, \ldots, n_t \rangle$ be primitive with minimal set of generators $\{n_1, \ldots, n_t\}$. Let t be the Frobenius number of S. Define F, $L_S(F)$, and $T(k)$ as follows:*

- $F = \langle t + 1, t + 2, \ldots, 2(t + 1) - 1 \rangle$,
- $L_S(F) = \{L_S(x) \ : \ x \in F \setminus \{0\}\}$ *(where $L_S(x)$ denotes the longest length of x in S),*
- *For any positive integer k, let $T(k)$ be the set of all x in $\{t + 1, t + 2, \ldots, 2(t + 1) - 1\}$ such that $k \geq L_S(x)$.*

A positive integer k is not in $\omega(S)$ if all of the following are true:

(1) $k \notin \omega(S \setminus F)$,
(2) $k \notin \omega(T(k))$, *and*
(3) *If $T(k)$ is not empty, then $k < 2 \cdot \min L_S(T(k))$.*

Proof. Let S, t, F, $L_S(F)$, and $T(k)$ be as defined above. It is clear that $F = \{t+1, t+2, t+3, \dots\} \subset S$. Suppose k is a positive integer that satisfies the above criteria. Then it is clear from criterion (1) that $k \notin \omega(S \setminus F)$. Using criteria (2) and (3) we will show that $k \notin \omega(F)$.

The minimal set of generators for F is $\{t+1, t+2, \dots, 2(t+1)-1\}$. Since every $x \in F$ is a linear combination of the generators of F, it is clear that $L_S(x) \geq \min L_S(\{t+1, t+2, \dots, 2(t+1)-1\})$. Thus we have $\min L_S(\{t+1, t+2, \dots, 2(t+1)-1\}) = \min L_S(F)$. Since $\omega(x) \geq L_S(x)$, we have $\omega(x) \geq \min L_S(\{t+1, t+2, \dots, 2(t+1)-1\}) = \min L_S(F)$ for any $x \in F$.

If $T(k)$ is empty, then for all $x \in \{t+1, t+2, \dots, 2(t+1)-1\}$ we have $k < L_S(x)$. Thus, $k < \min L_S(\{t+1, t+2, \dots, 2(t+1)-1\}) = \min L_S(F) \leq \omega(x)$ for all $x \in F$. Hence, $k \notin \omega(F)$.

Now, suppose $T(k)$ is not empty. We have that $k \notin \omega(\{t+1, t+2, \dots, 2(t+1)-1\} \setminus T(k))$ because $k < L_S(x) \leq \omega(x)$ for any $x \in \{t+1, t+2, \dots, 2(t+1)-1\} \setminus T(k)$. From criterion (2), we have $k \notin \omega(T(k))$, and so $k \notin \omega(\{t+1, t+2, \dots, 2(t+1)-1\})$. Now, for $x \in F \setminus \{t+1, t+2, \dots, 2(t+1)-1\}$, it is clear that $L_S(x) \geq 2 \cdot \min L_S(\{t+1, t+2, \dots, 2(t+1)-1\})$. Criterion (3) gives us $k < 2 \cdot \min L_S(T(k)) = 2 \cdot \min L_S(\{t+1, t+2, \dots, 2(t+1)-1\})$, which implies $k \notin \omega(F \setminus \{t+1, t+2, \dots, 2(t+1)-1\})$. Hence, $k \notin \omega(F)$. □

Example 3.2. Let $S = \langle 9, 29, 39\rangle$. Computer data indicates that 4 and 5 are not in $\omega(S)$, so let $k \in \{4, 5\}$. The Frobenius number of S is 127. Using the definitions given in the above proposition, we have:

- $F = \langle 128, 129, \dots, 255\rangle = \{128, 129, 130, \dots\}$,
- $L_S(\{128, 129, \dots, 255\}) = \{4, 5, 6, \dots, 28\}$,
- $T(4) = \{136\}$ and $T(5) = \{136, 145\}$.

We check the three criteria of the above proposition on 4 and 5:

(1) $4, 5 \notin \omega(S \setminus F) = \{3, 6, 7, 8, \dots, 22, 23, 24, 27, 28, 29\}$,
(2) $4 \notin \omega(T(4)) = \omega(\{136\}) = \{27\}$ and $5 \notin \omega(T(5)) = \omega(\{136, 145\}) = \{27, 28\}$,
(3) $T(4)$ and $T(5)$ are not empty and we have $4 < 2 \cdot \min L_S(T(4)) = 2 \cdot L_S(136) = 8$ and $5 < 2 \cdot \min L_S(T(5)) = 2 \cdot \min\{136, 145\} = 2 \cdot L_S(136) = 8$.

Since 4 and 5 satisfy the criteria, we have that $4, 5 \notin \omega(\langle 9, 29, 39\rangle)$, but as indicated above, $3 \in \omega(S)$.

4. ω-Measure in 2-generator Numerical Monoids

Throughout this section, we will be dealing with the special class of numerical monoids generated by two elements. Thus, we let $S = \langle n_1, n_2\rangle$ be primitive with $1 < n_1 < n_2$. We determine an exact formula for $\omega(x)$ in Theorem 4.4, but we first require three Lemmas.

Lemma 4.1. *Let $x \in S \backslash S^{\times}$, and let $(d_1, d_2) \in \mathcal{D}(x)$ such that $(d_1, d_2) \notin \mathcal{F}(x) \cup \{(m_1(x), 0), (0, m_2(x))\}$. Then, at least one of the following is true:*

(1) *There exists $(x_1, x_2) \in \mathcal{F}(x)$ such that (d_1, d_2) subsumes (x_1, x_2),*
(2) *(d_1, d_2) subsumes $(m_1(x), 0)$, or*
(3) *(d_1, d_2) subsumes $(0, m_2(x))$.*

Proof. Let $x \in S \setminus S^{\times}$, and let $(d_1, d_2) \in \mathcal{D}(x)$ such that $(d_1, d_2) \notin \mathcal{F}(x) \cup \{(m_1(x), 0), (0, m_2(x))\}$. Then $d_1n_1 + d_2n_2 - x = c_1n_1 + c_2n_2 \in S$. If there is an $(x_1, x_2) \in \mathcal{F}(x)$ such that $x_1 \leq d_1$ and $x_2 \leq d_2$, then we are done. So suppose that there is no such (x_1, x_2). Then for any given $(x_1, x_2) \in \mathcal{F}(x)$, either $x_1 > d_1$ and $x_2 < d_2$, or $x_2 > d_2$ and $x_1 < d_1$.

First, suppose that $x_1 > d_1$ and $x_2 < d_2$. Then $d_1 - x_1 < 0$ and $d_2 - x_2 > 0$. Since $(d_1 - x_1)n_1 + (d_2 - x_2)n_2 = c_1n_1 + c_2n_2 \in S$, there exists a positive integer k such that $c_1 = d_1 - x_1 + kn_2 \geq 0$ and $c_2 = d_2 - x_2 - kn_1 \geq 0$. Then $d_2 = c_2 + x_2 + kn_1$, and so $d_2n_2 - x = (c_2 + x_2 + kn_1)n_2 - (x_1n_1 + x_2n_2) = c_2n_2 + (kn_2 - x_1)n_1$. Because $c_2 \geq 0$, if we can show that $kn_2 - x_1 \geq 0$, then we get $d_2n_2 - x \in S$. Suppose $kn_2 - x_1 < 0$. Let $a_1 = x_1 - kn_2$ and $a_2 = x_2 + kn_1$. Then $(a_1, a_2) \in \mathcal{F}(x)$ such that $d_1 - a_1 \geq 0$ and $d_2 - a_2 \geq 0$, which is a contradiction. Thus, we have $kn_2 - x_1 > 0$, and so $d_2n_2 - x \in S$. From the definition of $m_2(x)$, it follows that $m_2(x) \leq d_2$, and so (d_1, d_2) subsumes $(0, m_2(x))$.

Now, suppose that $x_2 > d_2$ and $x_1 < d_1$. An argument similar to the above gives us $m_1(x) \leq d_1$, and thus (d_1, d_2) subsumes $(m_1(x), 0)$. □

Lemma 4.2. *Let $x \in S \setminus S^\times$. Then* $\max\{x_1 + x_2 : \ (x_1, x_2) \in \mathcal{F}(x)\} \leq m_1(x)$.

Proof. Let $x \in S \setminus S^\times$. Let $(x_1, x_2) \in \mathcal{F}(x)$ such that $\max\{x_1 + x_2 : \ (x_1, x_2) \in \mathcal{F}(x)\} = x_1 + x_2$. Then $x_1 + x_2 = L(x)$, the longest factorization length of x, and $x_2 < n_1$. If $x_2 = 0$, then $x = x_1n_1$ and $L(x) = x_1$. By the definition of $m_1(x)$, we have $m_1(x) = x_1 = L(x)$. Suppose that $x_2 > 0$. Since $0 < n_1 - x_2$ and $m_1(x)n_1 - x = m_1(x)n_1 - (x_1n_1 + x_2n_2) = (m_1(x) - x_1)n_1 + (-x_2)n_2 = (m_1(x) - x_1 - n_2)n_1 + (n_1 - x_2)n_2 \in S$, we have $m_1(x) - x_1 - n_2 \geq 0$. It follows from the minimality of $m_1(x)$ that $x_1 + n_2 = m_1(x)$. Now, since $x_2 < n_1 < n_2$, we get $L(x) = x_1 + x_2 < x_1 + n_2 = m_1(x)$. □

Lemma 4.3. *Let $x \in S \setminus S^\times$. Then $\omega(x) = \max\ \{m_1(x),\ m_2(x)\}$.*

Proof. Let $x \in S \setminus S^\times$. It is clear that $\mathcal{F}(x) \cup \{(m_1(x), 0), (0, m_2(x))\}$ is a subset of $\mathcal{Y}(x)$. Let $(d_1, d_2) \in \mathcal{D}(x)$ such that $(d_1, d_2) \notin \mathcal{F}(x) \cup \{(m_1(x), 0), (0, m_2(x))\}$. By Lemma 4.1, (d_1, d_2) subsumes some element of $\mathcal{F}(x) \cup \{(m_1(x), 0), (0, m_2(x))\}$. This implies that $(d_1, d_2) \notin \mathcal{Y}(x)$, and so $\mathcal{F}(x) \cup \{(m_1(x), 0), (0, m_2(x))\} = \mathcal{Y}(x)$. Finally, we get $\omega(x) = \max\ \{m_1(x),\ m_2(x)\}$ by applying Lemma 4.2. □

Theorem 4.4. *Let $x \in S \setminus S^\times$ and $(x_1, x_2) \in \mathcal{F}(x)$. Then,*

$$\omega(x) = \max\left\{ \left\lceil \frac{x_2}{n_1} \right\rceil n_2 + x_1,\ \ \left\lceil \frac{x_1}{n_2} \right\rceil n_1 + x_2 \right\}.$$

Proof. Let $x \in S \setminus S^\times$ and $(x_1, x_2) \in \mathcal{F}(x)$. Then $x = x_1n_1 + x_2n_2$. It follows from the definition of $m_1(x)$ that $m_1(x)n_1 - x = y_2n_2$, where $y_2 < n_1$. Then, $m_1(x)n_1 - x = m_1(x)n_1 - (x_1n_1 + x_2n_2) = (m_1(x) - x_1)n_1 - x_2n_2 = y_2n_2 \in S$ implies $m_1(x) - x_1 \geq 0$. Since $n_2 \mid (m_1(x) - x_1)n_1$ and $\gcd(n_1, n_2) = 1$, we have $n_2 \mid (m_1(x) - x_1)$. Thus, there exists a nonnegative integer q such that $m_1(x) - x_1 = qn_2$, and so $m_1(x) = qn_2 + x_1$. Then $m_1(x)n_1 - x = (qn_2 + x_1)n_1 - (x_1n_1 + x_2n_2) = (qn_1 - x_2)n_2 \in S$ implies $qn_1 - x_2 \geq 0$. Since $q \geq \frac{x_2}{n_1}$ and q is an integer, we get $q = \lceil \frac{x_2}{n_1} \rceil$. Hence, $m_1(x) = qn_2 + x_1 = \lceil \frac{x_2}{n_1} \rceil n_2 + x_1$. A similar proof shows that $m_2(x) = \lceil \frac{x_1}{n_2} \rceil n_1 + x_2$. □

Theorem 4.5. *Let $x \in S \setminus S^\times$. Then $\omega(x) = x$ if and only if $x \in \{n_1, n_2\}$.*

Proof. Theorem 4.4 implies $\omega(n_1) = m_2(n_1) = n_1$ and $\omega(n_2) = m_1(n_2) = n_2$. For the converse, suppose that $m_1(x) = x$. We have $m_1(x) = \lceil \frac{x_2}{n_1} \rceil n_2 + x_1 = x_2n_2 + x_1n_1$. We can write $x_2 = kn_1 + r$ for unique integers $k \geq 0$ and $0 \leq r \leq n_1 - 1$. Then $\lceil \frac{x_2}{n_1} \rceil n_2 + x_1 = (k + \lceil \frac{r}{n_1} \rceil)n_2 + x_1 = (kn_1 + r)n_2 + x_1n_1$. We have $(k(n_1 - 1) + r - \lceil \frac{r}{n_1} \rceil)n_2 + x_1(n_1 - 1) = 0$. Therefore $k = x_1 = 0$ and $r = \lceil \frac{r}{n_1} \rceil$. Since $x \neq 0$ we cannot have $r = 0$ as well; so $r = 1$ and $x = n_2$.

A very similar argument shows $m_2(x) = x$ implies $x = n_1$. So $x \notin \{n_1, n_2\}$ implies $\omega(x) \neq x$. □

Theorem 4.7 will describe the set $\omega(S)$. Its proof follows immediately from the following Lemma.

Lemma 4.6. *Let S be a primitive numerical monoid of the form $S = \langle n_1, n_2 \rangle$.*

(i) *If $x \in S$ and $\omega(x) < n_1$, then $x = 0$.*
(ii) *If k is an integer such that $0 \le k \le n_2 - n_1$, then $\omega((n_1 + k)n_1) = n_1 + k$.*
(iii) *If $k \ge 0$, then $\omega(kn_1 + n_2) = n_2 + k$.*

Proof. (i) Theorem 4.4 implies $\omega(x) \ge \lceil \frac{x_2}{n_1} \rceil n_2 + x_1$; so $\omega(x) < n_1$ implies $x_2 = 0$. Similarly $\omega(x) \ge \lceil \frac{x_1}{n_2} \rceil n_1 + x_2$; so $\omega(x) < n_1$ implies $x_1 = 0$. Therefore, $\omega(x) < n_1$ implies $x = 0$.

(ii) Let k be an integer such that $0 \le k \le n_2 - n_1$. We will consider elements of the form $(n_1+k)n_1$ in S.

Let $x_1 = n_1 + k$ and $x_2 = 0$; then $(x_1, x_2) \in \mathcal{F}((n_1 + k)n_1)$. According to Theorem 4.4, we have

$$m_1((n_1 + k)n_1) = \left\lceil \frac{x_2}{n_1} \right\rceil n_2 + x_1 = \left\lceil \frac{0}{n_1} \right\rceil n_2 + (n_1 + k) = n_1 + k,$$
$$m_2((n_1 + k)n_1) = \left\lceil \frac{x_1}{n_2} \right\rceil n_1 + x_2 = \left\lceil \frac{n_1 + k}{n_2} \right\rceil n_1.$$

We note that $\lceil \frac{n_1}{n_2} \rceil \le \lceil \frac{n_1+k}{n_2} \rceil \le \lceil \frac{n_2}{n_2} \rceil$; so $m_2((n_1 + k)n_1) = n_1$ and $\omega((n_1 + k)n_1) = n_1 + k$.

(iii) Let $k \ge 0$ be an integer. Consider the element $kn_1 + n_2 \in S$. Let $x_1 = k$ and $x_2 = 1$. Then $(x_1, x_2) \in \mathcal{F}(kn_1 + n_2)$. According to Theorem 4.4, we have

$$m_1(kn_1 + n_2) = \left\lceil \frac{x_2}{n_1} \right\rceil n_2 + x_1 = \left\lceil \frac{1}{n_1} \right\rceil n_2 + k = n_2 + k,$$
$$m_2(kn_1 + n_2) = \left\lceil \frac{x_1}{n_2} \right\rceil n_1 + x_2 = \left\lceil \frac{k}{n_2} \right\rceil n_1 + 1.$$

Since $\left\lceil \frac{k}{n_2} \right\rceil n_1 + 1 < \left(\frac{k}{n_2} + 1 \right) n_1 + 1 < k + (n_1 + 1) \le k + n_2$, we have $\omega(kn_1 + n_2) = n_2 + k$. □

The next theorem follows immediately.

Theorem 4.7. $\omega(S) = \{n_1, n_1 + 1, n_1 + 2, \dots\}$.

We close by showing that in the 2-generator case, the ω function has nice asymptotic behavior. We will first require a lemma. The existence of the limit below is guaranteed by [1, Theorem 2.8].

Lemma 4.8. *Consider $tn_1 \in S \setminus S^\times$ and $tn_2 \in S \setminus S^\times$ for $t \in \mathbb{N}$.*

(1) $\lim_{t\to\infty} \frac{\omega(tn_1)}{t} = 1$, *and there exists a $T \in \mathbb{N}$ such that $\omega(tn_1) = t$ for all $t > T$.*
(2) $\lim_{t\to\infty} \frac{\omega(tn_2)}{t} = \frac{n_2}{n_1}$.

Proof. For the first statement, consider elements of the form tn_1, where $t \in \mathbb{N}$. It follows from the definition that $m_1(tn_1) = t$. Applying Theorem 4.4, we have $m_2(tn_1) = \lceil \frac{t}{n_2} \rceil n_1 \le (t+1)\frac{n_1}{n_2}$, which is less than t whenever $t > \frac{n_1}{n_2 - n_1}$. So $\lim_{t\to\infty} \frac{\omega(tn_1)}{t} = \lim_{t\to\infty} \frac{m_1(tn_1)}{t} = 1$.

For the second statement, consider elements of the from tn_2, where $t \in \mathbb{N}$. It follows from the definition that $m_2(tn_2) = t$. Applying Theorem 4.4, we have $m_1(tn_2) = \lceil \frac{t}{n_1} \rceil n_2 \ge \frac{tn_2}{n_1} > t$. We have $\frac{n_2}{n_1} \le \frac{m_1(tn_2)}{t} \le \frac{n_2}{n_1} + \frac{n_2}{t}$. So $\lim_{t\to\infty} \frac{\omega(tn_2)}{t} = \lim_{t\to\infty} \frac{m_1(tn_2)}{t} = \frac{n_2}{n_1}$. □

Theorem 4.9. *Let $x \in S \setminus S^\times$. Then* $\lim_{x\to\infty} \frac{\omega(x)}{x} = \frac{1}{n_1}$.

Proof. We may ignore all $x \in S \setminus S^{\times}$ except for $x > n_1n_2 + n_1$.

Let $q(x) = \left\lfloor \frac{x-n_1n_2}{n_1} \right\rfloor$. From $x > n_1n_2 + n_1$, we know that $q(x) \geq 1$. We also know that $x - n_1n_2 = q(x)n_1 + r(x)$ for some $r(x)$ satisfying $0 \leq r(x) < n_1$. Thus, $x = q(x)n_1 + (n_1n_2 + r(x))$. Since $n_1n_2 + r(x)$ is greater than the Frobenius number of S, which is $n_1n_2 - n_1 - n_2$, it is clear that $n_1n_2 + r(x) \in S$ and $q(x)n_1 \in S$. By Proposition 2.1, we have $\omega(x) \leq \omega(q(x)n_1) + \omega(n_1n_2 + r(x))$.

Now let $q'(x) = \left\lfloor \frac{x+n_1n_2}{n_1} \right\rfloor$. We likewise see that $q'(x) \geq 2n_2+1$ and that $x+n_1n_2 = q'(x)n_1+r'(x)$ for some $r'(x)$ satisfying $0 \leq r'(x) < n_1$. Thus, $q'(x)n_1 = x + (n_1n_2 - r'(x))$. We need only note that $n_1n_2 - r'(x)$ is again greater than the Frobenius number of S, and thus we have $\omega(q'(x)n_1) \leq \omega(x) + \omega(n_1n_2 - r'(x))$.

We now set $B = \max\ \{\omega(x) \mid x \in S,\ x \leq n_1n_2 + n_1\}$. Then we have

$$\omega(q'(x)n_1) - \omega(n_1n_2 - r'(x)) \leq \omega(x) \leq \omega(q(x)n_1) + \omega(n_1n_2 + r(x)).$$

Thus, $\omega(q'(x)n_1) - B \leq \omega(x) \leq \omega(q'(x)n_1) + B$, and so we have the following

$$\frac{\omega(q'(x)n_1)}{x} - \frac{B}{x} \leq \frac{\omega(x)}{x} \leq \frac{\omega(q(x)n_1)}{x} + \frac{B}{x}. \tag{1}$$

We observe that $\lim\limits_{x\to\infty} \dfrac{q(x)}{x} = \lim\limits_{x\to\infty} \dfrac{q'(x)}{x} = \dfrac{1}{n_1}$. We also see that $q(x), q'(x) \to \infty$ as $x \to \infty$, and thus by Lemma 4.8, $\lim\limits_{x\to\infty} \dfrac{\omega(q(x)n_1)}{q(x)} = \lim\limits_{x\to\infty} \dfrac{\omega(q'(x)n_1)}{q'(x)} = \lim\limits_{x\to\infty} \dfrac{\omega(xn_1)}{x} = 1$. We then see that

$$\lim_{x\to\infty} \frac{\omega(q'(x)n_1)}{x} - \frac{B}{x} = \lim_{x\to\infty} \left(\frac{\omega(q'(x)n_1)}{q'(x)}\right)\left(\frac{q'x}{x}\right) - \frac{B}{x} = (1)\left(\frac{1}{n_1}\right) - 0 = \frac{1}{n_1}.$$

Likewise, we have

$$\lim_{x\to\infty} \frac{\omega(q(x)n_1)}{t} + \frac{B}{x} = \lim_{x\to\infty} \left(\frac{\omega(q(x)n_1)}{q(x)}\right)\left(\frac{q(x)}{x}\right) + \frac{B}{x} = (1)\left(\frac{1}{n_1}\right) + 0 = \frac{1}{n_1}.$$

By (1), we then see that $\lim\limits_{x\to\infty} \dfrac{\omega(x)}{x} = \dfrac{1}{n_1}$. □

References

[1] D. F. Anderson and S. T. Chapman, How far is an element from being prime?, to appear in *J. Algebra Appl.*

[2] C. Bowles, S. T. Chapman, N. Kaplan and D. Reiser, On Delta sets of numerical monoids, *J. Algebra Appl.* **5**(2006), 1–24.

[3] S. T. Chapman, M. Holden and T. Moore, On full elasticity in atomic monoids and integral domains, *Rocky Mountain J. Math.* **36**(2006), 1437–1455.

[4] S. T. Chapman, N. Kaplan and R. Hoyer, Delta sets of numerical monoids are eventually periodic, *Aequationes Math* **77**(2009), 273–279.

[5] S. T. Chapman, N. Kaplan, T. Lemburg, A. Niles and C. Zlogar, Shifts of generators and delta sets of numerical monoids, to appear in *J. Comm. Algebra*.

[6] P.A. García-Sánchez and J.C. Rosales, Finitely Generated Commutative Monoids, Nova Science Publishers, Commack, New York, 1999.

[7] A. Geroldinger and F. Halter-Koch, *Non-unique Factorizations: Algebraic, Combinatorial and Analytic Theory*, Pure and Applied Mathematics, vol. 278, Chapman & Hall/CRC, 2006.

[8] A. Geroldinger and W. Hassler, Local tameness of v-Noetherian monoids, *J. Pure Applied Algebra* **212**(2008), 1509–1524.

[9] M. Omidali, The catenary and tame degree of certain numerical monoids, preprint.

[10] W. Schmid, Differences in sets of lengths in Krull monoids with finite class group, *J. Théor. Nombres Bordeaux* **17**(2005), 323–345.

University of Tennessee, Department of Mathematics, Knoxville, TN 37996-1300

E-mail address: anderson@math.utk.edu

Sam Houston State University, Department of Mathematics and Statistics, Box 2206, Huntsville, TX 77341-2206

E-mail address: scott.chapman@shsu.edu

Princeton University, Department of Mathematics, Princeton NJ 08544-1000

E-mail address: NKaplan@math.harvard.edu

Current address: Harvard University, Department of Mathematics, One Oxford Street, Cambridge MA 02138

University of Richmond, Department of Mathematics, 28 Westhampton Way, Richmond, Virginia 23173

E-mail address: dtork@berkeley.edu

Current address: University of California at Berkeley, Department of Industrial Engineering and Operations Research, 4141 Etcheverry Hall, Mail Code 1777, Berkeley, CA 94720-1777